DOUBLE SHELL AS A MODEL OF A MICRO-INHOMOGENEOUS POROELASTIC MEDIUM

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ABSTRACT

Although most rocks are complex multi-mineralic aggregates, quantitative interpretation workflows usually ignore this complexity and employ Gassmann equation and effective stress laws that assume a micro-homogeneous (mono-mineralic) rock. Even though the Gassmann theory and effective stress concepts have been generalized to micro-inhomogeneous rocks, they are seldom if at all used in practice because they require a greater number of parameters, which are difficult to measure or infer from data. Furthermore, the magnitude of the effect of micro-heterogeneity on fluid substitution and on effective stress coefficients is poorly understood. In particular, it is an open question whether deviations of the experimentally measurements from theoretical predictions of the effective stress coefficients for drained and undrained elastic moduli can be explained by the effect of micro-heterogeneity. In an attempt to bridge this gap, we consider an idealized model of a micro-inhomogeneous medium: a Hashin assemblage of double spherical shells. Each shell consists of a spherical pore surrounded by two concentric spherical layers of two different isotropic minerals. The elasticity problem for spherically symmetric deformation for this geometry allows for an exact analytical solution. By analyzing this solution, we show that the results are exactly consistent with the equations of Brown and Korringa (which represent an extension of Gassmann's equation to micro-inhomogeneous media). We also show that the effective stress coefficients for bulk volume α, for porosity n_ϕ and for drained n_K^{dry} and undrained n_K^{ud} moduli are quite sensitive to the degree of heterogeneity (contrast between the moduli of the two mineral components). For instance, while for micro-homogeneous rocks the theory gives $n_\phi = 1$, for strongly micro-inhomogenous rocks n_ϕ may span a range of values from $-\infty$ to ∞ (depending on the contrast between moduli of inner and outer shells). Furthermore, contrary to a popular view, the effective stress coefficient for pore volume (Biot-Willis coefficient) α can be smaller than the porosity ϕ. Since in the double shell geometry the minerals form a regular and ordered geometric pattern (so that the fluid is always in touch with the material of the

inner shell and never with the material of the outer shell), the effect of heterogeneity in this geometry is probably amplified compared to random mineral assemblages. Thus our results can be considered as a kind of upper bound for these effects. Although the exact results are limited to the idealized and unrealistic concentric sphere geometry with isolated spherical pores, simple intuitive arguments suggest that the main conclusions are applicable to more general geometries with interconnected pore space. Further studies are required to understand the applicability of the results to realistic rock geometries.

KEYWORDS:

rock physics, elastics, gassmann theory

INTRODUCTION

Real rocks are complex heterogeneous materials consisting of a solid matrix and pore space filled with a fluid. Effect of pore pressure P_{fl} and fluid bulk modulus K_{fl} on physical properties of rocks are important issues for many practical purposes such as pore pressure prediction (Zhang, 2011; Dutta, 2002), seismic reservoir characterization (Avseth *et al.*, 2005) and quantitative interpretation of time-lapse seismic data.

Consider a physical property of the rock, denoted F, which depends only on the current stress state irrespective of the stress history and stress path. This means that the property is a function of confining stress (tensor of the second rank) $\boldsymbol{\sigma}_{\mathbf{C}}$ and pore pressure $F(\boldsymbol{\sigma}_{\mathbf{C}}, P_{fl})$. Its differential can be written

$$dF(\boldsymbol{\sigma}_{\mathbf{C}}, P_{fl}) = \left(\frac{\partial F}{\partial P_{fl}}\right)_{\boldsymbol{\sigma}_{\mathbf{C}}} dP_{fl} + \left(\frac{\partial F}{\partial \boldsymbol{\sigma}_{\mathbf{C}}}\right)_{P_{fl}} : d\boldsymbol{\sigma}_{\mathbf{C}}, \tag{1}$$

where "**:**" denotes the final product of the second rank tensors.

Usually, confining stress and fluid pressure have a similar effect but with the opposite signs on a given rock property (e.g. porosity, permeability, elastic constants etc.), and thus, every property is controlled by some linear combination of a form

$$P_e = P_C - n_F P_{fl}, \tag{2}$$

where $P_C = -\ 1/3\ \mathrm{Tr}[\boldsymbol{\sigma}_{\mathbf{C}}]$ is called confining pressure and n_F is so-called effective stress coefficient for the property.

Use of such a linear combination (2) with a constant effective stress coefficient is strictly valid only for small perturbations of the pressures. In general, effective stress coefficient n_F is a nonlinear function of and P_C and P_{fl} and can be defined as follows:

$$n_F = -\frac{\left(\dfrac{\partial F}{\partial P_{fl}}\right)_{P_C}}{\left(\dfrac{\partial F}{\partial P_C}\right)_{P_{fl}}}, \tag{3}$$

An in-depth analysis of the general concept of effective stress has been performed by Robin (1973), Carroll and Katsube (1983), Zimmerman (1991), and Berryman (1992) among others. These studies show that there is no universal effective stress coefficient for all rock properties, and different values apply for different physical quantities *F*. In particular, a simple theoretical analysis for rocks with a homogeneous, isotropic and linearly elastic frame, gives n_F=1 for all scale independent properties (that is, properties that don't change when the sample with all its pore space is subjected to a uniform (self-similar) deformation), see e.g., Wyllie *et al.* (1958), Berryman (1992), Gurevich (2004). This means that differential pressure $P_d = P_C - P_{fl}$ is the only controlling factor for such properties including porosity and drained elastic constants but excluding such scale-dependent properties as permeability or density. The result n_F=1 for scale independent properties is derived under the following assumptions:

- the dry frame of the porous rock behaves as a linear elastic solid within the considered range of confining pressures P_C; with bulk moduli K_{dry} and K_{ud};
- the solid constituent of the porous material is microhomogeneous and isotropic (consists of one isotropic mineral);
- the fluid pressure is equilibrated inside the pore space;
- no chemical or physical interactions between solid and fluid take place during the process of deformation.

The simple result n_F=1 for porosity and elastic moduli is often used in applications, but is in variance with some laboratory measurements, which often show values of n_F smaller than 1. This raises the question as to which of the assumptions behind the result n_F=1 is violated in real rocks (or measurements). An obvious candidate is violation of the assumption of microhomogeneity – almost all real rocks are heterogeneous, as they consist of a number of minerals. To explore the effect of micro-heterogeneity on effective stress coefficients, we analyse deformations of a simple double shell model depicted in Figure 1 (in composite mechanics such a model is also known as a doubly coated sphere (Milton, 2002), but we prefer the term double shell because coating a pore, which may be empty, sounds awkward). A double shell is the simplest example of an inhomogeneous porous sample, and is amenable to exact analysis of the elasticity problem. Such an analysis was previously performed by Ciz *et al*. (2008). Here we expand on that study, present an explicit analytical solution for the bulk deformation of the double shell, and explore implications for effective stress coefficients.

A closely related to the problem of effective stress coefficients is a problem of fluid substitution – finding an effective bulk modulus of the saturated rock as a function of its fluid modulus. For rocks satisfying the assumptions listed above, the bulk modulus is given by Gassmann equation, which expresses the bulk modulus of the saturated rock as a function porosity and compressibilities of the dry frame, solid grain material (of which the frame is made)and fluid. If the rock is not homogeneous, Gassmann equation is no longer valid. Brown and Korringa (1975) extended the Gassmann theory to the case of an inhomogeneous matrix. However the theory of Brown and Korringa (BK) finds little use in practice, since it requires the knowledge of parameters, which are difficult to measure or infer from data Furthermore, there are experimental measurements of undrained modulus that deviate from theoretical predictions, which give ground to theoretical investigations about their validity (Bennethum, 2006; Sahay, 2013). For this case too, double shell model, which allows for the exact solution, is useful in exploring consistency with predictions of poroelasticity theory.

THEORETICAL BACKGROUND

To make the presentation more consistent and clear, here we give brief a derivation of the basic equations of static linear poroelasticity and define the main parameters, which are necessary for utilizing the exact solution for the double shell model.

Brown and Korringa (1975) showed that the following four compressibilities fully define volume deformation of porous material subjected to confining and pore pressures:

$$\frac{1}{K_{dry}} = \frac{1}{V_0}\left(\frac{\partial V_0}{\partial P_d}\right)_{P_{fl}}, \tag{4}$$

$$\frac{1}{K_M} = \frac{1}{V_0}\left(\frac{\partial V_0}{\partial P_{fl}}\right)_{P_d}, \tag{5}$$

$$\frac{1}{K_\phi} = \frac{1}{V_\phi}\left(\frac{\partial V_\phi}{\partial P_{fl}}\right)_{P_d}, \tag{6}$$

$$\frac{1}{K_p} = \frac{1}{V_\phi}\left(\frac{\partial V_\phi}{\partial P_d}\right)_{P_{fl}}. \tag{7}$$

where V_0 and V_ϕ stand for total and pore space volumes of the porous sample. For the analysis of the values of these parameters in different rock structures, see Berryman and Milton (1991), Berge and Berryman (1995); Makarynska et al. (2007), and Mavko and Mukerji (2013).

Deformations caused by an increment of fluid pressure while differential pressure is held constant can be conceived as a volumetric strain of either bulk or pore volume when the sample is fully plunged into the fluid with constant pressure. That is why such conditions are called un-jacketed.

K_P is not independent and could be expressed through K_{dry} and K_M with the use of the reciprocity theorem of linear elasticity:

$$\frac{1}{K_P} = \frac{1}{\phi}\left(\frac{1}{K_{dry}} - \frac{1}{K_M}\right), \tag{8}$$

where

$$\phi = V_\phi / V \tag{9}$$

is the porosity of the rock. Proof of this theorem for inhomogeneous linear elastic bodies could be found in Lomakin (1973). The study also concerns with the particular case of the radially heterogeneous sphere

The four compressibilities help define effective stress coefficients for various rock characteristics. They are also key parameters for the fluid-substitution technique. This is a method to evaluate the dependency of the bulk volumetric strain on the fluid compressibility if the pore fluid is restricted to flow through the boundary of the porous sample – undrained experiment. Apparent bulk modulus corresponding to such boundary conditions reads

$$\frac{1}{K_{ud}}=\frac{1}{V_0}\left(\frac{\partial V_0}{\partial P_C}\right)_{fl}, \tag{10}$$

where subscript *fl* designates that the fluid mass is held constant during compression.

In particular, we can find an expression for the porosity effective stress coefficient n_ϕ. To this end, we write a general expression for the differential of V_ϕ:

$$dV_\phi=\phi dV_0+V_0 d\phi\,. \tag{11}$$

Differentiating equation (9) and using (4)-(7) we can write an equation for the porosity variation

$$\frac{d\phi}{\phi}=\left(\frac{1}{K_P}-\frac{1}{K_{dry}}\right)dP_d+\left(\frac{1}{K_\phi}-\frac{1}{K_M}\right)dP_{fl}\,. \tag{12}$$

If the porosity is constant then left-hand side of (12) is zero. Using some elementary Jacobian technique (e.g., Weber and Arfken, 2004) we could express the effective stress coefficient defined by equation (3) as follows:

$$n_\phi=\frac{\partial\left(\phi,P_C\right)}{\partial\left(P_{fl},P_C\right)}\frac{\partial\left(P_{fl},P_C\right)}{\partial\left(\phi,P_{fl}\right)}=\left(\frac{\partial P_C}{\partial P_{fl}}\right)_\phi=\left(\frac{\partial P_d}{\partial P_{fl}}\right)_\phi+1, \tag{13}$$

Equation (13) makes it possible to express n_ϕ.in terms of introduced compressibilities (see e.g., Berryman, 1992)

:

$$n_\phi=1-\frac{\dfrac{1}{K_\phi}-\dfrac{1}{K_M}}{\dfrac{1}{K_p}-\dfrac{1}{K_{dry}}}\,. \tag{14}$$

For self-similar deformations, from the definitions (5) and (6) we get $\frac{1}{K_\phi}=\frac{1}{K_M}$ and recover the result n_ϕ=1.

Another important parameter for poroelasticity is an effective stress coefficient for overall volume of the micro-inhomogeneous sample - α',which appears explicitly in Biot's theory of poroelasticity (e.g., Biot and Willis, 1957). Writing a differential of bulk volume dV_0 as a function of applied pressures and expressing corresponding partial derivatives with the use of (4)-(5), we obtain

$$\frac{dV_0}{V_0}=\frac{1}{K_{dry}}dP_d+\frac{1}{K_M}dP_{fl}=\frac{1}{K_{dry}}\left[dP_C-K_{dry}\left(\frac{1}{K_{dry}}-\frac{1}{K_M}\right)dP_{fl}\right], \tag{15}$$

so that

$$\alpha'=1-\frac{K_{dry}}{K_M}. \tag{16}$$

The parameters introduced above are enough to evaluate effect of fluid compressibility on the effective bulk compressibility during undrained deformation of the rock. To this end, we write an equation similar to (15) for the pore volume change

$$\frac{dV_\phi}{V_\phi}=\frac{1}{K_P}dP_d+\frac{1}{K_\phi}dP_{fl}=\frac{1}{K_P}\left[dP_C-\beta' dP_{fl}\right], \tag{17}$$

where $\beta'=1-K_P/K_\phi$ is the effective stress coefficient for pore volume. At the same time, the pore space is fully saturated with the fluid. Thus, we can relate dV_ϕ to the volume change of fluid that has already been trapped inside the porous sample plus additional volume of the fluid due to exchange with the surrounding medium. If the fluid is not allowed to flow through the boundaries of the sample, then the volumetric strain in the fluid is given by

$$\frac{dV_\phi}{V_\phi}=\frac{1}{K_{fl}}dP_{fl}, \tag{18}$$

Using equations (17)-(18) we can define a ratio of dP_{fl} to dP_C under undrained conditions, so-called, Skempton coefficient (Detournay and Cheng, 1993):

$$B \equiv \left(\frac{\partial P_{fl}}{\partial P_C}\right)_{fl} = \frac{\dfrac{1}{K_{dry}} - \dfrac{1}{K_M}}{\dfrac{1}{K_{dry}} - \dfrac{1}{K_M} + \phi\left(\dfrac{1}{K_{fl}} - \dfrac{1}{K_\phi}\right)}, \tag{19}$$

Coefficient B makes it possible to express the increment of the fluid pressure through the change of confining pressure, specifically: $dP_{fl}=B\ dP_C$ and $dP_d=d(P_C–P_{fl})=(1\text{-}B)dP_C$. Substituting these expressions into equation (15) we can obtain an expression for the undrained bulk modulus (10). In this way, we recover the result of Brown and Korringa - the equation for K_{ud} through K_{dry}, K_M, K_ϕ, ϕ and bulk modulus of the fluid K_{fl}. For the sake of simplicity we write this equation with the use of concise notation proposed by Berryman and Milton (1991):

$$K_{ud} = K_{dry} + \alpha'^2 M', \tag{20}$$

where M is the fluid storage coefficient:

$$\frac{1}{M'} = \phi\left(\frac{1}{K_{fl}} - \frac{1}{K_\phi}\right) + \frac{\alpha'}{K_M}, \tag{21}$$

In the following sections, we explore the behavior of the parameters defined above for a particular case of a double shell.

ANALYTICAL SOLUTIONS

As discussed above, there are several assumptions in the theoretical derivations of undrained bulk modulus and effective stress coefficients for velocities. One possible factor that may cause deviations from the Gassmann theory is micro-inhomogeneity. To analyse this effect, we compute effective stress coefficients for a simple spherical shell configuration, where a spherical pore is surrounded by spherical shells of two linear elastic and isotropic minerals - a double-shell model, Figure 1.

Ciz *et al.* (2008) presented an analytical solution for the deformation of double-layered spherical shell subjected to different pressures P_{IN} on internal (radius R_p) and P_C on external (radius R_{OUT}) surface; boundary between layers has radius – R_{IN}; material of the layers is linear elastic; moduli of the inner layer – K_{IN}, μ_{IN} , outer layer – K_{OUT}, μ_{OUT}. Unfortunately, the paper by Ciz *et al.* (2008) contains a number of typos, which makes it impossible to analyse their results. Thus in Appendix A we present a brief derivation of the solution for the displacement field inside the model. Using the solution, we are able to calculate the bulk moduli defined in (4)-(6) which are sufficient for calculation of the effective stress coefficients and undrained bulk modulus.

The deformation of a double shell is totally the same as for an assemblage of the identical shells of different sizes and effective bulk modulus of such structure corresponds to the Hashin-Shtrikman bounds (Hashin, 1962; Milton, 2002). Hence, we are able to consider fluid pressure being equilibrated inside the pore volume and the experiment being held under the undrained conditions, despite the fact that the pores are not connected (the fluid-substitution method does not require interconnected pore system). In addition, macrohomogeneity of the porous sample is not required in the BK approach.

Firstly, we check analytically coincidence between the undrained bulk moduli calculated according to the BK method and the exact analytical expression for linearized volumetric strain of the fluid-filled double shell spherical pore. To do this, we substitute equations (A-10)-(A-12) into (16), (20) and (21) and then find that the result is equal to the exact one - the expression (10). Single-shell model yields the expected results for microhomogeneous porous volume: the Gassmann equation for fluid-substitution and unit effective stress coefficients for porosity and K_{dry}.

EFFECTIVE STRESS COEFFICIENTS

If fluid pressure and confining pressure acting on a double shell are increased by the same amount, deformations of the inner and outer shell can be significantly different from each other. We illustrate this with a simple thought experiment. Assume that the inner layer of the double-shell is made of fluid (μ_{IN}=0) meaning the pressure P_{IN} is constant throughout this layer and equals to the one applied on the internal surface. This means that outer layer is subjected to P_{IN} on the inner boundary. If the differential pressure $P_d = 0$ then the actual pressure throughout the sample is P_{IN}. At the same time deformation is different due to the difference of bulk moduli, thus, volume fractions of the components will change. This means that, in general, the porosity has changed and, at the same time, response to the subsequent pressures increments will be different from the initial one. Thus, we see that differential pressure is no longer the controlling factor for the porosity and compressibilities of the model.

The explicit expressions for the 'apparent' compressibilities are cumbersome (nevertheless, they are listed in Appendix – equations (A10)-(A-12)), and thus, are best analysed numerically. In the following subsections we compare these results with existing theoretical analyses and attempt to give them a physical interpretation.

Effective stress coefficient for porosity

In Figure 2 we present a typical dependence of the porosity, bulk and pore volume effective stress coefficients The model parameters used for calculations presented in the Figures 2-4 are following:

- one of the spherical shells (inner or outer) has moduli K_0 = 37 GPa, μ_0 = 43GPa corresponding to quartz;
- the other shell has the same ratio $K_1/\mu_1 = 37/43$ as the first shell; this means that it has the same constant Poisson ratio $\nu \approx 0.081$ and variable shear modulus μ_1. The effective stress coefficients are plotted against the normalized shear modulus $\hat{\mu} = \dfrac{\mu_1}{\mu_0}$;
- radii of the shells are shown in captions to the figures, We use dimensionless length parameters, expressing all the radii in units of R_P, this corresponds to unit pore radius;

The value $\hat{\mu}=1$ corresponds to a case where inner and outer shell materials are the same, that is, we have a single homogeneous shell. We have found that the plots for different Poisson's ratios look similar.

The most interesting feature of the curves is a singularity, which occurs for a particular combination of the parameters when the outer shell material is softer than the inner one. On the face of it, such behavior appears to be unphysical (Müller and Sahay, 2013b). Yet this result has a rather simple physical explanation. Equation (12) can be rewritten in a form

$$\frac{d\phi}{\phi}=\left(\frac{1}{K_P}-\frac{1}{K_{dry}}\right)dP_C+\left(\frac{1}{K_\phi}-\frac{1}{K_M}-\frac{1}{K_P}+\frac{1}{K_{dry}}\right)dP_{fl}\,. \tag{22}$$

If a material is such that the drained pore and bulk moduli K_P and K_{dry} are equal then an increment of the confining pressure does not change the porosity of the medium and the fluid pressure is the only factor affecting the porosity:

$$\phi=f\left(P_c,P_{fl}\right)\equiv f\left(P_{fl}\right), \tag{23}$$

and its weight in the linear combination P_e , and effective stress coefficient become infinite. Note that the standard definition of effective stress $P_e=P_C-n_\phi P_{fl}$ is not applicable in this case because $P_C-n_\varphi P_{fl}$ is not equal P_{fl} for any n_φ, and it is more convenient to use an alternative definition of effective stress $P_e'=n_{\phi i}P_C-P_{fl}$, where $n_{\phi i}=n_\phi^{-1}$. When $K_P=K_{dry}$, $n_{\phi i}=0$ and $n_\phi=\infty$.

But is the case $K_P=K_{dry}$, which gives $n_{\phi i}=0$ realistic? Consider the following thought experiment. First, let the inner and outer shell be made of the same material (so that effectively, we have a single shell). If for this shell, confining and pore pressure are increased by the same (small and positive) amount *dP*, then the shell experiences a uniform (self-similar) stretch. It follows that if the pore pressure is kept constant, and only confining pressure increased, then the porosity will decrease. Now let us gradually increase bulk and shear moduli of the inner shell (keeping the Poisson's ratio constant). When these inner shell moduli become very large, the increment *dP* of the confining pressure will only compress the outer shell, but will not deform the inner shell appreciably. Hence the pore volume will remain (almost) the same, while the total volume will decrease. Hence, the porosity will increase. Since for such a shell both pore pressure and confining pressure tend to increase the porosity, the effective stress coefficient is negative. Furthermore, we see that the (relative) effect of confining pressure on porosity will vary gradually as the bulk and shear moduli increase. Thus $n_{\phi i}$ will change gradually from 1 to a negative value and at some point will cross the point where $\left(\frac{\partial \phi}{\partial P_C}\right)_{P_{fl}}$ =0, $n_{\phi i} = 0$ and n_φ has a singularity of 1/*x* type.

As can be seen from Figure 2, n_ϕ could be smaller than unity and even becomes negative. This result is in contradiction to the conclusion of Sahay (2013) where the author asserts that the effective stress coefficient should obey the condition

$$\phi \le n_\phi \le \frac{\phi K_{fl} + (1-\phi) K_{sol}}{K_{fl}}, \tag{24}$$

where K_{sol} is an 'apparent' bulk modulus of the solid frame in the unjacketed experiment:

$$\frac{1}{K_{sol}} = \frac{1}{V_0 - V_\phi}\left(\frac{\partial\left(V_0 - V_\phi\right)}{\partial P_{fl}}\right)_{P_d}, \tag{25}$$

Our results show that the condition (24) is not a rigorous bound for n_ϕ. A probable reason for this is that the numerator of the right-hand-side of (24) is not a true Voigt bound because K_{sol} is not a constituent modulus but some effective parameter, which depends not only on constituent moduli but also on the geometry of the pore space. We should mention that our results are consistent with the general relations between effective stress coefficients presented in Berryman (1992) (his section 4, e.g. equation (35)). Also, we see that the singularities occur at $\hat{\mu}$ corresponding to $\alpha'=\phi$ as follows from the equation (21) from Berryman (1992).

Effective stress coefficient for drained and undrained bulk moduli

As discussed, compared to homogeneous porous media, calculation of the undrained bulk modulus for a micro-inhomogeneioous rock involves two more compressibilities, which characterize the unjacketed response of the porous medium. These values are rarely available for real rocks, which creates an obstacle in practical use of the BK approach. A double shell gives us an opportunity to compare the effective stress coefficient for drained n_K^{dry} and undrained n_K^{ud} bulk moduli.

To calculate analytical values of these coefficients we should express them through the known displacement field inside the double shell. Since we consider the materials to be linearly elastic, the moduli of the model constituents are constant during the deformation process. Hence, pressures increments dP_{fl} and/or dP_C affect only the radii of the layers

$$dR_k = \left(\frac{\partial u_r}{\partial P_C}\right)_{P_{fl}}\Bigg|^{r=R_k} dP_C + \left(\frac{\partial u_r}{\partial P_{fl}}\right)_{P_C}\Bigg|^{r=R_k} dP_{fl} = c_k dP_C + f_k dP_{fl}\,, \tag{26}$$

where R_k corresponds to subscripts R_p, R_{IN}, R_{OUT}; all partial derivatives are calculated on the boundary specified by the superscript. For convenience, we introduce new coefficients c_k and f_k. K_{ud} and K_{dry} depend explicitly on the geometry of the double shell (equation (A-10) etc.) which, in turn, is a function of applied pressures (see equation (26)). Thus, effective stress coefficient can be computed by evaluating the partial derivatives in equation (3) using the chain rule

$$n_K^{dry(ud)} = -\frac{\sum_k \frac{\partial K_{dry(ud)}}{\partial R_k} f_k}{\sum_k \frac{\partial K_{dry(ud)}}{\partial R_k} c_k}\,, \tag{27}$$

Typical behavior of n_K^{dry} and n_K^{ud} is plotted on the Figure 3-4. The structure of the composite matrix has a significant effect on the effective stress coefficients: if the outer shell (load-bearing material) is soft then both coefficients are sufficiently smaller than in the opposite case. Another notable feature is proximity of n_K^{dry} and n_K^{ud} within a wide range of values except for a very compliant inner layer in the water-saturated case. Since the double-shell model is a structure with the strongest effect of the micro-inhomogeneity, for more realistic cases the difference should be expected to be even smaller.

Deviation of n_K^{dry} from n_K^{ud} appears to be roughly linear-dependent on the type of the pore-filler (K_{fl}). In principle, this feature can be useful for identifying the type of pore-filling material from seismic data.

FLUID-SUBSTITUTION

As was discussed by (Borwn and Korringa, 1975), their theory could be deduced from some of the results of Biot's theory (e.g. Geertsma and Smit, 1961). Berryman and Milton (1991) presented the constitutive equations corresponding to the BK theory. These equations are the same as the Biot equations (Detournay and Cheng, 1993). Sahay (2013) proposed a linear theory of porous media deformation different from Biot's theory. The difference is due to a different treatment of porosity perturbation in the process of deformation. This fact leads to introduction of new fluid storage coefficient $M*$ and effective stress coefficient for overall volume of the porous sample $\alpha *$. The expression for the undrained bulk modulus K_{ud}^{*} in terms of these parameters takes the form:

$$K_{ud}^{*} = K_{dry} - \alpha\alpha * M *, \tag{28}$$

where the parameters are introduced as follows:

$$\begin{aligned} &\alpha = 1 - \frac{K_{dry}}{K_{sol}}, \\ &\alpha * = \alpha - \left(1 - \frac{n_{\phi} - \phi}{1-\phi}\right)(\alpha - \phi), \\ &\frac{1}{M *} = \frac{\phi}{K_{fl}} + \frac{\alpha * - \phi}{K_{sol}}. \end{aligned} \tag{29}$$

Müller and Sahay (2013a, b) and Sahay (2013) claim that the BK approach is in error and should not be used. They give some theoretical substantiation to this end and assert that equation (7) of the BK is incorrect. In addition to theoretical arguments, Müller and Sahay (2013a) demonstrate that their approach is in better agreement with experimental data of Hart and Wang (2010).

We do not find the arguments of Müller and Sahay (2013a) convincing. The equation (7) seems to us correct too. It is instructive to compare this solution to the exact solution for an exactly solvable example, such as the double shell. For simplicity, we consider a particular case of the double shell such that the bulk modulus of the inner solid constituents equals the bulk modulus of the fluid. In Figure 5 the undrained bulk modulus of such a shell computed from equation (28) (blue line) along with the exact solution (red line) is plotted against the *shear* modulus of the inner layer. Substituting $K_{IN} = K_{fl}$ into boundary conditions on the shell interfaces (A-8) we see that the deformation of the considered model is the same as of a single shell (with a modified porosity). Hence, Gassmann equation should apply and the exact solution for K_{ud} does not depend on the shear modulus of the inner layer. However K^*_{ud} computed with equation (28) differs from the exact solution significantly. Furthermore, for a certain combination of the model parameters $K^*_{ud} \to \infty$ corresponding to infinite M^*.

Another argument of Müller and Sahay (2013b) is that K_{ud} calculated using the BK approach violates Voigt upper bound (*Hill, 1963*) for the mixture of solid and fluid constituents. However, the for the micro-inhomogeneous porous solids, the expression given by Müller and Sahay (2013b) as the Voigt upper bound is not a rigorous bound, because it involves the 'apparent' bulk modulus of the solid frame - K_{sol}. The latter depends not only on the elastic moduli of the constituents and their volume fractions but also on the structure of the material.

To understand qualitatively the reason why $K_{ud} > K_{sol}$ for some configurations consider the case of the double shell with a very soft inner and virtually incompressible outer layers. Calculation of K_{sol} requires application of equal pressures to both surfaces of the shell. Then the soft material is subjected to the same pressure as the stiff one and thus the resultant volumetric change is dominated by the compression of the soft shell. Under undrained conditions fluid pressure is defined by the Skempton coefficient (see equation (19)). This coefficient could be significantly smaller than unity. Hence, compliant material of the inner layer and the fluid are virtually not compressed at all and thus the porous volume is nearly constant and K_{ud} tends to infinity together with K_{OUT} exceeding K_{sol}. In Figure 6 we show that such configuration is realizable. An interesting feature of the plot is that the $K_{ud} > K_{sol}$ occurs nearly simultaneously for the shells with the soft material inside and outside.

DISCUSSIONS AND CONCLUSIONS

At the beginning of the study, we have formulated the basics of linear poroelasticity for the case of micro-inhomogeneous solid frame. This allowed us to show why the BK approach is not widely used in practice: the values of the constants K_M and K_ϕ are rarely available. In their original paper, Brown and Korringa (1975) proposed a hypothesis that quantitative difference between the Gassmann and their theories is negligible when *'a constituent with a markedly different-from-average compressibility is not preferentially in positions in the elastic framework where this constituent would be subjected to more or less stress than it would in the average positions in the elastic framework'.* The double-shell model represents very strong deviation from such a situation yet it satisfies all of the assumptions and limitations of the BK theory. Thus the double shell model makes possible to judge how crucial the error of keeping to the homogeneous frame assumption is.

The double shell model allowed us to obtain an exact analytical solution for displacements under hydrostatic confining and fluid pressures presented in Appendix A. Thus, the double shell is a good test example for a range of concepts of linear static poroelasticity.

We have presented application of the proposed model to several problems: effective stress coefficients for porosity, drained and undrained bulk moduli and fluid-substitution technique. The most important results are:

1. Effective stress coefficient can take any value $-\infty \le n_\phi \le +\infty$. Analysis of equations (12)-(13) makes it possible to understand physical reasons for such behavior. Because of the

singularity, a reciprocal of effective stress coefficients $1/n_\phi$ is a more convenient parameter for practical use than n_ϕ.

2. Effective stress coefficients n_K^{dry} and n_K^{ud} are quite close to each other over a wide range of reasonable model parameters. The difference between them is controlled by the fluid compressibility. Absolute values of the coefficients usually lies between 0 and 1 but some configurations have n_K^{dry} and n_K^{ud} a little bit larger than 1 or even negative for some unrealistic cases.
3. The exact expression for K_{ud} of the assemblage of the identical double-shell spherical pores coincides with the BK result. At the same time, Sahay's (2013) equation gives result different from the exact ones. Usually this discrepancy is relatively small but for special cases as shown on Figure 4 it could be rather significant.

There are a number of possible applications of the proposed model being beyond the scope of the present paper. We only aimed to present the exact result for double-shell model and show its validity as a test example for some concepts of poroelasticity. However, there is still a question concerning validity of the obtained results to the case of general porous medium. In particular, we are interested to explore if similar results can be obtained for media with interconnected pore space.

The somewhat unexpected results are the range of possible values of n_ϕ and violation of 'Voigt-like' bound. To analyse whether these results are unique to concentric sphere geometry, we recall the case of the double shell with an outer material much stiffer than inner one. This example can be extended to a solid frame of arbitrary shape whose internal surface is covered with a soft material (so that the pore fluid is in contact with the soft matter only). Then, all the considerations that describe the case of a double shell with soft inner shell (giving $K_{ud} > K_{sol}$ for some combinations of parameters) are applicable here. Confining pressure is borne by rigid skeleton while K_{sol} is mostly affected by the compression of the inner soft shell (see equation (25)).

Considering the singularity in the porosity effective stress coefficient, we can perform a thought experiment similar to the one used to illustrate this singularity for a double shell: we again use a double shell with a soft outer shell and much stiffer inner shell but assume that the spherical pores are now connected to each other by a matrix micro-porosity of a small volume fraction. This model roughly corresponds to carbonates where relatively large vugs with, say, ϕ_v ~5-10% are connected to each other through microcracks with porosity ϕ_c ~0.1%. Even though matrix porosity may partially close under confining pressure, ϕ_v will not be affected because of the very stiff inner shell. This means that the relative pore volume strain is small $dV_\phi/V_\phi \sim 0.01$ which could be equal or even less than the volumetric strain dV_0/V_0 of the bulk volume due to an additional compression of the soft outer shell. In such a way, we can imagine the case where $1/n_\phi = 0$

These intuitive arguments suggest that the main findings of this paper should be applicable to some porous materials with interconnected pore space. Although these materials may be too complex for a direct analytical solution, quantitative analysis may be possible using approximate approaches. For instance, the effect of microporosity can be estimated using perturbation theory. This quantitative analysis is beyond the scope of this paper. Also, while these more general geometries are less restrictive than the concentric sphere geometry, they are still idealised. Further studies are required to understand the applicability of the results of this study to realistic rock geometries.

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APPENDIX A

DERIVATION OF SOLUTION FOR THE DOUBLE-SHELL/PORE MODEL

The equation for the stress $\boldsymbol{\sigma}$ equilibrium in elastic medium without body forces reads:

$$\nabla \cdot \boldsymbol{\sigma} = 0, \qquad \text{(A-1)}$$

where ∇ is a conventional notation for Hamilton differential operator and '$\cdot$' is a scalar product. For a perfect contact between elastic solids boundary conditions consist of continuities of displacements **u** and surface forces **($\boldsymbol{\sigma}$·n),** where **n** is a normal vector to the external surface of the body.

Keeping to the assumptions of linear elasticity, we can hold only first-order terms of the strain tensor - $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)$. Hooke's law represents stress-strain relationship:

$$\boldsymbol{\sigma} = \left(K - \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{g} + 2\mu\boldsymbol{\varepsilon}, \qquad \text{(A-2)}$$

where Tr($\boldsymbol{\varepsilon}$) denotes a trace of the strain tensor; **g** – is a unit tensor of second rank.

Combining (A-1) - (A-2) and expression for $\boldsymbol{\varepsilon}$ we obtain a familiar form of Navier's equation:

$$\left(\mathrm{K}+\frac{2}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times\nabla\times\mathbf{u} = 0, \qquad \text{(A-3)}$$

where '$\times$'is a cross product.

The double-shell model and configuration of applied forces are characterized by point symmetry relative to the center of the model. This fact leads to a conclusion that the field of elastic displacements **u** will be spherically symmetric – only radial component $u_r \neq 0$. Hence, the displacements field is curl free (∇×**u**=0). Thus, using spherical coordinates, we can rewrite (A-3):

$$\nabla\left(\nabla\cdot\mathbf{u}\right)=0,\Rightarrow\nabla\cdot\mathbf{u}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)=3A=\text{const}\,. \tag{A-4}$$

The general solution to equation (A-4) yields:

$$u_r=Ar+\frac{B}{r^2}, \tag{A-5}$$

where A and B are integration constants. These constants can be obtained from the boundary conditions:

$$\begin{cases} u_r^{-}=u_r^{+}, \\ \sigma_{rr}^{-}=\sigma_{rr}^{+}. \end{cases} \tag{A-6}$$

where superscripts '+' and '-' are used to distinguish the values on the different sides of the interface. Expression for radial component of stress reads:

$$\sigma_{rr}=\left(K+\frac{4}{3}\mu\right)\frac{\partial u_r}{\partial r}+2\left(K-\frac{2}{3}\mu\right)\frac{u_r}{r}, \tag{A-7}$$

Writing out the solution (A-5) for each layer and substituting into (A-7) and then (A-8) gives an exact system of linear equations

$$\begin{cases} A_pR_p=A_{IN}R_p+\dfrac{B_{IN}}{R_p}, \\ 3A_{IN}K_{IN}-\dfrac{4B_{IN}\mu_{IN}}{R_p{}^3}=3K_{fl}A_p, \\ A_{IN}R_{IN}+\dfrac{B_{IN}}{R_{IN}}=A_{OUT}R_{IN}+\dfrac{B_{OUT}}{R_{IN}}, \\ 3A_{IN}K_{IN}-\dfrac{4B_{IN}\mu_{IN}}{R_{IN}{}^3}=3A_{OUT}K_{OUT}-\dfrac{4B_{OUT}\mu_{OUT}}{R_{IN}{}^3}, \\ 3A_{OUT}K_{OUT}-\dfrac{4B_{OUT}\mu_{OUT}}{R_{OUT}{}^3}=P_C. \end{cases} \tag{A-8}$$

Calculations of effective stress coefficients for different physical properties require the solution for double shell subjected to fixed pressures on internal and external surfaces. In this case, the first equation from (A-8) is omitted and the right-hand side of the second one equals to P_f.

It is instructive to remind that we keep to linearity assumptions throughout the manuscript. Thus, we should use a linearized volumetric deformation of the body instead of exact one:

$$\frac{dV_k}{V_k} \xrightarrow{V_k=\frac{4}{3}\pi R_k^{\ 3}} 1-\left[\frac{R_k+u_r\left(R_k\right)}{R_k}\right]^3 \xrightarrow{\substack{linearity\\ hypothesis}} \approx -3\frac{u_r\left(R_k\right)}{R_k}. \tag{A-9}$$

Using definitions (4)–(6) and setting R_P to unity, we obtain following equations for apparent compressibilities:

$$\frac{1}{K_{dry}} = \frac{4\mu_{IN}R_{IN}^3\left(4\mu_{OUT}R_{OUT}^3+3K_{OUT}R_{IN}^3\right)+3K_{IN}\left(4\mu_{OUT}R_{OUT}^3+3K_{OUT}R_{IN}^3+4\mu_{IN}\left(R_{OUT}^3-R_{IN}^3\right)\left(R_{IN}^3-1\right)\right)}{4\left(4K_{OUT}\mu_{IN}\mu_{OUT}R_{IN}^3\left(R_{OUT}^3-R_{IN}^3\right)+K_{IN}\left(4\mu_{IN}\mu_{OUT}R_{IN}^3\left(R_{IN}^3-1\right)+3K_{OUT}\left(\mu_{OUT}\left(R_{OUT}^3-R_{IN}^3\right)+\mu_{IN}R_{OUT}^3\left(R_{IN}^3-1\right)\right)\right)\right)}, \tag{A-10}$$

$$\frac{1}{K_M} = \frac{\mu_{IN}R_{IN}^3\left(4\mu_{OUT}\left(R_{OUT}^3-1\right)+3K_{OUT}\left(R_{IN}^3-1\right)\right)+3K_{IN}\left(R_{OUT}^3-R_{IN}^3\right)\left(\mu_{OUT}+\mu_{IN}\left(R_{IN}^3-1\right)\right)}{4K_{OUT}\mu_{IN}\mu_{OUT}R_{IN}^3\left(R_{OUT}^3-R_{IN}^3\right)+K_{IN}\left(4\mu_{IN}\mu_{OUT}R_{IN}^3\left(R_{IN}^3-1\right)+3K_{OUT}\left(\mu_{OUT}\left(R_{OUT}^3-R_{IN}^3\right)+\mu_{IN}R_{OUT}^3\left(R_{IN}^3-1\right)\right)\right)}, \tag{A-11}$$

$$\frac{1}{K_\phi} = \frac{\mu_{OUT}R_{IN}^3\left(4\mu_{IN}\left(R_{OUT}^3-1\right)+3K_{IN}\left(R_{OUT}^3-R_{IN}^3\right)\right)+3K_{OUT}\left(R_{IN}^3-1\right)\left(\mu_{IN}R_{OUT}^3-\mu_{OUT}\left(R_{OUT}^3-R_{IN}^3\right)\right)}{4K_{OUT}\mu_{IN}\mu_{OUT}R_{IN}^3\left(R_{OUT}^3-R_{IN}^3\right)+K_{IN}\left(4\mu_{IN}\mu_{OUT}R_{IN}^3\left(R_{IN}^3-1\right)+3K_{OUT}\left(\mu_{OUT}\left(R_{OUT}^3-R_{IN}^3\right)+\mu_{IN}R_{OUT}^3\left(R_{IN}^3-1\right)\right)\right)}, \tag{A-12}$$

By substituting these moduli into equations (14) and (16), we obtain effective stress coefficients for total volume and porosity for the double-shell model. Undrained bulk modulus can be calculated using either the BK equation (20) or by evaluation the displacements of external surface of fluid-filled double shell under unit pressure.

LIST OF FIGURES

1.

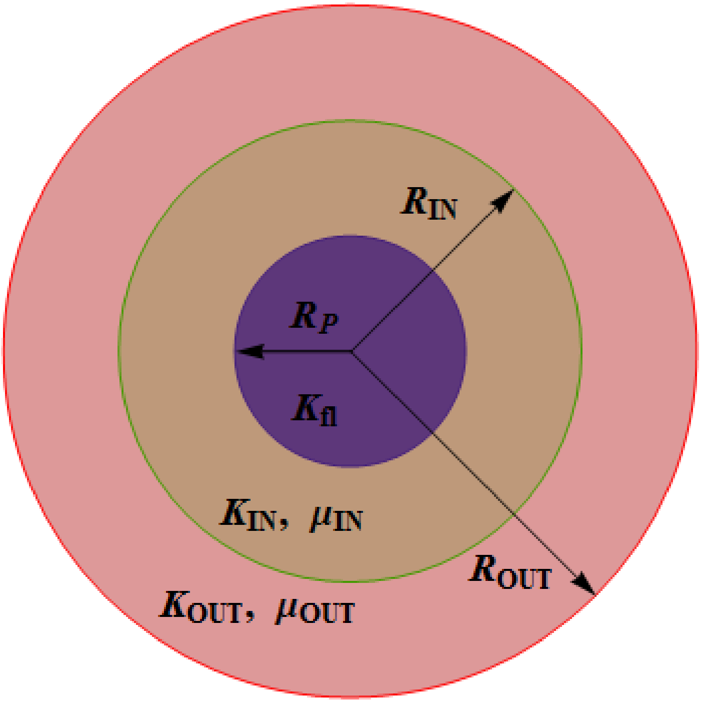

Figure 1. Sketch of the double-shell spherical pore.

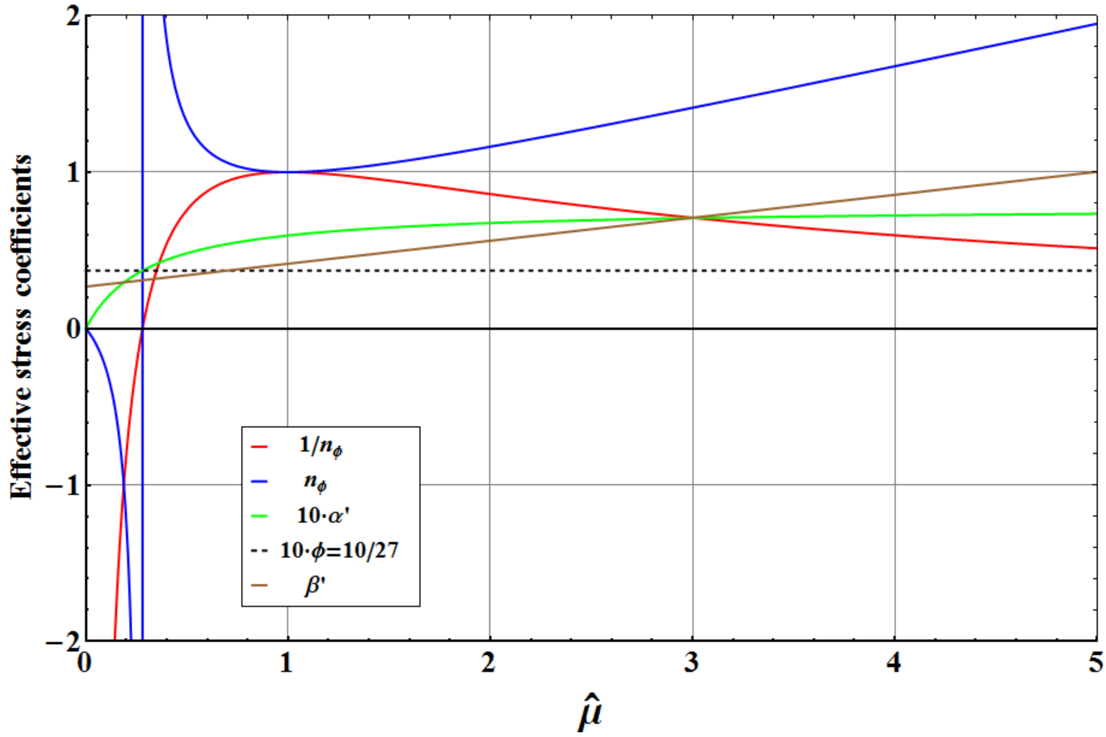

Figure 2 Effective stress coefficient for porosity n_ϕ and its inverse, effective stress coefficients for bulk α' and pore β' volumes for a double-shell with R_{IN} = 2, R_{OUT} = 3 versus normalised shear modulus of the outer shell (the inner shell has moduli of quartz). The case of the normalised shear modulus equal 1 corresponds to the single-shell and thus n_ϕ=1. Range of values of α' is significantly smaller than for other parameters due to small porosity. We plot $10\alpha'$ to improve readability of the plot.

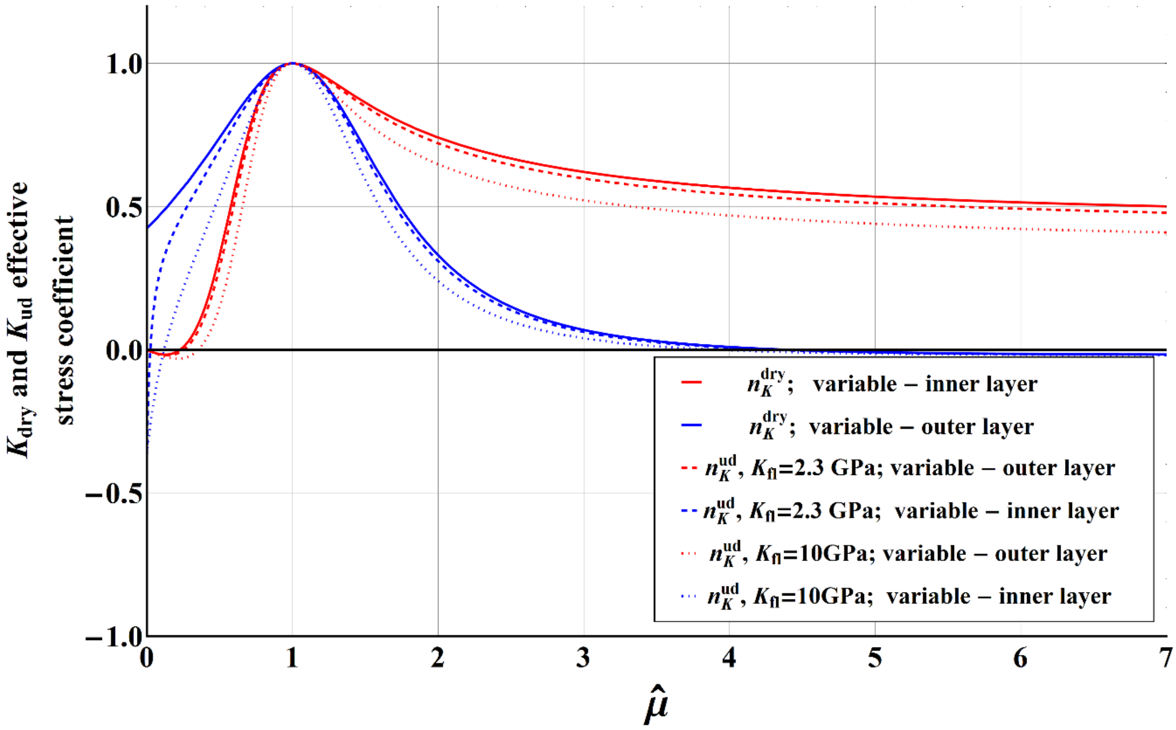

Figure 3 Effective stress coefficient for drained and undrained bulk moduli for the double-shell with $R_{IN} = 2$, $R_{OUT} = 3$. The case of the unit reduced shear modulus corresponds to the single-shell and thus $n_K=1$. Fluid-compressibility takes values from the range: *0, 2.3GPa, 10GPa* – which roughly corresponds to dry, water- and heavy-oil-saturated rocks.

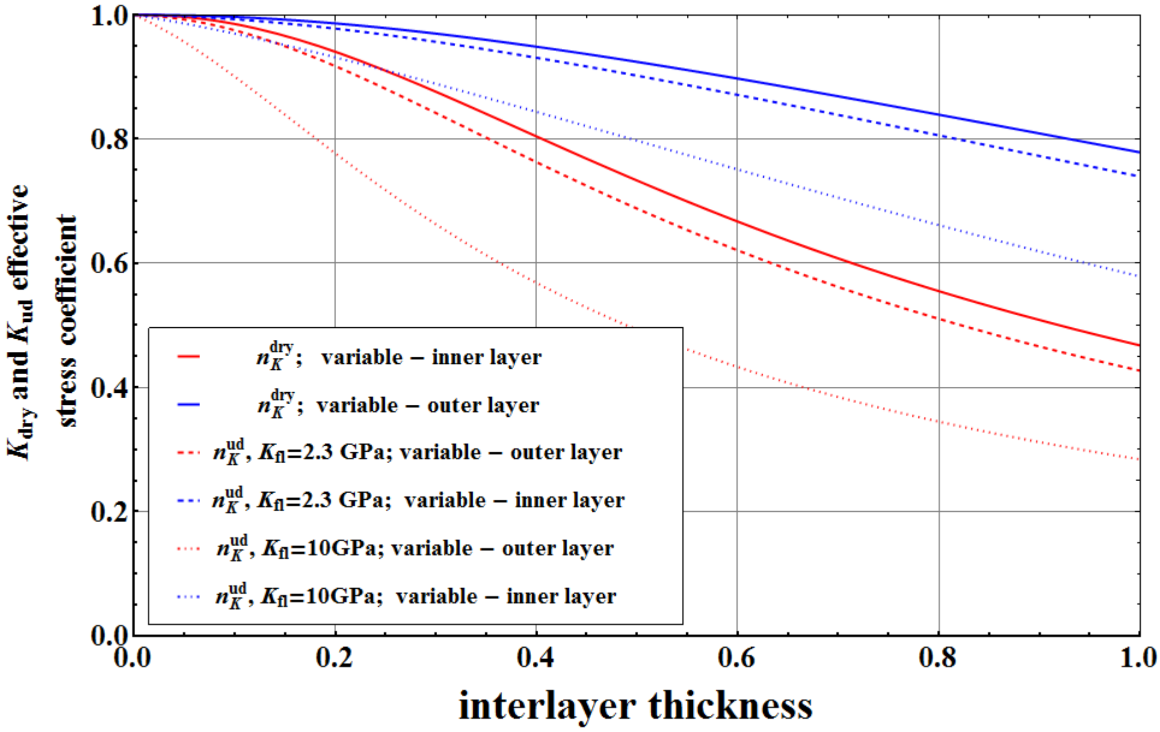

Figure 4 Effective stress coefficient for drained and undrained bulk moduli for the double-shell with fixed thickness of the outer layer R_{OUT} - R_{IN} = 0.5, interlayer thickness is varied. The case of the zero interlayer corresponds to the single-shell and thus $n_K=1$. Fluid-compressibility takes the same values as on Figure 3.

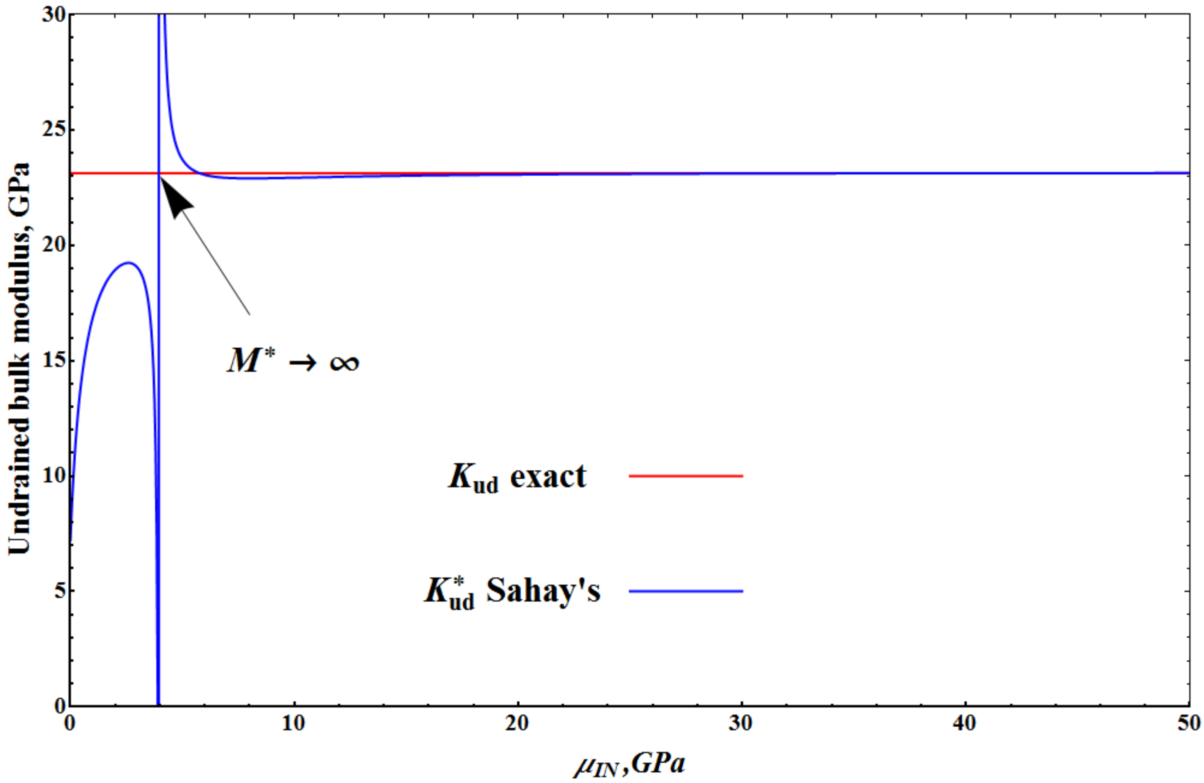

Figure 5 Exact and Sahay's undrained bulk moduli for the double shell model where $K_{IN} = K_{fl.}$=2.3 GPa

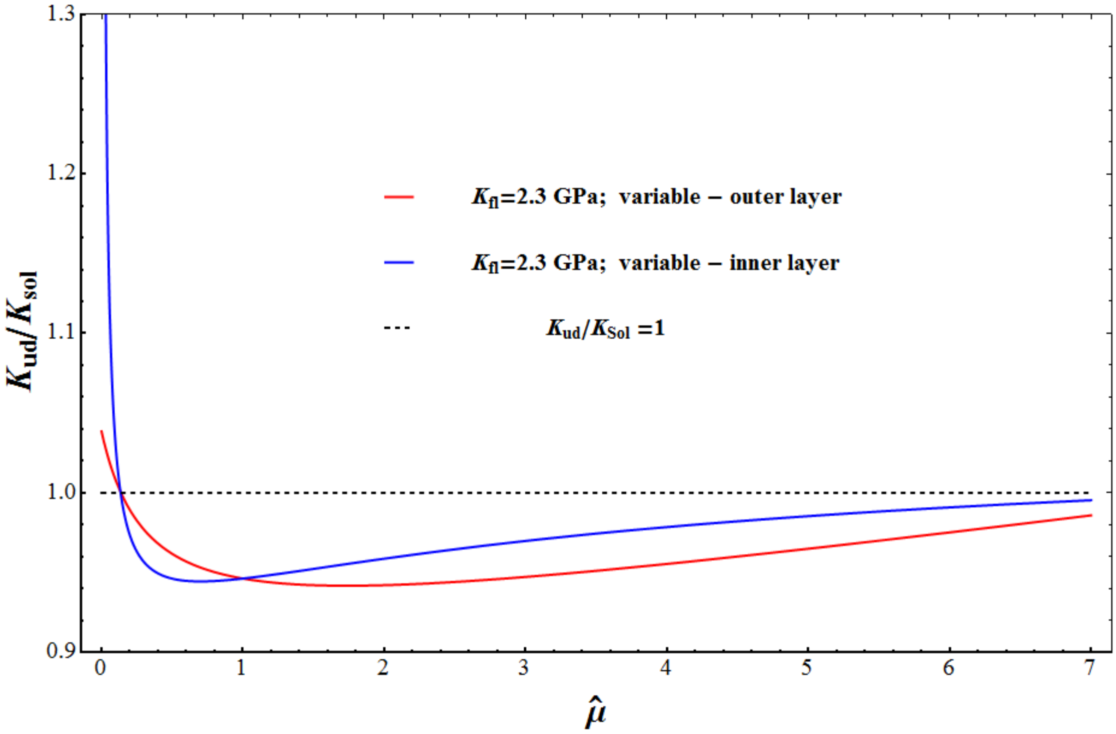

Figure 6 .Ratio of the undrained bulk modulus to the 'apparent' bulk modulus of the solid frame defined by equation (19). $K_{fl.}$*=2.3 GPa*.