Multiple Models – Fixed, Switching, Interacting

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Abstract. In dynamic models the dynamic and the observation equations are based on a known system model. The multiple model approach introduces uncertainties about the system model by a set of possible system models. In the multiple model approach for fixed models the true system does not change during the whole observation process, wheareas in the approach for switching models a jump from one model to another is allowed. In the later case the state estimation usually has to be approximated, e.g. by so-called interacting multiple models. The multiple model approach for fixed, switching and interacting models are presented and their application for GNSS ambiguity resolution is discussed, but open questions still remain.

Keywords. Bayesian statistics, ambiguity resolution, model uncertainty, recursive estimation, multiple models, interacting multiple models

1 Introduction

The multiple model is a generalization of a dynamic system. In the dynamic system the system model, that defines the dynamic and the observation equations, is considered as known, so that the dynamic system can be interpreted as a single model approach. Compared to it the multiple model approach provides the dynamic system with model uncertainties by offering a whole set of possible system models. In case of fixed models we assume that the true model is one of the offered system models and that it does not change during the measurement process. Further model uncertainty is modeled by the switching model approach that takes dynamic system models into account that can switch from one to another. In figure 1 the possible model sequencies for a single model approach and multiple model approaches with two possible models are shown from time $k-2$ to time k. For the state estimation all possible model sequencies have to be taken into consideration. In case of switching models the number of possibilities rapidly grows with time, so that usually an approximation is necessary, e.g. the interacting multiple model technique. The multiple model approach was originally presented by Magill (1965). The approach for fixed models was extended to switching models in the 70's. The approximation by interacting models was proposed in (Blom 1984) and (Blom and Bar-Shalom 1988). For GNSS ambiguity resolution the integer ambiguities define the set of system models. The technique of fixed multiple models was applied for ambiguity resolution already by Brown and Hwang (1983) as multiple model adaptive estimation or Magill adaptive filter and modified in Henderson (2001). Wolfe et al. (2001) refer to the multiple model approach for fixed models as multiple hypothesis Wald sequential probability ratio test. Estimating the GNSS ambiguities in the multiple model approach for fixed models leads recursively to the same results as the Bayesian approach, see (Betti et al. 1993) and (Gundlich, Koch 2002), that introduces the ambiguities as discrete integer random variables. In the context with ambiguity resolution the multiple model approach for switching models is used to model cycle slips, see Chen and Harigae (2001) and Wolfe et al. (2001). In the following an overview of the multiple model approaches is given in the context of ambiguity resolution.

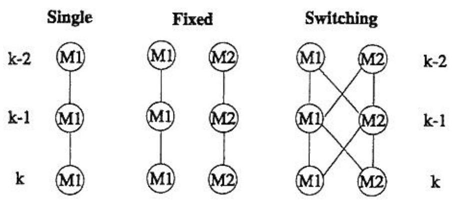

Fig. 1 Single Model, Fixed Models, Switching Models

2 Theory

2.1 Bayesian Statistics

In the Bayesian approach the unknown parameters $\boldsymbol{x}$ are interpreted as random variables. Their probability density function describes the information of the unknown parameters: the prior density $p(\boldsymbol{x})$ expresses the knowledge of the unknown parameters $\boldsymbol{x}$ without considering the observations $\boldsymbol{y}$, the posterior density $p(\boldsymbol{x}|\boldsymbol{y})$ of the parameters $\boldsymbol{x}$ summarizes all information given the observations $\boldsymbol{y}$. The posterior density results from the Bayes' theorem, see for example (Koch 2000, p. 32),

$$p(\boldsymbol{x}|\boldsymbol{y}) = \frac{p(\boldsymbol{x})p(\boldsymbol{y}|\boldsymbol{x})}{p(\boldsymbol{y})} \propto p(\boldsymbol{x})p(\boldsymbol{y}|\boldsymbol{x}) \tag{1}$$

that follows immediately from

$$p(\boldsymbol{x},\boldsymbol{y}) = p(\boldsymbol{y})p(\boldsymbol{x}|\boldsymbol{y}) = p(\boldsymbol{x})p(\boldsymbol{y}|\boldsymbol{x}) \ .$$

Maximizing the posterior density function leads to the MAP(maximum a posteriori) estimate

$$\hat{\boldsymbol{x}}_{MAP} = \arg\max_{\boldsymbol{x}\in\boldsymbol{X}} p(\boldsymbol{x}|\boldsymbol{y}) \ . \tag{2}$$

If a quadratic loss function is defined, see e.g. (Koch 2000, p. 65f), the Bayes estimate results in the expectation

$$\hat{\boldsymbol{x}}_B = E\{\boldsymbol{x}|\boldsymbol{y}\} \ . \tag{3}$$

2.2 Single Model

The linear dynamic system at time k consists of the dynamic equation

$$\boldsymbol{x}(k) = \boldsymbol{\Phi}(k)\boldsymbol{x}(k-1) + \boldsymbol{d}(k-1) \tag{4}$$

and the observation equation

$$\boldsymbol{y}(k) = \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{e}(k) \tag{5}$$

with the state $\boldsymbol{x}(.)$, the known transition matrix $\boldsymbol{\Phi}(.)$, the Gaussian system noise $\boldsymbol{d}(.) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_d(.))$, the observations $\boldsymbol{y}(.)$, the known design matrix $\boldsymbol{A}(.)$ and the Gaussian measurement noise $\boldsymbol{e}(.) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}(.))$. The Gaussian system and measurement noise are not time correlated and not correlated to each other. State estimation in linear dynamic systems is recursively solved by the Kalman filter, that we shortly derive in a Bayesian approach.

Initialization: Using a non-informative prior, $p(\boldsymbol{x}(1)) \propto const.$, the Bayes' theorem (1) leads to the normal distributed state $\boldsymbol{x}(1)$ given the observations $\boldsymbol{y}(1)$, short $\boldsymbol{x}(1)|\boldsymbol{y}(1)$,

$$\boldsymbol{x}(1)|\boldsymbol{y}(1) \sim \mathcal{N}\Big(\hat{\boldsymbol{x}}(1|1), \boldsymbol{Q}_x(1|1)\Big)$$

with covariance matrix

$$\boldsymbol{Q}_x(1|1) = \Big(\boldsymbol{A}(1)'\boldsymbol{Q}(1)^{-1}\boldsymbol{A}(1)\Big)^{-1}$$

and expectation

$$\hat{\boldsymbol{x}}(1|1) = \boldsymbol{Q}_x(1|1)\boldsymbol{A}(1)'\boldsymbol{Q}(1)^{-1}\boldsymbol{y}(1) \ .$$

Time update (t-update): For the recursive state estimation the normal distributed state $\boldsymbol{x}(k-1)|\boldsymbol{Y}(k-1)) \sim \mathcal{N}(\hat{\boldsymbol{x}}(k-1|k-1), \boldsymbol{Q}_x(k-1|k-1))$ with $\boldsymbol{Y}(k-1) = \{\boldsymbol{y}(1), \ldots, \boldsymbol{y}(k-1)\}$ is time updated. Because of the linear dynamic equation (4) the prediction leads to the normal density

$$\boldsymbol{x}(k)|\boldsymbol{Y}(k-1) \sim \mathcal{N}\Big(\hat{\boldsymbol{x}}(k|k-1), \boldsymbol{Q}_x(k|k-1)\Big)$$

with expectation and covariance matrix,

$$\hat{\boldsymbol{x}}(k|k-1) = \boldsymbol{\Phi}(k)\hat{\boldsymbol{x}}(k-1|k-1) \ ,$$

$$\boldsymbol{Q}_x(k|k-1)$$
$$= \boldsymbol{\Phi}(k)\boldsymbol{Q}_x(k-1|k-1)\boldsymbol{\Phi}(k)' + \boldsymbol{Q}_d(k) \ .$$

Measurement update (m-update): The measurement update is obtained from the recursive Bayes' theorem, see (1),

$$p\Big(\boldsymbol{x}(k)|\boldsymbol{y}(k), \boldsymbol{Y}(k-1)\Big) \propto \tag{6}$$
$$p\Big(\boldsymbol{y}(k)|\boldsymbol{x}(k), \boldsymbol{Y}(k-1)\Big)p\Big(\boldsymbol{x}(k)|\boldsymbol{Y}(k-1)\Big)$$

with the density of the time updated state $p(\boldsymbol{x}(k)|\boldsymbol{Y}(k-1))$ acting as prior and the Gaussian likelihood function $\boldsymbol{y}(k)|\boldsymbol{x}(k) \sim \mathcal{N}(\boldsymbol{A}(k)\boldsymbol{x}(k), \boldsymbol{Q}(k))$, see (5). That leads to the normal distribution

$$\boldsymbol{x}(k)|\boldsymbol{Y}(k) \sim \mathcal{N}\Big(\hat{\boldsymbol{x}}(k|k), \boldsymbol{Q}_x(k|k)\Big) \tag{7}$$

with expectation $\hat{\boldsymbol{x}}(k|k)$ and covariance matrix $\boldsymbol{Q}_x(k|k)$ computed as follows

$$\boldsymbol{v}(k) = \boldsymbol{y}(k) - \boldsymbol{A}(k)\hat{\boldsymbol{x}}(k|k-1) \ , \tag{8}$$

$$\boldsymbol{Q}_v(k) = \boldsymbol{A}(k)\boldsymbol{Q}_x(k|k-1)\boldsymbol{A}(k)' + \boldsymbol{Q}(k) \ , \tag{9}$$

$$\boldsymbol{K}(k) = \boldsymbol{Q}_x(k|k-1)\boldsymbol{A}(k)'\boldsymbol{Q}_v(k)^{-1} \ ,$$

$$\hat{\boldsymbol{x}}(k|k) = \hat{\boldsymbol{x}}(k|k-1) + \boldsymbol{K}(k)\boldsymbol{v}(k) \ ,$$

$$\boldsymbol{Q}_x(k|k) = \Big(\boldsymbol{I} - \boldsymbol{K}(k)\Big)\boldsymbol{A}(k)\boldsymbol{Q}_x(k|k-1)$$

with the predicted residuals $v(k)$, their covariance matrix $Q_v(k)$ and the filter gain $K(k)$.

Results: Because of the normal distribution (7) the MAP (2) and the Bayes estimate (3) are the same,

$$\hat{x}_{MAP}(k|k) = \hat{x}_B(k|k) = \hat{x}(k|k) .$$

2.3 Fixed Models

In the single model case, that is solved via the Kalman filter, the system model defined by the transition and design matrix and noise is known. The multiple model approach deals with a model uncertainty about several system models. In the case of fixed models it is assumed that the true model M is one of the known system models $M_i, i = 1, \ldots, r$,

$$M \in \{M_1, \ldots, M_r\} .$$

That means for the linear dynamic model that transition matrix, design matrix and the noise depend on an unknown model M, thus

$$x(k) = \Phi(k, M)x(k-1) + u(k-1, M)$$

and

$$y(k) = A(k, M)x(k) + e(k, M) .$$

The common posterior density $p(x(k), M_i|Y(k))$ of state and model has to be derived. It can also be expressed as following

$$p\Big(x(k), M_i|Y(k)\Big)$$

$$= p\Big(x(k)|M_i, Y(k)\Big)P\Big(M_i|Y(k)\Big) .$$

Then we get the marginal state density

$$p\Big(x(k)|Y(k)\Big)$$

$$= \sum_{i=1}^{r} p\Big(x(k)|M_i, Y(k)\Big)P\Big(M_i|Y(k)\Big) \qquad (10)$$

as weighted sums of normal densities: Given the model M_i the state $x(k)$ is normal distributed, see the Kalman filter results in section 2.2,

$$x(k)|M_i, Y(k) \sim \mathcal{N}\Big(\hat{x}^i(k|k), Q_x^i(k|k)\Big) .$$

The weights are given by the model probabilities, that are estimated recursively.

Initialization: The starting probabilities $P(M_i|y(1)), i = 1, \ldots, r$ have to be chosen. If no other information is given and the state initialization provides no information over the probabilities, usually equal start probabilities are used.

Time update (t-update): No time update of the model probabilities is necessary, because the system models are assumed to be constant.

Measurement update (m-update): Using the Bayes' theorem (6) for the model probabilities we find

$$P\Big(M_i|Y(k)\Big) \propto$$

$$p\Big(y(k)|M_i, Y(k-1)\Big)P\Big(M_i|Y(k-1)\Big) \qquad (11)$$

which is normalized to fulfill the condition

$$\sum_{j=1}^{r} P\Big(M_j|Y(k)\Big) = 1 . \qquad (12)$$

The likelihood function $p(y(k)|M_i, Y(k-1))$ is replaced by the corresponding density value of the predicted residuals $v^i(k)$ with, see (8) and (9),

$$v(k)|M_i, Y(k-1) \sim \mathcal{N}\Big(0, Q_v^i(k)\Big) .$$

Results: The posterior state density (10) leads to the Bayes estimate (3)

$$\hat{x}_B(k|k) = \sum_{i=1}^{r} \hat{x}^i(k|k)P\Big(M_i|Y(k)\Big)$$

that is a mixture of all single model results weighted by their model probabilities. Maximizing the model probability leads to the MAP estimate (2)

$$\hat{x}_{MAP}(k|k) = \hat{x}^i(k|k),$$

with

$$i = \arg\max_{j=1,\ldots,r} P\Big(M_j|Y(k)\Big) .$$

The covariance matrix of the state follows from (10),

$$Q_x(k|k) = \sum_{i=1}^{r} P\Big(M_i|Y(k)\Big)\Big(Q_x^i(k|k)+$$

$$(\hat{x}_B(k|k) - \hat{x}^i(k|k))(\hat{x}_B(k|k) - \hat{x}^i(k|k))'\Big) .$$

2.4 Switching Models

The multiple model approach in now extended to switching models. The linear dynamic model is based on a system model that is allowed to change during the observation process. The true unknown system model can vary with time, but still is assumed to be one of a known model set,

$$M(k) \in \{M_1, \ldots, M_r\} . \qquad (13)$$

The probability of a jump from one model to another is given by the transition probability of the event $M(k) = M_j$, short $M_j(k)$, given $M(k-1) = M_i$ or $M_i(k-1)$,

$$p_{ij} = P\Big(M_j(k)|M_i(k-1)\Big) . \qquad (14)$$

It is assumed to be independent of time and state. The linear dynamic model for switching models is

$$\boldsymbol{x}(k) = \boldsymbol{\Phi}(k, M(k))\boldsymbol{x}(k-1) + \boldsymbol{u}(k-1, M(k)),$$

$$\boldsymbol{y}(k) = \boldsymbol{A}(k, M(k))\boldsymbol{x}(k) + \boldsymbol{e}(k, M(k)) .$$

In contrast to the multiple model approach for fixed models, we do not only have the uncertainty of one system model but the uncertainty of a whole sequence. At time k r^k different model sequences $S_l(k)$ are possible,

$$S_l(k) = \{M_{l_1}(1), \ldots, M_{l_k}(k)\}$$

with $l_k \in \{1, \ldots, r\}$ and $l = 1, \ldots, r^k$, see figure 2.

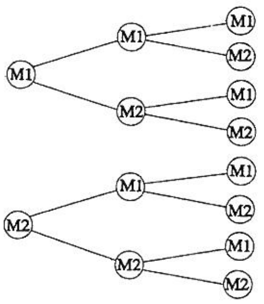

Fig. 2 Possible model sequences for $r = 2$, $k = 1, .., 3$

In analogy to (10) but with respect to all possible model sequences the posterior state density is expressed as marginal density

$$p\Big(\boldsymbol{x}(k)|\boldsymbol{Y}(k)\Big) =$$

$$\sum_{l=1}^{r^k} p\Big(\boldsymbol{x}(k)|S_l(k), \boldsymbol{Y}(k)\Big) P\Big(S_l(k)|\boldsymbol{Y}(k)\Big) . \qquad (15)$$

The conditional state densities given the model sequence $S_l(k)$ are Gaussian

$$\boldsymbol{x}(k)|S_l(k), \boldsymbol{Y}(k) \sim \mathcal{N}\Big(\hat{\boldsymbol{x}}^l(k|k), \boldsymbol{Q}_x^l(k|k)\Big)$$

whose parameters result from corresponding Kalman filters, see section 2.2. Besides a bank of r^k Kalman filters the update of the probabilities of the model sequences is necessary.

Initialization: In analogy to the fixed model approach the starting probabilities $P(M_i(1)|\boldsymbol{y}(1))$, $i = 1, \ldots, r$ have to be chosen. Further the transition probabilities p_{ij} (14) have to be introduced as priors.

Time update (t-update): The probabilities of sequences $S_l(k-1)$, $l = 1, \ldots, r^{k-1}$ at time $k-1$ are updated with the transition probability, thus

$$P\Big(S_l(k)|\boldsymbol{Y}(k-1)\Big) = p_{ij} P\Big(S_l(k-1)|\boldsymbol{Y}(k-1)\Big)$$

for $S_l(k-1) = \{M_{l_1}(1), \ldots, M_i(k-1)\}$ and $S_l(k) = \{S_l(k-1), M_j(k)\}$.

Measurement update (m-update): In analogy to (11) the Bayes' theorem is applied for the measurement update

$$P\Big(S_l(k)|\boldsymbol{Y}(k)\Big)$$

$$\propto p\Big(\boldsymbol{y}(k)|S_l(k), \boldsymbol{Y}(k-1)\Big) P\Big(S_l(k)|\boldsymbol{Y}(k-1)\Big),$$

which is normalized in analogy to (12). The density $p(\boldsymbol{y}(k)|S_l(k), \boldsymbol{Y}(k-1))$ is obtained from the density of the predicted residuals of the Kalman filter that corresponds to the model sequence $S_l(k)$.

Results: From the posterior state density (15) follows the Bayes estimate (3)

$$\hat{\boldsymbol{x}}_B(k|k) = \sum_{l=1}^{r^k} \hat{\boldsymbol{x}}^l(k|k) P\Big(S_l(k)|\boldsymbol{Y}(k)\Big)$$

and the MAP estimate (2)

$$\hat{\boldsymbol{x}}_{MAP}(k|k) = \hat{\boldsymbol{x}}^l(k|k),$$

with the maximal probability of a sequence

$$l = \arg \max_{i=1,\ldots,r^k} P\Big(S_i(k)|\boldsymbol{Y}(k)\Big) .$$

The covariance matrix of the state is also derived from the posterior density (15)

$$\boldsymbol{Q}_x(k|k) = \sum_{l=1}^{r^k} P\Big(S_l(k)|\boldsymbol{Y}(k)\Big) \Big(\boldsymbol{Q}_x^l(k|k) +$$

$$(\hat{\boldsymbol{x}}_B(k|k) - \hat{\boldsymbol{x}}^l(k|k))(\hat{\boldsymbol{x}}_B(k|k) - \hat{\boldsymbol{x}}^l(k|k))'\Big) .$$

Because of the fast increasing r^k, see figure 2, approximations of the posterior density and the estimate are necessary. An overview over the update of state and models for a single model, for fixed models and for switching models is shown in figure 3.

	state	model
Single		known
initialization	$p(x(1)\|y(1))$	–
	$p(x(k-1)\|Y(k-1))$	–
t – update	$\downarrow$	
	$p(x(k)\|Y(k-1))$	–
m – update	$\downarrow$	
	$p(x(k)\|Y(k))$	–
	(KF)	
Fixed $i=1,\dots,r$		$M \in \{M_1,\dots,M_r\}$
initialization	$p(x(1)\|M_i, y(1))$	$P(M_i\|y(1))$
	$p(x(k-1)\|M_i, Y(k-1))$	$P(M_i\|Y(k-1))$
t – update	$\downarrow$	$\|\|$
	$p(x(k)\|M_i, Y(k-1))$	$P(M_i\|Y(k-1))$
m – update	$\downarrow$	$\downarrow$
	$p(x(k)\|M_i, Y(k))$	$P(M_i\|Y(k))$
	(r **KF**)	
Switching $l=1,\dots r^k$		$M(k) \in \{M_1,\dots,M_r\}$
prior		p_{ij}
initialization	$p(x(1)\|S_l(1), y(1))$	$P(S_l(1)\|y(1))$
	$p(x(k-1)\|S_l(k-1), Y(k-1))$	$P(S_l(k-1)\|Y(k-1))$
t – update	$\downarrow$	$\downarrow$
	$p(x(k)\|S_l(k), Y(k-1))$	$P(S_l(k)\|Y(k-1))$
m – update	$\downarrow$	$\downarrow$
	$p(x(k)\|S_l(k), Y(k))$	$P(S_l(k)\|Y(k))$
	(l **KF**)	

Fig. 3 Single Model, Fixed Models, Switching Models

2.5 Interacting Models

The multiple model approach for switching models is approximated by interacting models. Instead of equation (15) the posterior state density is derived as marginal density of the common density of state and model $M(k)$,

$$p\Big(x(k)|\boldsymbol{Y}(k)\Big) = \qquad (16)$$

$$\sum_{i=1}^{r} p\Big(x(k)|M_i(k), \boldsymbol{Y}(k)\Big) P\Big(M_i(k)|\boldsymbol{Y}(k)\Big) \, .$$

In case of switching models, the conditional densities $p(x(k)|M_j(k), \boldsymbol{Y}(k))$, $j = 1, \dots, r$, are not Gaussian. The idea of interacting multiple models is to approximate the densities by normal distributions. This approximation is also used for the recursive estimation of the model probability. For explanation see the following considerations.

	state	model
IMM $i=1,\dots r$		$M(k) \in \{M_1,\dots,M_r\}$
prior		p_{ij}
initialization	$p(x(1)\|M_i(1), y(1))$	$P(M_i(1)\|y(1))$
	$p(x(k-1)\|M_i(k-1), Y(k-1))$	$P(M_i(k-1)\|Y(k-1))$
	interacting	mixing probabilities
	$p(x(k-1)\|M_i(k-1), Y(k-1))$	$P(M_i(k-1)\|Y(k-1))$
t – update	$\downarrow$	$\downarrow$
	$p(x(k)\|M_i(k), Y(k-1))$	$P(M_i(k)\|Y(k-1))$
m – update	$\downarrow$	$\downarrow$
	$p(x(k)\|M_i(k), Y(k))$	$P(M_i(k)\|Y(k))$
	(r interacting **KF**)	

Fig. 4 Interacting Multiple Models (IMM) approximating switching Models

Assume the normal distribution for

$$x(k-1)|M_i(k-1), \boldsymbol{Y}(k-1) \sim$$

$$\mathcal{N}\Big(\hat{x}^i(k-1|k-1), \boldsymbol{Q}_x^i(k-1|k-1)\Big),$$

which is valid for $k-1=1$.

Time update (t-update): The time update under the condition of $M_j(k)$ is derived with the Kalman filter and leads to the normal distribution

$$x(k)|M_i(k-1), M_j(k), \boldsymbol{Y}(k-1) \sim$$

$$\mathcal{N}\Big(\boldsymbol{x}^{i,j}(k|k-1), \boldsymbol{Q}_x^{i,j}(k|k-1)\Big) \, . \qquad (17)$$

The state density given only $M_j(k)$ is a Gaussian mixture

$$p\Big(\boldsymbol{x}(k)|M_j(k), \boldsymbol{Y}(k-1)\Big) = \qquad (18)$$

$$\sum_{i=1}^{r} p\Big(\boldsymbol{x}(k)|M_i(k-1), M_j(k), \boldsymbol{Y}(k-1)\Big)$$

$$P\Big(M_i(k-1)|M_j(k-1), \boldsymbol{Y}(k-1)\Big)$$

with the mixing probabilities $\mu_{i|j}(k-1) = P(M_i(k-1)|M_j(k-1), \boldsymbol{Y}(k-1))$ as weights. In analogy to the Bayes' theorem, see (1), and with $P(M_j(k)|M_i(k-1), \boldsymbol{Y}(k-1)) = p_{ij}$ we find the mixing probabilities

$$P\Big(M_i(k-1)|M_j(k-1), \boldsymbol{Y}(k-1)\Big) \propto$$

$$p_{ij} P\Big(M_i(k-1)|\boldsymbol{Y}(k-1)\Big) \, .$$

The Gaussian mixture (18) is now approximated by the Gaussian distribution

$$\boldsymbol{x}(k)|M_j(k), \boldsymbol{Y}(k-1)$$

$$\sim \mathcal{N}\Big(\hat{\boldsymbol{x}}^j(k|k-1), \boldsymbol{Q}_x^j(k|k-1)\Big) \qquad (19)$$

with parameters

$$\hat{\boldsymbol{x}}^j(k|k-1) = \sum_{i=1}^{r} \hat{\boldsymbol{x}}^{i,j}(k|k-1)\mu_{i|j}(k-1)$$

$$\boldsymbol{Q}_x^j(k|k-1) = \sum_{i=1}^{r} \mu_{i|j}(k-1)\Big(\boldsymbol{Q}_x^{i,j}(k|k-1) +$$

$$(\boldsymbol{\Delta x}^{i,j})(\boldsymbol{\Delta x}^{i,j})'\Big),$$

with $\boldsymbol{\Delta x}^{i,j} = (\hat{\boldsymbol{x}}^{i,j}(k|k-1) - \hat{\boldsymbol{x}}^j(k|k-1))$.

Measurement update (m-update): The measurement update for the state results from the corresponding Kalman filter and leads to the normal distribution

$$\boldsymbol{x}(k)|M_j(k), \boldsymbol{Y}(k) \sim \mathcal{N}\Big(\hat{\boldsymbol{x}}^j(k|k), \boldsymbol{Q}_x^j(k|k)\Big) \, .$$

Given the model probability $P(M_i(k-1)|\boldsymbol{Y}(k-1))$ the model probability $P(M_j(k)|\boldsymbol{Y}(k))$ is obtained from the time update, see (11),

$$P(M_j(k)|\boldsymbol{Y}(k)) \propto$$

$$p\Big(\boldsymbol{y}(k)|M_j(k), \boldsymbol{Y}(k-1)\Big) P\Big(M_j(k)|\boldsymbol{Y}(k-1)\Big) \, .$$

As likelihood function $p(\boldsymbol{y}(k)|M_j(k), \boldsymbol{Y}(k-1))$ the normal density of the predicted residuals computed in the Kalman filter is used, given the approximated normal distribution (19).

Results: Finally the posterior density (16) is approximated by a sum of r normal densities that leads to the Bayes estimate

$$\hat{\boldsymbol{x}}_B(k|k) = \sum_{j=1}^{r} \hat{\boldsymbol{x}}^j(k|k) P\Big(M_j(k)|\boldsymbol{Y}(k)\Big)$$

and to the covariance matrix of the state

$$\boldsymbol{Q}_x(k|k) = \sum_{j=1}^{r} P\Big(M_j(k)|\boldsymbol{Y}(k)\Big)\Big(\boldsymbol{Q}_x^j(k|k) +$$

$$(\hat{\boldsymbol{x}}^j(k|k) - \hat{\boldsymbol{x}}_B(k|k))(\hat{\boldsymbol{x}}^j(k|k) - \hat{\boldsymbol{x}}_B(k|k))'\Big) \, .$$

In the algorithm for interacting models the order of conditional time update (17) for a certain model and the mixing, see (19), is changed. In figure 4 the interacting models technique is summarized.

3 Ambiguity Resolution

We start the ambiguity resolution with the float model. That means, we consider the ambiguities as real valued parameters. The state vector is divided into the real valued ambiguities $\boldsymbol{a}(k)$ and the remaining parameters $\boldsymbol{b}(k)$. If we assume, that during the observation process the same satellites are observed and no cycle slips occur, then the ambiguities are constant. We get the dynamic equations

$$\boldsymbol{a}(k+1) = \boldsymbol{a}(k) = \boldsymbol{a} \, ,$$

$$\boldsymbol{b}(k+1) = \boldsymbol{\Phi}_b(k) + \boldsymbol{u}_b(k)$$

and the observation equation

$$\boldsymbol{y}(k) = \boldsymbol{A}_a(k)\boldsymbol{a} + \boldsymbol{A}_b(k)\boldsymbol{b}(k) + \boldsymbol{e}(k) \, .$$

If we treat the ambiguities as real valued parameters, the posterior densities are normal distributed,

$$\boldsymbol{a}(k)|\boldsymbol{Y}(k) \sim \mathcal{N}\Big(\hat{\boldsymbol{a}}(k|k), \boldsymbol{Q}_a(k|k)\Big) \, , \qquad (20)$$

$$\boldsymbol{b}(k)|\boldsymbol{Y}(k) \sim \mathcal{N}\Big(\hat{\boldsymbol{b}}(k|k), \boldsymbol{Q}_b(k|k)\Big) \, . \qquad (21)$$

The state estimate and the covariance matrix are recursively computed with the Kalman filter. In the following single and multiple model approaches integer ambiguities are interpreted as system models.

3.1 Single Model

If the ambiguities are supposed to be the known integers $a \in \mathbb{Z}$, then these integers represent a system model that defines the dynamic and observation equations

$$b(k) = \Phi_b b(k-1) + d_b(k-1) ,$$

$$y(k) = A_a(k)a + A_b(k)b(k) + e(k) .$$

The remaining parameters $b(k)$ given the ambiguities a are normal distributed

$$b(k)|a, Y(k) \sim \mathcal{N}\Big(\hat{b}_a(k|k), Q_{b|a}(k|k)\Big). \quad (22)$$

If the integer ambiguities are the integer least squares solution,

$$a = \arg\min_{z \in \mathbb{Z}} (\hat{a}(k|k) - z)' Q_a(k|k)^{-1} (\hat{a}(k|k) - z),$$

the posterior distribution leads to the well known fixed solution.

3.2 Fixed Models

For the fixed multiple model approach we introduce an uncertainty for the integer ambiguities. Different integer candidates a_i, $i = 1, \ldots r$ define the system models. We assume that the ambiguities a lie in this set,

$$a \in \{a_1, \ldots, a_r\} \subset \mathbb{Z} .$$

The infinite set of integers $\mathbb{Z}$ has to be restricted to a finite one. This leads to r linear dynamic systems

$$b(k) = \Phi_b b(k-1) + d_b(k-1) ,$$

$$y(k) = A_a(k)a_i + A_b(k)b(k) + e(k).$$

with normal distributed state

$$b(k)|a_i, Y(k) \sim \mathcal{N}\Big(\hat{b}^i(k|k), Q_{b|a}(k|k)\Big).$$

The posterior densities under the condition of the different integers are solved by the corresponding Kalman filters or derived as conditional densities from the posterior density of the float solution. They differ only concerning their expectations. In analogy to (10) the posterior density that considers all possible system models is

$$p(b(k)|Y(k)) = \sum_{i=1}^{r} p(b(k)|a_i, Y(k)) P(a_i|Y(k)).$$

The probability of the integers is recursively computed. The initialization of the float solution contains information about the starting probabilities. If the initial real valued ambiguities are normal distributed with

$$a(1)|y(1) \sim \mathcal{N}\Big(\hat{a}(1|1), Q_a(1,1)\Big) ,$$

the start probabilities are

$$P(a_i|y(1)) \propto$$

$$\exp\Big(-\frac{1}{2}(\hat{a}(1|1) - a_i)' Q_a(1|1)^{-1} (\hat{a}(1|1) - a_i)\Big) ,$$

see (Betti et al. 1993), Gundlich and Koch (2002). The recursive computation of the probabilities according to the fixed model approach leads to identical results as the batch solution

$$P\Big(a_i|Y(k)\Big) \propto$$

$$\exp\Big(-\frac{1}{2}(\hat{a}(k|k) - a_i)' Q_a(k|k)^{-1} (\hat{a}(k|k) - a_i)\Big) .$$

Thus the fixed multiple model approach has the same posterior density as the Bayesian approach in (Betti et al. 1993) or (Gundlich, Koch 2002), with the Bayes estimate

$$\hat{b}_B(k|k) = \sum_{i=1}^{r} \hat{b}^i(k|k) P\Big(a_i|Y(k)\Big)$$

and the MAP estimate

$$\hat{b}_{MAP}(k|k) = b^i(k|k)$$

with the maximal probability of the ambiguities

$$i = \arg\max_{j=1,\ldots,r} P\Big(a_j|Y(k)\Big) .$$

The covariance matrix of $b(k)$ is

$$Q_b(k|k) = \sum_{i=1}^{r} P\Big(a_i|Y(k)\Big)\Big(Q_{b|a}$$

$$+(\hat{b}^i(k|k) - \hat{b}_B(k|k))(\hat{b}^i(k|k) - \hat{b}_B(k|k))'\Big) .$$

The MAP estimate represents the state estimate of the fixed solution (22), whereas the Bayes estimate is a mixture of different solutions that belong to different integer ambiguities. The ambiguity probabilities provide a criterion for the choice between the two estimates. The MAP estimate is used for a high maximal probability. In case of a lower probability, if one does not want to rely on a single integer ambiguity vector, the Bayes estimate as a combination of integer vectors can be chosen. In contrast to the float solution, it still considers the ambiguities as integers.

3.3 Switching and Interacting Models

In analogy to (13) we allow jumps from one integer candidate to another one within the ambiguity set

$$a(k) \in \{a_1, \ldots, a_r\} \in \mathbb{Z}\,.$$

These jumps are possible integer cycle slips. The probability of a cycle slip has to be defined a priori by the transition probability (14). Using switching models we have not only the problem of approximating an infinite set by a finite set, but the chosen set should also take possible cycle slips into account. We get the linear dynamic system

$$b(k) = \Phi_b b(k-1) + d_b(k-1)$$

$$y(k) = A_a(k)a(k) + A_b(k)b(k) + e(k)\,.$$

In the posterior density

$$p\Big(b(k)|Y(k)\Big) = \sum_{i=1}^{r} p\Big(b(k)|a_i(k), Y(k)\Big)P\Big(a_i(k)|Y(k)\Big) \tag{23}$$

the probability $P(a_i(k)|Y(k))$ can be updated without an approximation, but the density $p(b(k)|a_i(k), Y(k))$ has to be approximated by interacting models.

4 Conclusions

The multiple model approach enables to take model uncertainties into consideration, that are introduced at the expense of computing time. In case of switching models an approximation is necessary, besides further information in the form of the transition probabilities has to be provided. For ambiguity resolution the fixed model approach is the recursive version of the Bayesian approach that introduces integer ambiguities as random parameters. Cycle slips are taken into account, if the switching model approach is applied. Practical problems arise with the choice of the integer ambiguities as system models. In case of ambiguity resolution with fixed models the integer set of ambiguities can be derived from the initial float solution: All integers within a confidence region of the float solution of the ambiguities form the integer set. An efficient search for those integer ambiguities is possible with the LAMBDA method, see (Teunissen 1994). The definition of the integer sets in a multiple model approach for switching models is more complicated, because the new ambiguities after a cycle slip should be contained in the set. If they belong to the set, then the multiple model algorithm for switching or interacting models is robust to the cycle slips. But we can not expect, that after a cycle slip the new integer ambiguities lie in the once chosen set. Therefore the detection of cycle slips and the use of time dependent integer sets are necessary. For the detection of cycle slips the ambiguity probabilities $P(a_i(k)|Y(k))$ in (23) could be used. This approach however needs further study. Not yet solved is also the question, how the transition probabilities have to be defined.

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