Comparison of the low-frequency predicitons of Biot's and de Boer's poroelasticity theories with Gassmann's equation

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Predictions of Biot's theory (BT) of poroelasticity [J. Acoust. Soc. Am. **28**, 168 (1956)] and de Boer's theory of porous media (TPM) [*Theory of Porous Media* (Springer, Berlin, 2000)] for the low-frequency bulk modulus of a fluid-saturated porous medium are compared with the Gassmann equation [Vierteljahrsschr. Naturforsch. Ges. Zur. **96**, 1 (1951)]. It is shown that BT is consistent with the Gassmann equation, whereas TPM is not. It is further shown that the bulk modulus of a suspension of solid particles in a fluid as predicted by TPM is only correct if the particles are incompressible. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2778763]

Many natural and man-made materials, such as rocks, soils, foams, biological tissues, and construction materials, can be described as porous media consisting of an elastic frame and a network of empty or fluid-filled pores. The theory of elastic wave propagation of such materials (theory of poroelasticity) was proposed by Biot more than 50 years ago and is still widely used today. Biot[1–3] derived poroelasticity equations from thermodynamic considerations. Later exactly the same equations were obtained by two other methods: statistical volume averaging[4,5] and asymptotic homogenization of periodic structures.[6–8] This latter method is based on the rigorous mathematical framework of periodic homogenization[9,10] and yields precise validity conditions for the equations of poroelasticity.

Despite these rigorous derivations, the acceptance of Biot's theory is not universal. In recent decades, a number of authors have claimed certain inconsistencies in Biot's original derivations and proposed alternative theories of elastic properties of porous materials.[11–14] One of these theories is the so-called theory of porous media proposed by de Boer.[11,15,16] Comparison of different theories is difficult due to the use of different notations and different forms of equations of motions and constitutive equations. However, recently Schanz and Diebels[17] performed a detailed analysis of Biot's theory (BT) of poroelasticity and theory of porous media (TPM) and managed to write governing equations of the two theories in a similar form which allows for easy comparison. They show that, with certain simplifying assumptions, the two theories give identical predictions for porous media with incompressible constituents but differing predictions in case of compressible constituents.

Schanz and Diebels[17] do not address the question as to which of the two predictions (in the case of compressible constituents) is more plausible or indeed correct. In this letter I attempt to address this very question by comparing the predictions of the two theories in the limit of low frequency, that is, in the static limit. It is well known that in the static limit the bulk modulus of a fluid-saturated medium is given by the exact Gassmann[18] equation. By comparing the predictions of both theories with Gassmann's exact result I show that the prediction of BT is consistent with it, whereas the prediction of TPM is not.

According to Schanz and Diebels,[17] in the absence of body forces, one-dimensional propagation of a compressional elastic wave along the x_1 axis in the Laplace domain both in BT and TPM can be written in the form

$$L\frac{d^2}{dx^2}u - A\frac{d}{dx}p - s^2Bu = 0,$$

$$\frac{d^2}{dx^2}p - s^2Cp - s^2D\frac{d}{dx}u = 0, \tag{1}$$

where $u \equiv u_1$ is displacement along x_1 axis, p is the fluid pressure, $L=K+4G/3, K$ and G are bulk and shear moduli of the dry frame, ρ is bulk density of the saturated material, s is Laplace parameter, and A, B, C, and D are coefficients that have different meanings in the two theories. In BT,

$$A=\alpha, \quad B=\rho, \quad C=\frac{\phi^2\rho_F}{\beta R^B}, \quad D=\frac{\alpha\rho_F}{\beta}.$$

Here $\alpha=1-K/K_S$ is the Biot-Willis coefficient, ϕ is porosity, and ρ_F is fluid density,

$$R^B=\frac{\phi^2 K_F K_S^2}{K_F(K_S-K)+\phi K_S(K_S-K_F)}$$

and

$$\beta=\frac{\kappa\phi^2\rho_F s}{\phi^2+s\kappa(\rho_a+\phi\rho_F)},$$

where κ is the permeability and ρ_a the so-called apparent density. For small frequencies $s\to 0$ and we have $\beta=\kappa\rho_F s$, so that

$$A=\alpha, \quad B=\rho, \quad C=\frac{\phi^2}{\kappa s R^B}, \quad D=\frac{\alpha}{\kappa s}. \tag{2}$$

In TPM,

$$A=\phi+z^S(1-\phi), \quad B=\rho, \quad C=\frac{\phi}{\beta R\vartheta}=\frac{\phi}{\kappa\rho_F sR\vartheta} \tag{3}$$

and

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$$D = \frac{\rho_0^{FR}}{\beta}A = \frac{A}{\kappa s},$$

where $\rho_0^{FR} = \rho_F$ is the fluid density, R is the universal gas constant, ϑ is the absolute temperature, and $z^S = \alpha = 1 - K/K_S$. Schanz and Diebels[17] showed that with the parameter definitions [Eqs. (2) and (3)], the predictions of the two theories differ. In order to assess which of the two theories is correct, below I compare the low-frequency predictions of each of the theories with the prediction of the Gassmann equation.

By considering the plane-wave solution of the form $\exp(ikx_1)$, we can rewrite Eq. (1) in the form

$$-k^2Lu - Aikp - s^2Bu = 0,$$

$$-k^2p - s^2Cp - s^2Diku = 0, \tag{4}$$

where k can be considered a wave number. The system [Eq. (4)] is a system of two linear equations which has nontrivial solutions if and only if its determinant is zero, that is,

$$\begin{vmatrix} -(k^2L + s^2B) & -Aik \\ -s^2Dik & -k^2 - s^2C \end{vmatrix} = 0, \tag{5}$$

or, in frequency domain, $\omega = is$,

$$\begin{vmatrix} -(k^2L - \omega^2B) & -Aik \\ \omega^2Dik & -k^2 + \omega^2C \end{vmatrix} = 0. \tag{6}$$

This gives a quadratic equation in k^2

$$Lk^4 - k^2\omega^2(LC + B + AD) + \omega^4BC = 0.$$

Substituting $k = S\omega$ yields biquadratic equation for the slowness S,

$$LS^4 - S^2(LC + B + AD) + BC = 0.$$

Noting that for small frequencies C and D scale with ω^{-1}, while A and B scale with $O(1)$, we can write

$$LS^4 - S^2(LC + AD) + BC = 0. \tag{7}$$

Equation (7) has two roots,

$$S^2 = \frac{(LC + AD) \pm \sqrt{(LC + AD)^2 - 4BCL}}{2L},$$

or, again for small ω,

$$S^2 = \frac{(LC + AD)(1 \pm 1 \mp 2BCL/(LC + AD)^2)}{2L}.$$

Thus the two roots are

$$S_+^2 = \frac{LC + AD}{L}$$

and

$$S_-^2 = \frac{BC}{LC + AD},$$

or, for two velocities,

$$c_+^2 = \frac{L}{LC + AD}$$

and

$$c_-^2 = \frac{LC + AD}{BC} = \frac{K + (4/3)G + AD/C}{\rho}$$

(note that $B = \rho$ in both theories). Since C and D scale with ω^{-1}, $c_+ \to 0$ for $\omega \to 0$, and therefore corresponds to Biot's slow wave,[1] while c_- corresponds to the fast or normal compressional wave. At low frequencies the velocity of normal compressional wave has the form

$$c_-^2 = \frac{K_{\text{sat}} + (4/3)G}{\rho}. \tag{8}$$

Thus in both theories we have

$$K_{\text{sat}} = K + \frac{AD}{C}. \tag{9}$$

In BT we have

$$K_{\text{sat}} = K + \frac{\alpha^2 R^B}{\phi^2}$$

or

$$K_{\text{sat}} = K + \frac{\alpha^2}{\alpha/K_S + \phi(1/K_F - 1/K_S)}. \tag{10}$$

Equation (10) is the familiar Gassmann equation. Thus, we have obtained the well known fact that in the low-frequency limit, BT gives the velocity consistent with Gassmann equation.

Before deriving the corresponding result in TPM, we note that it contains two constants, R and ϑ, which are absent in BT. This is not an original feature of TPM, but results from the way Schanz and Diebels[17] defined the equation of state for the fluid. Namely, they assume an ideal gas law,

$$\rho^{FR}(p) = \frac{\rho^{FR}}{p_0}p = \frac{p}{R\vartheta}$$

or

$$\frac{\partial \rho^{FR}(p)}{\partial p} = \frac{1}{R\vartheta},$$

where R is the universal gas constant and ϑ the absolute temperature. This law is, however, restricted to ideal gases whose effect on the overall compressibility of the porous medium would be negligible anyway. To make the equation of state for the fluid consistent with that for BT, we write it in a general linear form,

$$\frac{\rho^{FR} - \rho_0^{FR}}{\rho_0^{FR}} = \frac{p - p_0}{K_F}$$

or

$$\frac{\partial \rho^{FR}(p)}{\partial p} = \frac{\rho_0^{FR}}{K_F}.$$

Thus to make the equation of state for the fluid in TPM consistent with that used in BT, we should replace $R\vartheta$ with K_F/ρ_0^{FR}. This gives

$$A = \phi + z^S(1-\phi), \quad B = \rho, \quad C = \frac{\phi}{i\omega\kappa K_F}, \quad D = \frac{\phi}{i\omega\kappa}. \tag{11}$$

Substitution of Eq. (11) into the general expression [Eq. (9)] yields

$$K_{\mathrm{sat}} = K + [\phi + z^S(1-\phi)]^2 \frac{K_F}{\phi}$$

or

$$K_{\mathrm{sat}} = K + [\phi + \alpha(1-\phi)]^2 \frac{K_F}{\phi}. \tag{12}$$

Clearly, Eq. (12) is not, in general, consistent with the Gassmann equation [Eq. (10)]. In particular, the former is linear in fluid bulk modulus, while the latter is not. Since in the case of an isotropic frame made of an isotropic and homogeneous solid constituent, Gassmann's equation is exact; this clearly shows that the prediction of TPM is incorrect. To make it even more apparent, consider a suspension of solid particles in a fluid. In this case the bulk modulus of the dry frame $K=0$, $\alpha=1$, and BT gives

$$K_{\mathrm{sat}} = \frac{1}{(1-\phi)/K_S + \phi/K_F}. \tag{13}$$

This is classical Wood's equation, which shows that the modulus of the suspension is harmonic average of the constituent bulk moduli of the solid and fluid. On the other hand, TPM gives $K_{\mathrm{sat}} = K_F/\phi$. Generally speaking, this result is incorrect as it suggests that the compressibility of the suspension is independent of the compressibility of the solid particles. The TPM result is only correct if the particles are incompressible.

I conclude that in the limit of low frequencies the prediction of Biot's theory is consistent with the Gassmann equation, whereas the prediction of de Boer's theory of porous media is not.

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[1] M. A. Biot, J. Acoust. Soc. Am. **28**, 168 (1956).
[2] M. A. Biot, J. Acoust. Soc. Am. **28**, 179 (1956).
[3] M. A. Biot, J. Appl. Phys. **33**, 1482 (1962).
[4] V. N. Nikolayevskiy, K. S. Basniyev, A. T. Gorbunov, and G. A. Zotov, *Mechanics of Saturated Porous Media* (Nedra, Moscow, 1970) [in Russian].
[5] S. R. Pride, A. F. Gangi, and F. D. Morgan, J. Acoust. Soc. Am. **92**, 3278 (1992).
[6] T. Levy, Int. J. Eng. Sci. **17**, 1005 (1979).
[7] J.-L. Auriault, Int. J. Eng. Sci. **18**, 775 (1980).
[8] R. Burridge and J. B. Keller, J. Acoust. Soc. Am. **70**, 1140 (1981).
[9] E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory* (Springer, New York, 1980).
[10] N. Bakhvalov and G. Panasenko, *Homogenisation: Averaging Processes in Periodic Media, Mathematical Problems in the Mechanics of Composite Materials* (Kluwer, Dordrecht, 1989).
[11] R. de Boer, *Theory of Porous Media* (Springer, Berlin, 2000).
[12] S. L. Lopatnikov and A. H. D. Cheng, J. Mech. Phys. Solids **52**, 2801 (2004).
[13] K. Wilmanski, Soil Dyn. Earthquake Eng. **26**, 509 (2006).
[14] V. Delacruz and T. J. T. Spanos, Geophysics **50**, 1556 (1985).
[15] R. de Boer, Appl. Mech. Rev. **49**, 201 (1996).
[16] R. de Boer, Int. J. Solids Struct. **35**, 3446 (1998).
[17] M. Schanz and S. Diebels, Acta Mech. **161**, 213235 (2003).
[18] F. Gassmann, Vierteljahrsschr. Natforsch. Ges. Zur. **96**, 1 (1951).