On the computation of the fundamental subspaces for descriptor systems*

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Abstract

In this paper, we investigate several theoretical and computational aspects of fundamental subspaces for linear, time-invariant (LTI) descriptor systems, which appear in the solution of many control and estimation problems. Different types of reachability and controllability for descriptor systems are described and discussed. The Rosenbrock system matrix pencil is employed for the computation of supremal output-nulling subspaces and supremal output-nulling reachability subspaces for descriptor systems.

1 Introduction

In the last few decades, there has been a growing interest in the study of descriptor systems, also known as singular or generalized or implicit systems. Descriptor systems have many applications in circuit theory, large-scale systems, biological systems, neurology, power systems, robotics, aircraft modeling, see e.g. [4], [20], [23], [30], [12] and the references cited therein. The difficulty associated with the extension of classical control and estimation techniques to the descriptor case lies in the fact that descriptor systems have a richer and more articulated structure than the standard linear time-invariant (LTI) systems, see e.g. [6], [9]-[11], [16], [21]-[22], [24], [28]-[34]. For a survey on descriptor systems and the geometric analysis of LTI descriptor systems we refer the reader to [19] and [20], respectively.

There is no obvious and unique way to extend concepts such as reachability and controllability to descriptor systems. Indeed, different types of reachability and controllability have been defined for descriptor systems, see for example the important survey [5]. All these different types of reachability/ controllability coincide with the standard notions of reachability and controllability in the case of LTI systems. Roughly, two fundamental frameworks to deal with these issues were proposed by Rosenbrock in [32] and by Verghese et al. in [33]. It was stated in [33] that Rosenbrock was the first to point out the difficulties in his own definitions given in [32], due to unnecessary restrictions on the part of the system with no dynamical significance. The main difference between these two frameworks

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was in the definition of controllability at infinity [9]. The concept has later been generalized for general differential-algebraic equations (DAE) by Geerts in [17]; Frankowska in [14] treated the controllability of DAE systems with the theory of differential inclusions, see also [3]. Bonilla et al. in [7] studied the reachability notion in the sense of [14], showing some important connections with [17].

In the same years, several papers focussed on the generalization of the fundamental concepts of geometric control for descriptor systems, such as the characterization and computation of fundamental subspaces, see e.g. [6], [14], [16]-[17], [19]-[22], [24], [28]-[31]. The main subspaces of classical geometric theory for LTI descriptor systems are the so-called (A, E, B)-controlled invariant and restricted (E, A, B)-invariant subspaces, see e.g. [24], [28]. These subspaces turn out to be very important in the descriptor case, because they appear to be the building blocks used to characterize the reachable subspace of a descriptor system. Lewis and Özçaldiran in [22] defined and investigated the properties of the output-nulling subspaces for descriptor systems. These are subspaces of initial states for which there exists a control input that maintains the output identically at zero. Output-nulling subspaces are used to determine solvability conditions for problems such as disturbance decoupling with static and dynamic feedback, model matching, and noninteracting control to name a few. In [21], the notions of conditioned invariant and input-containing subspaces have been introduced for descriptor systems within the context of unknown-input observation. Geerts in [16] gave definitions in terms of distributions for output-nulling, input-containing subspaces and output-nulling reachability subspaces, and extended the classic standard LTI algorithms for their computation.

As already mentioned, two types of controllability at infinity were defined in the literature. Although it has been extensively acknowledged that the definition of controllability at infinity by Verghese et al. is more natural − as it does not present the restrictions of the one given by Rosenbrock − as also pointed out in [5], most of the existing literature in the area of geometric control for descriptor systems has so far been hinging on the definition given by Rosenbrock in [32]. Thus, the first aim of this paper is to clarify the different types of reachability and controllability for descriptor systems and introduce a new definition for the reachable subspace. The second aim is to show the connections between these different types of reachability and controllability with the fundamental subspaces of the geometric approach in the descriptor case. The third objective is to extend a famous result by Moore and Laub [25], which has also been expressed in polynomial terms in [13], to descriptor systems. This result has been used in the literature to devise numerically robust techniques to compute bases for the aforementioned output-nulling, reachability and input-containing subspaces as also shown in [26]. The approach in [25] and [13] has also been used to solve noninteracting, model matching and input detection problems and, more recently, for the solution of the monotonic tracking control problem in the multi-input, multi-output (MIMO) case [27]. Thus, we envisage that the extension of this fundamental result to descriptor systems will open the door to the possibility of appropriately formulating and providing a solution to these problems in the singular case.

In this paper, the geometric analysis of square descriptor systems is studied based on the framework of Verghese et al. and Geerts in [33] and [16], respectively. More specifically, we firstly give the definitions of the so-called restricted system equivalence and the dynamics decomposition form, see e.g. [15], [32], [11], [12, Ch.2], [34]. This equivalent form will be used in this paper for clarity of arguments. Next, different types of reachability and controllability for descriptor systems are described, which will then be used for the analysis of the fundamental subspaces for descriptor systems.

The paper is structured as follows. In Section 2, some preliminary key concepts are presented for descriptor systems. In Section 3, different types of reachability and controllability for descriptor systems are described and discussed. Section 4 deals with the fundamental subspaces for descriptor systems, namely controlled invariant, output-nulling and input-containing subspaces. In Section 5, computational methods are provided for obtaining reachability and output-nulling subspaces via the Rosenbrock system matrix pencil in the same spirit of the Moore-Laub method for the standard case. The considerations are illustrated with a numerical example in Section 6. Finally, some concluding remarks are offered in Section 7.

Notation. The origin of a vector space is denoted by $\{0\}$. The image and the kernel of a matrix A are represented by $\operatorname{im} A$ and $\ker A$, respectively. For convenience, a linear mapping between finite-dimensional spaces and a matrix representation with respect to a particular basis are not distinguished notationally. The spectrum of a square matrix A is denoted by $\sigma(A)$. Given a linear map $A : \mathcal{X} \longrightarrow \mathcal{Y}$ and a subspace $\mathcal{S}$ of $\mathcal{Y}$, the symbol $A^{-1}\,\mathcal{S}$ represents the inverse image of $\mathcal{S}$ with respect to the linear map A, i.e., $A^{-1}\,\mathcal{S} = \{x \in \mathcal{X} \mid A\,x \in \mathcal{S}\}$. If $\mathcal{J} \subseteq \mathcal{X}$, the restriction of the map A to $\mathcal{J}$ will be denoted by $A\,|\mathcal{J}$. If $\mathcal{X} = \mathcal{Y}$ and $\mathcal{J}$ is A-invariant, the eigenvalues of A restricted to $\mathcal{J}$ will be denoted by $\sigma\,(A\,|\mathcal{J})$. The symbol $\oplus$ will stand for the direct sum of subspaces. Finally, the symbol $\mathfrak{i}$ represents the imaginary unit, i.e., $\mathfrak{i} = \sqrt{-1}$, while the symbol $\overline{\lambda}$ represents the complex conjugate of $\lambda \in \mathbb{C}$.

2 Preliminaries

Consider a linear, time-invariant, continuous-time descriptor system Σ governed by

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $E \in \mathbb{R}^{n\times n}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. For all $t \geq 0$, the vector $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the descriptor variable, $u(t) \in \mathcal{U} = \mathbb{R}^m$ is the control input and $y(t) \in \mathcal{Y} = \mathbb{R}^p$ is the output. In this paper, we identify the system governed by (1) with the quintuple (E, A, B, C, D). Matrix E is allowed to be singular with $\ell \doteq \operatorname{rank} E \leq n$.

We introduce the *dynamics decomposition form*, which is the most important restricted equivalent form for linear descriptor systems. First, recall that two descriptor systems, described by the quintuples (E, A, B, C, D) and $(\overline{E}, \overline{A}, \overline{B}, \overline{C}, \overline{D})$, with state vectors $x(t)$ and $\overline{x}(t)$, respectively, are called *restricted system equivalent* under the transformation (Q, P) if there exist two non-singular matrices

$Q, P \in \mathbb{R}^{n\times n}$ such that $QEP = \overline{E},\ QAP = \overline{A},\ QB = \overline{B},\ CP = \overline{C},\ D = \overline{D},\ x(t) = P\,\overline{x}(t)$, see e.g. [15], [32]. Given a descriptor linear system described by (E, A, B, C, D) there exist non-singular matrices Q and P such that (E, A, B, C, D) and (QEP, QAP, QB, CP, D) are restricted system equivalent under (Q, P) with $QEP = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}$, see e.g. [11], [12, Ch.2], [34].[1] Consider such pair (Q, P). The matrices and the state vector of (QEP, QAP, QB, CP, D) are partitioned conformably as

$$QAP = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\quad QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\quad CP = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\quad P^{-1}x(t) = \begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix},$$

so that the restricted equivalent descriptor system is described by the following equations

$$\dot{\tilde{x}}(t) = A_{11}\tilde{x}(t) + A_{12}z(t) + B_1u(t), \tag{2a}$$

$$0 = A_{21}\tilde{x}(t) + A_{22}z(t) + B_2u(t), \tag{2b}$$

$$y(t) = C_1\tilde{x}(t) + C_2z(t) + Du(t). \tag{2c}$$

Equation (2a) is the so-called *dynamic* subsystem, while equation (2b) is the so-called *static* or *algebraic* subsystem. Thus, no generality is lost by assuming that the system (1) is already in the equivalent form (2), so that it can be written as

$$\begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{3a}$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix} + Du(t). \tag{3b}$$

In other words, we assume with no loss of generality that the matrices E, A, B, C of the descriptor system Σ are already in the block form $E = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

The matrix pencil $\lambda E - A$ is called *regular* if $\det(\lambda E - A)$ is not identically zero, see e.g. [15], [6], [33], [22], and the degree of $\det(\lambda E - A)$ will be denoted by q. Regularity is a desirable property for a descriptor system, because, if a system is regular, the solution exists and is unique given $x(0^-)$ and $u(t)$, see e.g. [33], [22].[2] In the regular case, a matrix pencil $\lambda E - A$ has η *finite generalized eigenvalues*, which are the η roots of $\det(\lambda E - A)$ with multiplicities $m_1, m_2, \ldots, m_\eta$ such that $m_1 + m_2 + \ldots + m_\eta = q$, and a *generalized eigenvalue at* ∞ with multiplicity $n - q$. The finite generalized eigenvalues and the generalized eigenvalue at ∞ of $\lambda E - A$ are the *generalized eigenvalues* of the matrix pencil $\lambda E - A$. The finite spectrum of a square pair (E, A) of a descriptor system is denoted by $\sigma(E, A)$. The generalized

[1] The pair (Q, P) can be obtained, for example, by computing the singular value decomposition of E

$$E = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^\top = U\begin{bmatrix} \Sigma & 0 \\ 0 & I_{n-\ell} \end{bmatrix}\begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}V^\top,$$

where U and V are orthogonal and Σ is a diagonal matrix containing the non-zero singular values of E. Then we may compute $Q = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{n-\ell} \end{bmatrix}U^{-1},\ P = \left(V^\top\right)^{-1} = V$.

[2] The generalized eigenvalues can also be defined for a singular descriptor system as the roots of the greatest common divisor of the minors of order equal to the normal rank of $\lambda E - A$, [15]. In the sequel, the notion of *impulse controllability* is introduced, which allows a singular descriptor system to be transformed into a regular one.

eigenvalue at ∞ of multiplicity $n-q$ can be thought of as being given by the product of a generalized eigenvalue at ∞ of multiplicity $\ell-q$ associated with the impulse response of the open-loop system at $t=0$ and a generalized eigenvalue at ∞ of multiplicity $n-\ell$ associated with a non-dynamic response, see e.g. [15], [22].

The finite generalized eigenvalues can be at most ℓ, i.e., $q \le \ell$, see e.g. [33], [12, Ch.3], and if the descriptor system (1) has ℓ finite poles, then it is called *impulse-free*, see e.g. [17], [18], [36], [12]. An impulse-free system is also sometimes called *internally proper*, see e.g. [2], [6], [8]. If a descriptor system is impulse-free, then it is always regular, because $\ker A_{22} = \{0\}$, see e.g. [8], [10], [12, Ch.7]. Since E and A are assumed to be square, the condition $\ker A_{22} = \{0\}$ is equivalent to the invertibility of A_{22}. In this case, $\lambda E - A$ is invertible as a polynomial matrix, see e.g. [15], [8].

Impulsive modes in descriptor systems are typically not desired, because they may cause performance degradation and damage or even destroy an engineering system, see e.g. [10]-[12, Ch.7], [22]. The so-called *impulse controllability* guarantees that there exists a state feedback, such that the closed-loop system is impulse-free, see e.g. [2], [6], [9]-[12, Ch.9], [17], [18], [36]. Consequently, impulse controllability implies regularizability, which guarantees that there is a feedback control such that the closed-loop system is regular, see e.g. [12, Ch.4], and the regularity assumption is not necessary.

In this paper we make the following standing assumptions:

(i) $\operatorname{rank} \begin{bmatrix} E & A & B \end{bmatrix} = n$,

(ii) the columns of $\begin{bmatrix} B \\ D \end{bmatrix}$ and the rows of $\begin{bmatrix} C & D \end{bmatrix}$ are linearly independent,[3]

(iii) $\operatorname{rank} \begin{bmatrix} E & AE_\infty & B \end{bmatrix} = n$, where E_∞ is a basis matrix for $\ker E$.

The first assumption is made to avoid linear dependence on the descriptor equations, see e.g. the discussion in Bonilla et al. [7]. The third assumption is the criterion for the impulse controllability, see e.g. [17], [7], [5]. Notice that *(iii)* implies *(i)*. However, we write these two conditions separately for consistency with the results in [17], [7].

Under assumption *(iii)*, we are able to apply a preliminary state feedback $u(t) = H_1 \tilde{x}(t) + H_2 z(t) + v(t)$ to the impulse controllable system Σ as in (3), so that the closed-loop system is impulse-free and thus regular, i.e., such that $\det(A_{22} + B_2 H_2) \neq 0$, see e.g. [10], [12, Ch.7]. It is clear from this consideration that, with no loss of generality, H_1 can be taken to be the zero matrix. The closed-loop system $\hat{\Sigma}$ under the state feedback $u(t) = H_2 z(t) + v(t)$ is governed by

$$E\dot{x}(t) = \hat{A}x(t) + Bv(t), \tag{4a}$$

$$y(t) = \hat{C}x(t) + Dv(t), \tag{4b}$$

[3]If $\begin{bmatrix} B \\ D \end{bmatrix}$ has non-trivial kernel, a subspace $\mathcal{U}_0$ of the input space exists that does not influence the local state dynamics. By performing a suitable (orthogonal) change of basis in the input space, we may eliminate $\mathcal{U}_0$ and obtain an equivalent system for which this condition is satisfied. Likewise, if $[\, C \;\; D \,]$ is not surjective, there are some outputs that result as linear combinations of the remaining ones, and these can be eliminated using a dual argument by performing a change of coordinates in the output space.

where $\hat{A} \doteq \begin{bmatrix} A_{11} & \hat{A}_{12} \\ A_{21} & \hat{A}_{22} \end{bmatrix}$, $\hat{C} \doteq \left[\begin{array}{cc} C_1 & \hat{C}_2 \end{array}\right]$, $\hat{A}_{12} \doteq A_{12} + B_1 H_2$, $\hat{A}_{22} \doteq A_{22} + B_2 H_2$, $\hat{C}_2 \doteq C_2 + DH_2$. We observe that, using the feedback $u(t)$, the submatrices A_{11}, A_{21} and C_1 have not changed and $v(t)$ can be regarded as the new input function.

3 Reachability and controllability of descriptor systems

This section is devoted to recalling the different types of reachability and controllability for descriptor systems and the two main corresponding frameworks. First, the following definitions are needed. The space of *consistent initial states*, denoted by $\mathcal{V}_{[E,A,B]}$, is defined as the set of initial states $x_0 \in \mathcal{X}$ for which there exists a solution (x, u) of (1a) such that $x(0) = x_0$, see e.g. [17], [7], [5]. The condition for the so-called *C-solvability in the function sense* of Geerts in [17] is that $E\mathcal{V}_{[E,A,B]} = E\mathcal{X}$, see also [7]. The space of *consistent initial differential variables*, denoted by $\mathcal{V}^{\text{diff}}_{[E,A,B]}$, is defined as the set of initial states $x_0 \in \mathcal{X}$ for which there exists a solution (x, u) of (1a) such that $Ex(0) = Ex_0$, [5]. There holds $\mathcal{V}^{\text{diff}}_{[E,A,B]} = \mathcal{V}_{[E,A,B]} + \ker E$, see e.g. [5].

We recall now a definition given in [5]. Let x and u be such that

$$x, \dot{x}, u \text{ are locally Lebesgue measurable and } (x, u) \text{ satisfies (1a) for almost all } t \in \mathbb{R}. \tag{5}$$

Definition 3.1 *System* (1a) *is called:*

- *controllable within the reachable states (R-controllable) if for all $x_0, x_f \in \mathcal{V}_{[E,A,B]}$ there exist $t > 0$ and (x, u) as in* (5) *such that $x(0) = x_0$ and $x(t) = x_f$;*
- *controllable at infinity if for all $x_0 \in \mathcal{X}$ there exists (x, u) as in* (5) *such that $x(0) = x_0$;*
- *completely reachable (C-reachable) if there exists $t > 0$ such that for all $x_f \in \mathcal{X}$ there exists (x, u) as in* (5) *such that $x(0) = 0$ and $x(t) = x_f$;*
- *completely controllable (C-controllable) if there exists $t > 0$ such that for all $x_0, x_f \in \mathcal{X}$ there exists (x, u) as in* (5) *such that $x(0) = x_0$ and $x(t) = x_f$;*
- *impulse controllable (I-controllable) if for all $x_0 \in \mathcal{X}$ there exists (x, u) as in* (5) *such that $Ex(0) = Ex_0$;*
- *strongly reachable (S-reachable) if there exists $t > 0$ such that for all $x_f \in \mathcal{X}$ there exists (x, u) as in* (5) *such that $Ex(0) = 0$ and $Ex(t) = Ex_f$;*
- *strongly controllable (S-controllable) if there exists $t > 0$ such that for all $x_0, x_f \in \mathcal{X}$ there exists (x, u) as in* (5) *such that $Ex(0) = Ex_0$ and $Ex(t) = Ex_f$.*

R-controllability is controllability in the regular sense and it is associated with the finite generalized eigenvalues. The criterion for R-controllability states that the system (1a) is R-controllable if and only if the *controllability pencil* $\begin{bmatrix} \lambda E - A & B \end{bmatrix}$ has full row rank for all finite generalized eigenvalues λ, see e.g. [33], [29].

Controllability at infinity is associated with the infinite generalized eigenvalue of multiplicity $n-q$ and it was defined by Rosenbrock in [32]. System (1a) is controllable at infinity if and only if $\mathcal{V}_{[E,A,B]} = \mathcal{X}$, see e.g. [5], or, equivalently, if and only if rank $\begin{bmatrix} E & B \end{bmatrix} = n$, see e.g. [29].

Complete reachability is equivalent to complete controllability and implies controllability at infinity. A descriptor system as in (1a) is C-controllable if and only if it is R-controllable and controllable at infinity, see e.g. [9], [5].

Impulse controllability is associated with the generalized eigenvalue at ∞ with multiplicity $\ell - q$ corresponding to impulsive modes and it was defined as controllability at infinity by Verghese et al. in [33]. System (1a) is I-controllable if and only if $\mathcal{V}^{\text{diff}}_{[E,A,B]} = \mathcal{V}_{[E,A,B]} + \ker E = \mathcal{X}$, see e.g. [5], or, equivalently, if and only if rank $\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \operatorname{rank} E$, [10].

Strong reachability is equivalent to strong controllability and implies impulse controllability. A descriptor system as in (1a) is S-controllable if and only if it is R-controllable and I-controllable [33]. Consequently, under the assumption of I-controllability, in order to have S-controllability, we only need to have R-controllability, or, equivalently, modal controllability.

Complete controllability implies strong controllability as $x(0) = x_0$ implies $Ex(0) = Ex_0$. Clearly, strong controllability does not imply complete controllability. As observed in the introduction, Verghese et al. in [33] noted that Rosenbrock himself was the first to point out the difficulties with his definitions and showed that they resulted from unnecessary restrictions on parts of the system that have no dynamical role. Indeed, only the property $Ex(0) = Ex_0$ is needed, which can also hold when $x(0) \neq x_0$.[4] Thus, we will focus our attention on strong controllability. Note that, when $E = I_n$, the notions of C-controllability, S-controllability and R-controllability coincide with controllability in the standard case.

4 Fundamental subspaces for descriptor systems

We now recall some concepts of classical geometric control theory for descriptor systems. A subspace $\mathcal{J}$ of $\mathcal{X}$ is called (A,E)*-invariant* for a descriptor system Σ if $A\mathcal{J} \subseteq E\mathcal{J}$, see also [6]. Notice that when $E = I_n$, this definition reduces to the standard definition of A-invariance. If J is a basis matrix for $\mathcal{J}$, the subspace $\mathcal{J}$ is (A,E)-invariant if and only if $\operatorname{im}(AJ) \subseteq \operatorname{im}(EJ)$. The sum of (A,E)-invariant subspaces is clearly (A,E)-invariant. Using an argument based on duality, it is easily seen that the intersection of (A,E)-invariant subspaces is also (A,E)-invariant, i.e., $A(\mathcal{J}_1 \cap \mathcal{J}_2) \subseteq E(\mathcal{J}_1 \cap \mathcal{J}_2)$. Hence, the *Grassmannian* of all (A,E)-invariant subspaces of $\mathcal{X}$, here denoted by $\mathfrak{Gr}_{A,E}(\mathcal{X})$, is closed under subspace addition and intersection, and thus the set $(\mathfrak{Gr}_{A,E}(\mathcal{X}), +, \cap; \subseteq)$ is a lattice. Its minimum element is $\{0\}$ and its maximum element is the sum of all (E,A)-invariant subspaces of $\mathcal{X}$ and it

[4] Assuming that $E = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}$, $x(0) \doteq \begin{bmatrix} \tilde{x}(0) \\ z(0) \end{bmatrix}$, $x_0 \doteq \begin{bmatrix} \tilde{x}_0 \\ z_0 \end{bmatrix}$, then $Ex(0) = Ex_0$ or, equivalently, $\begin{bmatrix} \tilde{x}(0) \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{x}_0 \\ 0 \end{bmatrix}$, which holds true for $\tilde{x}(0) = \tilde{x}_0$.

is called the *characteristic subspace* of (E, A), see also [6]. This subspace is computed as the last term of the monotonically non-increasing sequence $\mathcal{J}_0 = \mathcal{X},\ \mathcal{J}_i = A^{-1}E\mathcal{J}_{i-1},\ i \in \{1, \dots, n-1\}$. The sequence converges to $\mathcal{J}^\star$ in at most $n-1$ steps, i.e., $\mathcal{J}^\star = \mathcal{J}_k$, where $k \le n-1$ is such that $\mathcal{J}_{k+1} = \mathcal{J}_k$.

Since the matrix E in (3) is idempotent, given two (A, E)-invariant subspaces $E\mathcal{J}_1, E\mathcal{J}_2$ we have $A(E\mathcal{J}_1 \cap E\mathcal{J}_2) \subseteq AE\mathcal{J}_1 \cap AE\mathcal{J}_2 \subseteq EE\mathcal{J}_1 \cap EE\mathcal{J}_2 = E\mathcal{J}_1 \cap E\mathcal{J}_2 = E(E\mathcal{J}_1 \cap E\mathcal{J}_2)$. Therefore, the set of all (A, E)-invariant subspaces of $E\mathcal{X}$ is also closed under subspace addition and intersection. Since $AE\mathcal{J} \subseteq EE\mathcal{J} = E\mathcal{J}$, an (A, E)-invariant subspace of the form $E\mathcal{J}$ is also A-invariant. The following simple result holds.

Lemma 4.1 *Let $\mathcal{J}$ be an r-dimensional subspace and let J be a basis matrix for $\mathcal{J}$. Then $\mathcal{J}$ is (A, E)-invariant if and only if there exists $X \in \mathbb{R}^{r\times r}$ such that $AJ = EJX$.*

Proof: The equation $AJ = EJX$ is equivalent to

$$\begin{aligned} A[\,J_1 \ \ J_2 \ \ \dots \ \ J_r\,] &= E[\,J_1 \ \ J_2 \ \ \dots \ \ J_r\,]X \\ &= [\,E[\,J_1 \ \ J_2 \ \ \dots \ \ J_r\,]X_1 \ \ E[\,J_1 \ \ J_2 \ \ \dots \ \ J_r\,]X_2 \ \ \dots \ \ E[\,J_1 \ \ J_2 \ \ \dots \ \ J_r\,]X_r\,], \end{aligned}$$

which implies $AJ_j = E[\,J_1 \ \ J_2 \ \ \dots \ \ J_r\,]X_j = EJ_1x_{1,j} + EJ_2x_{2,j} + \ldots + EJ_rx_{r,j}$, where we have partitioned $X_j = [\ x_{1,j} \ \ x_{2,j} \ \ \dots \ \ x_{r,j}\ ]^\top$. This equation means that A transforms a basis vector of $\mathcal{J}$ into a linear combination of $E\mathcal{J}$, i.e., into a vector of $E\mathcal{J}$. This is equivalent to saying that $\mathcal{J}$ is (A, E)-invariant. ■

Lemma 4.2 *Let $\mathcal{J}$ be an r-dimensional (A, E)-invariant subspace such that $\mathcal{J} \cap \ker E = \{0\}$. Then, there exists an $n\times n$ non-singular matrix $T = [\ T_1 \ \ T_2\ ]$ with $\operatorname{im} T_1 = \mathcal{J}$ and an $n \times n$ non-singular matrix $\overline{T} = [\ \overline{T}_1 \ \ \overline{T}_2\ ]$ with $\overline{T}_1 \doteq ET_1$ such that*

$$A' = \overline{T}^{-1}AT = \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix}, \tag{6}$$

where $A'_{11} \in \mathbb{R}^{r\times r}, A'_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$. Conversely, if there exist $n \times n$ non-singular matrices $T, \overline{T}$ such that (6) *holds, then the subspace* $\operatorname{im} T_1$ *is an r-dimensional (A, E)-invariant subspace.*

Proof: Let us partition $A' = \overline{T}^{-1}AT$ as $A' = \left[\begin{smallmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{smallmatrix}\right]$ and let us show that $A'_{21} = 0$. Let $x \in \mathcal{J}$ and consider the non-singular matrices $T, \overline{T}$ constructed as stated above. Since T is adapted to $\mathcal{J}$, we can write x with respect to the new basis as $x' = T^{-1}x = \left[\begin{smallmatrix} x'_1 \\ 0 \end{smallmatrix}\right]$ for some vector $x'_1 \in \mathbb{R}^r$. Thus

$$A'x' = \big(\overline{T}^{-1}AT\big)\big(T^{-1}x\big) = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}\begin{bmatrix} x'_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A'_{11}x'_1 \\ A'_{21}x'_1 \end{bmatrix}.$$

We must have $A'x' \in E\mathcal{J}$ because $\mathcal{J}$ is (A, E)-invariant. Notice that $r \le \ell$, because $\mathcal{J} \cap \ker E = \{0\}$ and $E\left[\begin{smallmatrix} x'_1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x'_1 \\ 0 \end{smallmatrix}\right]$ because $\left[\begin{smallmatrix} I_r & 0 & 0 \\ 0 & I_{\ell-r} & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x'_1 \\ 0 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x'_1 \\ 0 \\ 0 \end{smallmatrix}\right]$. Consequently, we have that $A'_{21}x'_1 = 0$ and from the arbitrariness of x_1' we have $A_{21}'=0$. Conversely, suppose $T,\overline{T}$ are $n\times n$ non-singular matrices such that (6) holds. Then, clearly

$$A'\begin{bmatrix} I_r \\ 0 \end{bmatrix}=\overline{T}^{-1}AT\begin{bmatrix} I_r \\ 0 \end{bmatrix}=E\begin{bmatrix} I_r \\ 0 \end{bmatrix}A_{11}'=\begin{bmatrix} I_r \\ 0 \end{bmatrix}A_{11}'.$$

Pre-multiplying both sides of the above identity by $\overline{T}$ yields

$$A\left[\begin{array}{cc} T_1 & T_2 \end{array}\right]\begin{bmatrix} I_r \\ 0 \end{bmatrix}=\left[\begin{array}{cc} ET_1 & \overline{T}_2 \end{array}\right]\begin{bmatrix} I_r \\ 0 \end{bmatrix}A_{11}',$$

which implies that $AT_1=ET_1A_{11}'$ and therefore im T_1 is (A,E)-invariant. ■

Remark 4.1 The matrix T_1 contains the finite generalized eigenvectors of the descriptor system (1) corresponding to the finite spectrum of the descriptor system restricted to $\mathcal{J}$. We denote this finite spectrum by $\sigma(E,A|\mathcal{J})$. A direct consequence of the above lemmas is that then $\sigma(E,A|\mathcal{J})=\sigma(X)=\sigma(A_{11}')$. If $\dim\mathcal{J}=q$, then the matrix T_1 contains the finite generalized eigenvectors of (1) corresponding to the finite spectrum $\sigma(E,A)$.

A subspace $\mathcal{V}$ of $\mathcal{X}$ is called *controlled invariant* or (A,E,B)-*controlled invariant* for a descriptor system Σ if $A\mathcal{V}\subseteq E\mathcal{V}+\text{im } B$, see e.g. [22], [5]. Notice that when $E=I_n$, the definition reduces to the classic (A,B)-controlled invariance, [35]. A controlled invariant subspace contains the initial states x_0 of Σ for which there exists a control input such that the entire trajectory remains in $E\mathcal{V}$. The set of all (A,E,B)-controlled invariant subspaces is closed under subspace addition, so there exists a maximum element $\mathcal{V}^\star$, which can be computed by the monotonically non-increasing sequence $\mathcal{V}_0=\mathcal{X}$, $\mathcal{V}_i=A^{-1}(E\mathcal{V}_{i-1}+\text{im } B)$, $i\in\{1,2,\dots,n-1\}$. The sequence converges to $\mathcal{V}^\star$ in at most $n-1$ steps, i.e., $\mathcal{V}^\star=\mathcal{V}_k$ where $k\le n-1$ is such that $\mathcal{V}_{k+1}=\mathcal{V}_k$. There holds $\mathcal{V}_{[E,A,B]}=\mathcal{V}^\star$, see e.g. [4], [5]. A descriptor system is controllable at infinity if and only if $\mathcal{V}^\star=\mathcal{X}$, while it is impulse controllable if and only if $E\mathcal{V}^\star=E\mathcal{X}$ or, equivalently, if and only if $\mathcal{V}^\star+\ker E=\mathcal{X}$, [5].

A subspace $\mathcal{W}$ that satisfies $\mathcal{W}=E^{-1}(A\mathcal{W}+\text{im } B)$ is called *restricted* (E,A,B)-*invariant* [22], [5]. The set of all restricted (E,A,B)-invariant subspaces is closed under intersection, so there exists a minimum element $\mathcal{W}^\star$, which can be computed by the monotonically non-decreasing sequence $\mathcal{W}_0=\ker E$, $\mathcal{W}_i=E^{-1}(A\mathcal{W}_{i-1}+\text{im } B)$, $i\in\{1,2,\dots,\ell-1\}$. The sequence converges to $\mathcal{W}^\star$ in at most $\ell-1$ steps, i.e., $\mathcal{W}^\star=\mathcal{W}_k$, where $k\le\ell-1$ is such that $\mathcal{W}_{k+1}=\mathcal{W}_k$.

Since we have essentially two different definitions of reachability for descriptor systems, which are the complete reachability and the strong reachability, we need two definitions for the reachable subspace. It is evident from the definitions of complete and strong reachability that the completely reachable subspace $\mathcal{R}_C$ is a subspace of $\mathcal{X}$ and the strongly reachable subspace $\mathcal{R}_S$ is contained in $\text{im}\,E$. The strongly reachable subspace $\mathcal{R}_S$ represents the states of $E\mathcal{X}$ that are reachable from the origin, while the reachable subspace $\mathcal{R}$ represents the states of $\mathcal{X}$ that are reachable from the origin in the sense of Verghese et al. in [33].

Controllability type	Criteria
R-controllability	$\mathcal{W}^\star = \mathcal{X},\ E\mathcal{W}^\star = E\mathcal{X}$
Controllability at infinity	$\mathcal{V}^\star = \mathcal{X}$
C-controllability	$\mathcal{V}^\star \cap \mathcal{W}^\star = \mathcal{X}$
I-controllability	$E\mathcal{V}^\star = E\mathcal{X},\ \mathcal{V}^\star + \ker E = \mathcal{X}$
S-controllability	$E(\mathcal{V}^\star \cap \mathcal{W}^\star) = E\mathcal{X},\ (\mathcal{V}^\star \cap \mathcal{W}^\star) + \ker E = \mathcal{X}$

Table 1: Criteria for types of controllability

Theorem 4.1
The completely reachable (C-reachable) subspace is

$$\mathcal{R}_C = \mathcal{V}^\star \cap \mathcal{W}^\star.$$

The strongly reachable (S-reachable) subspace is

$$\mathcal{R}_S = E(\mathcal{V}^\star \cap \mathcal{W}^\star).$$

The reachable subspace is given by

$$\mathcal{R} = (\mathcal{V}^\star + \ker E) \cap \mathcal{W}^\star.$$

The above equalities have been proved in [28] and [30].[5] Notice that $\mathcal{R} = \mathcal{R}_C + \ker E = \mathcal{R}_S \oplus \ker E$ because $\ker E \subseteq \mathcal{W}^\star$. The descriptor system (1a) is completely controllable if and only if $\mathcal{R}_C = \mathcal{X}$ and strongly controllable if and only if $\mathcal{R}_S = E\mathcal{X}$ or, equivalently, $\mathcal{R}_C + \ker E = \mathcal{X}$, see e.g. [5], or if and only if $\mathcal{R}_S \oplus \ker E = \mathcal{X}$. The C-reachable subspace represents the states of $\mathcal{X}$ that are reachable from the origin in the sense of Rosenbrock in [32]. The subspace $\mathcal{R}_S$ is equal to $\overline{\mathcal{V}}^\star \cap \overline{\mathcal{W}}^\star$, where $\overline{\mathcal{V}}^\star \doteq E\mathcal{V}^\star$ and $\overline{\mathcal{W}}^\star \doteq E\mathcal{W}^\star$, because $(\mathcal{V}^\star + \mathcal{W}^\star) \cap \ker E = (\mathcal{V}^\star \cap \ker E) + (\mathcal{W}^\star \cap \ker E)$, in view of the modular distributive rule, since $\ker E \subseteq \mathcal{W}^\star$. From the definitions of complete controllability and controllability at infinity or strong controllability and impulse controllability and the corresponding geometric criteria, it is clear that the criterion for R-controllability states that the descriptor system (1a) is R-controllable if and only if $\mathcal{W}^\star = \mathcal{X}$ or, equivalently, $\overline{\mathcal{W}}^\star = E\mathcal{X}$.

Proposition 4.1 *If a descriptor system Σ is I-controllable, then it is S-controllable if and only if it is R-controllable, i.e., $\mathcal{W}^\star = \mathcal{X}$, or $E\mathcal{W}^\star = E\mathcal{X}$.*

Proof: If Σ is I-controllable, then $\mathcal{V}^\star + \ker E = \mathcal{X}$, or, equivalently, $\overline{\mathcal{V}}^\star = E\mathcal{X}$ and if $\mathcal{W}^\star = \mathcal{X}$, then $\overline{\mathcal{W}}^\star = E\mathcal{X}$, so that $\mathcal{R}_S = \overline{\mathcal{V}}^\star \cap \overline{\mathcal{W}}^\star = E\mathcal{X}$ and Σ is S-controllable. Conversely, if Σ is S-controllable and I-controllable, then $\overline{\mathcal{V}}^\star \cap \overline{\mathcal{W}}^\star = E\mathcal{X}$ and $\overline{\mathcal{V}}^\star = E\mathcal{X}$ which implies that $\overline{\mathcal{W}}^\star = E\mathcal{X}$, or, equivalently, $\mathcal{W}^\star = \mathcal{X}$ because $\ker E \subseteq \mathcal{W}^\star$. ∎

[5]The C-reachable subspace was called reachable subspace in [28] and [30]. The S-reachable subspace was called controllable subspace in [28]. The reachable subspace was called controllable subspace in [30]. The subspaces were renamed and denoted accordingly in order to maintain consistency with the definition of controllability types for descriptor systems.

Remark 4.2 The reachable subspace is computed by the monotonically non-decreasing sequence for $\mathcal{W}^\star$ and since $\mathcal{W}_i = E^{-1}(A\mathcal{W}_{i-1} + \operatorname{im} B) \supseteq E^{-1}(A\mathcal{W}_{i-1}) + E^{-1}(\operatorname{im} B) \supseteq E^{-1}(\operatorname{im} B)$ for every i, the reachable subspace contains $E^{-1}(\operatorname{im} B)$. Thus, the first term of the sequence can be taken as $\mathcal{W}_0 = E^{-1}(\operatorname{im} B)$ and the reachable subspace is the smallest restricted (E, A, B)-invariant subspace containing $E^{-1}(\operatorname{im} B)$.

In the case $E = I_n$, there are no infinite generalized eigenvalues and the subspace $\mathcal{W}^\star = E^{-1}(A\mathcal{W}^\star + \operatorname{im} B)$ becomes $\mathcal{W}^\star = A\mathcal{W}^\star + \operatorname{im} B$. This implies that $A\mathcal{W}^\star \subseteq \mathcal{W}^\star$ and $\operatorname{im} B \subseteq \mathcal{W}^\star$. Thus, in the regular case, $\mathcal{W}^\star$ is A-invariant, contains $\operatorname{im} B$ and is such that $\dim \mathcal{W}^\star = \dim(A\mathcal{W}^\star + \operatorname{im} B)$. Additionally, $\mathcal{W}^\star = E^{-1}(A\mathcal{W}^\star + \operatorname{im} B) = E^{-1}(A\mathcal{W}^\star) + E^{-1}(\operatorname{im} B)$ if and only if the modular distributive rule $(A\mathcal{W}^\star + \operatorname{im} B) \cap \operatorname{im} E = (A\mathcal{W}^\star \cap \operatorname{im} E) + (\operatorname{im} B \cap \operatorname{im} E)$ holds. In [6], it is shown that if the condition $\operatorname{im} B \subseteq E\mathcal{J}^\star$, where $\mathcal{J}^\star$ is the characteristic subspace of (E, A), is not satisfied, then $u(t)$ may be restricted to belong to the subspace $\mathcal{U}_{ad} \doteq A^{-1}(E\mathcal{J}^\star)$. Consequently, it may always be assumed that the restriction has been performed and $\operatorname{im} B \subseteq E\mathcal{J}^\star$ holds. Under that assumption, we also have $\operatorname{im} B \subseteq \operatorname{im} E$ and then the modular distributive rule applies.

Theorem 4.2 *The reachable subspace $\mathcal{R}$ of a descriptor system Σ with $u(t) \in \mathcal{U}_{ad}$ for all $t \geq 0$ is the smallest (E, A)-invariant subspace containing $E^{-1}(\operatorname{im} B)$ and is denoted by $\langle E, A | E^{-1}(\operatorname{im} B)\rangle$.*

It should be noted that in [14] a method based on differential inclusions was used to derive a formula for the reachable subspace, which was later generalized in [31], see also [3], [7].

An *output-nulling* subspace $\mathcal{V}$ for the descriptor system (1) is a subspace of $\mathcal{X}$ which satisfies the inclusion

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq \big(E\mathcal{V} \oplus \{0\}\big) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{7}$$

see e.g. [22]. The subspace $\mathcal{V}^\star$ represents the set of initial states for which there exist smooth state and control functions (x, u) such that the corresponding output is identically zero and $x(0) = x_0$, [16]. It follows from (7) that $\mathcal{V}$, with basis matrix V, is an output-nulling subspace of a descriptor system, if and only if there exist matrices Λ, W of suitable dimensions such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} EV \\ 0 \end{bmatrix} \Lambda.$$

The set of output-nulling subspaces is closed under subspace addition, so there exists a maximum element which is denoted by $\mathcal{V}^\star$ and can be computed using the monotonically non-increasing sequence of subspaces

$$\mathcal{V}_0 = \mathcal{X}, \tag{8a}$$

$$\mathcal{V}_i = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\big(E\mathcal{V}_{i-1} \oplus \{0\}\big) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \right), \; i \in \{1, 2, \ldots, n-1\}. \tag{8b}$$

This sequence converges to $\mathcal{V}^\star$ in at most $n-1$ steps, i.e., $\mathcal{V}^\star = \mathcal{V}_k$ where $k \le n-1$ is such that $\mathcal{V}_{k+1} = \mathcal{V}_k$.

An *input-containing* subspace $\mathcal{S}$ for the descriptor system (1) is a subspace of $\mathcal{X}$ which satisfies

$$E^{-1}\left[\begin{array}{cc} A & B \end{array}\right]\left((\mathcal{S} \oplus \mathcal{U}) \cap \ker\left[\begin{array}{cc} C & D \end{array}\right]\right) \subseteq \mathcal{S},$$

[21], [16]. The subspace $\mathcal{S}^\star$ represents the set of initial states for which there exist impulsive state and control trajectories (x,u) such that $y=0$, [16]. The set of input-containing subspaces is closed with respect to subspace intersection, so there exists a minimum element, which is denoted by $\mathcal{S}^\star$, and can be computed using the monotonically non-decreasing sequence of subspaces

$$\mathcal{S}_0 = \ker E, \tag{9a}$$

$$\mathcal{S}_i = E^{-1}\left(\left[\begin{array}{cc} A & B \end{array}\right]\left((\mathcal{S}_{i-1} \oplus \mathcal{U}) \cap \ker\left[\begin{array}{cc} C & D \end{array}\right]\right)\right),\ i \in \{1,2,\ldots,\ell-1\}. \tag{9b}$$

There holds $\mathcal{S}^\star = \mathcal{S}_k$, where $k \le \ell-1$ is such that $\mathcal{S}_{k+1} = \mathcal{S}_k$.

The dual of the sequence (9) is the monotonically non-increasing sequence of subspaces

$$\begin{aligned} \underline{\mathcal{V}}_0 &= \operatorname{im} E, \\ \underline{\mathcal{V}}_i &= E\left[\begin{array}{c} A \\ C \end{array}\right]^{-1}\left(\left(\underline{\mathcal{V}}_{i-1} \oplus \{0\}\right) + \operatorname{im}\left[\begin{array}{c} B \\ D \end{array}\right]\right),\ i \in \{1,2,\ldots,n-\ell-1\}, \end{aligned}$$

[24], so that, if we define by $\underline{\mathcal{V}}^\star$ the maximum element of the above sequence, then $\mathcal{S}^\star$ is the dual of $\underline{\mathcal{V}}^\star$ and it holds true that $\underline{\mathcal{V}}^\star = E\mathcal{V}^\star$.

The *output-nulling reachability* subspace $\mathcal{R}^\star$ represents the set of initial states for which there exists an impulsive input and a trajectory from the origin such that $y=0$ and $Ex(0) = Ex_0$, [16]. The subspace $\mathcal{R}^\star$ on $\mathcal{V}^\star$ is computed by

$$\mathcal{R}^\star = (\mathcal{V}^\star + \ker E) \cap \mathcal{S}^\star,$$

[16] or $\mathcal{R}^\star = (\mathcal{V}^\star \cap \mathcal{S}^\star) + \ker E$, because, from (9), $\ker E \subseteq \mathcal{S}^\star$. The subspace $\mathcal{V}^\star + \ker E$ represents the set of initial states for which there exists a smooth state and control function pair (x,u) such that $y=0$, [16].[6]

Since $\ker E \subseteq \mathcal{R}^\star$, we can write $\mathcal{R}^\star$ as $\mathcal{R}^\star = \mathcal{R}_S^\star \oplus \ker E$ such that $\mathcal{R}_S^\star$ is orthogonal to $\ker E$. If we denote by r the dimension of $\mathcal{R}_S^\star$, then the dimension of $\mathcal{R}^\star$ is equal to $\dim \mathcal{R}^\star = r + \dim(\ker E) = r + n - \ell$. We can also write $\mathcal{V}^\star + \ker E = \mathcal{V}_S^\star \oplus \ker E$ such that $\mathcal{V}_S^\star$ is orthogonal to $\ker E$. We denote by v the dimension of $\mathcal{V}_S^\star$ and the dimension of $\mathcal{V}^\star + \ker E$ is equal to $\dim(\mathcal{V}^\star + \ker E) = v + \dim(\ker E) = v + n - \ell$. Finally, notice that $\mathcal{R}_S^\star = E\mathcal{R}^\star, \mathcal{V}_S^\star = E\mathcal{V}^\star = \underline{\mathcal{V}}^\star$.

[6]Notice that the subspaces $\mathcal{V}^\star, \mathcal{S}^\star, \mathcal{R}^\star$ have been denoted respectively by $\mathcal{V}_C(\Sigma), \mathcal{W}(\Sigma), \mathcal{R}(\Sigma)$ in [16].

5 Computation of fundamental subspaces

We now focus our attention on impulse-free systems. The main aim of this section is to provide the generalization to descriptor systems of the relationship between reachability and output-nulling subspaces in terms of the Rosenbrock system matrix pencil. Moreover, the S-reachable and reachable subspaces are computed.

The first step in our approach is to apply a preliminary state feedback $u(t) = H_2 z(t) + v(t)$ to the impulse controllable system Σ as in (3), so that $\det(A_{22} + B_2 H_2) \neq 0$. Consider the impulse-free, closed-loop system $\hat{\Sigma}$ as in (4). Another equivalent form of $\hat{\Sigma}$ is given by

$$\tilde{Q}E\tilde{P} = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}, \tilde{Q}\hat{A}\tilde{P} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & I_{n-\ell} \end{bmatrix}, \tilde{Q}B = \begin{bmatrix} \tilde{B}_1 \\ B_2 \end{bmatrix}, \hat{C}\tilde{P} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}, \tilde{P}^{-1} \begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \tilde{x}(t) \\ \tilde{z}(t) \end{bmatrix},$$

where

$$\tilde{Q} = \begin{bmatrix} I_\ell & -\hat{A}_{12}\hat{A}_{22}^{-1} \\ 0 & I_{n-\ell} \end{bmatrix}, \ \tilde{P} = \begin{bmatrix} I_\ell & 0 \\ -\hat{A}_{22}^{-1}A_{21} & \hat{A}_{22}^{-1} \end{bmatrix}, \ \tilde{P}^{-1} = \begin{bmatrix} I_\ell & 0 \\ A_{21} & \hat{A}_{22} \end{bmatrix},$$

and $\tilde{A}_{11} \doteq A_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}A_{21}$, $\tilde{B}_1 \doteq B_1 - \hat{A}_{12}\hat{A}_{22}^{-1}B_2$, $\tilde{C}_1 \doteq C_1 - \hat{C}_2\hat{A}_{22}^{-1}A_{21}$, $\tilde{C}_2 \doteq \hat{C}_2\hat{A}_{22}^{-1}$, see e.g. [34], [11], [8], so that the restricted equivalent system can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}_{11}\tilde{x}(t) + \tilde{B}_1 v(t), \tag{10a}$$

$$0 = \tilde{z}(t) + B_2 v(t), \tag{10b}$$

$$y(t) = \tilde{C}_1\tilde{x}(t) + \tilde{C}_2\tilde{z}(t) + Dv(t). \tag{10c}$$

Now if we replace $\tilde{z}(t) = -B_2 v(t)$ from (10b) to (10c), we obtain the standard system $\tilde{\Sigma}$

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}v(t), \tag{11a}$$

$$y(t) = \tilde{C}\tilde{x}(t) + \tilde{D}v(t), \tag{11b}$$

where $\tilde{A} \doteq \tilde{A}_{11} \in \mathbb{R}^{\ell\times\ell}$, $\tilde{B} \doteq \tilde{B}_1 \in \mathbb{R}^{\ell\times m}$, $\tilde{C} \doteq \tilde{C}_1 \in \mathbb{R}^{p\times\ell}$, $\tilde{D} \doteq D - \tilde{C}_2 B_2 \in \mathbb{R}^{p\times m}$, see also [33].

5.1 Rosenbrock system matrix pencil

The *Rosenbrock system matrix pencil* of a descriptor system $\hat{\Sigma}$ as in (4) is defined as

$$P_{\hat{\Sigma}}(\lambda) \doteq \begin{bmatrix} \hat{A} - \lambda E & B \\ \hat{C} & D \end{bmatrix} = \left[\begin{array}{cc|c} A_{11} - \lambda I_\ell & \hat{A}_{12} & B_1 \\ A_{21} & \hat{A}_{22} & B_2 \\ \hline C_1 & \hat{C}_2 & D \end{array}\right],$$

see e.g. [32], [16]. The *invariant zeros* of $\hat{\Sigma}$ are the values of $\lambda \in \mathbb{C}$ for which $\operatorname{rank} P_{\hat{\Sigma}}(\lambda) < n + \operatorname{normrank} G(\lambda) = \operatorname{normrank} P_{\hat{\Sigma}}(\lambda)$, where $G(\lambda) \doteq \hat{C}(\lambda E - \hat{A})^{-1}B + D$, see e.g. [1]. The Rosenbrock system matrix pencil of the associated standard system $\tilde{\Sigma}$ in (11) is $P_{\tilde{\Sigma}}(\lambda) \doteq \begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$, where $\tilde{A} = A_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}A_{21}$, $\tilde{B} = B_1 - \hat{A}_{12}\hat{A}_{22}^{-1}B_2$, $\tilde{C} = C_1 - \hat{C}_2\hat{A}_{22}^{-1}A_{21}$, $\tilde{D} = D - \hat{C}_2\hat{A}_{22}^{-1}B_2$.

The following lemma shows the relation between the Rosenbrock system matrix pencil of an impulse-free descriptor system (4) and the Rosenbrock system matrix pencil of the associated standard system (11).

Lemma 5.1 *The Rosenbrock system matrix pencil of an impulse-free descriptor system* $\hat{\Sigma}$ *as in* (4) *can be decomposed as*

$$P_{\hat{\Sigma}}(\lambda) = P_1 \begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell} \end{bmatrix} P_2,$$

where

$$P_1 = \left[\begin{array}{cc|c} I_\ell & 0 & \hat{A}_{12}\hat{A}_{22}^{-1} \\ 0 & 0 & I_{n-\ell} \\ \hline 0 & I_p & \hat{C}_2\hat{A}_{22}^{-1} \end{array}\right], \quad P_2 = \left[\begin{array}{cc|c} I_\ell & 0 & 0 \\ 0 & 0 & I_m \\ \hline A_{21} & \hat{A}_{22} & B_2 \end{array}\right].$$

Proof: We prove this by direct computation:

$$\begin{aligned} P_1 \begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell} \end{bmatrix} P_2 &= P_1 \begin{bmatrix} A_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}A_{21} - \lambda I_\ell & 0 & B_1 - \hat{A}_{12}\hat{A}_{22}^{-1}B_2 \\ C_1 - \hat{C}_2\hat{A}_{22}^{-1}A_{21} & 0 & D - \hat{C}_2\hat{A}_{22}^{-1}B_2 \\ A_{21} & \hat{A}_{22} & B_2 \end{bmatrix} \\ &= \left[\begin{array}{cc|c} A_{11} - \lambda I_\ell & \hat{A}_{12} & B_1 \\ A_{21} & \hat{A}_{22} & B_2 \\ \hline C_1 & \hat{C}_2 & D \end{array}\right] = P_{\hat{\Sigma}}(\lambda). \end{aligned}$$

■

Remark 5.1 Notice that the Rosenbrock system matrix pencil of Σ in (1) can be decomposed as $P_\Sigma(\lambda) = P_{\hat{\Sigma}}(\lambda) \left[\begin{smallmatrix} I_n & 0 \\ H & I_m \end{smallmatrix}\right]^{-1}$, since

$$\begin{bmatrix} A + BH - \lambda E & B \\ C + DH & D \end{bmatrix} = \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} \begin{bmatrix} I_n & 0 \\ H & I_m \end{bmatrix}.$$

The decomposition established in Lemma 5.1 can be used to determine a relation between the null-spaces of $P_{\hat{\Sigma}}(\lambda)$ and $P_{\tilde{\Sigma}}(\lambda)$, as the following lemma shows.

Lemma 5.2 *Let* $\hat{\Sigma}$ *be an impulse-free descriptor system as in* (4)*. There holds*

$$\ker P_{\hat{\Sigma}}(\lambda) = P_2^{-1}\big(\ker P_{\tilde{\Sigma}}(\lambda) \oplus \{0\}\big).$$

Proof: First, observe that $\ker\left(P_1 \left[\begin{smallmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell} \end{smallmatrix}\right]\right) = \ker P_{\tilde{\Sigma}}(\lambda) \oplus \{0\}$. In order to show that, let $\left[\begin{smallmatrix} v' \\ w' \\ z' \end{smallmatrix}\right] \in \ker\left(P_1 \left[\begin{smallmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell} \end{smallmatrix}\right]\right)$, then

$$P_1 \begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell} \end{bmatrix} \begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} = \begin{bmatrix} \tilde{A} - \lambda I_\ell & \tilde{B} & \hat{A}_{12}\hat{A}_{22}^{-1} \\ 0 & 0 & I_{n-\ell} \\ \tilde{C} & \tilde{D} & \hat{C}_2\hat{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} = 0,$$

from which $z'=0$. Thus, $P_{\tilde{\Sigma}}(\lambda)\begin{bmatrix} v' \\ w' \end{bmatrix}=0$. This shows that $\ker\left(P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}\right)\subseteq \ker P_{\tilde{\Sigma}}(\lambda)\oplus\{0\}$. Now let $\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix}\in \ker P_{\tilde{\Sigma}}(\lambda)\oplus\{0\}$. Then $\begin{bmatrix} v' \\ w' \end{bmatrix}\in\ker P_{\tilde{\Sigma}}(\lambda)$ and $z'=0$, so that $\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix}=0$. Also $P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix}=0$ and therefore $\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix}\in\ker\left(P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}\right)$.

We show that $\ker P_{\hat{\Sigma}}(\lambda)\subseteq P_2^{-1}\big(\ker P_{\tilde{\Sigma}}(\lambda)\oplus\{0\}\big)$. Let $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in\ker P_{\hat{\Sigma}}(\lambda)$, then $P_{\hat{\Sigma}}(\lambda)\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}=0$ or, from Lemma 5.1, $P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}P_2\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}=0$, which is satisfied for $P_2\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in\ker\left(P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}\right)$ and implies that

$$\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in P_2^{-1}\ker\left(P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}\right)=P_2^{-1}\left(\ker P_{\tilde{\Sigma}}(\lambda)\oplus\{0\}\right).$$

We show the opposite inclusion $P_2^{-1}\big(\ker P_{\tilde{\Sigma}}(\lambda)\oplus\{0\}\big)\subseteq\ker P_{\hat{\Sigma}}(\lambda)$. Let $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in P_2^{-1}\big(\ker P_{\tilde{\Sigma}}(\lambda)\oplus\{0\}\big)$. Then, $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in P_2^{-1}\ker\left(P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}\right)$ or, equivalently, $P_2\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in\ker\left(P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}\right)$. Thus, $P_1\begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell}\end{bmatrix}P_2\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}=0$, or, equivalently, $P_{\hat{\Sigma}}(\lambda)\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}=0$, so that $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in\ker P_{\hat{\Sigma}}(\lambda)$. ■

Remark 5.2 Let $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in\ker P_{\hat{\Sigma}}(\lambda)$. In view of Lemma 5.2, we also have $P_2\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}=\begin{bmatrix} \hat{v} \\ w \\ A_{21}\hat{v}+\hat{A}_{22}z+B_2w \end{bmatrix}\in\ker P_{\tilde{\Sigma}}(\lambda)\oplus\{0\}$. Let now $\begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}\in\ker P_{\tilde{\Sigma}}(\lambda)$. Comparing the above, it follows that $\begin{bmatrix} \hat{v} \\ w \end{bmatrix}=\begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}$ and $A_{21}\hat{v}+\hat{A}_{22}z+B_2w=0$. Thus, $\ker P_{\hat{\Sigma}}(\lambda)=\left\{\begin{bmatrix} \tilde{v} \\ -\hat{A}_{22}^{-1}(A_{21}\tilde{v}+B_2\tilde{w}) \\ \tilde{w} \end{bmatrix}:\ \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}\in\ker P_{\tilde{\Sigma}}(\lambda)\right\}$.

We now compute $\ker P_{\hat{\Sigma}}(\lambda)$. Let $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}\in\ker P_{\hat{\Sigma}}(\lambda)$. Then

$$\left[\begin{array}{cc|c} A_{11}-\lambda I_\ell & \hat{A}_{12} & B_1 \\ A_{21} & \hat{A}_{22} & B_2 \\ \hline C_1 & \hat{C}_2 & D \end{array}\right]\left[\begin{array}{c} \hat{v} \\ z \\ \hline w \end{array}\right]=0. \tag{12}$$

In view of Remark 5.2, we can write $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix}=P_2^{-1}\begin{bmatrix} \tilde{v} \\ \tilde{w} \\ 0 \end{bmatrix}$ or, equivalently,

$$\left[\begin{array}{c} \hat{v} \\ z \\ \hline w \end{array}\right]=\hat{P}_2^{-1}\left[\begin{array}{c} \tilde{v} \\ 0 \\ \hline \tilde{w} \end{array}\right], \tag{13}$$

where $\hat{P}_2^{-1}=\begin{bmatrix} I_\ell & 0 & 0 \\ -\hat{A}_{22}^{-1}A_{21} & \hat{A}_{22}^{-1} & -\hat{A}_{22}^{-1}B_2 \\ 0 & 0 & I_m \end{bmatrix}$, $\hat{P}_2=\begin{bmatrix} I_\ell & 0 & 0 \\ A_{21} & \hat{A}_{22} & B_2 \\ 0 & 0 & I_m \end{bmatrix}$. We replace (13) in (12) and multiply on the left by $\begin{bmatrix} I_\ell & -\hat{A}_{12}\hat{A}_{22}^{-1} & 0 \\ 0 & I_{n-\ell} & 0 \\ 0 & -\hat{C}_2\hat{A}_{22}^{-1} & I_p \end{bmatrix}$, so that

$$\left[\begin{array}{cc|c} \tilde{A}-\lambda I_\ell & 0 & \tilde{B} \\ 0 & I_{n-\ell} & 0 \\ \hline \tilde{C} & 0 & \tilde{D} \end{array}\right]\left[\begin{array}{c} \tilde{v} \\ 0 \\ \hline \tilde{w} \end{array}\right] = 0. \tag{14}$$

5.2 Computation of reachability and output-nulling subspaces

Consider the standard system $\tilde{\Sigma}$ in (11). The following lemma provides the way to compute the supremal output-nulling reachability subspace $\tilde{\mathcal{R}}^\star$, see [25], [26].

Lemma 5.3 *Let $\tilde{r} = \dim \tilde{\mathcal{R}}^\star$ and $\lambda_1, \lambda_2, \ldots, \lambda_{\tilde{r}}$ be distinct complex numbers all different from the invariant zeros of the system and such that, if $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$, there exists a $j \in \{1, 2, \ldots, \tilde{r}\} \setminus \{i\}$ such that $\lambda_j = \overline{\lambda}_i$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{\tilde{r}}$ be ordered in such a way that the first $2s$ values are complex, while the remaining are real and for all odd $k < 2s$ we have $\lambda_{k+1} = \overline{\lambda}_k$. For each $k \in \{1, 2, \ldots, \tilde{r}\}$, let $\begin{bmatrix} \tilde{V}_k' \\ \tilde{W}_k' \end{bmatrix}$ be a basis for $\ker P_{\tilde{\Sigma}}(\lambda_k)$, so that*

$$\begin{bmatrix} \tilde{A}-\lambda_k I_\ell & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}\begin{bmatrix} \tilde{V}_k' \\ \tilde{W}_k' \end{bmatrix} = 0.$$

Let

$$\begin{bmatrix} \tilde{V}_k \\ \tilde{W}_k \end{bmatrix} = \begin{cases} \begin{bmatrix} \tilde{V}_k' \\ \tilde{W}_k' \end{bmatrix} + \begin{bmatrix} \tilde{V}_{k+1}' \\ \tilde{W}_{k+1}' \end{bmatrix} & \text{if } k < 2s \text{ is odd,} \\ \mathrm{i}\left(\begin{bmatrix} \tilde{V}_k' \\ \tilde{W}_k' \end{bmatrix} - \begin{bmatrix} \tilde{V}_{k-1}' \\ \tilde{W}_{k-1}' \end{bmatrix}\right) & \text{if } k \le 2s \text{ is even,} \\ \begin{bmatrix} \tilde{V}_k' \\ \tilde{W}_k' \end{bmatrix} & \text{if } k > 2s. \end{cases}$$

Then for each $k \in \{1, 2, \ldots, \tilde{r}\}$, the columns of $\tilde{V}_k$ are real and linearly independent and $\tilde{\mathcal{R}}^\star = \operatorname{im}\begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 & \ldots & \tilde{V}_{\tilde{r}} \end{bmatrix}$.

We now generalize the classic Moore-Laub algorithm to descriptor systems.

Theorem 5.1 *Let r be the dimension of $\mathcal{R}_S^\star$ and let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be distinct complex numbers all different from the invariant zeros of the system and such that, if $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$, there exists a $j \in \{1, 2, \ldots, r\} \setminus \{i\}$ such that $\lambda_j = \overline{\lambda}_i$. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be ordered in such a way that the first $2s$ values are complex while the remaining are real and for all odd $k < 2s$ we have $\lambda_{k+1} = \overline{\lambda}_k$. For each $k \in \{1, 2, \ldots, r\}$, let $\begin{bmatrix} V_k' \\ W_k' \end{bmatrix}$ be a basis for $\ker P_{\hat{\Sigma}}(\lambda_k)$, so that*

$$\begin{bmatrix} \hat{A}-\lambda_k E & B \\ \hat{C} & D \end{bmatrix}\begin{bmatrix} V_k' \\ W_k' \end{bmatrix} = 0. \tag{15}$$

Let

$$\begin{bmatrix} V_k \\ W_k \end{bmatrix} = \begin{cases} \begin{bmatrix} V_k' \\ W_k' \end{bmatrix} + \begin{bmatrix} V_{k+1}' \\ W_{k+1}' \end{bmatrix} & \textit{if } k < 2s \textit{ is odd,} \\ \mathrm{i}\left(\begin{bmatrix} V_k' \\ W_k' \end{bmatrix} - \begin{bmatrix} V_{k-1}' \\ W_{k-1}' \end{bmatrix} \right) & \textit{if } k \leq 2s \textit{ is even,} \\ \begin{bmatrix} V_k' \\ W_k' \end{bmatrix} & \textit{if } k > 2s. \end{cases}$$

Then $r = \tilde{r}$, *for each* $k \in \{1, 2, \ldots, r\}$, *the columns of* V_k *are real and linearly independent and* $\mathcal{R}^\star = \mathrm{im} \begin{bmatrix} V_1 & V_2 & \ldots & V_r \end{bmatrix} + \ker E$, $\mathcal{R}_S^\star = \mathrm{im}\,(E \begin{bmatrix} V_1 & V_2 & \ldots & V_r \end{bmatrix})$.

Proof: For the basis $\begin{bmatrix} V_k' \\ W_k' \end{bmatrix}$ of $\ker P_{\hat{\Sigma}}(\lambda_k)$ there holds

$$\left[\begin{array}{cc|c} A_{11} - \lambda I_\ell & \hat{A}_{12} & B_1 \\ A_{21} & \hat{A}_{22} & B_2 \\ \hline C_1 & \hat{C}_2 & D \end{array}\right] \left[\begin{array}{c} \hat{V}_k' \\ Z_k' \\ \hline W_k' \end{array}\right] = 0, \tag{16}$$

where $\begin{bmatrix} \hat{V}_k' \\ Z_k' \end{bmatrix} = V_k'$ for each $k \in \{1, \ldots, r\}$ or from (14)

$$\left[\begin{array}{cc|c} \tilde{A} - \lambda I_\ell & 0 & \tilde{B} \\ 0 & I_{n-\ell} & 0 \\ \hline \tilde{C} & 0 & \tilde{D} \end{array}\right] \left[\begin{array}{c} \tilde{V}_k' \\ 0 \\ \hline \tilde{W}_k' \end{array}\right] = 0 \tag{17}$$

or

$$\begin{bmatrix} \tilde{A} - \lambda I_\ell & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{V}_k' \\ \tilde{W}_k' \end{bmatrix} = 0. \tag{18}$$

The above equation provides a basis for the kernel of the Rosenbrock system matrix pencil of the associated standard system $\tilde{\Sigma}$ in (11). Applying Lemma 5.3, we have that for each $k \in \{1, 2, \ldots, \tilde{r}\}$, the columns of $\tilde{V}_k$ are real and linearly independent and $\tilde{\mathcal{R}}^\star = \mathrm{im} \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 & \ldots & \tilde{V}_{\tilde{r}} \end{bmatrix}$. Comparing equations (16)-(18), it follows that, for each $k \in \{1, 2, \ldots, \tilde{r}\}$, the columns of $\begin{bmatrix} \tilde{V}_k \\ 0 \end{bmatrix}$ are real and linearly independent and the same holds for $\begin{bmatrix} \tilde{V}_k \\ Z_k \end{bmatrix}$, where $Z_k = -\hat{A}_{22}^{-1}(A_{21}\tilde{V}_k + B_2\tilde{W}_k)$. Finally, from (16) and since $\mathcal{R}^\star$ contains $\ker E$, we find that $\mathcal{R}^\star$ is equal to $\mathrm{im} \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 & \ldots & \tilde{V}_{\tilde{r}} \\ Z_1 & Z_2 & \ldots & Z_{\tilde{r}} \end{bmatrix} + \ker E$, so that $\mathcal{R}_S^\star = \mathrm{im} \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 & \ldots & \tilde{V}_{\tilde{r}} \\ 0 & 0 & \ldots & 0 \end{bmatrix} = \tilde{\mathcal{R}}^\star \oplus \{0\}$, and therefore $r = \tilde{r}$. ■

Remark 5.3 The same result of Theorem 5.1 holds for the computation of $\mathcal{V}^\star + \ker E, \mathcal{V}_S^\star$ when we consider $\lambda_1, \lambda_2, \ldots, \lambda_r, z_1, z_2, \ldots, z_{v-r}$ distinct complex numbers, where $z_1, z_2, \ldots, z_{v-r}$ are the invariant zeros of $\hat{\Sigma}$, which coincide with the invariant zeros of the associated standard system $\tilde{\Sigma}$. The output-nulling subspace $\mathcal{V}^\star$ is computed by $\mathrm{im} \begin{bmatrix} V_1 & \ldots & V_r & V_{r+1} & \ldots & V_v \end{bmatrix}$, where $V_{r+1}, \ldots, V_v$ are computed as in Theorem 5.1 for $z_1, z_2, \ldots, z_{v-r}$.

Remark 5.4 Notice that the preliminary feedback H does not affect the computation of the reachability and output-nulling subspaces. Indeed, (15) can be written as

$$\begin{bmatrix} A-\lambda_k E & B \\ C & D \end{bmatrix}\begin{bmatrix} I_n & 0 \\ H & I_m \end{bmatrix}\begin{bmatrix} V_k' \\ W_k' \end{bmatrix} = \begin{bmatrix} A-\lambda_k E & B \\ C & D \end{bmatrix}\begin{bmatrix} V_k' \\ HV_k'+W_k' \end{bmatrix} = 0$$

and since the upper submatrices in $\begin{bmatrix} V_k' \\ W_k' \end{bmatrix}$ and $\begin{bmatrix} V_k' \\ HV_k'+W_k' \end{bmatrix}$ are the same, the image of the upper blocks of $\begin{bmatrix} V_k' \\ W_k' \end{bmatrix}$ and $\begin{bmatrix} V_k' \\ HV_k'+W_k' \end{bmatrix}$ is the same for every k.

5.3 Computation of S-reachable and reachable subspaces

Before we proceed to the computation, we introduce the *standard decomposition form* or *Kronecker form* for regular descriptor systems, see e.g. [15], [9], [19], [28], [12, Ch.2]. A regular descriptor system is restricted system equivalent to a system described by the following equations

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 v(t), \\ N\dot{x}_2(t) &= x_2(t) + B_2 v(t), \\ y(t) &= C_1 x_1(t) + C_2 x_2(t) + Dv(t), \end{aligned}$$

where $x_1(t) \in \mathbb{R}^q$, $x_2(t) \in \mathbb{R}^{n-q}$ and N is a nilpotent matrix with *index of nilpotency* α, where $\alpha \doteq \min\{k \in \mathbb{N} \mid N^k = 0\}$. The C-reachable subspace with respect to the standard decomposition form is given by $\mathcal{R}_C = \langle A_1 | \mathrm{im}\, B_1 \rangle \oplus \langle N | \mathrm{im}\, B_2 \rangle$, where $\langle A_1 | \mathrm{im}\, B_1 \rangle = \mathrm{im} \begin{bmatrix} B_1 & A_1 B_1 & \dots & A_1^{q-1} B_1 \end{bmatrix}$ and $\langle N | \mathrm{im}\, B_2 \rangle = \mathrm{im} \begin{bmatrix} B_2 & NB_2 & \dots & N^{\alpha-1} B_2 \end{bmatrix}$, see e.g. [19], [28], [12, Ch.4].

Proposition 5.1 *The S-reachable subspace for an impulse-free descriptor system as in* (10a)-(10b) *is equal to* $\mathcal{R}_S = \tilde{\mathcal{R}}_0 \oplus \{0\}$ *and the reachable subspace is equal to* $\mathcal{R} = (\tilde{\mathcal{R}}_0 \oplus \{0\}) \oplus \ker E$, *where* $\tilde{\mathcal{R}}_0$ *is the reachable subspace of* $(\tilde{A}, \tilde{B})$ *in* (11a), *i.e.,* $\tilde{\mathcal{R}}_0 = \langle \tilde{A} | \mathrm{im}\, \tilde{B} \rangle$.

Proof: The standard decomposition form coincides with the form (10) and $N = 0$ because the descriptor system is impulse-free. Therefore the S-reachable subspace is given by $\mathcal{R}_S = E\mathcal{R}_C = \langle \tilde{A}_{11} | \mathrm{im}\, \tilde{B}_1 \rangle \oplus \{0\} = \langle \tilde{A} | \mathrm{im}\, \tilde{B} \rangle \oplus \{0\}$. Consequently, the S-reachable subspace is equal to $\mathcal{R}_S = \tilde{\mathcal{R}}_0 \oplus \{0\}$ and the reachable subspace is equal to $\mathcal{R} = (\tilde{\mathcal{R}}_0 \oplus \{0\}) \oplus \ker E$. ■

Remark 5.5 Note that the preliminary feedback H does not affect the computation of the reachable and S-reachable subspace, since $\mathcal{W}^\star = E^{-1}((A+BH)\mathcal{W}^\star + \mathrm{im}\ B) = E^{-1}(A\mathcal{W}^\star + \mathrm{im}\ B)$, because $BH\mathcal{W}^\star \subseteq \mathrm{im}\ B$.

6 Numerical example

Consider a continuous-time descriptor system Σ described by the matrices

$$E = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \; A = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right], \; B = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] = \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right],$$
$$C = \left[\begin{array}{cc|cc} 0 & 0 & -1 & 4 \end{array}\right] = \left[\begin{array}{cc} C_1 & C_2 \end{array}\right], \; D = \left[\begin{array}{cc} 0 & 1 \end{array}\right].$$

The system is not regular but it is I-controllable, since $\operatorname{rank}\left[\begin{array}{ccc} E & AE_\infty & B \end{array}\right] = 4$. We apply the state feedback $u(t) = Hx(t) + v(t)$, where

$$H = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc} H_1 & H_2 \end{array}\right],$$

so that the closed-loop system $\hat{\Sigma}$ is impulse-free and described by the quintuple $(E, \hat{A}, B, \hat{C}, D)$, where

$$\hat{A} = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc} A_{11} & \hat{A}_{12} \\ A_{21} & \hat{A}_{22} \end{array}\right], \; \hat{C} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 4 \end{array}\right] = \left[\begin{array}{cc} C_1 & \hat{C}_2 \end{array}\right].$$

Denoting by $\mathbf{e}_i$ the i-th canonical basis vector of $\mathbb{R}^4$ and from (8) and (9), we compute $\mathcal{V}^\star = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, 4\mathbf{e}_3 + \mathbf{e}_4\}$, $\ker E = \operatorname{span}\{\mathbf{e}_3, \mathbf{e}_4\}$, $\mathcal{S}^\star = \operatorname{span}\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, so that $\mathcal{R}^\star = (\mathcal{V}^\star + \ker E) \cap \mathcal{S}^\star = \operatorname{im}\left[\begin{array}{c|cc} \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \end{array}\right] = \mathcal{R}^\star_S \oplus \ker E$. The dimension of $\mathcal{R}^\star_S$ is 1 and so $r = 1$. Let us choose $\lambda = -2$ and compute $\ker P_{\hat{\Sigma}}(-2) = \ker \left[\begin{smallmatrix} \hat{A}-(-2)E & B \\ \hat{C} & D \end{smallmatrix}\right] = \operatorname{span}\left\{\left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right]\right\}$, where $V = \left[\begin{array}{cccc} 0 & 2 & -4 & -1 \end{array}\right]^\top$, $W = \left[\begin{smallmatrix} -3 \\ 4 \end{smallmatrix}\right]$.

We compute the matrices $\tilde{A} = A_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}A_{21} = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\tilde{B} = B_1 - \hat{A}_{12}\hat{A}_{22}^{-1}B_2 = \left[\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right]$, $\tilde{C} = C_1 - \hat{C}_2\hat{A}_{22}^{-1}A_{21} = \left[\begin{array}{cc} 0 & 4 \end{array}\right]$, $\tilde{D} = D - \hat{C}_2\hat{A}_{22}^{-1}B_2 = \left[\begin{array}{cc} 4 & 1 \end{array}\right]$ and obtain a quadruple of the associated standard system $\tilde{\Sigma}$. The Rosenbrock system matrix pencil of $\tilde{\Sigma}$ is

$$P_{\tilde{\Sigma}}(\lambda) = \left[\begin{array}{cc} \tilde{A} - \lambda I_2 & \tilde{B} \\ \tilde{C} & \tilde{D} \end{array}\right] = \left[\begin{array}{cc|cc} -\lambda - 1 & 0 & 0 & 0 \\ 0 & -\lambda + 1 & 2 & 0 \\ \hline 0 & 4 & 4 & 1 \end{array}\right],$$

so that $\tilde{\Sigma}$ has an invariant zero at $z = -1$, which is also the invariant zero of Σ. From (8) and (9) with $E = I_2$ we compute $\tilde{\mathcal{V}}^\star = \operatorname{im}\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\tilde{\mathcal{S}}^\star = \operatorname{span}\left\{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right\}$, so that $\tilde{\mathcal{R}}^\star = \tilde{\mathcal{V}}^\star \cap \tilde{\mathcal{S}}^\star = \operatorname{span}\left\{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right\}$. The dimension of $\tilde{\mathcal{R}}^\star$ is 1, so we choose for example $\lambda = -2$ and compute $\ker P_{\tilde{\Sigma}}(\lambda) = \ker \left[\begin{smallmatrix} \tilde{A}-(-2)I_2 & \tilde{B} \\ \tilde{C} & \tilde{D} \end{smallmatrix}\right] = \operatorname{span}\left\{\left[\begin{smallmatrix} \tilde{V} \\ \tilde{W} \end{smallmatrix}\right]\right\}$, where $\tilde{V} = \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$, $\tilde{W} = \left[\begin{smallmatrix} -3 \\ 4 \end{smallmatrix}\right]$. Then $Z = -\hat{A}_{22}^{-1}(A_{21}\tilde{V} + B_2\tilde{W}) = \left[\begin{smallmatrix} -4 \\ -1 \end{smallmatrix}\right]$ and $V = \left[\begin{smallmatrix} \tilde{V} \\ Z \end{smallmatrix}\right] = \left[\begin{array}{cc|cc} 0 & 2 & -4 & -1 \end{array}\right]^\top$, $W = \tilde{W} = \left[\begin{smallmatrix} -3 \\ 4 \end{smallmatrix}\right]$, which coincide with the V and W computed above.

Alternatively, in view of (13), we compute $\hat{P}_2 \begin{bmatrix} \tilde{V} \\ Z \\ \tilde{W} \end{bmatrix}$ as

$$\begin{bmatrix} \hat{V} \\ \hat{Z} \\ \hat{W} \end{bmatrix} = \hat{P}_2 \begin{bmatrix} \tilde{V} \\ Z \\ \tilde{W} \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 0 \\ 2 \\ \hline -4 \\ -1 \\ \hline -3 \\ 4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 2 \\ \hline 0 \\ 0 \\ \hline -3 \\ 4 \end{array}\right]$$

which has $\hat{Z} = 0$. Basis matrices for $\mathcal{R}^\star$ and $\mathcal{R}_S^\star$ are given respectively by $\text{span}\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and $\text{span}\{\mathbf{e}_2\}$.

We compute $\ker P_{\hat{\Sigma}}(-1) = \ker \begin{bmatrix} \hat{A}-(-1)E & B \\ \hat{C} & D \end{bmatrix} = \text{im} \begin{bmatrix} V' \\ W' \end{bmatrix}$, where $V' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^\top$, $W' = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$. It follows that $\mathcal{V}^\star = \text{im} \, [\, V \;\; V' \,] = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, 4\mathbf{e}_3 + \mathbf{e}_4\}$, $\mathcal{V}^\star + \ker E = \text{im} \, [\, V \;\; V' \,] + \ker E = \mathcal{X}$.

The S-reachable subspace $\mathcal{R}_S$ is equal to $E\mathcal{W}^\star$ and the reachable subspace $\mathcal{R}$ is equal to $\mathcal{W}^\star$. Alternatively, we may compute $\mathcal{R}_S, \mathcal{R}$ via the reachable subspace of $\tilde{\Sigma}$, which is $\tilde{\mathcal{R}}_0 = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, and we find

$$\mathcal{R}_S = \tilde{\mathcal{R}}_0 \oplus \{0\} = \text{span}\left\{\left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \end{array}\right]\right\}, \; \mathcal{R} = \big(\tilde{\mathcal{R}}_0 \oplus \{0\}\big) \oplus \ker E = \text{im} \left[\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

7 Conclusions

In this paper, the geometric structure of square LTI descriptor systems has been investigated. We described and discussed different types of reachability and controllability for descriptor systems. Since descriptor systems may exhibit impulsive modes, impulse controllability was also assumed. However, regularity was not assumed, since impulse controllability implies regularizability. We analyzed the two main frameworks on reachability and controllability for descriptor systems, given by Rosenbrock and Verghese et al., which leads to a new definition of the reachable subspace for descriptor systems based on the framework by Verghese et al. Finally, it was shown that the Rosenbrock system matrix pencil can be employed to compute the supremal output-nulling subspace and the supremal output-nulling reachability subspace of a descriptor systems.

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