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41 FRACTIONAL BLACK-SCHOLES MODELS: COMPLETE MLE WITH APPLICATION TO FRACTIONAL OPTION PRICING

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Abstract: Geometric fractional Brownian motion (GFBM) is an extended model of the traditional geometric Brownian motion that is widely used for Black-Scholes option pricing. By considering GFBM, we are now able to capture the memory dependency. This method will enable us to derive the estimators of the drift, μ, volatility, σ^2, and also the index of self similarity, H, simultaneously. This will enable us to use the fractional Black-Scholes model with all the needed parameters. Simulation outcomes illustrate that our methodology is efficient and reliable. Empirical application to stock exchange index with option pricing under GFBM is also made.

Key words: maximum likelihood estimation; geometric fractional Brownian motion; long memory; option pricing.

1 INTRODUCTION

GFBM is a geometric version of the Fractional Brownian motion (FBM), denoted by $B_H(t)$, which has first been studied by Kolmogorov as early as in 1940. Mandelbrot and Van Ness (1968) gave the name and actively developed its statistical properties. Since then, many works have been conducted in a variety of fields, ranging from network traffic (c.f., Abry, Flandrin, Taqqu and Veitch, 2000) to economics and finance (c.f., Mandelbrot and Van Ness 1968; Shiryaev 1999; Cajueiro and Barbachan 2005). The problem generally deals with self similarity, and one of the favorite problems being still debated in the literature is to find a good method to estimate the index of self-similarity, i.e., the Hurst parameter H, which is named after the English hydrologist H.E. Hurst, who first developed index of self similarity when studying the Nile River in 1951.

Application of FBM and GFBM to financial option pricing is a very natural idea to extend the famous Black-Scholes option theory as based on BM and GBM. However, works regarding the FBM in the early years by using the pathwise integration theory showed that the mathematical markets based on $B_H(t)$ could have arbitrage opportunity that proved to be no use in financial modelling (Rogers, 1997). This problem has discouraged further investigation in this field for many years. Only recently have researchers worked on $B_H(t)$ using ordinary product pathwise as an alternative approach. This exhibits promising results in producing no arbitrage situation and further stimulates active works by taking $B_H(t)$ as an underlying process in mathematical markets models. Hu and Øksendal (2003) proved that the white noise calculus based on $B_H(t)$ with $\frac{1}{2} < H < 1$, corresponding to Ito type fractional Black Sholes market, has no arbitrage and the market is complete. Elliott and van der Hoek (2003) extended H to the range of $[0, 1]$. They worked on option pricing and consider FBM as the driving noise process for this kind of problems. Though there is some criticism regarding this approach[1], the option pricing under GFBM has been well developed based on this new framework, with the Black-Scholes option pricing as a special case (taking $H = 0.5$). A variety of researches have taken account of long memory property in the option pricing, see, for example, Aldabe et. al (1998) for regularized fractional Brownian motion, Bertrand (2005) for multiscale fractional Brownian motion with European option, Elliott and Chan (2004) for valuation of perpetual American options, and Jumarie (2005) for Merton's optimal portfolio. In this paper, we adopt the recent outcome by Mishura (2008), who gives a properly defined formula for fractional Black-Scholes market for European option.

A crucial problem with the applications of these option pricing formulae in the fractional Black-Scholes markets in practice is how to obtain the unknown values of the parameters in GFBM. In particular there are two key parameters, the volatility σ and the long memory parameter H, that play a crucially important role in valuing, say, European option in Mishura (2008); see section

4 for detail. However, in the literature, to the best of our knowledge, there seems very few work addressing this problem. An exception is the paper by Kukush *et al.* (2005) who developed an incomplete maximum likelihood estimation (IMLE) of the volatility σ, separated from the estimation of the long memory parameter H which is made in advance by some estimation methods specially designed only for H, such as the R/S analysis, variation analysis, etc..

In this paper, we study the problem of estimating the unknown parameters, including the drift μ, volatility σ and Hurst index H, involved in the GFBM based on the discrete observations in the setting of $0 < H < 1$. Unlike Kukush *et al.* (2005), we propose an approach of complete maximum likelihood estimation (CMLE), which enables us not only to derive the estimators of μ and σ^2, and also the estimate of the long memory parameter, H, simultaneously, with large sample distributions offered, for the risky assets in the fractional Black-Scholes market governed by GFBM. Our simulation outcome will illustrate that our methodology by CMLE is efficient and reliable for the model of GFMB, while the separating method of estimating σ^2 and H by IMLE together with the widely used R/S analysis may lead to poor estimate of them. Empirical application to stock exchange index with option pricing under GFBM also shows that our method can make reasonable outcome of the European option prices, while the traditional Black-Scholes formula seems undervalue the option and the IMLE method with R/S analysis for GFBM may lead to overvaluation of the option.

2 MODEL OF GEOMETRIC FRACTIONAL BROWNIAN MOTION

In this section, we give a brief background introduction to GFBM, which is necessary below.

2.1 *Fractional Brownian Motion,* $B_H(t)$

The FBM first came to limelight in the financial world due to Mandelbrot and van Ness (1968), who generalized the traditional Brownian motion with $H = \frac{1}{2}$ to FBM $B_H(t)$ for $0 < H < 1$. $B_H(t)$ is a self-similar Gaussian process, with index $0 < H < 1$ and stationary increments defined on a probability space with the properties that $B_H(0) = 0$, $E[B_H(t)] = 0$ for every $t \geq 0$, and its covariance is defined in the form

$$C_H(t) = E[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \tag{2.1}$$

The self-similarity means that for any $\alpha > 0$, $B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$. Clearly, when $H = \frac{1}{2}$, $B_H(t)$ reduces to a standard Brownian motion $B(t)$. For further details, the reader is referred to Biagini, Hu, Øksendal and Zhang (2008).

We need the following property on the increment of the FBM. Set

$$e_j = B_H(j+1) - B_H(j) \tag{2.2}$$

for $j \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. Then the covariance of e_j can be presented as follows:

$$r(k) = Ee_{j+k}e_j = \frac{1}{2}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}) \tag{2.3}$$

for $k \in \mathbb{Z}$. Note that when $H < \frac{1}{2}$, the increments are negatively correlated whereas $H > \frac{1}{2}$ shows the positive correlation. This increment is a stationary process, which is often called as fractional Gaussian noise. It is easily showed that

$$r(k) \sim H(2H-1)k^{2H-2}, \quad \text{as } k \to \infty, \tag{2.4}$$

which implies that if $H > \frac{1}{2}$ then the summation of correlations diverges, that is $\sum_{k=0}^{\infty} r(k) = \infty$, often referred to as long memory or long range dependence property.

2.2 Geometric Fractional Brownian Motion

We are concerned with fractional Black-Scholes markets, in which the risky asset price process, $S(t)$, driven by FBM is modelled by GFBM, in the form

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t), \tag{2.5}$$

where $S(0) = s > 0$, and μ and $\sigma > 0$ are the drift and volatility, respectively. The solution to this fractional differential equation (Hu and Øksendal, 2000) is given by

$$S(t) = s\ \exp\{\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\}. \tag{2.6}$$

The estimation of the Hurst index H and the volatility σ in this model is particularly important in financial asset pricing (see Section 5 below). How to estimate them is what we aim at in next section.

3 METHODOLOGY OF ESTIMATION

3.1 Model simplification

We begin with a review of the incomplete likelihood estimation proposed by Kukush *et al.* (2005).

Let $X(t) = \ln(\frac{S(t)}{s})$. Then it follows from (2.6) that $X(t) = \sigma B_H(t) + \mu t - (\frac{\sigma^2}{2})t^{2H}$, $t \geq 0$. As in Kukush *et al.* (2005), we assume that the historical data are observed at discrete times $t_k = \frac{kT}{n}$, $k = 0, 1, ..., n$, over the time interval

$[0, T]$. By setting $X_k = X(t_k)$ and $B_{Hk} = B_H(t_k)$ and considering $k = 1, ..., n$, we can have

$$\Delta X_k = \sigma \Delta B_{Hk} + \mu \Delta t_k - \frac{\sigma^2}{2} \Delta (t^{2H})_k, \tag{3.1}$$

where $\Delta X_k = X_k - X_{k-1}$, ΔB_{Hk} and Δt_k are defined similarly, and $\Delta(t^{2H})_k = t_k^{2H} - t_{k-1}^{2H}$.

Kukush *et al.* (2005) developed an incomplete maximum likelihood estimation (IMLE) procedure. First, they assumed H can be estimated in advance by some estimation methods well designed in the literature for H, such as the R/S analysis, variation analysis, etc. Then they considered $Y_k = \frac{n^H \Delta X_k}{T^H}$ and wrote (3.1) as

$$Y_k = \sigma \varepsilon_k + \frac{n^H \mu \Delta t_k}{T^H} - \frac{1}{2} \sigma^2 T^H n^H \Delta \tau_k^{2H} \tag{3.2}$$

for $k = 1, ..., n$, where $\Delta \tau_k^{2H} = (\frac{k}{n})^{2H} - (\frac{k-1}{n})^{2H}$, and $\varepsilon_k = \frac{n^H \Delta B_{Hk}}{T^H}$. Simple calculation shows that ε_k is normally distributed with $E\varepsilon_k = 0$, $E\varepsilon_k^2 = 1$ and the covariance of ε_k the same as in (3.7). Using (3.2), Kukush *et al.* (2005) then suggested an IMLE of the volatility σ, based on Y_k, with

$$\hat{\sigma}^2_{\text{IMLE}} = \frac{1}{n} \sum_{k=1}^{n} (Y_k - \bar{Y})^2, \tag{3.3}$$

where $\bar{Y} = \frac{1}{n} \sum_{k=1}^{n} Y_k$. This estimation method was applied in option pricing by Cajueiro and Barbachan (2005). Note that $\hat{\sigma}^2_{\text{IMLE}}$ is just the usual sample variance of Y_k. It is essentially as assumed that the Y_k in the model (3.2) is stationary. This however may not be true in general owing to $\Delta \tau_k^{2H}$ depending on k if $H \neq 0.5$. Under some constrained conditions imposed on n and T tending to infinity, $\hat{\sigma}^2_{\text{IMLE}}$ can be showed to be consistent (c.f., Kukush *et al.* (2005)); however it may not be efficient particularly when the sample sizes are finite in practice.

In this paper, we study the problem of estimating the unknown parameters, including the drift μ, volatility σ and Hurst index H, involved in the GFBM based on the discrete observations in the setting of $0 < H < 1$. Unlike Kukush *et al.* (2005), we propose an approach of complete maximum likelihood estimation (CMLE), which enables us to estimate μ, σ^2 and H, simultaneously. We will alternatively consider the returns series $Z_k = \Delta X_k$, rather than Y_k, as follows (following from (3.2)):

$$\begin{aligned} Z_k = \Delta X_k &= (\frac{T}{n})^H Y_k \\ &= (\frac{T}{n})^H \sigma \varepsilon_k + \mu \frac{T}{n} - \frac{1}{2} \left\{ (\frac{T}{n})^H \sigma \right\}^2 n^{2H} \Delta \tau_k^{2H} \\ &\equiv \sigma_1 \varepsilon_k + \mu_1 - \frac{1}{2} \sigma_1^2 n^{2H} \Delta \tau_k^{2H}, \end{aligned} \tag{3.4}$$

where $\sigma_1 = (\frac{T}{n})^H \sigma$ and $\mu_1 = \frac{\mu T}{n}$. We construct our complete maximum likelihood estimation based on (3.4).

3.2 Complete Maximum Likelihood Estimation (CMLE)

In this subsection, we are concerned with the estimation of $\theta = (\sigma_1^2, \mu_1, H)'$ by using the method of CMLE. Here, A' stands for the transpose of a vector or a matrix A.

3.3 Likelihood function of $\theta = (\sigma_1^2, \mu_1, H)'$

Based on (3.4), our observations are $Z = (Z_1, ..., Z_n)'$, and for notational convenience, set $\mathbf{x}_H = n^{2H}(\Delta\tau_1^{2H}, ..., \Delta\tau_n^{2H})'$ and $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)'$.

Then, the vector form of (3.4) is as follows:

$$Z = \sigma_1\varepsilon + \mu_1\mathbf{1} - \frac{1}{2}\sigma_1^2 x_H \tag{3.5}$$

where $\mathbf{1}$ is an n-dimensional vector of components 1's. Set

$$\Sigma = var(Z) = \sigma_1^2(E\varepsilon\varepsilon')_{n\times n} = \sigma_1^2\Sigma_0, \tag{3.6}$$

with $\Sigma_0 = \Sigma_0(H) = (\gamma_{ij})_{n\times n}$ given by

$$\gamma_{ij} = E\varepsilon_i\varepsilon_j = \frac{1}{2}(|i-j+1|^{2H} - 2|i-j|^{2H} + |i-j-1|^{2H}). \tag{3.7}$$

Since the process is Gaussian, the log likelihood for Z is given below.

$$\begin{aligned}\ell_n(\theta) &= -\frac{1}{2}\log|\Sigma| - \frac{1}{2}(Z - \mu_1\mathbf{1} + \frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma^{-1}(Z - \mu_1\mathbf{1} + \frac{1}{2}\sigma_1^2\mathbf{x}_H)\\ &= -\frac{1}{2}(n\log\sigma_1^2 + \log|\Sigma_0|) - \frac{1}{2\sigma_1^2}(Z - \mu_1\mathbf{1} + \frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_0^{-1}(Z - \mu_1\mathbf{1} + \frac{1}{2}\sigma_1^2\mathbf{x}_H).\end{aligned} \tag{3.8}$$

Therefore, the CMLE of $\theta = (\sigma_1^2, \mu_1, H)'$ is

$$\hat{\theta} = (\hat{\sigma_1^2}, \hat{\mu_1}, \hat{H})' = \arg\max_{\theta\in\Theta} \ell_n(\theta) \tag{3.9}$$

where Θ is a compact subset of $\mathbb{R}^+ \times \mathbb{R} \times (0,1)$, which contains the actual parameter vector $\theta_0 = (\sigma_{10}^2, \mu_{10}, H_0)'$.

We finally obtain the estimators of σ^2 and μ as follows:

$$\widehat{\sigma}^2 = \left(\frac{n}{T}\right)^{\hat{H}} \widehat{\sigma}_1^2, \quad \hat{\mu} = \frac{n}{T}\hat{\mu}_1. \tag{3.10}$$

3.4 Algorithm

Our aim now is to calculate $\hat{\theta}$ in (3.9). Maximizing (3.8) directly is quite involved. We suggest a profile method to simplify the calculation. For a given H, we can derive the maximum likelihood estimators for σ_1^2 and μ_1, by maximizing (3.8) with respect to σ_1^2 and μ_1. They are achieved by setting the first order partial derivatives of $\ell_n(\theta)$ with respect to σ_1^2 and μ_1 equal to 0, giving

$$\widehat{\sigma}_1^2 = \frac{2Z'\Sigma_1 Z}{\sqrt{n^2 + \mathbf{x}_H'\Sigma_1\mathbf{x}_H Z'\Sigma_1 Z} + n} \tag{3.11}$$

and

$$\hat{\mu}_1 = \frac{1}{\mathbf{1}'\Sigma_0^{-1}\mathbf{1}}(\mathbf{1}'\Sigma_0^{-1}Z + \frac{1}{2}\widehat{\sigma}_1^2\mathbf{1}'\Sigma_0^{-1}\mathbf{x}_H), \tag{3.12}$$

where

$$\Sigma_1 = \Sigma_0^{-1}\left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}'\Sigma_0^{-1}}{\mathbf{1}'\Sigma_0^{-1}\mathbf{1}}\right) \tag{3.13}$$

with $\mathbf{I}$ being an $n \times n$ identity matrix.

Now, in order to estimate H, we replace σ_1^2 and μ_1 in (3.8) by (3.11) and (3.12). Consequently, we obtain

$$\begin{aligned}\ell_{1n}(H) &= \ell(\widehat{\sigma}_1^2, \hat{\mu}_1, H)\\ &= -\frac{1}{2}(n\log\widehat{\sigma}_1^2 + \log|\Sigma_0|)\\ &- \frac{1}{2\widehat{\sigma}_1^2}(Z - \hat{\mu}_1\mathbf{1} + \frac{1}{2}\widehat{\sigma}_1^2\mathbf{x}_H)'\Sigma_0^{-1}(Z - \hat{\mu}_1\mathbf{1} + \frac{1}{2}\widehat{\sigma}_1^2\mathbf{x}_H).\end{aligned} \tag{3.14}$$

This is a function of H. Taking the differentiation of $\ell_{1n}(H)$ with respect to H is difficult. However, note that $\ell_{1n}(H)$ is a univariate profile likelihood function of H. There are many available numerical methods that can be used to maximize $\ell_{1n}(H)$ without using the information of differentiation, for example, Golden Section Search method. Thus, we can get the estimator $\hat{H}$ of H.

With this in mind, we propose an algorithm given below.

- Maximize (3.14) numerically to get the estimator, $\hat{H}$, of H.
- Calculate the estimators, $\hat{\sigma}_1^2$ and $\hat{\mu}_1$, by replacing H with $\hat{H}$ in (3.11) and (3.12), respectively.
- Compute the estimators of σ^2 and μ by (3.10).

4 SIMULATION STUDY

In order to examine the performance of the proposed estimators, we did some simulation experiments. Let us begin with how the data is simulated. We first consider the model (2.6). As in the last section, we take $t_k = \frac{kT}{n}$, and $B_{Hk} = B_H(t_k)$ has the property of Gaussian distribution with $EB_H(t_k) = 0$ and $E(B_H^2(t_k)) = t_k^{2H} = (\frac{kT}{n})^{2H}$. By considering this property, the equation (2.6) now becomes

$$S_k = S(t_k) = s \exp\left[\sigma \left(\frac{T}{n}\right)^H B_{Hk} + \mu \left(\frac{kT}{n}\right) - \frac{1}{2}\sigma^2 \left(\frac{kT}{n}\right)^{2H}\right]. \tag{4.1}$$

We take the values of the parameters $\mu = 0.2752908$, $\sigma^2 = 0.2554078$, $H = 0.549$ and the initial value of $s = 903.84$. We simulate the time series from this discrete time model and apply our methodology to estimate the parameters $\vartheta = (\sigma^2, \mu, H)$ using the simulated data set. The simulation is repeated one hundred times to look at the performance.

To have an idea on the performance of the estimators suggested by Kukush *et al.* (2005), we also considered the estimation method by Kukush *et al.* as a comparison. No doubt, R/S analysis is most widely used for estimation of Hurst index in the literature. We apply the Hurst value obtained from R/S analysis in Kukush *et al.*'s method. The simulated outcomes of the average value of estimates based on 100 replications, with bias and variance, are reported in Tables 1-4, for $T = 15$, $T = 30$, $T = 40$ and $T = 50$, respectively. The 5 cases of sample sizes $n = 100, 200, 300, 400, 500$ are considered in each table, where

- $\hat{H}_{CMLE}$ = Hurst index obtained by using the method proposed in this paper;
- $\hat{\mu}_{CMLE} = \mu$ obtained by the method proposed in this paper;
- $\hat{\sigma}^2_{CMLE} = \sigma^2$ obtained by the method proposed in this paper;
- $\hat{H}_{RS}$ = Hurst index obtained by using the method of R/S analysis;
- $\hat{\sigma}^2_{IMLE} = \sigma^2$ obtained by the method of Kukush et. al (2005) with $\hat{H}_{RS}$.

It is obvious from the results obtained in Tables 4.1–4.4 that our methodology performs considerably better. Most of the biases and variances obtained by

Table 4.1 Outcome of simulation with $T = 15$: average value of estimates based on 100 replications, with bias in ( ) and variance in [ ]

n	100	200	300	400	500
$\hat{H}_{CMLE}$	0.5378 (−0.0112) [0.0019]	0.5395 (−0.0095) [0.0011]	0.5446 (−0.0044) [0.0012]	0.5438 (−0.0052) [0.0008]	0.5409 (−0.0081) [0.0009]
$\hat{\mu}_{CMLE}$	0.2590 (−0.0163) [0.0321]	0.2593 (−0.0160) [0.1022]	0.3199 (0.0446) [0.0270]	0.2512 (−0.0240) [0.0254]	0.2697 (−0.0056) [0.0159]
$\hat{\sigma}^2_{CMLE}$	0.2439 (−0.0115) [0.0043]	0.2424 (−0.0131) [0.0025]	0.2520 (−0.0034) [0.0037]	0.2500 (−0.0054) [0.0029]	0.2448 (−0.0106) [0.0034]
$\hat{H}_{RS}$	0.6575 (0.1085) [0.0318]	0.6275 (0.0785) [0.0200]	0.6099 (0.0609) [0.0141]	0.6326 (0.0836) [0.0113]	0.5969 (0.0479) [0.0057]
$\hat{\sigma}^2_{IMLE}$	0.4739 (0.2185) [0.1158]	0.5127 (0.2573) [0.2747]	0.4748 (0.2194) [0.1925]	0.5717 (0.3163) [0.2226]	0.4060 (0.1506) [0.0524]

Table 4.2 Outcome of simulation with $T = 30$: average value of estimates based on 100 replications, with bias in ( ) and variance in [ ]

n	100	200	300	400	500
$\hat{H}_{CMLE}$	0.5392 (−0.0098) [0.0017]	0.5374 (−0.0116) [0.0010]	0.5448 (−0.0042) [0.0012]	0.5425 (−0.0065) [0.0009]	0.5454 (−0.0036) [0.0008]
$\hat{\mu}_{CMLE}$	0.2398 (−0.0355) [0.0194]	0.2788 (0.0035) [0.0172]	0.2475 (−0.0278) [0.0152]	0.2691 (−0.0062) [0.0163]	0.2841 (0.0088) [0.0184]
$\hat{\sigma}^2_{CMLE}$	0.2457 (−0.0097) [0.0019]	0.2457 (−0.0097) [0.0018]	0.2520 (−0.0034) [0.0024]	0.2527 (−0.0027) [0.0019]	0.2538 (−0.0016) [0.0021]
$\hat{H}_{RS}$	0.6165 (0.0675) [0.0291]	0.6302 (0.0812) [0.0165]	0.6175 (0.0685) [0.0104]	0.5950 (0.0460) [0.0113]	0.6155 (0.0665) [0.0073]
$\hat{\sigma}^2_{IMLE}$	0.3230 (0.0676) [0.0232]	0.3902 (0.1348) [0.0389]	0.3937 (0.1383) [0.0573]	0.3806 (0.1252) [0.0481]	0.4177 (0.1623) [0.0498]

using our method are within an acceptable tolerance. All of our estimates for H are obviously quite stable and less biased. The performance on the estimation of σ^2 is also fairly satisfactory. We can also see that the larger the sample size n, the better the estimation performs. Further, overall, with a larger T, the outcome become better for any n.

Table 4.3 Outcome of simulation with $T = 40$: average value of estimates based on 100 replications, with bias in ( ) and variance in [ ]

n	100	200	300	400	500
$\hat{H}_{MLE}$	0.539 (-0.0100) [0.0019]	0.5363 (-0.0127) [0.0011]	0.5409 (-0.0081) [0.0012]	0.5455 (-0.0035) [0.0007]	0.5433 (-0.0057) [0.0007]
$\hat{\mu}_{CMLE}$	0.2853 (0.0100) [0.0869]	0.2662 (-0.0091) [0.0130]	0.2841 (0.0089) [0.0153]	0.3083 (0.0330) [0.0155]	0.2799 (0.0046) [0.0133]
$\hat{\sigma}^2_{CMLE}$	0.2504 (-0.0050) [0.0021]	0.2512 (-0.0042) [0.0016]	0.2477 (-0.0077) [0.0016]	0.2510 (-0.0044) [0.0015]	0.2487 (-0.0067) [0.0016]
$\hat{H}_{RS}$	0.6113 (0.0623) [0.0289]	0.6258 (0.0768) [0.0226]	0.6275 (0.0785) [0.0139]	0.6282 (0.0792) [0.0091]	0.6095 (0.0605) [0.0095]
$\hat{\sigma}^2_{IMLE}$	0.3011 (0.0457) [0.0115]	0.3752 (0.1198) [0.0409]	0.3883 (0.1328) [0.0394]	0.4052 (0.1498) [0.0420]	0.3945 (0.1390) [0.0628]

Table 4.4 Outcome of simulation with $T = 50$: average value of estimates based on 100 replications, with bias in ( ) and variance in [ ]

n	100	200	300	400	500
$\hat{H}_{MLE}$	0.541 (-0.0080) [0.0018]	0.54 (-0.0090) [0.0014]	0.5423 (-0.0067) [0.0012]	0.5428 (-0.0062) [0.0008]	0.5438 (-0.0052) [0.0008]
$\hat{\mu}_{CMLE}$	0.2907 (0.0154) [0.0714]	0.2675 (-0.0078) [0.0150]	0.2867 (0.0114) [0.0141]	0.2690 (-0.0062) [0.0101]	0.2644 (-0.0109) [0.0113]
$\hat{\sigma}^2_{CMLE}$	0.2469 (-0.0085) [0.0013]	0.2478 (-0.0076) [0.0014]	0.2508 (-0.0046) [0.0015]	0.2520 (-0.0034) [0.0012]	0.2526 (-0.0028) [0.0018]
$\hat{H}_{RS}$	0.6249 (0.0759) [0.0327]	0.6167 (0.0677) [0.0127]	0.6134 (0.0644) [0.0100]	0.6212 (0.0722) [0.0095]	0.6008 (0.0518) [0.0101]
$\hat{\sigma}^2_{IMLE}$	0.2881 (0.0327) [0.0072]	0.3210 (0.0656) [0.0116]	0.3417 (0.0863) [0.0148]	0.3784 (0.1230) [0.0278]	0.3614 (0.1060) [0.0283]

5 EMPIRICAL RESULTS

5.1 Data

We used a data set from KLCI available online at *http://www.econstats.com*. The daily close price data set of KLCI from 3 January, 2005 to 29 December, 2006 is examined, with 494 observations. The return series is then calculated in logarithm. The return is considered to prevent the high volatility in the data. The changes in the price seem to be more practical as these changes are stationary. The figures of the price and return series are presented in Figures 5.1 and 5.2. A summary of the return series can be found in Table 5.1, where the mean of this series is 0.0003915 and the variance is 0.00002584.

Table 5.1 Summary of the return series of KLCI

Min.	1st Qu.	Median	Mean	Var	3rd Qu.	Max.
-0.0202000	-0.0023850	0.0005159	0.0003915	0.00002584	0.0029860	0.0190700

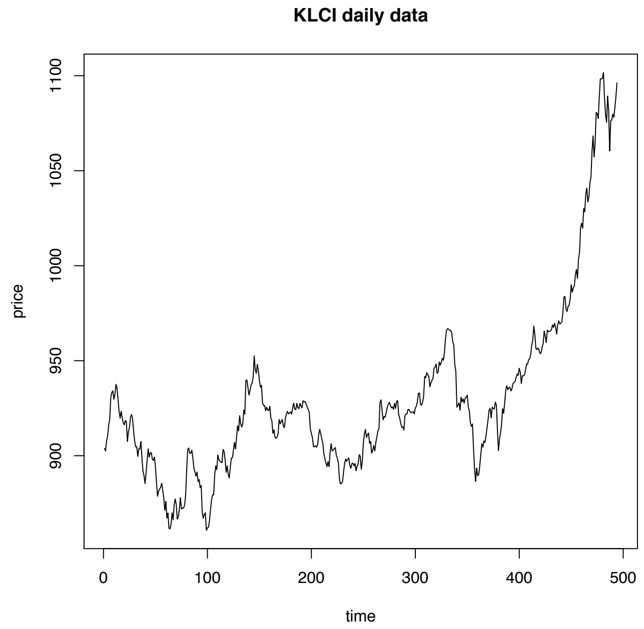

Figure 5.1 Daily close price series of KLCI from 3rd January 2005 to 29 December 2006

5.2 Estimation based on CMLE method

We present in this subsection the results of our study of modeling the data of KLCI by GFBM. We try to estimate the parameters of the risky asset model by using the proposed complete maximum likelihood estimation based on daily return series. The estimates are summarized in Table 5.2.

We can clearly see from Table 5.2 that the suggested estimates are $H = 0.575$, $\sigma^2 = 0.00002576$ and $\mu = 0.0004510$. This finding agrees with the work

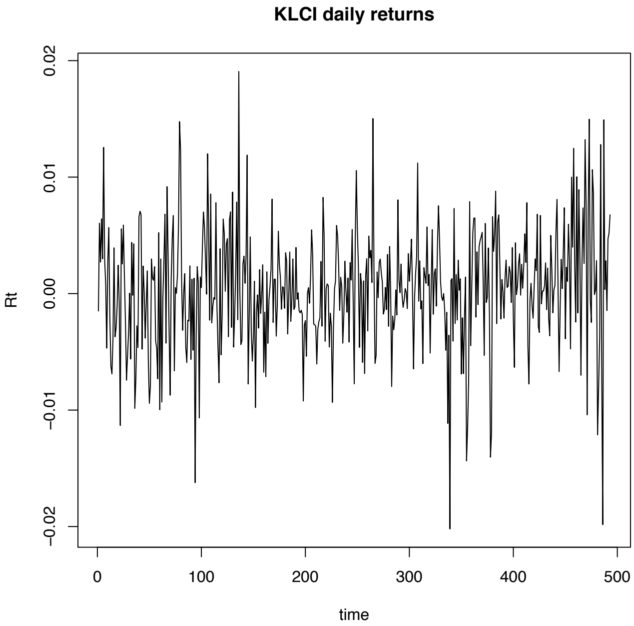

Figure 5.2 Daily return series of KLCI from 3rd January 2005 to 29 December 2006

by Sadique and Silvapulle (2001), where the presence of a weak long memory in Malaysia financial data is suggested.

Table 5.2 Likelihood value with respects to the model parameters

H value	$\widehat{\sigma^2}$	$\widehat{\mu}$	**likelihood value**
0.500	0.00002573	0.0004035	2357.961
0.570	0.00002571	0.0004470	2361.453
0.573	0.00002574	0.0004494	2361.464
0.574	0.00002575	0.0004502	2361.465
0.575	**0.00002576**	**0.0004510**	**2361.465**
0.576	0.00002577	0.0004518	2361.464
0.600	0.00002613	0.0004740	2361.115

5.3 Application to European Option Pricing

To calculate the appropriate value of European call option, we consider several maturity times (days) for an already traded option. The risk-free interest rate is fixed at 3.5% per annum in accordance with the actual Malaysia conventional interest rate on 29 December, 2006, and we are interested in the daily interest rate in this work. We select the underlying price at time as MYR1096.24, following the price on 29 December, 2006. The volatility and Hurst exponent are estimated based on our method for the historical daily return data of KLCI, with estimates listed in Table 5.2. For comparison, we also calculated the value of European call option using the estimates based on the method of Kukush et al. (2005), the R/S analysis is used to obtain an estimator of H in advance, as well as the traditional Black-Scholes European option price. The outcomes are listed in Table 5.3.

Table 5.3 Comparison of European call option prices using different methods: C_{CMLE} (this work), C_{IMLE} (Kukush *et al.* with R/S analysis) and C_{BS} (traditional Black Scholes)

$T_0 - t_0$	**K**	C_{CMLE} $(H = 0.575)$ $[\sigma^2 = 0.00002576]$	C_{IMLE} $(H = 0.6551)$ $[\sigma^2 = 0.00002590]$	C_{BS} $(H = 0.5)$ $[\sigma^2 = 0.00002589]$
15	1070	30.8566	35.2810	28.7439
	1080	23.2219	28.4503	20.2328
	1090	16.6880	22.4382	13.0493
	1100	11.3930	17.2847	7.5809
	1110	7.3561	12.9897	3.9079
30	1070	35.9385	43.3136	31.9585
	1080	28.9344	36.9983	24.1350
	1090	22.7585	31.2702	17.3923
	1100	17.4615	26.1410	11.8932
	1110	13.0511	21.6084	7.6796
40	1070	38.9955	47.9854	34.0120
	1080	32.2057	41.8453	26.4335
	1090	26.1415	36.2119	19.8239
	1100	20.8361	31.0917	14.2966
	1110	16.2947	26.4823	9.8847
50	1070	41.8534	52.3096	35.9756
	1080	35.2107	46.2922	28.5660
	1090	29.2202	40.7253	22.0404
	1100	23.9057	35.6126	16.4850
	1110	19.2709	30.9517	11.9273

From Table 5.3, we see that all cases exhibit somewhat differently in their call prices. Call prices valued by the traditional Black-Scholes provide us with the least values, where the long memory is not taken into account. Method proposed in this work prices the call in an intermediate value between those obtained by the traditional Black-Scholes and the method by Kukush et al. with the R/S analysis. Call prices valued by Kukush et. al. with the R/S analysis are the highest. Our method is based on rigorous theoretical reasoning (see results in the previous sections). It provides practically acceptable results, where the long-memory is taken into account. It is seen that the longer the time to expiry, the higher the value of call price becomes. In the case of "in the money", the call price reveals a higher value when compared with the case of "out of the money", as expected.

6 CONCLUSION

In this paper, we have proposed a CMLE method and investigate the performance of our method for the geometric fractional Brownian motion in financial modelling. We have also compared the performance of our method with the previous work in the literature.

From the simulation study, we learnt that our method performed significantly well in a comparison to the previous method. We also showed that by using this method, we can get a good estimation of all the parameters involved in the geometric fractional Brownian motion. These parameters are important in fractional Black Scholes markets. We are now able to price the European option using the fractional Black Scholes models by supplying the relevant parameters needed in the option pricing formula.

Based on the findings in this paper, we believe that geometric fractional Brownian motion is a good and promising tool to be further investigated in the financial world in order to provide better understanding on how the markets actually behave. We hope that this work will give an inspiration of more studies of geometric fractional Brownian motions in application in near future.

Notes

1. Hult and Bjork (2005) criticized on the meaning of self-financing in this framework, but agreed that the method used does not admit arbitrage.

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