MIXED SYMMETRIC DUALITY IN NONDIFFERENTIABLE MATHEMATICAL PROGRAMMING\(^1\)

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A mixed symmetric dual formulation is presented for a class of nondifferentiable nonlinear programming problems with multiple arguments. Weak, strong and converse duality theorems are established. The mixed symmetric dual formulation unifies the two existing symmetric dual formulations in the literature.

Key Words: Symmetric Duality; Nondifferentiable Nonlinear Programming; Generalized Convexity; Support Function

1. INTRODUCTION

Symmetric duality in nonlinear programming was introduced by Dorn in\(^7\). More precisely, a mathematical programming problem and its dual are said to be symmetric if the dual of the dual is the original problem. In other words, when the dual is recast in the form of the primal, its dual is the primal problem. Subsequently, Dantzig, Eisenberg and Cottle\(^6\) and Mond\(^9\) formulated a pair of symmetric dual programs for a scalar-valued function \(f(x, y)\) that is convex in the first variable and concave in the second variable. Then, Mond and Weir\(^10\) gave a different pair of symmetric dual nonlinear programs in which a weaker convexity assumption was imposed on \(f\).

Recently, Mond and Schechter\(^11\) studied nondifferentiable symmetric duality (of both Wolfe and Mond-Weir types) for a case in which the objective function contains a support function. Chandra, \textit{et al.}\(^5\) presented a mixed symmetric dual formulation for a nonlinear programming problem. Motivated by their research, we propose a pair of new mixed symmetric dual nondifferentiable nonlinear programs in this paper. The pair can be reduced to that of Mond and Schechter\(^11\) and that of Chandra \textit{et al.}\(^5\) as special cases. We then obtain the weak and strong duality theorems for the new pair of mixed symmetric dual nondifferentiable nonlinear programs under a weaker \(F\)-convexity condition.

2. PRELIMINARIES

Let \(C\) be a compact convex set in \(\mathbb{R}^n\). The support function of \(C\) is defined by

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\[ s(x \mid C) := \max \left\{ x^T y : y \in C \right\}. \]

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists \( z \) such that \( s(y \mid C) \geq s(x \mid C) + z^T (y - x) \) for all \( x \in C \). The subdifferential of \( s(x \mid C) \) is given by

\[ \partial s(x \mid C) := \left\{ z \in C : z^T x = s(x \mid C) \right\}. \]

For any set \( S \subset IR^n \) the normal cone to \( S \) at a point \( x \in S \) is defined by

\[ N_S(x) := \left\{ y \in IR^n : y^T (z - x) \leq 0 \text{ for all } z \in S \right\}. \]

It is readily verified that for a compact convex set \( C \), \( y \) is in \( N_C(x) \) if and only if \( s(y \mid C) = x^T y \) if and only if \( x \in \partial s(y \mid C) \).

Let \( f(x, y) \) be a real-valued twice differentiable function defined on \( IR^n \times IR^m \). Let \( \nabla_1 f(x, y) \) and \( \nabla_2 f(x, y) \) denote the partial derivatives of \( f \) with respect to \( x \) and \( y \), respectively. Also let \( \nabla_1^2 f(x, y) \) denotes the Hessian matrix of \( f \) evaluated at \( (x, y) \). The symbols \( \nabla_2^2 f(x, y), \nabla_1^2 f(x, y) \) and \( \nabla_2^1 f(x, y) \) are defined similarly.

We now introduce the following definitions, see Hanson and Monn.

**Definition 1** — Let \( X \subset IR^n \). A functional \( F : X \times X \times IR^n \to IR \) is said to be sublinear with respect to its third argument if, for any \( x, y \in X \)

(A) \( F(x, y; a_1 + a_2) \leq F(x, y; a_1) + F(x, y; a_2) \) for any \( a_1, a_2 \in IR^n \);

(B) \( F(x, y, \alpha a) = \alpha F(x, y; a) \), for any \( \alpha \in IR_+ \) and \( a \in IR^n \).

**Definition 2** — Let \( X \subset IR^n, Y \subset IR^m \) and \( F : X \times Y \times IR^n \to IR \) be sublinear with respect to its third component. \( f(., y) \) is said to be \( F \)-convex at \( x \in X \), for fixed \( y \in Y \), if

\[ f(x, y) - f(\bar{x}, y) \geq F(x, \bar{x}; \nabla_1 f(\bar{x}, y)), \ \forall x \in X. \]

**Definition 3** — Let \( X \subset IR^n, Y \subset IR^m \) and \( f : X \times Y \to IR \). Let \( F : X \times Y \times IR^n \to IR \) be sublinear with respect to its third component. \( f(x, .) \) is said to be \( F \)-concave at \( \bar{y} \in Y \), for fized \( x \in X \) if

\[ f(x, \bar{y}) - f(x, y) \geq F(y, \bar{y}; -\nabla_2 f(x, \bar{y})), \ \forall y \in Y. \]

**Definition 4** — Let \( X \subset IR^n, Y \subset IR^m \) and \( F : X \times Y \times IR^n \to IR \) be sublinear with respect to its third component. \( f(., y) \) is said to be \( F \)-pseudoconvex at \( \bar{x} \), for fixed \( y \in Y \), if

\[ F(x, \bar{x}; \nabla_1 f(\bar{x}, y)) \geq 0 \Rightarrow f(x, y) \geq f(\bar{x}, y), \ \forall x \in X. \]
Definition 5 — Let \( X \subset \mathbb{R}^n \), \( Y \subset \mathbb{R}^m \) and \( f : X \times Y \to \mathbb{R} \). Let \( F : X \times Y \times \mathbb{R}^n \to \mathbb{R} \) be sublinear with respect to its third component. \( f(x, \cdot) \) is said to be \( F \)-pseudoconcave at \( \bar{y} \), for fixed \( x \in X \), if

\[
F(y, \bar{y}; \nabla_x f(x, \bar{y})) \geq 0 \Rightarrow f(x, y) \geq f(x, \bar{y}), \quad \forall \ y \in Y.
\]

3. Mixed Type Symmetric Duality

For \( N = \{1, 2, \ldots, n\} \) and \( M = \{1, 2, \ldots, m\} \) let \( J_1 \subset N, K_1 \subset M \) and \( J_2 = N \setminus J_1, K_2 = M \setminus K_1 \). Let \( |J_1| \) denote the number of elements in the subject \( J_1 \). The other numbers \( |J_2|, |K_1| \) and \( |K_2| \) are defined similarly. It is clear that \( x \in \mathbb{R}^n \) can be written as \( x = (x^1, x^2), x^1 \in \mathbb{R}^{|J_1|}, x^2 \in \mathbb{R}^{|J_2|} \). Similarly, \( y \in \mathbb{R}^m \) can be written as \( y = (y^1, y^2), y^1 \in \mathbb{R}^{|K_1|}, y^2 \in \mathbb{R}^{|K_2|} \). Let \( f : \mathbb{R}^{|J_1|} \times \mathbb{R}^{|K_1|} \to \mathbb{R} \) and \( g : \mathbb{R}^{|J_2|} \times \mathbb{R}^{|K_2|} \to \mathbb{R} \) be twice differentiable. Here, if \( J_1 = 0 \), then \( J_2 = N \setminus J_1 = 0 \) and \( |J_2| = n \). So, in this \( \mathbb{R}^{|J_1|} \) and \( \mathbb{R}^{|J_2|} \) are 0 and \( \mathbb{R}^n \), respectively. The other cases \( K_1 = 0, K_2 = 0 \) and \( J_2 = 0 \) are defined similarly.

We now state the following pair of non-differentiable programs and discuss their duality results.

**Primal Problem (MP)** — Minimize \( f(x^1, y^1) + g(x^2, y^2) + s(x^1 \mid C_1) + s(x^2 \mid C_2) - (y^1)^T \nabla_y f(x^1, y^1) - (y^2)^T \nabla_y g(x^2, y^2) \) subject to \( (x^1, x^2, y^1, y^2, z^1, z^2) \in \mathbb{R}^{|J_1|} \times \mathbb{R}^{|J_2|} \times \mathbb{R}^{|K_1|} \times \mathbb{R}^{|K_2|} \times \mathbb{R}^{|K_1|} \times \mathbb{R}^{|K_2|} \)

\[
\nabla_2 f(x^1, y^1) - z^1 \leq 0,
\]

\[
\nabla_2 g(x^2, y^2) - z^2 \leq 0,
\]

\[
(y^2)^T (\nabla_2 g(x^2, y^2) - z^2) \geq 0,
\]

\[
x^1 \geq 0, x^2 \geq 0,
\]

\[
z^1 \in D_1, z^2 \in D_2
\]

**Dual problem (MD)**

Maximize :

\[
f(u^1, v^1) + g(u^2, v^2) - s(u^1 \mid D_1) - s(u^2 \mid D_2) - (u^1)^T \nabla_1 f(u^1, v^1) + (u^2)^T w^2
\]

subject to :

\[
(u^1, u^2, v^1, v^2, w^1, w^2) \in \mathbb{R}^{|J_1|} \times \mathbb{R}^{|J_2|} \times \mathbb{R}^{|K_1|} \times \mathbb{R}^{|K_2|} \times \mathbb{R}^{|K_1|} \times \mathbb{R}^{|K_2|}
\]

\[
\nabla_1 f(u^1, v^1) + w^1 \geq 0,
\]

\[
\nabla_1 g(u^2, v^2) + w^2 \geq 0,
\]

\[
(u^1)^T (\nabla_1 f(u^1, v^1) + w^1) \leq 0
\]
\[ u^1 \geq 0, \quad v^2 \geq 0, \quad \text{... (9)} \]
\[ w^1 \in C_1, w^2 \in C_2, \quad \text{... (10)} \]

where \( C_1, C_2, D_1 \) and \( D_2 \) are compact and convex sets of \( IR^{|J_1|}, IR^{|J_2|}, IR^{|K_1|} \) and \( IR^{|K_2|} \) respectively.

**Theorem 1 — (Weak Duality):** Let \( F_1, F_2, G_1 \) and \( G_2 \) be sublinear functionals, and let \((x^1, x^2, y^1, y^2, z^1, z^2)\) be feasible for problem \((MP)\) and \((u^1, u^2, v^1, v^2, w^1, w^2)\) be feasible for problem \((MD)\). If \( f(\cdot, y^1) \) is \( F_1 \)-convex for fixed \( y^1, f(x^1, \cdot) \) is \( F_2 \)-concave for fixed \( x^1, g(\cdot, y^2) + (\cdot)^T w^2 \) is \( G_1 \)-pseudoconvex for fixed \( y^2 \) and \( g(x^2, \cdot) - (\cdot)^T z^2 \) is \( G_2 \)-pseudoconcave for fixed \( x^2 \), and the following conditions are satisfied:

- (i) \( F_1(x^1, u^1; \nabla_1 f(u^1, v^1)) + (u^1)^T \nabla_1 f(u^1, v^1) + (x^1)^T w^1 \geq 0; \)
- (ii) \( G_1 x^2, u^2 \nabla_1 g(u^2, v^2) + w^2 \) + \( (u^2)^T (w^2 + \nabla_1 g(u^2, v^2)) \geq 0; \)
- (iii) \( F_2(y^1, u^1) \nabla_2 f(x^1, y^1)) + (y^1)^T \nabla_2 f(x^1, y^1) - (u^1)^T z^1 \leq 0; \) and
- (iv) \( G_2(y^2, u^2) \nabla_2 f(x^2, y^2)) + (y^2)^T \nabla_2 f(x^2, y^2) - z^2 \leq 0; \)

then \( \inf (MP) \geq \sup (MD) \).

**Proof:** Suppose that \((x^1, x^2, y^1, y^2, z^1, z^2)\) and \((u^1, u^2, v^1, v^2, w^1, w^2)\) are feasible for problems \((MP)\) and \((MD)\), respectively. Then using the \( F_1 \)-convexity of \( f(\cdot, y^1) \) and \( F_2 \)-concavity of function \( f(x^1, \cdot) \), we have

\[ f(x^1, y^1) - f(u^1, v^1) \geq F_1(x^1, u^1; \nabla_1 f(u^1, v^1)), \]

and

\[ f(x^1, v^1) - f(x^1, y^1) \geq F_2(v^1, y^1; \nabla_2 f(x^1, y^1)). \]

Rearranging the above two inequalities, and by using conditions (i) and (iii), we obtain

\[ f(x^1, y^1) - f(u^1, v^1) \geq - (u^1)^T \nabla_1 f(u^1, v^1) \]
\[ \quad - (x^1)^T w^1 + (y^1)^T \nabla_2 f(x^1, y^1) - (v^1)^T z^1. \]

Using \((u^1)^T z^1 \leq s(z^1 \mid D_1)\) and \((x^1)^T w^1 \leq s(x^1 \mid C_1)\) we have

\[ f(x^1, y^1) + s(x^1 \mid C_1) - (y^1)^T \nabla_2 f(x^1, y^1) \geq f(u^1, v^1) \]
\[ \quad - s(v^1 \mid D_1) - (u^1)^T \nabla_1 f(u^1, v^1) \quad \text{... (11)} \]

From condition (ii) and (8), we have

\[ G_1(x^2, u^2; \nabla_1 g(u^2, v^2) + w^2) \geq - (u^2)^T (w^2 + \nabla_1 g(u^2, v^2)) \geq 0. \]
By $G_1$-pseudoconvexity of $g(\cdot, y^2)+ (\cdot)^T w^2$, we get
\[ g(x^2, u^2) + (x^2)^T w^2 \geq g(u^2, u^2) + (u^2)^T w^2. \] ... (12)

In a similar fashion, from condition (iv) and (3), we have
\[ G_2(y^2, u^2; \nabla_2 f(x^2, y^2) - z^2) \leq - (y^2)^T (\nabla_2 f(x^2, y^2) - (z^2)^T \leq 0. \]

By $G_2$-pseudoconcavity of $g(x^2, \cdot) - (\cdot)^T z^2$, we get
\[ g(x^2, u^2) - (u^2)^T z^2 \leq g(x^2, y^2) - (y^2)^T z^2. \] ... (13)

From eq. (12) and (13) we can conclude that
\[ g(x^2, u^2) + (x^2)^T w^2 - (y^2)^T z^2 \geq g(u^2, u^2) - (u^2)^T z^2 + (u^2)^T w^2. \] ... (14)

Using $(x^2)^T w^2 \leq s(x^2 \mid C_2)$ and $(u^2)^T z^2 \leq s(u^2 \mid D_2)$, we have
\[ g(x^2, u^2) + s(x^2 \mid C_2) - (y^2)^T z^2 \geq g(u^2, u^2) - s(u^2 \mid D_2) + (u^2)^T w^2. \] ... (15)

Finally, (11) and (15) give
\[
\begin{align*}
&f(x^1, y^1) + g(x^2, u^2) + s(x^2 \mid C_1) + s(x^2 \mid C_2) - (y^1)^T \nabla_2 f(x^1, y^1) - (y^2)^T z^2 \\
&\geq f(u^1, u^1) + g(u^2, u^2) - s(u^2 \mid D_1) - s(u^2 \mid D_2) - (u^2)^T \nabla_1 f(u^1, u^1) + (u^2)^T w^2. \quad \text{... (16)}
\end{align*}
\]

Thus, $\inf (MP) \geq \sup (MD)$. \hfill \Box

**Theorem 2 — (Strong duality)** Suppose that $(x^1, x^2, y^1, y^2, z^1, z^2)$ is optimal for problem $(MP)$ and that the Hessian matrix $\nabla_2^2 f(x^1, y^1)$ is nonsingular, that $\nabla_2^2 g(x^2, y^2)$ is positive definite or negative definite and that $\nabla_2^2 g(x^2, y^2) \neq z^2$. Then $(x^1, x^2, y^1, y^2, z^1, z^2)$ is feasible for problem $(MD)$ and the corresponding objective function value are equal. If in addition the hypotheses of Theorem 1 hold, then there exist $w^1, w^2$ such that $(u^1, u^2, v^1, v^2, w^1, w^2) = (c^1, x^1, y^1, y^2, w^1, w^2)$ is optimal for problem $(MD)$.

**Proof:** Let $q = (x^1, x^2, y^1, y^2, z^1, z^2)$ and
\[
\begin{align*}
F(q) &= f(x^1, y^1) + g(x^2, y^2) - (y^1)^T \nabla_2 f(x^1, y^1) - (y^2)^T z^2 + s(q \mid \{0\} \times \{0\} \times \{0\} \times C_1 \times C_2, \\
G(q) &= \nabla_2 f(x^1, y^1) - z^1, \\
H(q) &= \nabla_2 g(x^2, y^2) - z^2, \\
I(q) &= -(y^2)^T (\nabla_2 g(x^2, y^2) - z^2),
\end{align*}
\]
\[ J(q) = -x^1, \]
\[ K(q) = -x^2, \]
\[ D = \mathbb{R}^{J_1} \times \mathbb{R}^{J_2} \times \mathbb{R}^{K_1} \times \mathbb{R}^{K_2} \times D_1 \times D_2. \]

Then Problem (MP) can be restated as follows:

\[ \text{minimize } F(q) \]

\[ \text{Subject to:} \]

\[ G(q) \leq 0, \]
\[ H(q) \leq 0, \]
\[ I(q) \leq 0, \]
\[ J(q) \leq 0, \]
\[ K(q) \leq 0, \]
\[ q \in D. \]

Since \((\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{z}^1, \bar{z}^2)\) is optimal for problem (MP) i.e., \(q = (x^1, x^2, y^1, y^2, z^1, z^2)\) is optimal for above programming, by the Fritz John conditions\(^{12}\) and note that \(N_C(q) = \{0\} \times \{0\} \times \{0\} \times \{0\} \times N_{D_1}(z^1) \times N_{D_2}(z^2)\) at any \(q \in C\), there exist

\[ \alpha \in \mathbb{R}, \alpha_1 \in \mathbb{R}^{J_1}, \alpha_2 \in \mathbb{R}^{K_1}, \lambda \in \mathbb{R}, \mu_1 \in \mathbb{R}^{J_1} \text{ and } \mu_2 \in \mathbb{R}^{J_2}, \] such that

\[ \alpha [V_1 f(\bar{x}^1, \bar{y}^1) - (\bar{y}^1) T V_{12}^2 f(\bar{x}^1, \bar{y}^1) + \bar{w}^1] + \alpha_1 T V_{12}^2 f(\bar{x}^1, \bar{y}^1) - \mu_1 = 0, \] \[ \alpha_2 \] \[ \alpha [V_1 g(\bar{x}^2, \bar{y}^2) + \bar{w}^2] + (\alpha_2 - \lambda \bar{y}^2) T V_{12}^2 g(\bar{x}^2, \bar{y}^2) - \mu_2 = 0, \]

\[ - \alpha V_{22}^2 f(\bar{x}^1, \bar{y}^1) \bar{y}^1 + \alpha_1 T V_{22}^2 g(\bar{x}^1, \bar{y}^1) = 0, \]

\[ (\alpha - \lambda) [V_2 g(\bar{x}^2, \bar{y}^2) - \bar{z}^2] + (\alpha_2 - \lambda \bar{y}^2) T g_{22}^2(\bar{x}^2, \bar{y}^2) = 0, \]

\[ \alpha_1 \in N_{D_1}(z^1), \]

\[ \alpha \bar{y}^2 + (\alpha_2 - \lambda \bar{y}^2) \in N_{D_2}(z^2), \]

\[ \bar{w}^1 \in C_1, (\bar{w}^1) T x^1 = s(x^1 | C_1), \]

\[ \bar{w}^2 \in C_2, (\bar{w}^2) T x^2 + s(x^2 | C_2), \]

\[ \alpha_1^T [V_2 f(\bar{x}^1, \bar{y}^1) - \bar{z}^1] = 0, \]
\[ \alpha_2^T [\nabla^2 g(\overrightarrow{x}, \overrightarrow{y}) - \overrightarrow{z}] = 0, \]  
... (26)

\[ \lambda (\overrightarrow{y})^T [\nabla^2 g(\overrightarrow{x}, \overrightarrow{y}) - \overrightarrow{z}] = 0, \]  
... (27)

\[ \mu_1^T \overrightarrow{x} = 0, \]  
... (28)

\[ \mu_2^T \overrightarrow{x} = 0, \]  
... (29)

\[ (\alpha, \alpha_2, \lambda, \mu_1, \mu_2) \geq 0 \text{ and } (\alpha, \alpha_2, \lambda, \mu_1, \mu_2) \neq 0. \]  
... (30)

From (19) and nonsingularity of the Hessian matrix \( \nabla^2_{x2} f(\overrightarrow{x}, \overrightarrow{y}) \), we have

\[ \alpha_1 = \alpha \overrightarrow{y}. \]  
... (31)

Multiplying (20) by \( \alpha_2 - \lambda \overrightarrow{y} \), and from (26) and (27), we obtain

\[ (\alpha_2 - \lambda \overrightarrow{y})^T \nabla^2_{x2} g(\overrightarrow{x}, \overrightarrow{y}) (\alpha_2 - \lambda \overrightarrow{y}) = 0. \]  
... (32)

Since \( \nabla^2_{x2} g(\overrightarrow{x}, \overrightarrow{y}) \) is positive or negative definite, we have

\[ \alpha_2 = \lambda \overrightarrow{y}. \]  
... (33)

From (20), (33) and the hypothesis \( \nabla^2 g(\overrightarrow{x}, \overrightarrow{y}) \neq \overrightarrow{z} \), we have

\[ \alpha = \lambda. \]  
... (34)

If \( \alpha = 0 \), then \( \lambda = 0 \) and from (33), \( \alpha_1 = 0 \), and from (17), \( \mu_1 = 0 \) and from (18), \( \mu_2 = 0 \). This contradicts (30). Hence \( \alpha > 0 \) and \( \lambda > 0 \). From (33) and (30), we have

\[ \overrightarrow{y} \geq 0. \]  
... (35)

From (31) and (30), we have

\[ \overrightarrow{y} \geq 0. \]  
... (36)

From (17), (31) and (30), we have

\[ \nabla^1 f(\overrightarrow{x}, \overrightarrow{y}) + \overrightarrow{w} \geq 0. \]  
... (37)

From (18), (33), (30) and \( \alpha > 0 \), we have

\[ \nabla^1 g(\overrightarrow{x}, \overrightarrow{y}) + \overrightarrow{w} \geq 0. \]  
... (38)

From (18), (33), (29) and \( \alpha > 0 \), we have

\[ \mu_2^T [\nabla^1 g(\overrightarrow{x}, \overrightarrow{y}) - \overrightarrow{w}] \leq 0. \]  
... (39)
Hence from (23), (24), (35), (36), (37), (38) and (39), \((x^1, \bar{x}^2, y^1, y^2, \bar{w}^1, \bar{w}^2)\) is feasible for (MD). Now from (17), (23), (28), (31) and \(\alpha > 0\), we have
\[
s(x^1 | C_1) = -(x^1)^T \nabla_1 f(x^1, y^1).
\]
... (40)

From (21) and (31), we know that \(y^1 \in N_{D_1}(\bar{z}^1)\), i.e.,
\[
(y^1)^T z^1 = s(y^1 | D_1).
\]
... (41)

From (22), (33) and \(\alpha > 0\), we have
\[
\bar{y} \in N_{D_2}(\bar{z}^2).
\]
That is,
\[
(y^2)^T z^2 = s(y^2 | D_2).
\]
... (42)

Finally, from (24), (40), (41) and (42), we give
\[
f(x^1, y^1) + g(x^2, \bar{y}^2) + s(x^1 | C_1) + s(x^2 | C_2) - (y^1)^T \nabla_2 f(x^1, y^1) - (y^2)^T \bar{z}^2
\]
\[
= f(x^1, y^1) + g(x^2, \bar{y}^2) - s(y^1 | D_1) - s(y^2 | D_2) - (x^1)^T \nabla_1 f(x^1, y^1) + (x^2)^T \bar{w}^2.
\]
... (43)

By the weak duality and (43), \((x^1, x^2, y^1, y^2, \bar{w}^1, \bar{w}^2)\) is an optimal solution of (MD).

By the similar method of Theorem 2, we can prove the following converse duality theorems

**Theorem 3 — (Converse duality)** Suppose that \((x^1, x^2, y^1, y^2, \bar{w}^1, \bar{w}^2)\) is optimal for problem (MD) and that the Hessian matrix \(\nabla_1^2 f(x^1, y^1)\) is nonsingular, that \(\nabla_1^2 g(x^2, y^2)\) is positive definite or negative definite and that \(\nabla_1 g(x^2, y^2) \neq \bar{w}^2\). Then \((x^1, x^2, y^1, y^2, \bar{w}^1, \bar{w}^2)\) is feasible for problem (MP) and the corresponding objective function value are equal. If in addition the hypotheses of Theorem 1 hold, then there exist \(\bar{z}^1, \bar{z}^2\) such that \((x^1, x^2, y^1, y^2, \bar{z}^1, \bar{z}^2) = (x^1, x^2, y^1, y^2, \bar{z}^1, \bar{z}^2)\) is optimal for problem (MP).

4. SPECIAL CASES

In this section we consider some special cases of problem (MP) and problem (MD) by choosing particular forms of sublinear functionals \(F_1, F_2, G_1\) and \(G_2\) and the compact convex sets \(C_1, C_2, D_1\) and \(D_2\).

(i) If \(C_1 = C_2 = \{0\}\), \(D_1 = D_2 = \{0\}\), then (MP) and (MD) reduce to a pair of primal problem and dual problem programs studied in Chandra, et al.\(^5\)

Primal Problem (MP)\(_1\):

Minimize
\[
f(x^1, y^1) + g(x^2, y^2) - (y^1)^T \nabla_2 f(x^1, y^1)
\]

subject to:
\[ \nabla_2 f(x^1, y^1) \leq 0, \]
\[ \nabla_2 g(x^2, y^2) \leq 0, \]
\[ (y^2)^T \nabla_2 g(x^2, y^2) \geq 0, \]
\[ x^1 \geq 0, x^2 \geq 0. \]

Dual Problem \((MD)_1\):

Maximize
\[ f(u^1, v^1) + g(u^2, v^2) - (u^1)^T \nabla_1 f(u^1, v^1) \]
subject to:
\[ \nabla_1 f(u^1, v^1) \geq 0, \]
\[ \nabla_1 g(u^2, v^2) \geq 0, \]
\[ (u^2)^T \nabla_1 f(u^2, v^2) \leq 0, \]
\[ v^1 \geq 0, v^2 \geq 0. \]

\((ii)\) If \(J^2 = 0\) and \(K^2 = 0\) the symmetric dual pair \((MP)\) and \((MD)\) reduces to the pair \((P)\) and \((D)\) of Mond and Schechter\(^\text{11}\)

Primal Problem \((MP)_2\):

Minimize
\[ f(x^1, y^1) + s(x^1 \mid c_1) - (y^1)^T \nabla_2 f(x^1, y^1) \]
subject to:
\[ \nabla_2 f(x^1, y^1) - z^1 \leq 0, \]
\[ x^1 \geq 0, \]
\[ z^1 \in D_1. \]

Dual Problem \((MD)_2\):

Maximize
\[ f(u^1, v^1) - s(v^1 \mid D_1) - (u^1)^T \nabla_1 f(u^1, v^1) \]
subject to:
\[ \nabla_1 f(u^1, v^1) - z^1 \leq 0, \]
\[ v^1 \geq 0, \]
where $C_1$ and $D_1$ are compact and convex sets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

If $J_1 = 0$ and $K_1 = 0$ the symmetric dual pair (MP) and (MD) reduces to the pair $(P_1)$ and $(D_1)$ of Mond and Schechter\cite{11}

Primal Problem $(MP)_3$

Minimize

$$g(x^2, y^2) + s(x^2 | C_2) - (y^2)^T z^2$$

subject to:

$$\nabla_2 g(x^2, y^2) - z^2 \leq 0,$$

$$(y^2)^T (\nabla_2 g(x^2, y^2) - z^2) \geq 0,$$

$$x^2 \geq 0,$$

$$z^2 \in D_2.$$

Dual problem $(MD)_3$:

Maximize

$$g(u^2, v^2) - s(v^2 | D_2) + (u^2)^T w^2$$

subject to:

$$\nabla_1 g(u^2, v^2) + w^2 \geq 0,$$

$$(u^2)^T (\nabla_1 f(u^2, v^2) + w^2) \leq 0,$$

$$v^2 \geq 0,$$

$$w^2 \in C_2,$$

where $C_2$ and $D_2$ are compact and convex sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Chandra, Husain and Abha (see [5]) proved the weak and strong duality theorems for $(MP)_1$ and $(MD)_1$ and Mond and Schechter (see [11]) proved the weak and strong duality theorems for $(MP)_2$ and $(MD)_2$ and $(MP)_3$ and $(MD)_3$ under convex-concave functions. Here we prove our weak, strong and converse duality theorem under $F$-convex and $F$-concave functions. So our theorem 1 and 2 generalize the main results in [5], and improve, extend and unified Mond and Schechter's work in\cite{11}.
(iii) From the symmetric dual models (MP) and (MD), we can construct other symmetric dual pairs. For example, if we take \( C_i = \left\{ A_i y : y^T A_i y \leq 1 \right\} \) \( (i = 1, 2) \) and \( D_i = \left\{ B_i x : x^T B_i x \leq 1 \right\} \) \( (i = 1, 2) \) where \( A_i \) and \( B_i \) are positive semi-definite, then it can be readily verified that \((x^T A_i x)^{1/2} = s \ (x \mid C_i)\) \( (y^T B_i y)^{1/2} = s \ (y \mid D_i)\) and thus a number of symmetric dual pairs and duality results are obtained. In particular, \((MP)_2\) and \((MD)_2\) reduce to the symmetric dual pair S. Chandra and I. Husain\(^4\).

(iv) These results in this paper can also be extended to multiobjective programming, and integer programming on the line of [2, 3] under various types of generalized convexity assumption.

REFERENCES