

# Validation of Vincenty's formulae for the geodesic using a new fourth-order extension of Kivioja's formula

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## Abstract

Vincenty's (1975) formulae for the direct and inverse geodetic problems (i.e., in relation to the geodesic) have been verified by comparing them with a new formula developed by adapting a fourth-order Runge-Kutta scheme for the numerical solution of ordinary differential equations (ODEs), advancing the work presented by Kivioja (1971). A total of 3801 lines of varying distances (10 km to 18,000 km) and azimuths ( $0^\circ$  to  $90^\circ$ ; because of symmetry) were used to compare these two very different techniques for computing geodesics. In every case, the geodesic distances agreed to within 0.115 mm, and the forward and reverse azimuths agreed to within  $5 \times 10^{-6}$  arc-seconds, thus verifying Vincenty's formula. If one wishes to plot the trajectory of the geodesic, however, the fourth-order Runge-Kutta extension of Kivioja's formula is recommended as a numerically efficient and convenient approach.

*Keywords:* geodesic; forward geodetic problem; reverse geodetic problem; distance; azimuth

## 1 Introduction

The *geodesics* on a surface are [locally at least] the curves of shortest distance on that surface between any two points. In geodesy, the geodesic is well understood to refer to the shortest surface distance between two points on the surface of the *ellipsoid* or *spheroid*. Over the years, there has been a considerable amount of work done on providing formulae that allow one to calculate geodesics, or normal sections as an approximation of the geodesic; see, e.g., Vincenty (1975), Sqaito (1979), Pittman (1986), Robbins (1962), Bowring (), Sodano (1965), Bowring

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(1981), Rainsford (19) Kivioja (1971), Krakiwsky and Thomson (1974, Bomford (1980). This paper will only consider the geodesic.

Two main problems are of geodetic interest. They are:

**The Direct Problem:** Given the latitude and longitude  $(\varphi_1, \lambda_1)$  of a point on the ellipsoid, along with the starting azimuth  $\alpha_1$  and geodesic distance  $s$ , find the finishing point  $(\varphi_2, \lambda_2)$  and azimuth  $\alpha_2$  of the geodesic at  $(\varphi_2, \lambda_2)$ .

**The Inverse Problem:** Given two points  $(\varphi_1, \lambda_1)$  and  $(\varphi_2, \lambda_2)$  on the ellipsoid, find the geodesic distance  $s$  between them, and the azimuths  $\alpha_1$  and  $\alpha_2$  of the geodesic at  $(\varphi_1, \lambda_1)$  and  $(\varphi_2, \lambda_2)$ .

Previous studies comparing various formulae for solution to the direct and inverse geodetic problems (i.e., the calculation of geodesic lengths and forward and reverse azimuths) have been made by, for example, Dupuy (1958) and by the Aeronautical Chart and Information Centre (1959 and 1960). However, these and other studies tend to rely on the assumption that one formula is more precise than the others, with this usually being Vincenty's. Of course, this begs the question: 'How accurate are Vincenty's algorithms?' To provide some sort of answer to this, modifications have been made to a method proposed by Kivioja (1971) for the solution of the direct problem.

Kivioja's (1971) approach is actually an application of a very basic method for the numerical solution of ordinary differential equations (ODEs). As such, it has quite a different 'flavour' to the other formulae for the solution of the direct problem. As originally proposed, however, Kivioja's method is not suitable for the solution of the direct problem over very long lines, for reasons outlined in Section 3.1. However, the modified version presented here is able to solve the direct problem extremely accurately and over arbitrarily long distances.

The results obtained from this method were compared with those obtained from Vincenty's direct algorithm over more than 3,800 lines with lengths ranging from 10 km to 18,000 km. In every case, they agree to within less than 0.115 mm. While this is not an unequivocal proof that Vincenty's algorithm provides sub-mm accuracy over all lines of up to 18,000 km, it is certainly strong evidence for thinking so.

All programming for this study was done in Java. Java was chosen for convenience, because of its simplicity, portability and the fact that almost all modern personal computers and workstations have a Java compiler. While Java is not considered a fast language, it has proven more than adequate for the computing required for the geodesic. Java source code is available from the first-named author.

All the numerical computations for comparison between the formulae were carried out using the International ellipsoid ( $a = 6378388.000$  m,  $b = 6356911.946$  m). While the GRS80 ellipsoid (Moritz, 1980) is more common in current use, for comparison of geodesic formulae the actual choice of ellipsoid will make no difference to the results.

## Notation

$a$  = major semi-axis of ellipsoid

$b$  = minor semi-axis of ellipsoid

$f$  =  $1 - b/a$  (flattening)

$\varphi$  = latitude

$\lambda$  = longitude

$\alpha$  = azimuth (measured clockwise from north)

$e'^2$  =  $(a^2 - b^2)/b^2$  (square of second eccentricity of ellipsoid)

$c$  =  $a^2/b$  (radius of curvature at the poles)

$V$  =  $\sqrt{1 + e'^2 \cos^2 \varphi}$

$M$  =  $c/V^3$  (radius of curvature in the meridian)

$N$  =  $c/V$  (radius of curvature in the prime vertical)

$C_c$  =  $N \cos \varphi \sin \alpha$  (Clairaut's constant for a geodesic)

$s$  = geodesic length

## 2 The Differential Equations of a Geodesic

A geodesic is a solution of a certain system of ODEs. Numerical algorithms can be used to trace the solutions of such systems from given initial conditions. For geodesics, this means tracing the curve from a given starting latitude, longitude and azimuth. In other words, solving the direct problem. This is the basis for the methods considered in Sections 3 and 4.

One advantage of these methods is that they not only produce the coordinates and azimuth of the end point of the geodesic, but also the coordinates and azimuth at many intermediate points along the geodesic. The major disadvantage of this approach is that it is not possible to 'invert' the formula and thereby obtain a closed solution to the inverse problem.

Two of these algorithms will be considered in some detail. First, an approach proposed by Kivioja (1971) will be examined. This method has a problem in that, as it stands, it is not able to trace a geodesic past a point of maximum or minimum latitude (the so called *vertices* of the geodesic). This leads to a modified version that provides extremely accurate solutions of the direct problem for lines of any length. This method is used in Section 5 to perform a comprehensive test of Vincenty's direct algorithm.

Figure ?? shows a small portion  $\delta s$  of a curve on the ellipsoid. The length of the meridian  $\lambda$  between  $\varphi - \delta\varphi$  and  $\varphi$  is approximately  $M\delta\varphi$  where  $M = M(\varphi)$  is the radius of curvature of the meridian. It can be shown (e.g., Bomford, 1980) that the radius of the parallel at  $\varphi$  is  $N \cos \varphi$ , so the length of that parallel between  $\lambda$  and  $\lambda + \delta\lambda$  is  $N \cos \varphi \times \delta\lambda$ . If  $\delta s$  is small, then the curvilinear triangle in Figure ?? is approximately a plane, right triangle.

Applying simple plane trigonometry and taking the limit as  $\delta s \rightarrow 0$  yields

$$\frac{d\varphi}{ds} = \frac{\cos \alpha}{M(\varphi)} \quad (1)$$

$$\frac{d\lambda}{ds} = \frac{\sin \alpha}{N(\varphi) \cos \varphi} \quad (2)$$

Any curve on the ellipsoid satisfies Equations (1) and (2). If the curve is a geodesic then it can be shown (e.g., Bomford, 1980) to satisfy the additional constraint

$$N \cos \varphi \sin \alpha = C_c. \quad (3)$$

This is known as *Clairaut's Equation*. The quantity  $C_c$  is constant for any particular geodesic, and is known as the *Clairaut Constant* of the geodesic.

Substitution of Equation (3) into Equations (1) and (2) leads to the system of ODEs

$$\frac{d\varphi}{ds} = \frac{\pm \sqrt{N^2 \cos^2 \varphi - C_c^2}}{MN \cos \varphi} \quad (4)$$

$$\frac{d\lambda}{ds} = \frac{C_c}{N^2 \cos^2 \varphi} \quad (5)$$

where the sign of the square root in Equation (4) is the same as the sign of  $\cos \alpha$ .

Equation (3) yields an expression for  $\alpha$  in terms of  $\varphi$

$$\alpha = \pm \arcsin \left( \frac{C_c}{N \cos \varphi} \right). \quad (6)$$

Equations (4) and (5) completely describe the path of a geodesic, and enable one to numerically calculate  $\varphi$  and  $\lambda$  at distances along the curve. For any particular geodesic, Equation (6) determines  $\alpha$  for a given  $\varphi$ .

Other systems of differential equations are possible. For example one can replace Equation (4) with an equation for  $\alpha$ ; see Section 4.

There are many numerical algorithms for the solution of systems of ordinary differential equations like Equations (4) and (5). Among the simplest and most commonly applied are the Runge-Kutta methods of orders two and four. See, for example, Butcher (1987) for a description of these and other algorithms.

### 3 Kivioja's Method for the Solution of the Direct Problem

The solution to the direct problem proposed in Kivioja (1971) is nothing more than an application of the Runge-Kutta method of second order to Equations (4) and (5). When applied to these equations, the second-order Runge-Kutta method involves the following stages:

1. The number of steps  $n$  is decided upon, and the step size  $\delta s$  is calculated as  $\delta s = s/n$ . The choice of  $n$  represents a compromise between speed (small  $n$ ) and accuracy (small  $\delta s$ ). For mm accuracy Jank and Kivioja (1980) recommend  $\delta s$  be about 100 m to 200 m, and for cm accuracy about one or two kilometres.
2. The initial point  $(\varphi_1, \lambda_1)$  and azimuth  $\alpha_1$  are used to calculate  $C_c$  using Equation (3).
3.  $(\varphi_i, \lambda_i)$  is set to  $(\varphi_1, \lambda_1)$ .
4. A temporary estimate of  $\varphi$  at a distance  $\delta s/2$  along the curve is calculated as  $\varphi_t = \varphi_i + 0.5 \times \delta s \times d\varphi/ds|_{\varphi=\varphi_i}$ .
5. The values of  $\varphi$  and  $\lambda$  at a distance  $\delta s$  past  $(\varphi_i, \lambda_i)$  are estimated as  $\varphi_f = \varphi_i + \delta s \times d\varphi/ds|_{\varphi=\varphi_t}$  and  $\lambda_f = \lambda_i + \delta s \times d\lambda/ds|_{\varphi=\varphi_t}$ . If it is required, the azimuth at  $(\varphi_f, \lambda_f)$  can be calculated from Equation (6) using the constant  $C_c$  calculated in stage 2.
6.  $\varphi_f$  and  $\lambda_f$  are then used as the initial points  $\varphi_i$  and  $\lambda_i$  in stage 4, and stages 4 and 5 are iterated  $n$  times to find the final coordinates  $(\varphi_2, \lambda_2)$ .
7. The azimuth  $\alpha_2$  of the geodesic at  $(\varphi_2, \lambda_2)$  can be calculated from Equation (6), along with the back azimuth  $\alpha_{21} = \alpha_2 \pm 180$ .

This simple algorithm can be easily programmed into a computer or a programmable calculator.

### 3.1 The Accuracy and Applicability of Kivioja's Direct Method

Unfortunately, Kivioja's method as outlined above (essentially the same as in Kivioja (1971) and Jank and Kivioja (1980)) has several limitations. The most troublesome of these is the behaviour of the method for  $\alpha$  near 90 or 270 degrees, that is where  $N^2 \cos^2 \varphi - C_c^2 \approx 0$ . This occurs as  $\varphi$  approaches its maximum or minimum values  $\varphi = \pm\varphi_{\max}$  referred to here as the *vertices* of the geodesic. In fact, the parallels  $\varphi \equiv \varphi_{\max}$  and  $\varphi \equiv -\varphi_{\max}$  also satisfy Equations (4) and (5), but are not geodesics. The solution bifurcates at the vertices, and the algorithm will follow the constant branch. This is because  $d\varphi/ds = 0$  at  $\pm\varphi_{\max}$ , and so  $\varphi$  is not altered at stages 4 and 5 of the procedure outlined in Section 3. Because of this the method cannot, without modification (described later), be used to follow a geodesic past a vertex.

It should be noted that this problem is not confined to very long lines. For example, Kivioja's method would not give a result for the line with  $\varphi_1 = 35^\circ 00' 00''$ ,  $\alpha_1 = 89^\circ 10' 00''$  and  $s = 150$  km. Clearly, it is possible to contrive arbitrarily short lines for which Kivioja's method will fail. For this reason, it is difficult to recommend its use, particularly as in the next section it will be shown how a few simple modifications to the method enable it to provide extremely accurate solutions to the direct problem for any lines whatsoever.

Equation (3) yields a simple expression for  $\varphi_{\max}$ , since  $\sin \alpha = 1$  at these points and so

$$N \cos \varphi = C_c$$

$$\frac{c}{\sqrt{1 + e'^2 \cos^2 \varphi}} \cos \varphi = C_c$$

which upon rearrangement yields

$$\cos^2 \varphi_{\max} = \frac{C_c^2}{c^2 - C_c^2 e'^2}. \quad (7)$$

Using Equation (7) it is, in principle, possible to modify the above procedure to compute lines that pass through a vertex, say for example from  $P_1$  to  $P_2$  in Figure ???. Such a procedure is briefly outlined in Jank and Kivioja (1980). It amounts to computing the line separately from  $P_1$  to  $P_{\max}$  and from  $P_{\max}$  to  $P_2$ . In fact, it would be very difficult to obtain accurate results using Equation (4). This is because such a procedure would, in essence, have to calculate the geodesic distance to the vertex. That is, it would have to calculate  $s$  as a function of  $\varphi$  near the vertex, rather than  $\varphi$  as a function of  $s$ . This is difficult to do accurately because  $ds/d\varphi$  approaches infinity at the vertex, and so small errors in  $\varphi$ , such as those produced by rounding in a computer, will produce large errors in  $s$ .

Another limitation of Kivioja's technique is that, in practise, the calculation of azimuth near  $\pm\varphi_{\max}$ , that is near azimuths of 90 or 270 degrees, is likely to be inaccurate. This is because  $\alpha$  is calculated using Equation (6) and the derivative of the function  $\arcsin$  is large for arguments near  $\pm 1$ . Thus, small errors in  $C_c/(N \cos \varphi)$ , due to computer rounding of floating point numbers for example, will produce large errors in the calculation of  $\alpha$ .

Further criticisms of Kivioja's approach have been raised by Meade (1981). In particular, even for modern programmable calculators, the number of iterations that are necessary for accurate results over long distances means that the calculation will be very slow compared with other methods that are available, for example those due to Vincenty (1975) and Sodano (1965) for long lines.

Having said this, Kivioja's technique is a transparent and very accurate way of solving the direct problem for geodesic paths that do not approach the maximum or minimum latitude. In addition, it enables the calculation of intermediate coordinates and azimuths along the path. This auxiliary information may be of use to those who wish to plot a geodesic; for example, to plot the shortest trajectory of an aircraft.

## 4 Improving Kivioja's Method for the Direct Geodetic Problem

In this section, some improvements to Kivioja's (1971) method are detailed. These address some of the shortcomings outlined in Section 3.1.

One improvement that can be made is the use of a fourth-order Runge-Kutta method. This provides greater accuracy than the second-order method used by Kivioja (1971), but with a larger step size and therefore fewer iterations. Moreover, it is not significantly more difficult to program. This increases the speed of the method, particularly for use with programmable calculators. Ramana Murty et al. (1993) used a similar fourth-order scheme (a Runge-Kutta-Gills method) for the solution of the direct problem.

The use of a fourth-order Runge-Kutta method does not solve the problem of computing the geodesic past a vertex. A more substantial improvement can be made to Kivioja's (1971) method by altering the system of equations to which the numerical integration is applied. Rather than calculating  $\varphi$  and  $\lambda$  by applying the Runge-Kutta method to Equations (4) and (5), and then using Equation (6) to calculate  $\alpha$  as a function of  $\varphi$ , one may form a differential equation for  $\alpha$  and apply the Runge-Kutta method to that equation as well.

Differentiating  $V$  with respect to  $\varphi$  yields

$$\frac{dV}{d\varphi} = -\frac{e'^2 \cos \varphi \sin \varphi}{V}. \quad (8)$$

Differentiating  $N \cos \varphi$ , using Equation (8), gives

$$\begin{aligned} \frac{d}{d\varphi}(N \cos \varphi) &= -cV^{-2} \frac{dV}{d\varphi} \cos \varphi - cV^{-1} \sin \varphi \\ &= cV^{-1} \sin \varphi (V^{-2} e'^2 \cos^2 \varphi - 1) \\ &= cV^{-1} \sin \varphi \left( \frac{e'^2 \cos^2 \varphi}{1 + e'^2 \cos^2 \varphi} - 1 \right) \\ &= -cV^{-3} \sin \varphi \\ &= -M \sin \varphi. \end{aligned} \quad (9)$$

Differentiating Equation (3) with respect to  $s$  and substituting Equations (9) and (1) gives

$$-M \sin \varphi \frac{\cos \alpha}{M} \sin \alpha + N \cos \varphi \cos \alpha \frac{d\alpha}{ds} = 0,$$

and therefore

$$\frac{d\alpha}{ds} = \frac{\tan \varphi \sin \alpha}{N} = \sin \varphi \frac{d\lambda}{ds}. \quad (10)$$

Clairaut's Equation (3) was solved to give  $\alpha$  in Equation (6), but can also be solved for  $\varphi$  to give

$$\varphi = \pm \arccos \left( \sqrt{\frac{C_c^2}{c^2 \sin^2 \alpha - e'^2 C_c^2}} \right). \quad (11)$$

From Equations (2) and (3), one obtains

$$\frac{d\lambda}{ds} = \frac{\sin^2 \alpha}{C_c}, \quad (12)$$

and Equations (10), (11) and (12) give

$$\frac{d\alpha}{ds} = \pm \sqrt{\frac{c^2 \sin^2 \alpha - C_c^2 e'^2 - C_c^2}{c^2 \sin^2 \alpha - C_c^2 e'^2}} \times \frac{\sin^2 \alpha}{C_c}. \quad (13)$$

Either of Equations (4) or (13) can be used in the numerical integration of the geodesics. However it has already been noted, and this applies equally to the fourth-order method, that the Runge-Kutta algorithm applied to Equation (4) will not work when  $d\varphi/ds = 0$ . This occurs when  $|\varphi| = \varphi_{\max}$ . For similar reasons it will not work for Equation (13) when  $d\alpha/ds = 0$ . This occurs at the equator. The solution is to use Equation (13) when  $|\varphi|$  is near  $\varphi_{\max}$ , and to use Equation (4) when  $\varphi$  is near 0.

Thus, for  $|\varphi| \leq \varphi_{\max}/2$ , the (fourth-order) Runge-Kutta method is applied to Equations (4) and (5), using Equation (6) to find  $\alpha$ . If  $|\varphi| > \varphi_{\max}/2$ , then the Runge-Kutta method is applied to Equations (12) and (13), and Equation (11) is used to find  $\varphi$ . In the course of a calculation, if the geodesic crosses one of the parallels  $\varphi = \pm\varphi_{\max}/2$ , then the system of equations is changed from Equations (4), (5) and (6) to Equations (11), (12) and (13), or *vice versa*. This scheme allows geodesics to be traced for arbitrarily large distance  $s$ , with great accuracy and without encountering problems at the equator or at the vertices.

There is a family of fourth order Runge-Kutta methods, and they are described in Butcher (1987). However, one in particular is usually described as *the* fourth order Runge-Kutta method. The following shows how it is applied in the case at hand.

For  $|\varphi| \leq \varphi_{\max}/2$  the basic procedure is similar to that outlined in Section 3. Only stages 4 and 5 need modification; they are replaced by:

4'. The quantities  $j_1, \dots, j_4$  and  $k_1, \dots, k_4$  are calculated using Equations (4) and (5):

$$\begin{aligned} j_1 &= \delta s \times d\varphi/ds|_{\varphi=\varphi_i}, & k_1 &= \delta s \times d\lambda/ds|_{\varphi=\varphi_i} \\ j_2 &= \delta s \times d\varphi/ds|_{\varphi=\varphi_i+j_1/2}, & k_2 &= \delta s \times d\lambda/ds|_{\varphi=\varphi_i+j_1/2} \\ j_3 &= \delta s \times d\varphi/ds|_{\varphi=\varphi_i+j_2/2}, & k_3 &= \delta s \times d\lambda/ds|_{\varphi=\varphi_i+j_2/2} \\ j_4 &= \delta s \times d\varphi/ds|_{\varphi=\varphi_i+j_3}, & k_4 &= \delta s \times d\lambda/ds|_{\varphi=\varphi_i+j_3} \end{aligned}$$

5'. The values of  $\varphi$  and  $\lambda$  at a distance  $\delta s$  along the geodesic past  $(\varphi_i, \lambda_i)$  are estimated as

$$\begin{aligned} \varphi_f &= \varphi_i + j_1/6 + j_2/3 + j_3/3 + j_4/6 \\ \lambda_f &= \lambda_i + k_1/6 + k_2/3 + k_3/3 + k_4/6 \end{aligned}$$

The value of  $\alpha$  can be obtained from Equation (6).

If  $|\varphi| > \varphi_{\max}/2$ , then the procedure is entirely analogous, but instead uses Equations (12) and (13), which increment  $\alpha$  and  $\lambda$ . Simply replace, with  $\alpha$ , all occurrences of  $\varphi$  in steps 4' and 5'. The value of  $\varphi$  is then obtained from Equation (11).

Not only does this method allow the accurate computation of arbitrarily long geodesics, it has substantial performance benefits over the second-order Runge-Kutta method that forms the basis of Kivioja's formula, particularly if it is to be used with a hand-held calculator. It allows the use of a larger step size, while still maintaining sufficient accuracy, thereby speeding up calculations.

This scheme also addresses another of the shortcomings of Kivioja’s method mentioned in Section 3.1. In the critical region where  $|\varphi| \approx \varphi_{\max}$ , that is, where  $\alpha$  is near 90 or 270 degrees, one is not relying on Equation (6), with its sensitive dependence on  $\varphi$  in this region, to calculate  $\alpha$ . Instead, whenever  $|\varphi| > \varphi_{\max}/2$ ,  $\alpha$  is being incremented using Equation (13).

#### 4.1 Modified Kivioja’s Method for the Solution of the Inverse Problem

Kivioja’s method for the solution of the direct problem is of interest because of the possibility of using this general technique to provide an independent check on other formulae, specifically Vincenty’s formula. The solutions proposed for the inverse problem in Kivioja (1971) and Janks and Kivioja (1980) appear to rely largely on a trial and error procedure, using Kivioja’s approach to the direct problem. As such they do not appear to be viable alternatives to other (closed) formulae in general use, and therefore they have not been investigated further.

## 5 Vincenty’s Formulae

Vincenty (1975) presented formulae for the solution of the direct and inverse problems that have now become widely used and are easily programmed into a computer or hand-held calculator. For example, Geoscience Australia (2003) provide free copies of a spreadsheet that implements Vincenty’s direct and inverse formulae. Vincenty’s formulae are based on an iterative procedure, but the number of iterations required for a solution is small, typically three or four. Even using a relatively slow programmable calculator, the time to obtain a solution is very short.

Vincenty’s formulae is re-presented here. The following notation is used, in addition to that introduced in Section 1:

- $L$  = difference in longitude, positive east
- $\alpha$  = azimuth of geodesic at the equator
- $U$  = reduced latitude,  $\tan U = (1 - f) \tan \varphi$
- $\Lambda$  = difference in longitude on auxiliary sphere
- $\sigma$  = angular distance  $P_1, P_2$  on the sphere

$\sigma_1$  = angular distance on the sphere from the equator to  $P_1$

$\sigma_m$  = angular distance on sphere from equator to line midpoint

The direct formula is as follows:

$$\tan \sigma_1 = \tan U_1 / \cos \alpha_1$$

$$\sin \alpha = \cos U_1 \sin \alpha_1$$

$$A = 1 + \frac{u^2}{16384} (4096 + u^2 (-768 + u^2 (320 - 175u^2))) \quad (14)$$

$$B = \frac{u^2}{1024} (256 + u^2 (-128 + u^2 (74 - 47u^2))) \quad (15)$$

$$2\sigma_m = 2\sigma_1 + \sigma \quad (16)$$

$$\Delta\sigma = B \sin \sigma \left( \cos 2\sigma_m + \frac{B}{4} [\cos \sigma (-1 + 2 \cos^2 2\sigma_m) - \frac{B}{6} \cos 2\sigma_m (-3 + 4 \sin^2 \sigma) (-3 + 4 \cos^2 2\sigma_m)] \right) \quad (17)$$

$$\sigma = \frac{s}{bA} + \Delta\sigma \quad (18)$$

The first term of Equation (18) is used as a first approximation to  $\sigma$ , then Equations (16), (17) and (18) are iterated until there is negligible change in  $\sigma$ . One then computes

$$\tan \varphi_2 = \frac{\sin U_1 \cos \sigma + \cos U_1 \sin \sigma \cos \alpha_1}{(1-f)\sqrt{\sin^2 \alpha + (\sin U_1 \sin \sigma - \cos U_1 \cos \sigma \cos \alpha_1)^2}}$$

$$\tan \Lambda = \frac{\sin \sigma \sin \alpha_1}{\cos U_1 \cos \sigma - \sin U_1 \sin \sigma \cos \alpha_1}$$

$$C = \frac{f}{16} \cos^2 \alpha (4 + f(4 - 3 \cos^2 \alpha)) \quad (19)$$

$$L = \Lambda - (1 - C)f \sin \alpha \times$$

$$(\sigma + C \sin \sigma [\cos 2\sigma_m + C \cos \sigma (-1 + 2 \cos^2 2\sigma_m)]) \quad (20)$$

$$\tan \alpha_2 = \frac{\sin \alpha}{-\sin U_1 \sin \sigma + \cos U_1 \cos \sigma \cos \alpha_1}$$

This solves the direct problem.

For the inverse formula one uses

$$\Lambda = L$$

as a first approximation, and then computes

$$\sin^2 \sigma = (\cos U_2 \sin \Lambda)^2 + (\cos U_1 \sin U_2 - \sin U_1 \cos U_2 \cos \Lambda)^2 \quad (21)$$

$$\begin{aligned}
\cos \sigma &= \sin U_1 \sin U_2 + \cos U_1 \cos U_2 \cos \Lambda \\
\tan \sigma &= \frac{\sin \sigma}{\cos \sigma} \\
\sin \alpha &= \frac{\cos U_1 \cos U_2 \sin \Lambda}{\sin \sigma} \\
\cos 2\sigma_m &= \cos \sigma - \frac{2 \sin U_1 \sin U_2}{\cos^2 \alpha}
\end{aligned}$$

A new estimate of  $\Lambda$  is then computed from Equations (19) and (20). This procedure is iterated, starting with Equation (21), until there is negligible change in  $\Lambda$ . One then computes

$$s = bA(\sigma - \Delta\sigma)$$

where  $\Delta\sigma$  is calculated using Equations (14), (15) and (17), and finally

$$\begin{aligned}
\tan \alpha_1 &= \frac{\cos U_2 \sin \Lambda}{\cos U_1 \sin U_2 - \sin U_1 \cos U_2 \cos \Lambda} \\
\tan \alpha_2 &= \frac{\cos U_1 \sin \Lambda}{-\sin U_1 \cos U_2 + \cos U_1 \sin U_2 \cos \Lambda}
\end{aligned}$$

For this study, Vincenty's (1975) algorithm was implemented precisely as detailed above.

## 6 Validation of Vincenty's Solution for the Direct Geodetic Problem

The results obtained from Vincenty's direct formula were compared to those obtained using the modified Runge-Kutta method described in Section 4, with a step size of 100 m. This step size was chosen because it represents a reasonable compromise between accuracy, obtained with small step sizes, and speed, obtained with larger step sizes and correspondingly fewer iterations. Because of symmetry, only starting points  $(\varphi_1, \lambda_1)$  with  $\varphi_1 \geq 0$  and  $\lambda_1 = 0$  were tested, using azimuths  $\alpha_1$  with  $0 \leq \alpha_1 \leq 180$  degrees. If  $\varphi_1 = 0$  then azimuths only ranged between 0 and 90 degrees. The following combinations of  $\varphi_1$ ,  $\alpha_1$  and  $s$  were tested:

- $\varphi_1$  ranging from zero to 85 degrees in increments of five degrees.
- $\alpha_1$  taking values 0, 1, 2, 5, 10, 20, 30, 40, 50, 60, 70, 80, 85, 88, 89, 90, 91, 92, 95, 100, 110, 120, 130, 140, 150, 160, 170, 175, 178, 179 and 180 degrees (but for  $\varphi_1 = 0$  only those values between 0 and 90 were considered).
- $s$  taking values 10, 100, 500, 1000, 5000, 10000 and 18000 kilometres.

Vincenty's direct formula	
compared with	
modified Runge-Kutta algorithm	
$\Delta\varphi_2$	$3.75'' \times 10^{-6}$
$\Delta\lambda_2$	$4.61'' \times 10^{-6}$
$\Delta\alpha_2$	$4.62'' \times 10^{-6}$
$\Delta s$	0.115 mm

Table 1: Maximum differences observed between Vincenty's direct formula and the modified Runge-Kutta algorithm from Section 4 over the 3801 lines described in the text. A step size of 100 metres was used for the Runge-Kutta algorithm.  $\Delta s$  is calculated as  $\Delta s = \sqrt{M^2(\Delta\varphi)^2 + N^2 \cos^2 \varphi (\Delta\lambda)^2}$ .

This gives a total of 3,801 combinations. Despite the fact that the Runge-Kutta method required up to 180,000 iterations for each combination, this entire test was completed in about one hour of processing time using a 2.2 GHz Athlon processor.

The results were written to an ASCII text file by the program only if the differences in the values of  $\varphi_2$ ,  $\lambda_2$  or  $\alpha_2$  produced by the two algorithms exceeded  $1.0 \times 10^{-6}$  seconds of arc. This occurred for some 405 combinations, a few of which were merely anomalies in the description of angles. For example, after a run over the north pole along a meridian, the modified fourth order Runge-Kutta method, as implemented, might report a latitude of 130 degrees, whereas Vincenty's algorithm would report 50 degrees. They clearly represent the same point. No attempt was made to remove these anomalies in the code, and they occur only in the case of a meridian.

The greatest differences found between Vincenty's direct formula and the modified Runge-Kutta algorithm are tabulated in Table 1.

In all cases tested, the solutions differ by less than 0.115 mm, even over lines of length 18,000 km. This indicates a remarkable agreement between the two methods, and provides an independent validation of Vincenty's direct solution, since the two algorithms use markedly different approaches.

Before it was realised by the first-named author that the Runge-Kutta algorithm could be modified to trace geodesics of arbitrary length, a similar test was conducted using a third-order Taylor series algorithm for numerical solution of ODEs. This test was run using a step size of 10 m and took about twenty hours of processor time

to complete, but no attempt was made to make the code for the Taylor series algorithm as efficient as possible. The greatest differences found between Vincenty and the Taylor series algorithm were  $1.2839'' \times 10^{-6}$  in  $\varphi_2$ ,  $1.18504'' \times 10^{-5}$  in  $\lambda_2$  and  $8.1776'' \times 10^{-6}$  in  $\alpha_2$ . These represent difference of less than 1 mm in length between the two methods.

## 7 Validation of Vincenty's Solution for the Inverse Geodetic Problem

Vincenty's formula for the inverse problem was tested against his formula for the direct problem by using the same set of combinations of starting points  $(\varphi_1, \lambda_1)$ , azimuths  $\alpha_1$  and distances  $s$  used in Section 6. For each combination, the direct formula was applied to compute  $(\varphi_2, \lambda_2)$  and  $\alpha_2$ , and the inverse formula was applied to  $(\varphi_1, \lambda_1)$  and  $(\varphi_2, \lambda_2)$  to compute  $\tilde{s}$ ,  $\tilde{\alpha}_1$  and  $\tilde{\alpha}_2$ . These values were then compared with  $s$ ,  $\alpha_1$  and  $\alpha_2$ . The formulae agree with each other to a very high level of accuracy:

$$\max |\Delta s| = 1.17 \times 10^{-6} \text{ metres}$$

$$\max |\Delta \alpha_1| = 5.29 \times 10^{-8} \text{ seconds of arc}$$

$$\max |\Delta \alpha_2| = 5.33 \times 10^{-8} \text{ seconds of arc.}$$

This is as expected, since "one formula was obtained by reversing the other." (Vincenty, 1975).

One shortcoming of Vincenty's inverse formula, however, is that it will fail for pairs of points that are antipodal or nearly antipodal. This occurs when  $|\Lambda|$ , as computed by Equation (20), is greater  $\pi$ .

## 8 Summary and Conclusions

Kivioja's formula for the direct geodetic problem, as given in Kivioja (1971) or Jank and Kivioja (1980), is not recommended because it is not universally applicable to any geodesic. For instance, it will not solve the problem for geodesics that pass through a vertex, and therefore may fail for lines of any length. Neither is it suitable for hand computation, and nor is it really suitable for use with programmable calculators because of the number of steps required for accuracy, and the fact that Vincenty's direct formula will give equally good results much more quickly.

Therefore, substantial improvements have been made to Kivioja's formula, and this improved formula has

been used to make a comprehensive test of Vincenty's formulae, finding agreement to within 0.115 mm over all 3,801 lines tested. These improvements involved: a fourth-order Runge-Kutta solution to the ODEs for the geodesic, as opposed to Kivioja's second-order approach; and a convenient reorganisation of the equations to avoid problems at the vertices. Importantly, the modified Kivioja formula now gives geodesics of arbitrary length. This improved technique is more suited to use with hand-held calculators because it can provide greater accuracy than Kivioja's original approach, while using larger step sizes and therefore fewer iterations.

If one is interested in tracing out a geodesic, that is knowing the coordinates and azimuth at many intermediate points along the geodesic (e.g., to plot the shortest trajectory of a flight), then the modified fourth order Runge-Kutta method, as presented in Section 4, is recommended. It will trace geodesics of arbitrary length. It is very easy to program (Java source code is available from the first-named author), and using a 100 m step size it gives sub-mm agreement with Vincenty's formula over all distances up to 18,000 km. Programmed in Java and running on a moderately fast desktop machine, it traces out an 18,000 km geodesic in about three seconds (using a 100 m step size).

Despite the availability of the modified Kivioja method presented here, Vincenty's formulae are still recommended for geodetic calculations. The results from the direct formula have been compared with those from the modified Kivioja method. Because these formulae arise from such different approaches to the problem, and because they agree so well, it is likely that Vincenty's formula gives sub-mm accuracy over lines of up to 18,000 km. Vincenty's inverse formula agrees almost perfectly with the direct formula. Both are suitable for use in a programmable scientific calculator, though neither are really suitable for hand calculations.

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