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Exact Tuning of PID Controllers in Control Feedback Design ⋆

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Abstract: In this paper, we introduce a range of techniques for the exact design of PID controllers for feedback control problems involving requirements on the steady-state performance and standard frequency domain specifications on the stability margins and crossover frequencies. These techniques hinge on a set of simple closed-form formulae for the explicit computation of the parameters of the controller in finite terms as functions of the specifications, and therefore they eliminate the need for graphical, heuristic or trial-and-error procedures. The relevance of this approach is *i)* theoretical, since a closed-form solution is provided for the design of PID-type controllers with standard frequency domain specifications; *ii)* computational, since the techniques presented here are readily implementable as software routines, for example using MATLAB®; *iii)* educational, because the synthesis of the controller reduces to a simple exercise on complex numbers that can be solved with pen, paper and a scientific calculator. These techniques also appear to be very convenient within the context of adaptive control and self-tuning strategies, where the controller parameters have to be calculated on-line.

1. INTRODUCTION

Countless tuning methods have been proposed for PID controllers over the last seventy years. Accounting for all of them goes beyond the possibilities of this paper. We limit ourselves to noticing that many surveys and textbooks have been entirely devoted to these techniques, that differ from each other in terms of the specifications, the amount of knowledge on the model of the plant and in terms of tools exploited, see e.g. Åström et al. (1995); Datta et al. (2000); Visioli (2006) and the references therein.

Recently, renewed interest has been devoted to design techniques for PID controllers under frequency domain specifications, see e.g. Yeung et al. (2000); Skogestad (2003); Keel et al. (2008); Ho et al. (1995). In particular, much effort has been devoted to computation of the parameters of the PID controllers that guarantee desired values of the gain/phase margins and of the crossover frequency. Specifications on the stability margins have always been extensively utilised in feedback control system design to ensure a robust control system. It is also common to encounter specifications on phase margin and gain crossover frequency, since these two parameters together often serve as a performance measure of the control system.

In the last fifteen years, three important sets of techniques have been proposed to deal with requirements on the phase/gain margins and on the gain crossover frequency, to the end of avoiding the trial-and-error nature of classical control methods based on Bode and Nichols diagrams. The first one is a graphical method hinging on design charts, and exploits an interpolation technique to determine the parameters of the PID controller. This method can deal with specification on the stability margins, crossover frequencies and on the steady-state performance, Yeung et al. (2000). A second important set of techniques that can handle specifications on the phase and gain margins relies on the approximation of the plant with a first (or second) order plus delay model, Ho et al. (1995), Skogestad (2003).

To overcome the difficulties and approximations of trial-and-error procedures on Bode and Nyquist plots, and of the three above described design methods, a unified design framework is presented in this paper for the closed-form solution of the feedback control problem with PID controllers. In this paper, simple closed-form formulae are easily established for the computation of the parameters of a PID controller that *exactly* meets specifications on the steady-state performance, stability margins and crossover frequencies, without the need to resort to approximations for the transfer function of the plant.

There are several advantages connected with the use of the methods presented here for the synthesis of PID controllers: *i)* unlike other analytical synthesis methods Wakeland (1976), steady-state performance specifications can be handled easily; Moreover, the desired phase/gain margins and crossover frequency can be achieved exactly, without the need for trial-and-error, approximations of the plant dynamics or graphical considerations; *ii)* a closed-form solution to the feedback control problem allows to analyse how the solution changes as a result of variations of the problem data; Moreover, the explicit formulae presented here can be exploited for the self-tuning of the controller; *iii)* very neat necessary and sufficient solvability conditions can be derived for each controller and each set of specifications considered, and reliable methods can be established to select the compensator structure to be employed depending on the specifications imposed; *iv)* The formulae presented here are straightforwardly implementable as MATLAB® routines. Furthermore, the calculation of the

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parameters of the controller is carried out via standard manipulations on complex numbers, and therefore appears to be very suitable for educational purposes; *v)* The closed-form formulae that deliver the parameters of the PID controller as a function of the specifications only depend on the magnitude and argument of the frequency response of the system to be controlled at the desired crossover frequency. As such, this method can be used in conjunction with a graphical method based on *any* of the standard diagrams for the representation of the dynamics of the frequency response, e.g., the Bode, Nyquist or Nichols diagrams; *vi)* In the case a mathematical model of the plant or a graphic representation of its frequency reponse are not available, the technique presented in this paper can be used on a first/second order plus delay approximation of the plant. The extra flexibility offered by the design method presented here consists in the fact that the formulae for the computation of the parameters are not linked to a particular plant structure. Thus, differently from other approaches based on first/second order approximations, when a more accurate mathematical model is available for the plant, the formulae presented here can still be used without modifications, and will deliver more reliable values for the parameters of the compensator.

This paper provides a unified and comprehensive exposition of this technique, not only for PID controllers in standard form, but also for PI and PD controllers.

2. PROBLEM FORMULATION

In this section we formulate the problem of the design of the parameters of a compensator belonging to the family of PID controllers such that different types of steady-state specifications are satisfied and such that the crossover frequency and the phase margin of the loop gain transfer function are equal to desired values ω_g and PM, respectively. Consider compensators described by the one of the following transfer functions:

i) PID controller in standard form:

$$C_{\rm PID}(s) = K_p\left(1+\frac{1}{T_i\,s}+T_d\,s\right); \tag{1}$$

ii) PI controllers:

$$C_{\rm PI}(s) = K_p\left(1+\frac{1}{T_i\,s}\right); \tag{2}$$

iii) PD controllers in standard form:

$$C_{\rm PD}(s) = K_p\,(1+T_d\,s); \tag{3}$$

with $K_p, T_i, T_d, \tau_d > 0$. The parameter K_p is the proportional sensitivity constant, while T_i and T_d are the time constants of the integral and derivative actions, respectively. The problem we are concerned with can be stated in precise terms as follows.

Problem 2.1. Consider the feedback control architecture in Figure 1, where $G(s)$ is the plant transfer function. Design

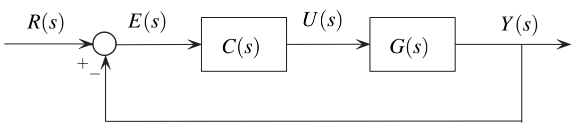

Fig. 1. Unity feedback control architecture.

a controller $C(s) \in \{C_{\rm PID}(s), C'_{\rm PID}(s), C_{\rm PI}(s), C_{\rm PD}(s), C'_{\rm PD}(s)\}$ such that the steady-state requirements on the tracking error $e(t) \stackrel{def}{=} r(t)-y(t)$ are satisfied, and such that the crossover frequency and the phase margin of the compensated system (loop gain transfer function) $L(s) \stackrel{def}{=} C(s)\,G(s)$ are ω_g and PM, respectively, i.e.,

$$|L(j\omega_g)| = 1, \tag{4}$$

$$arg\,L(j\omega_g) = PM-\pi. \tag{5}$$

If we are able to solve Problem 2.1, the desired compensator $C(s)$ ensures that the frequency response of the loop gain transfer function satisfies (45), but it does not automatically guarantee that the closed loop system is stable. In fact, the conditions (45) do not exclude the existence of other points where the polar plot of the loop gain frequency response $L(j\omega)$ intersects the unit circle. Therefore, an *a posteriori* verification is necessary to ensure that the closed loop is stable.

3. PID CONTROLLERS IN STANDARD FORM

Consider the classical PID controller (1). Our aim in this section is to solve Problem 2.1 with $C(s) = C_{\rm PID}(s)$. Here, we have to discriminate between two situations. The first is the one in which the steady-state specifications can be met by the use alone of a controller with a pole at the origin; consider e.g. the case of a type-0 plant and the steady-state performance criterion of zero position error. In this case, the fact itself of using a PID controller guarantees that the steady-state requirement is satisfied. The second case of interest is the one in which the imposition of the steady-state specifications gives rise to a constraint on the Bode gain $K_i \stackrel{\rm def}{=} K_p/T_i$ of $C_{\rm PID}(s)$. This situation occurs, for example, in the case of a type-0 plant when the steady-state specification not only requires zero position error, but also that the velocity error be equal to (or smaller than) a given non-zero constant. A similar situation arises with constraints on the acceleration error for type-1 plants.

3.1 Steady-State requirements do not constrain K_i

First, we consider the case where the steady-state specifications do not lead to a constraint on the integral constant of the PID controller. In order to compute the parameters of the PID controller, we write $G(j\omega)$ and $C_{\rm PID}(j\omega)$ in polar form as

$$G(j\omega) = |G(j\omega)|\,e^{j\arg G(j\omega)}, \quad C_{\rm PID}(j\omega) = M(\omega)\,e^{j\,\varphi(\omega)}.$$

The loop gain frequency response can be written as $L(j\omega) = |G(j\omega)|M(\omega)\,e^{j\,(\arg G(j\omega)+\varphi(\omega))}$. If the gain crossover frequency ω_g and the phase margin PM of the loop gain transfer function $L(s)$ are assigned, (4-5) lead to $|L(j\omega_g)| = 1$ and $\mathrm{PM} = \pi + \arg L(j\omega_g)$. These equations can be written as

(1) $M_g = 1/\,|G(j\omega_g)|$,
(2) $\varphi_g = \mathrm{PM}-\pi-\arg G(j\omega_g)$,

where $M_g \stackrel{\rm def}{=} M(\omega_g)$ and $\varphi_g \stackrel{\rm def}{=} \varphi(\omega_g)$. In order to find the parameters of the controller such that (4-5) are met,

$$M_g\,e^{j\,\varphi_g} = K_p\,\frac{1+j\,\omega_g\,T_i-\omega_g^2\,T_i\,T_d}{j\,\omega_g\,T_i} \tag{6}$$

must be solved in $K_p, T_i, T_d > 0$. In the solution to this problem there is a degree of freedom: by equating the real and imaginary parts of both sides of (6) we obtain two equations

$$\omega_g M_g T_i \cos\varphi_g = \omega_g K_p T_i, \tag{7}$$

$$-M_g \omega_g T_i \sin\varphi_g = K_p - K_p \omega_g^2 T_i T_d \tag{8}$$

in the three unknowns K_p, T_i and T_d.

A possibility to carry out the design of the PID controller in the case of unconstrained integral constant is to exploit the remaining degree of freedom so as to satisfy some further time or frequency domain requirements. Here, we consider two ways to exploit this freedom: the first is the one where the ratio T_d/T_i is chosen, so as to ensure, for example, that the zeros of the PID controller are real; the second is the one where a gain margin constraint is to be satisfied.

Imposition of the ratio T_d/T_i A very convenient way to exploit the degree of freedom in the solution of (6) is the imposition of the ratio $\sigma \stackrel{\text{def}}{=} T_d/T_i$. This is convenient since it is an easily established fact that when $T_i \geq 4\,T_d$, i.e., when $\sigma^{-1} \geq 4$, the zeros of the PID controller are real (and coincident when $\sigma^{-1} = 4$), and they are complex conjugate when $\sigma^{-1} < 4$. In the following theorem, necessary and sufficient conditions are given for the solvability of Problem 2.1 when σ is given. Moreover, closed-form formulae are provided for the parameters of the PID controller to meet the specifications on PM, ω_g and σ exactly.

Theorem 3.1. (Åström et al., 1995, p. 140). Let the ratio $\sigma = T_d/T_i$ be assigned. Equation (6) admits solutions in $K_p, T_i, T_d > 0$ if and only if

$$\varphi_g \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{9}$$

If (9) is satsfied, the solution of (6) with σ fixed is given by

$$K_p = M_g \cos\varphi_g \tag{10}$$

$$T_i = \frac{\tan\varphi_g + \sqrt{\tan^2\varphi_g + 4\,\sigma}}{2\,\omega_g\,\sigma} \tag{11}$$

$$T_d = T_i\,\sigma \tag{12}$$

Imposition of the Gain Margin Another possibility in the case of unconstrained K_i is to fix the gain margin to a certain value GM. To this end, the conditions $\arg L(j\omega_p) = -\pi$ and $\text{GM} = |L(j\omega_p)|^{-1}$ on the loop gain frequency response must be satisfied by definition of gain margin and phase crossover frequency ω_p. By writing again $C_{\text{PID}}(j\omega) = M(\omega)\,e^{j\,\varphi(\omega)}$, these conditions are equivalent to

$$M_p = \frac{1}{\text{GM}\,|G(j\omega_p)|}, \tag{13}$$

$$\varphi_p = -\pi - \arg G(j\omega_p), \tag{14}$$

where $M_p \stackrel{\text{def}}{=} M(\omega_p)$ and $\varphi_p \stackrel{\text{def}}{=} \varphi(\omega_p)$. Now $K_p, T_i, T_d > 0$ must be determined so that (6) and

$$M_p\,e^{j\,\varphi_p} = K_p\,\frac{1 + j\,\omega_p T_i - \omega_p^2 T_i T_d}{j\,\omega_p T_i} \tag{15}$$

are simultaneously satisfied. By equating the real and the imaginary part of (6) and (15) we obtain (7), (8) and the additional two equations

$$\omega_p M_p T_i \cos\varphi_p = \omega_p K_p T_i, \tag{16}$$

$$-M_p \omega_p T_i \sin\varphi_p = K_p - K_p \omega_p^2 T_i T_d. \tag{17}$$

From (7) and (16), we obtain the equation

$$M_g \cos\varphi_g = M_p \cos\varphi_p \tag{18}$$

in the unknown ω_p. For the control problem to be solvable, (18) must admit at least one strictly positive solution.

Lemma 3.1. Let $G(s)$ be a rational function in $s \in \mathbb{C}$. Then, equation (18) can be reduced to a polynomial equation in ω_p.

Remark 3.1. If the transfer function of the process is given by the product of a rational function $\hat{G}(s)$, and a delay $e^{-t_0 s}$, i.e., if $G(s) = \hat{G}(s)\,e^{-t_0 s}$, equation (18) is not polynomial in ω_p, and it needs to be solved numerically.

Theorem 3.2. Consider Problem 2.1 with the additional specification on the gain margin GM. Equations (6) and (15) admit solutions in $K_p, T_i, T_d > 0$ if and only if $\varphi_g \in (-\pi/2, \pi/2)$ and (18) admits a positive solution ω_p such that

$$\begin{cases} \omega_p < \omega_g \\ \omega_g \tan\varphi_g > \omega_p \tan\varphi_p \\ \omega_p \tan\varphi_g > \omega_g \tan\varphi_p \end{cases} \quad \text{or} \quad \begin{cases} \omega_p > \omega_g \\ \omega_g \tan\varphi_g < \omega_p \tan\varphi_p \\ \omega_p \tan\varphi_g < \omega_g \tan\varphi_p \end{cases} \tag{19}$$

If $\varphi_g \in (-\pi/2, \pi/2)$ and (19) is satisfied, the control problem admits solutions with a PID controller, whose parameters can be computed as

$$K_p = M_g \cos\varphi_g \tag{20}$$

$$T_i = \frac{\omega_g^2 - \omega_p^2}{\omega_g\,\omega_p\,(\omega_p \tan\varphi_g - \omega_g \tan\varphi_p)} \tag{21}$$

$$T_d = \frac{\omega_g \tan\varphi_g - \omega_p \tan\varphi_p}{\omega_g^2 - \omega_p^2} \tag{22}$$

Proof: A necessary condition for the problem to admit solutions is that ω_p is a solution of (18). From (7) and (8), and from (16) and (17), we obtain

$$-\omega_g T_i \tan\varphi_g = 1 - \omega_g^2 T_i T_d, \tag{23}$$

$$-\omega_p T_i \tan\varphi_p = 1 - \omega_p^2 T_i T_d. \tag{24}$$

By solving (23) and (24) in T_i and T_d, we obtain (20-22). For K_p to be positive, it is necessary that $\varphi_g \in (-\pi/2, \pi/2)$. Moreover, the time constants T_i and T_d are positive if ω_g and ω_p satisfy (19). ■

3.2 *Steady-State requirements constrain K_i*

Now, the Bode gain $K_i = K_p/T_i$ is determined via the imposition of the steady-state requirements; for example, for type-0 (resp. type-1) plants, K_i is computed via the imposition of the velocity error (resp. acceleration error).

As such, the factor K_i/s can be separated from $\tilde{C}_{\text{PID}}(s) = 1 + T_i s + T_i T_d s^2$, and viewed as part of the plant. In this way, the part of the controller to be designed is $\tilde{C}_{\text{PID}}(s)$.Denote $\tilde{G}(s) \stackrel{\text{def}}{=} \frac{K_p}{T_i s} G(s)$, so that the loop gain transfer function can be written as $L(s) = \tilde{C}_{\text{PID}}(s)\,\tilde{G}(s)$. Write $\tilde{G}(j\omega)$ and $\tilde{C}_{\text{PID}}(j\omega)$ in polar form as

$$\tilde{G}(j\omega) = |\tilde{G}(j\omega)|\,e^{j\arg\tilde{G}(j\omega)}, \qquad \tilde{C}_{\text{PID}}(j\omega) = M(\omega)\,e^{j\,\varphi(\omega)},$$

so that the loop gain frequency response can be written as $L(j\omega) = |\tilde{G}(j\omega)|\,M(\omega)\,e^{j\left(\arg\tilde{G}(j\omega) + \varphi(\omega)\right)}$. If the crossover frequency ω_g and the phase margin PM of $L(s)$ are assigned, the equations $|L(j\omega_g)| = 1$ and $\mathrm{PM} = \pi + \arg L(j\omega_g)$ must be satisfied. These can be written as

$$M_g = \left|\frac{K_p}{T_i\, j\,\omega_g}\, G(j\omega_g)\right|^{-1} = \frac{\omega_g}{K_i\,|G(j\omega_g)|} \tag{25}$$

$$\begin{aligned}\varphi_g &= \mathrm{PM} - \pi - \arg\left[\frac{K_p}{T_i\, j\,\omega_g}\, G(j\omega_g)\right] \\ &= \mathrm{PM} - \frac{\pi}{2} - \arg G(j\omega_g),\end{aligned} \tag{26}$$

where $M_g \stackrel{\text{def}}{=} M(\omega_g)$ and $\varphi_g \stackrel{\text{def}}{=} \varphi(\omega_g)$, since $K_p, T_i > 0$. In order to find the parameters of the controller such that *(i)-(ii)* are met, equation

$$M_g\, e^{j\,\varphi_g} = 1 + j\,\omega_g\, T_i - T_i\, T_d\, \omega_g^2 \tag{27}$$

must be solved in $T_i > 0$ and $T_d > 0$. The closed-form solution to this problem is given in the following theorem.

Theorem 3.3. Equation (27) admits solutions in $T_i > 0$ and $T_d > 0$ if and only if

$$0 < \varphi_g < \pi \qquad \text{and} \quad M_g \cos\varphi_g < 1. \tag{28}$$

If (28) are satisfied, the solution of (27) is given by

$$K_p = K_i\, \frac{1}{\omega_g}\, M_g \sin\varphi_g \tag{29}$$

$$T_i = \frac{1}{\omega_g}\, M_g \sin\varphi_g \tag{30}$$

$$T_d = \frac{1 - M_g\,\cos\varphi_g}{\omega_g\, M_g\, \sin\varphi_g}. \tag{31}$$

Example 3.1. Consider the plant transfer function

$$G(s) = \frac{1}{s\,(s+2)},$$

and consider the problem of determining the parameters of a PID controller that exactly achieves a phase margin of 45° and gain crossover frequency $\omega_g = 30\,\mathrm{rad/s}$ in the two situations:

- the acceleration error is equal to 0.005.
- the velocity error is equal to zero; first use the remaining degree of freedom to assign the ratio $T_i/T_d = 16$, and then to impose a gain margin equal to 3.

Consider the first problem. The acceleration error is

$$e_\infty^a = \frac{1}{\lim_{s\to 0} s^2\, L(s)} = \frac{1}{\lim_{s\to 0} s^2\, K_p\, \dfrac{1 + T_i\, s + T_i\, T_d\, s^2}{T_i\, s^2\,(s+2)}} = \frac{2\,T_i}{K_p}.$$

Then, $e_\infty^a = 0.005$ implies $K_i = K_p/T_i = 400$. Define $\tilde{G}(s) = K_i\, G(s)/s$. Compute $M_g = 30/K_i\,|G(30\,j)| = 9\sqrt{904}/4$ and $\varphi_g = \mathrm{PM} - \pi/2 - \arg G(30\,j) = \pi/4 + \arctan 15$. Using Theorem 3.3, the time constants of the PID controller are

$$T_i = \frac{M_g\,\sin\varphi_g}{30} = \frac{12}{5\sqrt{2}},\quad T_d = \frac{1 - M_g \cos\varphi_g}{30 M_g\, \sin\varphi_g} = \frac{\sqrt{2}+63}{2160}.$$

The zeros of the PID controller are real since T_i is greater than $4\,T_d$. From the ratio $K_p/T_i = 400$, we find $K_p = 400\,T_i = 960/\sqrt{2}$. The transfer function of the PID controller is

$$C_{\mathrm{PID}}(s) = \frac{960}{\sqrt{2}}\left(1 + \frac{5\sqrt{2}}{12\,s} + \frac{\sqrt{2}+63}{2160}\, s\right).$$

Now let us consider the second problem, where the ratio $K_i = K_p/T_i$ is not constrained. Now, the pole at the origin alone guarantees that the velocity error be equal to zero. We first consider the situation where $T_i/T_d = \sigma^{-1} = 16$. First, we compute $M_g = 1/|G(30\,j)| = 30\sqrt{904}$ and $\varphi_g = \mathrm{PM} - \pi - \arg G(30\,j) = \pi/4 - \pi + \pi/2 + \arctan 15 = -\pi/4 + \arctan 15$. Using (10) we find

$$K_p = 30\sqrt{904}\,\cos\left(-\frac{\pi}{4} + \arctan 15\right) = 480\sqrt{2}.$$

Moreover, a simple computation shows that $\tan\varphi_g = 7/8$, so that using the results in Theorem 3.1, and in particular (11) with $\sigma^{-1} = 16$, we find

$$T_i = \frac{7+\sqrt{65}}{30} \quad \text{and} \quad T_d = \frac{7+\sqrt{65}}{480}.$$

The transfer function of the PID controller in this case is

$$C_{\mathrm{PID}}(s) = 480\sqrt{2}\left(1 + \frac{30}{(7+\sqrt{65})\,s} + \frac{7+\sqrt{65}}{480}\, s\right).$$

Now we solve the same problem by imposing a gain margin equal to 3. We compute M_p and φ_p as functions of ω_p:

$$\begin{aligned} M_p &= \frac{1}{\mathrm{GM}|G(j\,\omega_p)|} = \frac{\omega_p\sqrt{\omega_p^2+4}}{3};\\ \varphi_p &= -\pi - \arg G(j\,\omega_p) = -\frac{\pi}{2} + \arctan\frac{\omega_p}{2}.\end{aligned}$$

Using these expression, (18) can be written as

$$\frac{\omega_p\sqrt{\omega_p^2+4}}{3}\,\sin\left(\arctan\frac{\omega_p}{2}\right) = M_g\,\cos\varphi_g,$$

whose unique real solution is $\omega_p = \sqrt{3\,M_g\,\cos\varphi_g} > 0$. Using the expressions for M_g and φ_g it is easily found that $\omega_p = \sqrt{720\sqrt{8}}\,\mathrm{rad/s}$. Hence,

$$\varphi_p = -\frac{\pi}{2} + \arctan\frac{\sqrt{720\sqrt{8}}}{2} = -\frac{\pi}{2} + \arctan\sqrt{180\sqrt{8}},$$

which gives $\tan\varphi_p = -1/\sqrt{180\sqrt{8}}$. It is easily verified that the conditions in Theorem 3.2 are not satisfied, since $\omega_p > \omega_g$ but $\omega_g \tan\varphi_g > \omega_p \tan\varphi_p = -2$. Let us consider the same problem with $\mathrm{PM} = \frac{2\pi}{3}$, $\omega_g = 3\,\mathrm{rad/s}$, and $\mathrm{GM} = 3$. In this case, we find $M_g = 3\sqrt{13}$, $\varphi_g = \frac{\pi}{6} + \arctan\frac{3}{2}$, and consequently $\tan\varphi_g = (2 + 3\sqrt{3})/(2\sqrt{3}-3)$. Using these values in (18) yields

$$\omega_p = \sqrt{3\,M_g\,\cos\varphi_g} = \sqrt{\frac{9}{2}\,(2\sqrt{3}-3)}.$$

As such,

$$\varphi_p = -\frac{\pi}{2} + \arctan\sqrt{\frac{9\,(2\sqrt{3}-3)}{8}},$$

which yields $\tan\varphi_p = -1/\sqrt{\frac{9}{8}(2\sqrt{3}-3)}$. In this case, the conditions in Theorem 3.2 are satisfied, and the parameters of the PID controller can be computed in closed form as

$$K_p = M_g \cos\varphi_g = \frac{3}{2}(2\sqrt{3}-3)$$

$$T_i = \frac{\omega_g^2 - \omega_p^2}{\omega_g\,\omega_p\,(\omega_p \tan\varphi_g - \omega_g \tan\varphi_p)} = \frac{5-2\sqrt{3}}{10+9\sqrt{3}}$$

$$T_d = \frac{\omega_g \tan\varphi_g - \omega_p \tan\varphi_p}{\omega_g^2 - \omega_p^2} = \frac{26\sqrt{3}}{144\sqrt{3}-243}.$$

Since $T_i < 4\,T_d$, the zeros of the PID controller are complex conjugate.

4. PI CONTROLLERS

The synthesis techniques presented in the previous sections can be adapted to the design of PI controllers for the imposition of phase margin and crossover frequency of the loop gain transfer function. This can be done, however, only when the steady-state specifications do not lead to the imposition of the Bode gain of the loop gain transfer function. The transfer function of a PI controller is given by (2). By defining $\tilde{G}(s) = G(s)/s$, we find

$$M_g = \frac{1}{|\tilde{G}(j\omega_g)|} = \frac{\omega_g}{|G(j\omega_g)|},$$
$$\varphi_g = \mathrm{PM} - \pi - \arg\tilde{G}(j\omega_g) = \mathrm{PM} - \frac{\pi}{2} - \arg G(j\omega_g).$$

To find the parameters of the PI controller, equation

$$M_g\, e^{j\,\varphi_g} = K_p\,\frac{j\,T_i\,\omega_g + 1}{T_i} \qquad (32)$$

must be solved in $K_p > 0$ and $T_i > 0$.

Theorem 4.1. Equation (32) admits solutions in $K_p > 0$ and $T_i > 0$ if and only if $\varphi_g \in (0, \pi/2)$. If this condition is satisfied, the solution of (32) is given by $K_p = M_g \sin\varphi_g/\omega_g$ and $T_i = \tan\varphi_g/\omega_g$.

As already observed, when the steady-state requirements lead to the imposition of the ratio K_p/T_i, the problem of assigning the phase margin and the crossover frequency of the loop gain transfer function cannot be solved in general. Indeed, if we define $\tilde{G}(s) = \frac{K_p}{T_i\,s}\,G(s)$ and $\tilde{C}_{\mathrm{PI}}(s) = 1 + T_i\,s$, the values $M_g = 1/|\tilde{G}(j\omega_g)|$ and $\varphi_g = \mathrm{PM} - \pi - \arg\tilde{G}(j\omega_g)$ are such that the identity $1 + j\,\omega_g\,T_i = M_g\,e^{j\,\varphi_g}$ must be satisfied. The latter yields $M_g \cos\varphi_g = 1$ and $M_g \sin\varphi_g = \omega_g\,T_i$, which are two equations in one unknown T_i; the first does not even depend on T_i, but only on the transfer function of the plant $G(s)$. As such, the problem is solvable only if $M_g \cos\varphi_g = 1$ is satisfied. If that constraint is satisfied, $T_i = M_g \sin\varphi_g/\omega_g$.

5. PD CONTROLLERS

As for the design of PI controllers, the synthesis techniques presented for the imposition of phase margin and crossover frequency of the loop gain transfer function can be used for PD controllers only when the steady-state specifications do not lead to the imposition of the proportional sensitivity K_p. The transfer function of a PD controller is given by (2). To find the parameters of the PD controller, the equation

$$M_g\, e^{j\,\varphi_g} = K_p\,(1 + j\,T_d\,\omega_g) \qquad (33)$$

must be solved in $K_p > 0$ and $T_d > 0$.

Theorem 5.1. Equation (33) admits solutions in $K_p > 0$ and $T_d > 0$ if and only if $\varphi_g \in (0, \pi/2)$. If this condition is satisfied, the solution of (33) is given by $K_p = M_g \cos\varphi_g$ and $T_d = \tan\varphi_g/\omega_g$.

When the imposition of the steady-state specifications lead to a sharp constraint on K_p, and define $\tilde{G}(s) = K_p\,G(s)$ and $\tilde{C}_{\mathrm{PD}}(s) = 1 + T_d\,s$, the values $M_g = 1/|\tilde{G}(j\omega_g)|$ and $\varphi_g = \mathrm{PM} - \pi - \arg\tilde{G}(j\omega_g)$ lead to the identity $1 + j\,\omega_g\,T_d = M_g\,e^{j\,\varphi_g}$, which in turn leads to the two equations $M_g \cos\varphi_g = 1$ and $M_g \sin\varphi_g = \omega_g\,T_d$, so that the problem admits solutions only if the constraint $M_g \cos\varphi_g = 1$ is satisfied. In that case, $T_d = M_g \sin\varphi_g/\omega_g$.

6. GRAPHICAL AND APPROXIMATE SOLUTION TO THE DESIGN PROBLEM

The synthesis methods developed in the previous sections are based on closed-form formulae for the computation of the parameters of the PID controller. Clearly, the strength of this approach – which provides an exact answer to the control problem design with standard steady-state and frequency domain specifications – can at first sight be also considered its weakness, because often a full knowledge of the dynamics of the plant is not available in practice. However, when this is the situation at hand, the approach presented in this paper still offers a solution to the design problem, either based on graphical considerations similar in spirit to those considered in the literature Yeung et al. (2000) (but that can be carried out on *any* of the standard diagrams for the frequency response), or on the approximation of the plant with a first or second order plus delay transfer function Ho et al. (1995). As already mentioned, a graphical version of the method presented in this paper can be implemented using any of the frequency domain plots usually employed in control to represent the frequency response dynamics, i.e., Bode diagrams, Nyquist diagrams or Nichols charts. This is due to the fact that the formulae used to derive the parameters of the PID controller are expressed in terms of the magnitude and the argument of the frequency response of the plant at a given crossover frequency, which is readable over any of these diagrams. Given the point of the Nyquist plot corresponding to the desired gain crossover frequency, i.e., we can estimate M_g and φ_g. These values lead to the parameters of the PID controller.

This approach may be employed even in absence of graphical descriptions of the plant transfer function. Indeed, in the very same spirit of the Ziegler and Nichols method, we may “perform an experiment” on the plant by feeding it with a sinusoidal input with frequency ω_g, i.e. the desired crossover frequency. From the steady-state output we can estimate $G(j\omega_g)$, and hence M_g and φ_g and we can thus readily apply the proposed method.

7. SECOND ORDER PLUS DELAY APPROXIMATION

Several tuning techniques proposed in the literature do not require exact knowledge of the mathematical model of the plant, but rely on its first or second order plus delay approximation, Ho et al. (1995); Åström et al. (1995). While in this paper the formulae for the parameters of the PID controller have been obtained under the assumption of exact knowledge of the transfer function of the plant, the procedure outlined can be used in conjuction with the heuristics or numerical methods based on these approximations. In this section we show that the formulae presented in this paper can be specialised to the case of a second order plus delay approximation of the plant dynamics. Consider the second order plus delay model

$$G(s) = \frac{K}{(1+\tau_1 s)(1+\tau_2 s)}\, e^{-T s},$$

where $\tau_1, \tau_2, T > 0$ and $K > 0$. A direct calculation shows that

$$M_g = \frac{\sqrt{(1+\omega_g^2\,\tau_1^2)\,(1+\omega_g^2\,\tau_1^2)}}{K},$$
$$\varphi_g = \theta - \pi + \arctan(\omega_g\,\tau_1) + \arctan(\omega_g\,\tau_2),$$

where $\theta = \mathrm{PM} + \omega_g\,T$, which lead to

$$M_g \cos(\varphi_g) = \frac{\omega_g\,(\tau_1+\tau_2)\,\sin\theta - (1-\omega_g^2\,\tau_1\,\tau_2)\,\cos\theta}{K}.$$

If the specifications are on the ratio $\sigma = T_d/T_i$, the parameters of the compensator can be computed using directly (10-12), that with this particular model yield

$$K_p = \frac{\omega_g\,(\tau_1+\tau_2)\,\sin\theta - (1-\omega_g^2\,\tau_1\,\tau_2)\,\cos\theta}{K},$$
$$T_i = \frac{(1-\omega_p^2\,\tau_1\,\tau_2)\,\tan\theta + \omega_p\,(\tau_1+\tau_2)}{(1-\omega_p^2\,\tau_1\,\tau_2) - \omega_p\,(\tau_1+\tau_2)\,\tan\theta},$$
$$T_d = T_i\,\sigma.$$

If the specifications are on both the phase and gain margin (and on the gain crossover frequency), we must also compute

$$M_p = \frac{\sqrt{(1+\omega_p^2\,\tau_1^2)\,(1+\omega_p^2\,\tau_1^2)}}{K},$$
$$\varphi_p = -\pi + \omega_p\,T + \arctan(\omega_p\,\tau_1) + \arctan(\omega_p\,\tau_2),$$

so that

$$M_p \cos(\varphi_p) = \frac{\omega_p\,(\tau_1+\tau_2)\sin(\omega_p\,T) - (1-\omega_p^2\tau_1\,\tau_2)\cos(\omega_p T)}{\mathrm{GM}\cdot K}.$$

Therefore, to achieve the desired margins at the desired gain crossover frequency, we need to find the roots ω_p of

$$\psi(\omega_p) = \omega_p\,(\tau_1+\tau_2)\,\sin(\omega_p\,T) - (1-\omega_p^2\,\tau_1\,\tau_2)\,\cos(\omega_p\,T) - \mathrm{GM}\cdot K\cdot M_g\cos(\varphi_g) = 0. \quad (34)$$

Notice that this time function $\psi(\omega_p)$ is not polynomial, since the transfer function of the model is not rational. The roots of this equation can be determined numerically. Of all the roots of $\psi(\omega_p)$, one satisfying the conditions of Theorem 3.2 must be determined (if no such roots exist, the problem does not admit solutions), and compute the parameters of the PID controller using (20-22). The closed-form formulae given in this paper for the parameters of the PID controller are given in finite terms. Hence, a remarkable advantage of this method is the fact that these formulae can applied to *any* plant approximation, even though in this section for the sake of comparison with the existing methods only the second order plus delay approximation has been considered. As such, the flexibility offered by the method presented here also extends to the case where the transfer function of the plant to be controlled is not exactly known, and necessarily guarantees a better performance when a better approximation of the plant dynamics is available. This also means that the method proposed in this paper outperformes any method constructed upon a plant approximation with a fixed structure.

8. CONCLUSIONS

A unified approach has been presented that enable the parameters of PID, PI and PD controllers to be computed in finite terms given appropriate specifications expressed in terms of steady-state performance, phase/gain margins and gain crossover frequency. The synthesis tools developed in this paper eliminate the need of trial-and-error and heuristic procedures in frequency-response design, and therefore they outperform the heuristic, trial-and-error and graphic approaches proposed so far in the literature in the case of perfect knowledge of the model of the plant. When the plant model is not exactly known – as is often the case in practice – the present method can still be employed for the design of the PID controller. Indeed, the formulae delivering the controller parameters only require the magnitude and the argument of the frequency response of the plant at the desired crossover frequency. These can be obtained graphically by direct inspection over any of the Nyquist, Bode and Nichols diagrams. Alternatively, the closed-form design techniques can be used jointly with a first or second order plus delay approximation of the plant to deliver the desired values of the stability margins and crossover frequency.

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