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# The Generalised Discrete Algebraic Riccati Equation arising in LQ optimal control problems: Part I

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**Abstract**—A geometric analysis is used to study the relationship existing between the solutions of the generalised Riccati equation arising from the classic infinite-horizon linear quadratic (LQ) control problem and the output-nulling and reachability subspaces of the underlying system. This analysis reveals the presence of a subspace that plays a crucial role in the solution of the related optimal control problem.

## I. INTRODUCTION

In the last fifty years Riccati equations have been found to arise in countless fields, starting from LQ optimal control problems and Kalman filtering problems, and including also linear dynamic games with quadratic cost criteria, spectral factorisation problems, singular perturbation theory, stochastic realization theory and identification, boundary value problems for ordinary differential equations, invariant embedding and scattering theory. For this reason, Riccati equations are universally regarded as a cornerstone of modern control theory. Several monographs have been entirely devoted to providing a general and systematic framework for the study of Riccati equations, see e.g. [19], [11], [10], [1].

In the discrete time, the classic solution of the infinite-horizon LQ problem is traditionally expressed in terms of the solution  $X$  of the Riccati equation

$$X = A^T X A - (A^T X B + S)(R + B^T X B)^{-1} (B^T X A + S^T) + Q, \quad (1)$$

where the matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $Q \in \mathbb{R}^{n \times n}$ ,  $S \in \mathbb{R}^{n \times m}$  and  $R \in \mathbb{R}^{m \times m}$  are such that the Popov matrix  $\Pi$  is symmetric and positive semidefinite, i.e.,

$$\Pi \triangleq \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \Pi^T \geq 0. \quad (2)$$

The set of matrices  $\Sigma = (A, B, Q, R, S)$  is often referred to as *Popov triple*, see e.g. [10]. Equation (1) is the so-called *discrete Riccati algebraic equation* DARE( $\Sigma$ ). Differently from the continuous case, it is not the inverse of  $R$  that explicitly appears in the Riccati equation but the inverse of the term  $R + B^T X B$ , which can be non-singular even when  $R$  is singular. Nevertheless, even though the distinction between the cases in which  $R$  is invertible or singular needs not be considered, very often even in the discrete time it is assumed

that  $R$  is non-singular because this assumption considerably simplifies several underlying mathematical derivations.

However, even the solution to the infinite-horizon LQ problem expressed in terms of matrices satisfying this equation is somehow restrictive. Indeed, an LQ problem may have solutions even if (1) has no solutions, and the optimal control can be written in this case as a state feedback written in terms of a matrix  $X$  such that  $R + B^T X B$  is singular and satisfies the more general Riccati equation

$$X = A^T X A - (A^T X B + S)(R + B^T X B)^\dagger (B^T X A + S^T) + Q, \quad (3)$$

$$\ker(R + B^T X B) \subseteq \ker(A^T X B + S), \quad (4)$$

where the matrix inverse in DARE( $\Sigma$ ) has been replaced by the Moore-Penrose pseudo-inverse, see [15]. Equation (3) is known in the literature as the *generalised discrete-time algebraic Riccati equation* GDARE( $\Sigma$ ). The GDARE( $\Sigma$ ) with the additional constraint on its solutions given by (4) is sometimes referred to as *constrained generalised discrete-time algebraic Riccati equation*, herein denoted by CGDARE( $\Sigma$ ). It is obvious that (3) constitutes a generalisation of the classic DARE( $\Sigma$ ). However, despite its generality, this type of Riccati equation has not yet received a great deal of attention in the literature. It has only been marginally studied in the monographs [16], [10], [1] and in the paper [3]. The only comprehensive contributions entirely devoted to the study of the solutions of this equation are [9] and [17]. The former investigates conditions under which the GDARE( $\Sigma$ ) admits a stabilising solution in terms of the deflating subspaces of the so-called extended symplectic pencil. The latter studies the connection between the solutions of this equation and the rank-minimising solutions of the so-called Riccati linear matrix inequality. In pursuing this task, the authors of [17] derived a series of important results on the structural properties of the solutions of the generalised Riccati equation, and in particular in the fundamental role played by the term  $R + B^T X B$ . The results presented in [17] are established in the very general setting in which the Popov matrix  $\Pi$  is not necessarily positive semidefinite as in (2).

In this paper we are interested in the connection of the use of the CGDARE( $\Sigma$ ) in the solution of optimal control or filtering problems. The aim is to provide a geometric picture describing the structure of the solutions of CGDARE( $\Sigma$ ) in terms of the output nulling subspaces of the original system  $\Sigma$  and the corresponding reachability subspaces. Indeed, under the usual assumption of positive semidefiniteness of the Popov matrix, the null-space of  $R + B^T X B$  is independent of the solution  $X$  of CGDARE( $\Sigma$ ). Even more importantly,

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this null-space is linked to the presence of a subspace – that will be identified in this paper – which plays an important role in the characterisation of the solutions of CGDARE( $\Sigma$ ), and also in the solution of the related optimal control problem. This subspace does not depend on the particular solution  $X$ , nor does the closed-loop matrix restricted to this subspace. This new geometric analysis reveals that the spectrum of the closed-loop system is divided into two parts: the first depends on the solution  $X$  of the CGDARE( $\Sigma$ ), while the second – coinciding exactly with the eigenvalues of the closed-loop restricted to this subspace – is independent of it. However, this fact does not constitute a limitation in the design of the optimal feedback, because when  $R + B^T X B$  is singular, the set of optimal controls presents a further degree of freedom – which is also identified in [16, Remark 4.2.3] – that allows to place all the poles of the closed-loop system at the desired locations without changing the cost.

*Notation.* Given the rational matrix  $M(z)$ , we define  $M^\sim(z) \triangleq M^T(z^{-1})$ . The normal rank of  $M(z)$  is defined as  $\text{normrank} M(z) \triangleq \max_{z \in \mathbb{C}} \text{rank} M(z)$ .

## II. LINEAR QUADRATIC OPTIMAL CONTROL AND CGDARE

In this section we analyse the connections between Linear Quadratic (LQ) optimal control and CGDARE. Consider the classic LQ optimal control problem. In particular, consider the discrete linear time-invariant system governed by

$$x(t+1) = Ax(t) + Bu(t), \quad (5)$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ , and let the initial state  $x_0 \in \mathbb{R}^n$  be given. The problem is to find a sequence of inputs  $u(t)$ , with  $t = 0, 1, \dots, \infty$ , minimising the cost function

$$J(x_0, u) \triangleq \sum_{t=0}^{\infty} \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \quad (6)$$

Since as aforementioned  $\Pi$  is assumed symmetric and positive semidefinite, we can factor it as

$$\Pi = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}, \quad (7)$$

where  $Q = C^T C$ ,  $S = C^T D$  and  $R = D^T D$ . We recall some classic linear algebra results which will be useful in the sequel, [4], [5].

*Lemma 2.1:* Consider the symmetric positive semidefinite matrix  $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ . Then, **(i)**  $\ker P_{12} \supseteq \ker P_{22}$ ; **(ii)**  $P_{12} P_{22}^\dagger P_{22} = P_{12}$ ; **(iii)**  $P_{12}(I - P_{22}^\dagger P_{22}) = 0$ ; **(iv)**  $P_{11} - P_{12} P_{22}^\dagger P_{12}^T \geq 0$ .

We now introduce some notation that will be used throughout the paper. First, to any matrix  $X = X^T \in \mathbb{R}^{n \times n}$  we

associate the following matrices:

$$Q_X \triangleq Q + A^T X A - X, \quad S_X \triangleq A^T X B + S, \quad (8)$$

$$R_X \triangleq R + B^T X B, \quad G_X \triangleq I - R_X^\dagger R_X, \quad (9)$$

$$K_X \triangleq R_X^\dagger S_X^T, \quad A_X \triangleq A - B K_X, \quad C_X \triangleq C - D R_X^\dagger S_X^T, \quad (10)$$

$$\Pi_X \triangleq \begin{bmatrix} Q_X & S_X \\ S_X^T & R_X \end{bmatrix}. \quad (11)$$

Note that  $\ker R_X = \text{im} G_X$ . When  $X$  is a solution of CGDARE( $\Sigma$ ), then  $K_X$  is the corresponding gain matrix,  $A_X$  the associated closed-loop matrix, and  $\Pi_X$  is the so-called *dissipation matrix*. All symmetric and positive semidefinite solutions of GDARE( $\Sigma$ ) satisfy (4), and are therefore solutions of CGDARE( $\Sigma$ ). In fact, if  $X$  is positive semidefinite,

$$\begin{bmatrix} Q_X + X & S_X \\ S_X^T & R_X \end{bmatrix} = \begin{bmatrix} A^T \\ B^T \end{bmatrix} X \begin{bmatrix} A & B \end{bmatrix} + \Pi \geq 0.$$

Therefore, applying Lemma 2.1 we find (4), that can be rewritten as  $\ker R_X \subseteq \ker S_X$  and also as  $S_X G_X = 0$ .

The following theorem illustrates the connection of CGDARE( $\Sigma$ ) and the solution of the standard infinite-horizon LQ optimal control problem.

*Theorem 2.1:* Suppose that for every  $x_0$  there exists an input  $u(t) \in \mathbb{R}^m$ , with  $t \in \mathbb{N}$ , such that  $J(x_0, u)$  is finite. Then:

- 1) CGDARE( $\Sigma$ ) admits symmetric solutions: A solution  $\bar{X} = \bar{X}^T \geq 0$  may be obtained as the limit of the sequence of matrices generated by iterating the *generalised Riccati difference equation*  $P_T(t) = \text{Ricc}\{P_T(t+1)\}$ , where  $\text{Ricc}\{\cdot\}$  is the Riccati operator defined as

$$\text{Ricc}\{P\} \triangleq A^T P A - (A^T P B + S)(R + B^T P B)^\dagger \cdot (B^T P A + S^T) + Q \quad (12)$$

with the zero terminal condition  $P_T(T) = 0$ .

- 2) The value of the optimal cost is  $x_0^T \bar{X} x_0$ .
- 3)  $\bar{X}$  is the minimum positive semidefinite solution of CGDARE( $\Sigma$ ).
- 4) The set of *all* optimal controls minimising (6) can be parameterised as

$$u(t) = -K_{\bar{X}} x(t) + G_{\bar{X}} v(t), \quad (13)$$

with arbitrary  $v(t)$ .

The proof of the first part of this theorem follows from the fact that the sequence of matrices obtained by iterating a generalised Riccati difference equation with the zero terminal condition is non-decreasing and bounded. For a complete proof, we refer to [6].

## III. PRELIMINARY TECHNICAL RESULTS

In this section, we present several technical results that will be used in the sequel. Most of these are ancillary results on the discrete Lyapunov equation and on spectral factorisation of independent interest.

### A. The discrete Lyapunov equation

In this section, we give some important results on the solutions  $X$  of the discrete Lyapunov equation:

$$X = A^T X A + Q, \quad (14)$$

where  $A, Q \in \mathbb{R}^{n \times n}$  and  $Q = Q^T \geq 0$ .

**Lemma 3.1:** Let  $X$  be a solution of the discrete Lyapunov equation (14). Then,  $\ker X$  is  $A$ -invariant and is contained in  $\ker Q$ .

**Proof:** Let  $\lambda \in \mathbb{C}$  be on the unit circle and such that  $(A + \lambda I_n)$  is invertible. We can re-write (14) as  $X = A^T X (A + \lambda I_n) - \lambda A^T X + Q$ , so that

$$(\lambda A^T + I_n) X = \lambda A^T X (\lambda^* A + I_n) + Q,$$

since  $\lambda^* = \lambda^{-1}$ . This is equivalent to

$$X (\lambda^* A + I_n)^{-1} = \lambda (\lambda A^T + I_n)^{-1} A^T X + (\lambda A^T + I_n)^{-1} Q (\lambda^* A + I_n)^{-1} \quad (15)$$

Let  $\xi \in \ker X$ . On pre-multiplying (15) by  $\xi^*$  and post-multiplying it by  $\xi$ , we obtain  $\xi^* (\lambda A^T + I_n)^{-1} Q (\lambda^* A + I_n)^{-1} \xi = 0$ , and since  $(\lambda A^T + I_n)^{-1} Q (\lambda^* A + I_n)^{-1}$  is Hermitian and positive semidefinite, we get

$$Q (\lambda^* A + I_n)^{-1} \xi = 0. \quad (16)$$

Let us now post-multiply (15) by the same vector  $\xi$ . We get  $X (\lambda^* A + I_n)^{-1} \xi = 0$ , which means that  $\ker X$  is  $(\lambda^* A + I_n)^{-1}$ -invariant. Hence, it is also  $(\lambda^* A + I_n)$ -invariant and therefore  $A$ -invariant. In view of (16),  $\ker X = (\lambda^* A + I_n)^{-1} \ker X$  is also contained in the null-space of  $Q$ . ■

We recall that (14) has a unique solution if and only if  $A$  is *unmixed*, i.e. for all pairs  $\lambda_1, \lambda_2 \in \sigma(A)$  we have  $\lambda_1 \lambda_2 \neq 1$ . In this case we have the following result.

**Lemma 3.2:** Let  $A$  be unmixed and  $X$  be the unique solution of (14) where  $Q = Q^T \geq 0$ . Then,  $\ker X$  is the unobservable subspace of the pair  $(A, Q)$ .

**Proof:** Let  $(A, Q)$  be in the observability form, i.e.,  $A = \begin{bmatrix} A_{11} & O \\ A_{21} & A_{22} \end{bmatrix}$  and  $Q = \begin{bmatrix} Q_1 & O \\ O & O \end{bmatrix}$ , where  $(A_{11}, Q_1)$  is an observable pair. Writing (14) in this basis gives

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^T & A_{21}^T \\ O & A_{22}^T \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} A_{11} & O \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} Q_1 & O \\ O & O \end{bmatrix} \\ = \begin{bmatrix} * & * \\ * & A_{22}^T X_{22} A_{22} \end{bmatrix} + \begin{bmatrix} Q_1 & O \\ O & O \end{bmatrix}. \quad (17)$$

Therefore,  $X_{22}$  satisfies the homogeneous discrete Lyapunov equation  $X_{22} = A_{22}^T X_{22} A_{22}$ . Since  $A$  is unmixed, the submatrix  $A_{22}$  is unmixed, and  $X_{22} = 0$  is the unique solution of  $X_{22} = A_{22}^T X_{22} A_{22}$ . As such, (17) can be simplified as

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & O \end{bmatrix} = \begin{bmatrix} * & A_{11}^T X_{12} A_{22} \\ A_{22}^T X_{12}^T A_{11} & O \end{bmatrix} + \begin{bmatrix} Q_1 & O \\ O & O \end{bmatrix}.$$

Again, since  $A$  is unmixed, for all  $\lambda_1 \in \sigma(A_{11})$  and  $\lambda_2 \in \sigma(A_{22})$  we have that  $\lambda_1 \lambda_2 \neq 1$ , and the top-right block of the latter equation yields the unique solution  $X_{12} = 0$ . Therefore, we get  $X_{11} = A_{11}^T X_{11} A_{11} + Q_1$ . In view of Lemma 3.1 the unique solution  $X_{11}$  of the latter equation has trivial kernel

because the pair  $(A_{11}, Q_1)$  is observable. This implies that  $\ker X = \text{im} \begin{bmatrix} O \\ I \end{bmatrix}$  where the partition is consistent with the block structure of  $X$ . This subspace is the unobservable subspace of  $(A, Q)$ . ■

The last result we need is the following.

**Lemma 3.3:** Let  $A \in \mathbb{R}^{n \times n}$ ,  $F \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and assume  $X \in \mathbb{R}^{n \times n}$  is such that

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} X F = \begin{bmatrix} X \\ O \end{bmatrix}. \quad (18)$$

Then,  $B^T X = 0$ .

**Proof:** Let us choose a basis in which  $F$  is written as  $F = \text{diag}\{N, F_I\}$ , where  $N$  is nilpotent and  $F_I$  is invertible. Let us decompose  $X$  accordingly, i.e.,  $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ . It is very easy to see that  $A^T X_1 N = X_1$  implies  $X_1 = 0$ . In fact, by multiplying such equation by  $A^T$  and  $N$  to the left and to the right, respectively, we obtain  $X_1 = (A^T)^k X_1 N^k$  for all  $k \geq 0$ . By choosing  $k$  to be greater than the nilpotency index of  $N$ , we get  $X_1 = (A^T)^k X_1 N^k = 0$ . From (18) we also obtain  $B^T X_2 F_I = 0$ , which implies  $B^T X_2 = 0$  since  $F_I$  is invertible. Therefore,  $B^T X = 0$ . ■

### B. Spectral Factorisation

Let us define the rational matrix  $W(z) \triangleq C(zI_n - A)^{-1} B + D$ . The spectrum  $\Phi(z) \triangleq W^{\sim}(z) W(z)$  associated with the Popov triple  $\Sigma$  can be written as

$$\Phi(z) = \begin{bmatrix} B^T(z^{-1}I_n - A^T)^{-1} & I_n \end{bmatrix} \Pi \begin{bmatrix} (zI_n - A)^{-1} B \\ I_n \end{bmatrix},$$

which is also referred to as *Popov function* associated with GDARE( $\Sigma$ ), [10]. The matrix inequality for an unknown matrix  $X = X^T$  of the form  $\Pi_X \geq 0$  is called the *discrete Riccati linear matrix inequality*, and is herein denoted by DRLMI( $\Sigma$ ). Let us also define

$$L(X) \triangleq \Pi_X - \Pi = \begin{bmatrix} A^T X A - X & A^T X B \\ B^T X A & B^T X B \end{bmatrix},$$

which is a linear function of  $X$ . The following result holds [17, p.322], see e.g. [2] for a detailed proof.

**Lemma 3.4:** For any  $X = X^T \in \mathbb{R}^{n \times n}$ , there holds

$$\Phi(z) = \begin{bmatrix} B^T(z^{-1}I - A^T)^{-1} & I \end{bmatrix} \Pi_X \begin{bmatrix} (zI - A)^{-1} B \\ I \end{bmatrix}. \quad (19)$$

**Theorem 3.1:** Let  $r$  denote the normal rank of the spectrum  $\Phi(z)$ . If  $X$  is a solution of CGDARE( $\Sigma$ ), the rank of  $R_X$  is equal to  $r$ . If  $X$  is a solution of DRLMI( $\Sigma$ ), the rank of  $R_X$  is at most equal to  $r$ .

For a proof of Theorem 3.1, see [6].

**Remark 3.1:** Theorem 3.1 is strictly related to Theorem 2.4 in [17]. The latter has been derived in the very general setting in which the Popov matrix may not be positive semidefinite. In that case,  $\text{rank} R_X = r$  for any solution  $X$  of CGDARE. Since we are assuming (2), a stronger result holds. Namely,  $\text{rank} R_X \leq r$  for any solution  $X$  of the linear matrix inequality.

#### IV. GEOMETRIC PROPERTIES OF THE SOLUTIONS OF GDARE

The first aim of this section is to show that, given a solution  $X$  of GDARE( $\Sigma$ ), the subspace  $\ker X$  is an output-nulling subspace for the quadruple  $(A, B, C, D)$ , i.e.,

$$\begin{bmatrix} A \\ C \end{bmatrix} \ker X \subseteq (\ker X \oplus 0_p) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix}, \quad (20)$$

and that  $-K_X$  is a *friend* of  $\ker X$ , i.e.,

$$\begin{bmatrix} A - BK_X \\ C - DK_X \end{bmatrix} \ker X \subseteq \ker X \oplus 0_p. \quad (21)$$

In the case  $X = X^T$  is the optimal solution of GDARE( $\Sigma$ ), it is very easy to see that  $\ker X$  is the largest output-nulling subspace of the quadruple  $(A, B, C, D)$ .

*Proposition 4.1:* Let  $X$  be the minimal positive semidefinite solution of GDARE( $\Sigma$ ). Then  $\ker X$  is the *largest* output-nulling subspace of the quadruple  $(A, B, C, D)$ . Moreover,  $-K_X$  is the corresponding friend.

**Proof:** Let  $x_0 \in \ker X$ . Since the corresponding optimal cost is  $J = x_0^T X x_0 = 0$ , the initial state  $x_0$  must belong to the largest output-nulling subspace of the quadruple  $(A, B, C, D)$ . *Vice-versa*, if we take a vector  $x_0$  of the largest output-nulling subspace  $\mathcal{V}^*$  of the quadruple  $(A, B, C, D)$ , by definition it is possible to find a control  $u(t)$  ( $t \geq 0$ ) such that the state trajectory lies on  $\mathcal{V}^*$  by maintaining the output at zero. This means that the corresponding value of the cost is zero. Hence,  $x_0^T X x_0 = 0$  implies  $x_0 \in \ker X$ . The fact that  $-K_X$  is a friend of  $\ker X$  follows straightforwardly from the fact that if the initial state of the system lies on  $\ker X$  and we assume by contradiction that  $(A - BK_X)x_0 \notin \ker X$ , then the corresponding trajectory is not optimal because it is associated with a strictly positive value of the cost. Moreover, since the optimal cost is zero, we must have  $(C - DK_X)\ker X = 0_p$ . ■

This result can be easily generalised to any positive semidefinite solution  $X = X^T \geq 0$  of GDARE( $\Sigma$ ). In fact, consider Problem A, which is a finite-horizon LQ problem on the interval  $\{0, \dots, T\}$  with  $X$  as the penalty matrix of the terminal state, i.e.,

$$J_A = x_T^T X x_T + \sum_{t=0}^{T-1} \begin{bmatrix} x^T(t) & u(t)^T \end{bmatrix} \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

The optimal solution of Problem A is obtained by solving a generalised Riccati difference equation with the terminal condition  $X_T = X \geq 0$ , see e.g. [15]. However, since  $X$  is a solution of GDARE( $\Sigma$ ), the solution of such difference equation is stationary and is equal to  $X$ , i.e.,  $X(t) = X$  for all  $t \in \{0, \dots, T\}$ . The optimal cost of Problem A is therefore  $J_A^* = x_0^T X x_0$ . If (21) does not hold, it is possible to choose  $x_0 \in \ker X$  for Problem A in such a way that  $x_1 = (A - BK_X)x_0 \notin \ker X$ . Let us now consider another finite-horizon LQ problem on the interval  $\{0, \dots, T\}$ , denoted by Problem B, with  $X$  as the penalty matrix of the terminal

state. Problem B is characterised by the cost function

$$J_B = x_T^T X x_T + \sum_{t=1}^{T-1} \begin{bmatrix} x^T(t) & u(t)^T \end{bmatrix} \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

Since  $x_1 \notin \ker X$ , the optimal value of  $J_B$  must be strictly positive, and since  $J_A \geq J_B \geq J_B^* > 0$ , it follows that  $J_A^* > 0$ , which is impossible because if the initial state of Problem A is on an output-nulling subspace of the quadruple  $(A, B, C, D)$ , a control maintaining the output function at zero always exists, and this control leads to zero cost, i.e.,  $J_A^* = 0$ .

Our aim now is to prove a deeper geometric result: (20) and (21) hold for any solution  $X = X^T$  of GDARE( $\Sigma$ ).

*Theorem 4.1:* Let  $X$  be a solution of GDARE( $\Sigma$ ). Then,  $\ker X$  is an output-nulling subspace of the quadruple  $(A, B, C, D)$  and  $-K_X$  is a friend of  $\ker X$ , i.e., (20) and (21) hold.

**Proof:** Since  $X$  is a solution of GDARE( $\Sigma$ ), the identity

$$X = A_X^T X A_X + Q_{0X} \quad (22)$$

holds, where  $Q_{0X} \triangleq [I \ -s_X R_X^+] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I \\ -R_X^+ S_X^T \end{bmatrix} \geq 0$ . In view of Lemma 3.1,  $\ker X$  is  $A_X$ -invariant and is contained in the null-space of  $Q_{0X}$ . By factoring  $\Pi$  as in (7), we get  $Q_{0X} = C_X^T C_X$ . Hence, the subspace  $\ker X$  is also contained in the null-space of  $C_X$  so that  $\ker X$  is output-nulling for the quadruple  $(A, B, C, D)$  and  $-K_X$  is a friend of  $\ker X$ . ■

We now provide a full characterisation of the reachable subspace on  $\ker X$ , because this subspace plays a crucial role in the solution of the associated optimal control problem. We focus our attention on the term  $R_X = R + B^T X B$ . When  $X$  is positive semidefinite, the null-space of  $R_X$  is given by the intersection of the null-space of  $R$  with that of  $X B$ . This result, which is very intuitive and easy to prove for positive semidefinite solutions of CGDARE( $\Sigma$ ), indeed holds for any solution  $X$ . However, in this case the proof – which is divided between Lemma 4.1 and Lemma 4.2 presented below – is much more involved, and requires the machinery constructed in the first part of the paper.

*Lemma 4.1:* Let  $X = X^T$  be a solution of CGDARE( $\Sigma$ ), and  $\mathcal{R}_0 \triangleq \text{im}[B G_X \ A_X B G_X \ A_X^2 B G_X \ \dots \ A_X^{n-1} B G_X]$ . Then,

$$\ker R_X \subseteq \ker R, \quad \text{and} \quad \mathcal{R}_0 \subseteq \ker C_X. \quad (23)$$

For a proof of Lemma 4.1, see [6, Lemma 4.1].

In Lemma 4.1 we have shown that  $\ker R_X \subseteq \ker R$ . Since  $R_X = R + B^T X B$ , it also follows that  $\ker R_X \subseteq \ker(B^T X B)$  for any solution  $X$  of CGDARE( $\Sigma$ ). However, a stronger result holds, which says that  $\ker R_X \subseteq \ker(X B)$ . This is an obvious consequence of Lemma 4.1 for any solution  $X \geq 0$ , while it is a quite surprising and deep geometric result in the general case.

*Lemma 4.2:* Let  $X = X^T$  be a solution of CGDARE( $\Sigma$ ). Then,

$$\ker R_X \subseteq \ker(X B). \quad (24)$$

For a proof of Lemma 4.1, see [6, Lemma 4.2].

*Proposition 4.2:* Let  $X = X^T$  be a solution of CGDARE( $\Sigma$ ) and  $\mathcal{R}_0$  be defined in Lemma 4.1. Then,  $X\mathcal{R}_0 = 0_n$ .

*Remark 4.1:* As an obvious corollary of Lemmas 4.1 and 4.2, we have that

$$\ker R_X = \ker(XB) \cap \ker R = \ker \begin{bmatrix} X & B \\ & R \end{bmatrix}. \quad (25)$$

*Theorem 4.2:* Let  $X = X^T$  be a solution of CGDARE( $\Sigma$ ). Let  $\mathcal{R}_{\ker X}^*$  is the largest reachability subspace on  $\ker X$ . Then,

$$\mathcal{R}_{\ker X}^* = \mathcal{R}_0. \quad (26)$$

**Proof:** Let us first show that

$$\text{im}(BG_X) = \ker X \cap B \ker D. \quad (27)$$

We recall that  $\text{im} G_X = \ker R_X$ . Moreover, from (25) we know that  $\ker R_X = \ker(XB) \cap \ker R$ . Then  $\text{im}(BG_X) = B \ker R_X = B(\ker(XB) \cap \ker R) = \ker X \cap B \ker R = \ker X \cap B \ker D$ .

Now we are ready to prove the statement of this theorem. Since  $\mathcal{R}_0$  is the reachable subspace from the origin of the pair  $(A_X, BG_X)$ , it is the smallest  $A_X$ -invariant subspace containing  $\text{im}(BG_X) = \ker X \cap B \ker D$ . On the other hand, the reachable subspace  $\mathcal{R}_{\ker X}^*$  on  $\ker X$  is characterised as follows [18], [12], [13]: Let  $F$  be an arbitrary friend of  $\ker X$ , i.e.,  $F$  is any feedback matrix such that  $(A + BF)\ker X \subseteq \ker X$  and  $(C + DF)\ker X = 0_p$ . Then  $\mathcal{R}_{\ker X}^*$  is the smallest  $(A + BF)$ -invariant subspace containing  $\ker X \cap B \ker D$ . Notice that  $\mathcal{R}_{\ker X}^*$  does not depend on the choice of the friend  $F$ , [18, Theorem 7.18]. We have seen in Theorem 4.1 that  $F = -K_X$  is a particular friend of  $\ker X$ . For this choice of  $F$ , we have  $A + BF = A - BK_X = A_X$ , so that  $\mathcal{R}_{\ker X}^*$  is the smallest  $A_X$ -invariant subspace containing  $\ker X \cap B \ker D$ , which is  $\mathcal{R}_0$  by definition. ■

In [17] it is proved that the inertia of  $R_X$  is independent of the particular solution  $X = X^T$  of CGDARE( $\Sigma$ ). Here, we want to show that much more is true when  $\Pi$  is positive semidefinite. Namely, the null-space of  $R_X$  is independent of the particular solution  $X = X^T$  of CGDARE( $\Sigma$ ).

*Theorem 4.3:* Let  $X_1, X_2$  be two solutions of CGDARE( $\Sigma$ ). Then,  $\ker R_{X_1} = \ker R_{X_2}$ .

A proof can be found in [6, Theorem 4.3].

Now we want to show that the subspace  $\mathcal{R}_{\ker X}^*$  is independent of the particular solution  $X = X^T$  of CGDARE( $\Sigma$ ). Moreover,  $A_X$  restricted to this subspace does not depend on the particular solution  $X = X^T$  of CGDARE( $\Sigma$ ).

*Theorem 4.4:* Let  $X$  and  $Y$  be two solutions of CGDARE( $\Sigma$ ). Let  $A_X$  and  $A_Y$  be the corresponding closed-loop matrices. Then,

$$\bullet \mathcal{R}_{\ker X}^* = \mathcal{R}_{\ker Y}^*, \text{ and}$$

$$\bullet A_X|_{\mathcal{R}_{\ker X}^*} = A_Y|_{\mathcal{R}_{\ker Y}^*}.$$

A proof of Theorem 4.4 can be found in [6, Theorem 4.4].

## V. STABILISATION

In the previous sections, we have observed that the eigenvalues of the closed-loop matrix  $A_X$  restricted to the subspace  $\mathcal{R}_0$  are independent of the particular solution  $X = X^T$  of CGDARE( $\Sigma$ ) considered. On the other hand, we have also observed that  $\mathcal{R}_0$  coincides with  $\mathcal{R}_{\ker X}^*$ , which is by definition the smallest  $(A - BK_X)$ -invariant subspace containing  $\ker X \cap B \ker D = \text{im}(BG_X)$ . It follows that it is always possible to find a matrix  $L$  that assigns all the eigenvalues of the map  $(A_X + BG_X L)$  restricted to the reachable subspace  $\mathcal{R}_{\ker X}^*$ , by adding a further term  $BG_X Lx(t)$  to the feedback control law, because this does not change the value of the cost with respect to the one obtained by  $u(t) = -K_X x(t)$ . Indeed, the additional term only affects the part of the trajectory on  $\mathcal{R}_{\ker X}^*$  which is output-nulling. However, in doing so it may stabilise the closed-loop if  $\ker X$  is externally stabilised by  $-K_X$ . We show this fact in the following example.

*Example 5.1:* Consider a Popov triple in which  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ ,  $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  and  $R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . The matrix  $X = \text{diag}\{0, 1\}$  is the only solution of GDARE( $\Sigma$ ) but not a solution of DARE( $\Sigma$ ), since  $R + B^T X B$  is singular. Hence, DARE( $\Sigma$ ) does not admit solutions. The corresponding closed-loop matrix is  $A_X = \text{diag}\{1, 0\}$ , so that the resulting closed-loop system is not asymptotically stable. However, the solution  $X$  of GDARE( $\Sigma$ ) is optimal for the LQ problem, because it leads to the cost  $J^* = x_2^2(0)$  which cannot be decreased. Now, consider the gain  $K = B^{-1}A$ . This gain leads to the closed-loop matrix  $A_{CL} = A - BK = 0$ , and the value of the performance index associated with this closed-loop is again  $J = x_2^2(0) = J^*$ . Therefore, this is another optimal solution of the LQ problem, which differently from  $X$  is also stabilising. However, this optimal solution is not associated with any solution of GDARE( $\Sigma$ ), since as aforementioned  $X$  is the only solution of GDARE( $\Sigma$ ). This example shows that there exists an optimal control which is stabilising, but no stabilising solutions of GDARE( $\Sigma$ ) exist. This fact can be explained on the basis of the fact that the set of all solutions of the infinite-horizon LQ problem is given by

$$\mathcal{U}(t) = \{-K_X x(t) + G_X v(t) \mid v(t) \in \mathbb{R}^m\},$$

where  $X$  is the optimizing solution of GDARE( $\Sigma$ ) and  $G_X = (I - R_X^\dagger R_X) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ . Therefore, the problem becomes that of using the degree of freedom given by  $v(t)$  in order to find a closed-loop solution that is optimal and also stabilising. In other words, we determine a matrix  $L$  in

$$x(t+1) = (A - BK_X)x(t) + BG_X Lx(t) = A_X x(t) + BG_X Lx(t)$$

such that the closed-loop  $A_{CL} = A_X + BG_X L$  is stabilised, see e.g. [13]. It is easy to see that, in general, the set of all optimal closed loop matrices  $A_{CL} = A_X + BG_X L$  are parameterised by  $A_{CL} = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$  where  $\alpha$  and  $\beta$  can be arbitrarily chosen by selecting a suitable  $L$ . In fact, since

$BG_X = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ , by choosing  $L = \begin{bmatrix} \alpha^{-1} & \beta \\ 0 & 0 \end{bmatrix}$  we obtain the desired form for the closed-loop matrix. Hence, in particular, we can obtain a zero or nilpotent closed-loop matrix. In both cases, the cost is the same and is equal to  $J^* = x_2^2(0)$ .

In other words, there is only one solution to GDARE( $\Sigma$ ) and is not stabilising, and all the optimal solutions of the optimal control problem are given by the closed-loop matrix  $A_X + BG_X L$ , where  $L$  is a degree of freedom. By using this degree of freedom, we have found solutions of the optimal control problem that are stabilising but which do not correspond to stabilising solutions of GDARE( $\Sigma$ ), because GDARE( $\Sigma$ ) does not have stabilising solutions.  $\square$

*Example 5.2:* Consider an LQ problem with

$$A = \begin{bmatrix} 2 & -6 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix}, \quad C = [1 \ 0], \quad D = [0 \ 5],$$

which lead to the penalty matrices  $Q = C^T C = \text{diag}\{1, 0\}$ ,  $S = C^T D = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$ ,  $R = D^T D = \text{diag}\{0, 25\}$ . The positive semi-definite matrix  $X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  is a solution of GDARE( $\Sigma$ ) but not of DARE( $\Sigma$ ), since  $R + B^T X B = R$  is singular. The corresponding gain is  $K_X = \begin{bmatrix} 0 & 0 \\ 1/5 & 0 \end{bmatrix}$  and the closed-loop matrix  $A_X$  coincides with  $A$ . Hence, the closed-loop system is not asymptotically stable. In this case,  $\ker X = \mathbb{R}^2$ , and the reachable subspace on  $\ker X$  coincides with  $\ker X$ . Therefore, even if  $X$  is not stabilising, using the degree of freedom given by matrix  $G_X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  we can find a solution of the infinite-horizon LQ problem with the same cost that is also stabilising. Here,  $\ker D = \text{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  so that  $B \ker D = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , and therefore also  $\text{im}(BG_X) = \ker X \cap B \ker D = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . By using the standard procedure for the assignment of the eigenvalues of  $(A_X + BG_X L)$  restricted to  $\mathcal{R}_{\ker X}^*$ , we can assign the closed-loop eigenvalues arbitrarily without destroying optimality. For example, if we choose to the values  $\{0.2, 0.3\}$  we obtain the matrix

$$L = \begin{bmatrix} 19/200 & -17/60 \\ -3/5 & 0 \end{bmatrix}.$$

The new closed-loop matrix becomes

$$A_X + BG_X L = \begin{bmatrix} 2 & -6 \\ 57/100 & -17/10 \end{bmatrix},$$

whose eigenvalues are exactly  $\{0.2, 0.3\}$ .  $\square$

#### CONCLUDING REMARKS

In this paper we analysed some structural properties of the generalised algebraic Riccati equation that arises in infinite-horizon discrete LQ optimal control. Important side results on discrete Lyapunov equations and on spectral factorisation have been established to the end of showing the fundamental role that the term  $R_X$  plays in the structure of the solutions of the CGDARE and of the corresponding LQ problem. The considerations that emerged from this analysis have in turn been used to show that a subspace can be identified that is independent of the particular solution of CGDARE considered. Even more importantly, it has been shown that

the closed-loop matrix restricted to this subspace does not depend on the particular solution of CGDARE. If such subspace is not zero, in the optimal control a further term can be added to the state-feedback generated from the solution of the Riccati equation that does not modify the value of the cost. This term can in turn be expressed in state-feedback form, and acts as a degree of freedom that can be employed to stabilise the closed-loop even in cases in which no stabilising solutions exists of the Riccati equation.

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