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Optimal Sensor and Actuator Locations in Linear Distributed Parameter Systems

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Abstract

A constructive method is developed to obtain optimal sensor and actuator locations for inverse optimal state estimation and control of a class of linear distributed parameter systems (DPSs). Given the inverse optimal state estimators and controllers for linear DPSs developed by the first author recently, it is shown that the performance index for optimal locations of sensors and actuators is the trace of the solution of the Bernoulli partial differential equations (PDEs), which are the optimal state estimation and control gain matrices. Thus, the optimal locations are designed so as to minimize the trace of the solution of the Bernoulli partial differential equations.

Keywords: Distributed parameter systems, Optimal locations, Bernoulli PDE

1 Introduction

In many physical processes, the dynamical system one wishes to estimate and control its states is described by PDEs, for example, chemical reactors, heat exchangers, transmission lines, vibrating beams and electrical, optical or acoustic waves. All of these are DPSs, which may be subject to disturbances, and the problem of estimating and controlling the states from noisy measurements is important. It is also important to distribute measurement sensors and actuators at proper locations to obtain the best estimate and control of the states of the DPSs since sensors and actuators placed at certain locations, such as at nodal ones, do not provide any useful information for the state estimation and actuating signals. The optimal state estimation, control, sensor and actuator location design techniques of the DPSs can be roughly classified into two main approaches.

The first approach referred to as the modal approach, see for example [31], [24], [28], [18], [27], [30], [35], [34], to the state estimation, control, and sensor location problems is first to obtain an approximate lumped parameter system (LPS) for a DPS by representing its state vector as a finite series of eigenfunctions of the partial differential operator modeling the DPS. Then the well-developed finite-dimensional techniques in [1], [20], [21], [14] are applied to design various observers for the approximated LPS. The optimal measurement location problem is solved by minimizing the trace of the estimate error covariance matrix. The modal approach benefits from the well-developed techniques for LPSs but can only observe a finite number of modes of the DPSs and has a significant drawback of computing appropriate gain matrices.

The second approach applies the calculus of variations in [6] and [19] or semigroup theory in [8] (used to represent PDEs as ODEs in Hilbert space) to derive a set of Euler-Lagrange (EL) equations in a form of the two-point-boundary value (TPBV) problem, of which the solution results in optimal filters/estimators and controllers. The EL equations can be solved by using a sweep algorithm in [19], which eventually results in Riccati nonlinear PDEs in the Euclidean space and operator Riccati equations in Hilbert space, see for example [23], [29], [2], [32], [34], [33], [7], [22] [5], [3], [15]. For the purpose of the optimal sensor and actuator location design, the solution of the Riccati nonlinear PDE is first approximated by a finite series of eigenfunctions of the partial differential operator modeling the DPS. The optimal sensor and actuator location problem is then solved by minimizing the trace of the error covariance matrix, i.e., the solution of the Riccati nonlinear PDE, see for example [4], [8], [25].

Difficulties in solving the Riccati nonlinear PDEs and TPBV problems, which are resulted from the design of optimal estimators/filters and controllers for DPSs, motivated the approach of designing inverse optimal filters/estimators and controllers in [11], [9], [10]. The difference between the direct and the inverse approaches is that the former designs a filter that minimizes a given cost, while the latter seeks a filter that minimizes a "meaningful" cost functional, a part of which is constructed from the solution of the Bernoulli PDE. This paper continues the works in [11], [9], [10] by development of techniques that provide design of optimal sensor and actuator locations for the filters/esimators and controllers proposed in [11], [9], [10]. We show that the performance index for optimal locations of sensors and actuators is the trace of the solution of the Bernoulli partial differential equations (PDEs), which are the optimal state estimation and control gain matrices. Then, the optimal locations are designed so as to minimize the trace of the solution of the Bernoulli partial differential equations.

Notations: For a $r \times r$ positive definite matrix $\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y})$, the notation $\boldsymbol{A}^{+}(\boldsymbol{x}, \boldsymbol{y})$ denotes its generalized inverse such that $\int_D \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{A}^{+}(\boldsymbol{y}, \boldsymbol{x}')d\boldsymbol{y} = \boldsymbol{I}\delta(\boldsymbol{x} - \boldsymbol{x}')$ with $\boldsymbol{I}$ being the $r \times r$ identity matrix, and $\delta(\boldsymbol{x} - \boldsymbol{x}')$ being the Dirac delta function of $(\boldsymbol{x} - \boldsymbol{x}')$. For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with the same size, the notation $< \boldsymbol{a}, \boldsymbol{b} >$ denotes their dot product. For a matrix operator $\boldsymbol{A}_{\boldsymbol{x}}$, the notation $\boldsymbol{A}_{\boldsymbol{x}}^*$ denotes its adjoint.

2 Optimal sensor location design

2.1 Distributed Parameter Systems for State Estimation

Let D be a open bounded set in Euclidean n-space E^n with piecewise smooth boundary S, and let t denote time defined on an interval $T = [t_0, t_f]$ with $t_f > t_0$. In this paper, we consider a class of DPSs governed by the following linear PDE:

$$\begin{aligned}
&\frac{\partial \boldsymbol{\chi}(\boldsymbol{x}, t)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}} \boldsymbol{\chi}(\boldsymbol{x}, t) + \boldsymbol{w}_d(\boldsymbol{x}, t), \quad \forall\, \boldsymbol{x} \in D, \\
&\boldsymbol{\chi}(\boldsymbol{x}, t_0) = \boldsymbol{\chi}_0(\boldsymbol{x}) + \boldsymbol{w}_0(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in D, \\
&\boldsymbol{\beta}_{\boldsymbol{\xi}} \boldsymbol{\chi}(\boldsymbol{\xi}, t) = 0, \quad \forall\, \boldsymbol{\xi} \in S,
\end{aligned} \tag{1}$$

where $\boldsymbol{x} = \mathrm{col}(x_1, ..., x_n) \in D$ is the n-dimensional spatial coordinate vector; $\boldsymbol{\chi}(\boldsymbol{x}, t) = \mathrm{col}(\chi_1(\boldsymbol{x}, t), ..., \chi_r(\boldsymbol{x}, t))$ is the r-dimensional vector state; $\boldsymbol{w}_d(\boldsymbol{x}, t)$ and $\boldsymbol{w}_0(\boldsymbol{x})$ are r-dimensional bounded disturbance vectors distributed over the interior. It is noted that we do not include a disturbance vector in the boundary equation since nonhomogeneous boundary condition can be essentially converted to a homogeneous one by adding a disturbance term in the PDE, see [17], [6].

We assume in addition that there are M_d of the m_d-dimensional measurement vectors $\boldsymbol{z}_d(\boldsymbol{x}_i, t)$ over the interior and M_b of the m_b-dimensional measurement vector $\boldsymbol{z}_b(\boldsymbol{x}, t)$ over the boundary are available in the form

$$\begin{aligned}
\boldsymbol{z}_d(\boldsymbol{x}_i, t) &= \boldsymbol{H}_d(\boldsymbol{x}_i, t) \boldsymbol{\chi}(\boldsymbol{x}_i, t) + \boldsymbol{\varepsilon}_d(\boldsymbol{x}_i, t), \\
\boldsymbol{z}_b(\boldsymbol{\xi}_i, t) &= \boldsymbol{H}_b(\boldsymbol{\xi}_i, t) \boldsymbol{\chi}(\boldsymbol{\xi}_i, t) + \boldsymbol{\varepsilon}_b(\boldsymbol{\xi}_i, t),
\end{aligned} \tag{2}$$

for $i = 1, ..., M_d$ in the first equation and $i = 1, ..., M_b$ in the second equation, where $\boldsymbol{H}_d(\boldsymbol{x}_i, t)$ is a $m_d \times r$ matrix defined for all $\boldsymbol{x}_i \in D$ and $t \in T$, $\boldsymbol{H}_b(\boldsymbol{\xi}_i, t)$ is a $m_b \times r$ matrix defined for all $\boldsymbol{\xi}_i \in S$ and $t \in T$, and $\boldsymbol{\varepsilon}_d(\boldsymbol{x}_i, t)$ and $\boldsymbol{\varepsilon}_b(\boldsymbol{\xi}_i, t)$ are the m_d-dimensional and m_b-dimensional vectors of bounded measurement disturbances, respectively.

In this paper, we impose the following assumption.

Assumption 2.1

1) The matrix operators $\boldsymbol{A}_{\boldsymbol{x}}$ and $\boldsymbol{\beta}_{\boldsymbol{\xi}}$ are given by

$$\begin{aligned}
\boldsymbol{A}_{\boldsymbol{x}}[\bullet] &= \sum_{i,j=1}^{n} \boldsymbol{A}_{ij}(\boldsymbol{x}, t) \frac{\partial^2 [\bullet]}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \boldsymbol{B}_i(\boldsymbol{x}, t) \frac{\partial [\bullet]}{\partial x_i} + \boldsymbol{C}(\boldsymbol{x}, t)[\bullet], \\
\boldsymbol{\beta}_{\boldsymbol{\xi}}[\bullet] &= \sum_{j=1}^{n} \boldsymbol{A}_j(\boldsymbol{\xi}, t) \frac{\partial [\bullet]}{\partial x_j} + \boldsymbol{F}(\boldsymbol{\xi}, t)[\bullet],
\end{aligned} \tag{3}$$

where the $\boldsymbol{A}_{ij}(\boldsymbol{x},t)$, $\boldsymbol{B}_i(\boldsymbol{x},t)$, $\boldsymbol{C}(\boldsymbol{x},t)$, and $\boldsymbol{F}(\boldsymbol{\xi},t)$ are $r\times r$ matrices, which are assumed to be self-adjoint, and $\boldsymbol{A}_j(\boldsymbol{\xi},t)=\sum_{i=1}^{n}\boldsymbol{A}_{ij}(\boldsymbol{\xi},t)\cos(\boldsymbol{n}_{\boldsymbol{\xi}},x_i)$, with $\boldsymbol{n}_{\boldsymbol{\xi}}$ being the outward normal to the boundary S at the point $\boldsymbol{\xi}\in S$, and $(\boldsymbol{n}_{\boldsymbol{\xi}},x_i)$ being the angle between the outward normal $\boldsymbol{n}_{\boldsymbol{\xi}}$ and the x_i-axis. Furthermore, the matrix $\boldsymbol{A}_{ij}(\boldsymbol{x},t)$ is symmetric, i.e., $\boldsymbol{A}_{ij}(\boldsymbol{x},t)=\boldsymbol{A}_{ji}(\boldsymbol{x},t)$. In the second equation of (28), we have denoted the notation $\frac{\partial[\bullet]}{\partial x_j}=\frac{\partial[\bullet(\boldsymbol{x},t)]}{\partial x_j}\Big|_{\boldsymbol{x}=\boldsymbol{\xi}}$.

2) There exist symmetric and positive definite matrices $\boldsymbol{Q}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)$ for all $\boldsymbol{x}_i\in D$ and $\boldsymbol{y}_j\in D$ with $i=1,..,M_d$ and $j=1,...,M_d$, and $\boldsymbol{Q}_b(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)$ for all $\boldsymbol{\xi_i}\in S$ and $\boldsymbol{\alpha}_j\in S$ with $i=1,..,M_b$ and $j=1,...,M_b$ such that the matrices $\bar{\boldsymbol{Q}}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)$ and $\bar{\boldsymbol{Q}}_b(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)$ defined by

$$\begin{aligned}\bar{\boldsymbol{Q}}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)&=\boldsymbol{H}_d^T(\boldsymbol{x}_i,t)\boldsymbol{Q}_d^+(\boldsymbol{x}_i,\boldsymbol{y}_j,t)\boldsymbol{H}_d(\boldsymbol{y}_j,t),\\ \bar{\boldsymbol{Q}}_b(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)&=\boldsymbol{H}_b^T(\boldsymbol{\xi}_i,t)\boldsymbol{Q}_b^+(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)\boldsymbol{H}_b(\boldsymbol{\alpha}_j,t)\end{aligned}\tag{4}$$

are bounded and symmetric and positive definite.

3) There exist a operator $\boldsymbol{L}_x(\boldsymbol{x},\boldsymbol{y},t)$ such that the system

$$\begin{aligned}&\frac{\partial\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t}=-\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t)\bar{\boldsymbol{A}}_{\boldsymbol{y}}-[\bar{\boldsymbol{A}}_{\boldsymbol{x}}]^T\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t),\\ &\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t_0)=\boldsymbol{Z}_0(\boldsymbol{x},\boldsymbol{y}),\ \boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{Z}(\boldsymbol{\xi},\boldsymbol{y},t)=0\end{aligned}\tag{5}$$

is exponentially stable at the origin, where

$$\bar{\boldsymbol{A}}_{\boldsymbol{x}}[\bullet]=\boldsymbol{A}_{\boldsymbol{x}}[\bullet]+\boldsymbol{L}_x[\bullet].\tag{6}$$

Moreover, the matrix $\boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)$ defined by

$$\boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)=\boldsymbol{L}_x\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)+\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{L}_y^T,\tag{7}$$

is symmetric and positive definite for a symmetric and positive definite matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$.

2.2 Optimal state estimate design

2.2.1 Optimal state estimation objective

Subject to the constraints defined by (1), the optimal state estimation objective is to design an estimate $\hat{\boldsymbol{\chi}}(\boldsymbol{x},t)$ for $\boldsymbol{\chi}(\boldsymbol{x},t)$ so as to minimize the following cost functional:

$$J=\int_{t_0}^{t_f}Ldt+J_0,\tag{8}$$

where

$$J_0=\frac{1}{2}\int_{D^2}\Big\langle\big[\boldsymbol{\chi}(\boldsymbol{x},t_0)-\boldsymbol{\chi}_0(\boldsymbol{x})\big],\boldsymbol{P}_0^+(\boldsymbol{x},\boldsymbol{y})\big[\boldsymbol{\chi}(\boldsymbol{y},t)-\boldsymbol{\chi}_0(\boldsymbol{y})\big]\Big\rangle d\boldsymbol{x}d\boldsymbol{y},\tag{9}$$

and

$$\begin{aligned}L=&\frac{1}{2}\int_{D^2}\left\langle\left(\frac{\partial\boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t}-\boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t)\right),\boldsymbol{R}_d^{+}(\boldsymbol{x},\boldsymbol{y},t)\left(\frac{\partial\boldsymbol{\chi}(\boldsymbol{y},t)}{\partial t}-\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{\chi}(\boldsymbol{y},t)\right)\right\rangle d\boldsymbol{x}d\boldsymbol{y}\\&+\frac{1}{2}\sum_{i,j=1}^{M_d}\Big\langle[\boldsymbol{z}_d(\boldsymbol{x}_i,t)-\boldsymbol{H}_d(\boldsymbol{x}_i,t)\boldsymbol{\chi}(\boldsymbol{x}_i,t)],\boldsymbol{Q}_d^{+}(\boldsymbol{x}_i,\boldsymbol{y}_j,t)[\boldsymbol{z}_d(\boldsymbol{y}_j,t)-\boldsymbol{H}_d(\boldsymbol{y}_j,t)\boldsymbol{\chi}(\boldsymbol{y}_j,t)]\Big\rangle\\&+\frac{1}{2}\sum_{i,j=1}^{M_b}\Big\langle[\boldsymbol{z}_b(\boldsymbol{\xi}_i,t)-\boldsymbol{H}_b(\boldsymbol{\xi}_i,t)\boldsymbol{\chi}(\boldsymbol{\xi}_i,t)],\boldsymbol{Q}_b^{+}(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)[\boldsymbol{z}_b(\boldsymbol{\alpha}_j,t)-\boldsymbol{H}_b(\boldsymbol{\alpha}_j,t)\boldsymbol{\chi}(\boldsymbol{\alpha}_j,t)]\Big\rangle,\end{aligned}\tag{10}$$

with the symmetric and positive definite matrices $\boldsymbol{Q}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)$ and $\boldsymbol{Q}_b(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)$ being defined in Item 2) of Assumption 3.1, and $\boldsymbol{P}_0(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)$ being symmetric and positive definite matrices.

2.2.2 Optimal state estimation design

In [9], the inverse optimal state estimator that minimizes the cost functional (8) is:

$$\begin{aligned}\frac{\partial\hat{\boldsymbol{\chi}}(\boldsymbol{x},t)}{\partial t}=&\boldsymbol{A}_{\boldsymbol{x}}\hat{\boldsymbol{\chi}}(\boldsymbol{x},t)+\sum_{i,j=1}^{M_d}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y}_i,t)\boldsymbol{H}_d^T(\boldsymbol{y}_i,t)\boldsymbol{Q}_d^{+}(\boldsymbol{y}_i,\boldsymbol{y}_j,t)\Big[\boldsymbol{z}_d(\boldsymbol{y}_j,t)-\boldsymbol{H}_d(\boldsymbol{y}_j,t)\hat{\boldsymbol{\chi}}(\boldsymbol{y}_j,t)\Big]\\&+\sum_{i,j=1}^{M_b}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi}_i,t)\boldsymbol{H}_b^T(\boldsymbol{\xi}_i,t)\boldsymbol{Q}_b^{+}(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)\Big[\boldsymbol{z}_b(\boldsymbol{\alpha}_j,t)-\boldsymbol{H}_b(\boldsymbol{\alpha}_j,t)\hat{\boldsymbol{\chi}}(\boldsymbol{\alpha}_j,t)\Big],\\\hat{\boldsymbol{\chi}}(\boldsymbol{x},t_0)=&\boldsymbol{\chi}_0(\boldsymbol{x}),\ \boldsymbol{\beta}_{\boldsymbol{\xi}}\hat{\boldsymbol{\chi}}(\boldsymbol{\xi},t)=0,\end{aligned}\tag{11}$$

where $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is

$$\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)=\int_{D^2}\boldsymbol{G}(\boldsymbol{x},t;\boldsymbol{x}',t_0)\Big[\boldsymbol{P}_0^{+}(\boldsymbol{x}',\boldsymbol{y}')+\boldsymbol{M}(\boldsymbol{x}',\boldsymbol{y}',t)\Big]^{+}\boldsymbol{G}^T(\boldsymbol{y},t;\boldsymbol{y}',t_0)d\boldsymbol{x}'d\boldsymbol{y}',\tag{12}$$

where

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},t)=&\int_{t_0}^{t}\sum_{i,j=1}^{M_d}\boldsymbol{G}^T(\boldsymbol{\eta}_i,\tau;\boldsymbol{x},t_0)\bar{\boldsymbol{Q}}_d(\boldsymbol{\eta}_i,\boldsymbol{\alpha}_j,\tau)\boldsymbol{G}(\boldsymbol{\alpha}_j,\tau;\boldsymbol{y},t_0)\\&+\int_{t_0}^{t}\sum_{i,j=1}^{M_b}\boldsymbol{G}^T(\boldsymbol{\xi}_i,\tau;\boldsymbol{x},t_0)\bar{\boldsymbol{Q}}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,\tau)\boldsymbol{G}(\boldsymbol{\gamma}_j,\tau;\boldsymbol{y},t_0),\end{aligned}\tag{13}$$

and the Green function $\boldsymbol{G}(\boldsymbol{x},t;\boldsymbol{x}',t')$ is such that

$$\frac{\partial\boldsymbol{G}(\boldsymbol{x},t;\boldsymbol{x}',t_0)}{\partial t}=-\bar{\boldsymbol{A}}_{\boldsymbol{x}}\boldsymbol{G}(\boldsymbol{x},t;\boldsymbol{x}',t_0),\boldsymbol{G}(\boldsymbol{x},t_0;\boldsymbol{x}',t_0)=\boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}'),\boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{G}(\boldsymbol{\xi},t;\boldsymbol{x}',t_0)=0.\tag{14}$$

2.2.3 Lyapunov stability analysis

Let $\boldsymbol{\chi}_e(\boldsymbol{x},t) = \boldsymbol{\chi}(\boldsymbol{x},t) - \hat{\boldsymbol{\chi}}(\boldsymbol{x},t)$ and $\boldsymbol{\chi}_e(\boldsymbol{\xi},t) = \boldsymbol{\chi}(\boldsymbol{\xi},t) - \hat{\boldsymbol{\chi}}(\boldsymbol{\xi},t)$ and consider the following Lyapunov functional candidate

$$V(t) = \int_{D^2} \Big\langle \boldsymbol{\chi}_e(\boldsymbol{x},t), \boldsymbol{P}^+(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}_e(\boldsymbol{y},t)\Big\rangle d\boldsymbol{x}d\boldsymbol{y}, \tag{15}$$

In [9], it is shown that

$$\begin{aligned}\frac{dV(t)}{dt} &= -\sum_{i,j=1}^{M_d} \boldsymbol{\chi}_e(\boldsymbol{x}_i,t), \bar{\boldsymbol{Q}}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)\boldsymbol{\chi}_e(\boldsymbol{y}_j,t) - \sum_{i,j=1}^{M_b} \boldsymbol{\chi}_e(\boldsymbol{\xi}_i,t), \bar{\boldsymbol{Q}}_b(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)\boldsymbol{\chi}_e(\boldsymbol{y}_j,t)\\
&- \int_{D^2} \Big\langle \boldsymbol{\chi}_e(\boldsymbol{x},t), \bar{\boldsymbol{R}}_d\boldsymbol{\chi}_e(\boldsymbol{y},t)\Big\rangle d\boldsymbol{x}d\boldsymbol{y} - 2\sum_{i,j=1}^{M_d} \boldsymbol{\varepsilon}_d(\boldsymbol{x}_i,t), \boldsymbol{Q}_d^+(\boldsymbol{x}_i,\boldsymbol{y}_j,t)\boldsymbol{H}_d(\boldsymbol{y}_j,t)\boldsymbol{\chi}_e(\boldsymbol{y}_j,t)\\
&+2\int_{D^2}\Big\langle \boldsymbol{w}_d(\boldsymbol{x},t), \boldsymbol{P}^+\boldsymbol{\chi}_e(\boldsymbol{y},t)\Big\rangle d\boldsymbol{x}d\boldsymbol{y} - 2\int_{S^2}\Big\langle \boldsymbol{\varepsilon}_b(\boldsymbol{\xi},t), \boldsymbol{Q}_b^+(\boldsymbol{\xi},\boldsymbol{\alpha},t)\boldsymbol{H}_b(\boldsymbol{\alpha},t)\boldsymbol{\chi}_e(\boldsymbol{\alpha},t)\Big\rangle dS_{\boldsymbol{\alpha}}dS_{\boldsymbol{\xi}},\end{aligned} \tag{16}$$

where

$$\bar{\boldsymbol{R}}_d(\boldsymbol{x},\boldsymbol{y},t) = \int_{D^2} \boldsymbol{P}^+(\boldsymbol{x},\boldsymbol{x}',t)\boldsymbol{R}_d(\boldsymbol{x}',\boldsymbol{y}',t)\boldsymbol{P}^+(\boldsymbol{y}',\boldsymbol{y},t)d\boldsymbol{x}'d\boldsymbol{y}'. \tag{17}$$

The equations (15) and (16) imply that the estimate errors $\boldsymbol{\chi}_e(\boldsymbol{x},t)$ and $\boldsymbol{\chi}_e(\boldsymbol{\xi},t)$ exponentially converge in L_2-norm to a ball centered at the origin as shown in [9]. The radius of the ball can be made arbitrarily small by a choice of sufficiently large $\bar{\boldsymbol{Q}}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)$, $\bar{\boldsymbol{Q}}_b(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)$, and $\boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)$ in the cost functional (10).

2.3 Optimal measurement locations

2.3.1 Optimality index

An optimality index should be chosen such that when it is minimized with respect to the measurement locations, the negative definiteness of the right hand side of (16) should be strengthened. This will result in a fast convergence and robustness of the observer (11). As such, let us analyze each term in the right hand side of (16). The terms containing the matrices $\bar{\boldsymbol{Q}}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)$ and $\bar{\boldsymbol{Q}}_b(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)$ are not desirable to be dealt with for choosing the measurement locations. This is because once the matrices $\boldsymbol{H}_d(\bullet)$, $\boldsymbol{H}_b(\bullet)$, $\boldsymbol{Q}_d(\bullet)$, and $\boldsymbol{Q}_b(\bullet)$ are specified, the matrices $\bar{\boldsymbol{Q}}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)$ and $\bar{\boldsymbol{Q}}_b(\boldsymbol{\xi}_i,\boldsymbol{\alpha}_j,t)$, see (4) for their expressions, cannot be changed. Therefore, we focus on the terms containing $\bar{\boldsymbol{R}}_d(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{P}^+(\boldsymbol{x},\boldsymbol{y},t)$, i.e., the third and fifth terms in the right hand side of (16). Substituting $\bar{\boldsymbol{R}}_d(\boldsymbol{x},\boldsymbol{y},t)$ given in (17) into the third term, denoted by T_3, in the right hand side of (16) results in

$$T_3 = -\int_{D^2} \Big\langle \boldsymbol{\chi}_e(\boldsymbol{x},t), \Big(\int_{D^2} \boldsymbol{P}^+(\boldsymbol{x},\boldsymbol{x}',t)\boldsymbol{R}_d(\boldsymbol{x}',\boldsymbol{y}',t)\boldsymbol{P}^+(\boldsymbol{y}',\boldsymbol{y},t)d\boldsymbol{x}'d\boldsymbol{y}'\Big)\boldsymbol{\chi}_e(\boldsymbol{y},t)\Big\rangle d\boldsymbol{x}d\boldsymbol{y}, \tag{18}$$

where the matrix $\boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)$ is defined in (7). On the other hand, once $\boldsymbol{L}_d(\boldsymbol{x},\boldsymbol{y},t)$ is chosen the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is fixed. Therefore, we can consider the matrix $\boldsymbol{R}_d(\boldsymbol{x}',\boldsymbol{y}',t)$ is fixed in (18). This combining with the fourth term in the right hand side of (16) results in a fact that the right hand side of (16) can be made more negative definite by increasing $\boldsymbol{P}^{+}(\boldsymbol{x},\boldsymbol{y},t)$, i.e., decreasing the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$. This coincides with the observation in [33], [7], [22], [5] where the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is considered as the covariance matrix.

It is seen from (12) that the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is a function of the measurement locations $\boldsymbol{x}_i$, $i=1,...,M_d$ over the interior D and $\boldsymbol{\xi}_i$, $i=1,...,M_b$ over the boundary S. In order to define optimal measurement locations, we need to choose an optimality index, i.e., a suitable measure of the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$. Since $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is a square, symmetric and positive definite matrix, the magnitude of $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is measured by its trace. Thus, the optimality index can be chosen as follows

$$I(t)=\int_{D^2}\mathrm{tr}(\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t))d\boldsymbol{x}d\boldsymbol{y}, \tag{19}$$

where $\mathrm{tr}(\bullet)$ denotes the trace of $\bullet$. Minimizing the function $I(t)$ will give optimal measurement locations.

2.3.2 Approximation of $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ in Finite Dimensional Subspace

For practical application, we need to compute the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ given by (12). This requires computation of the generalized inverse $\left[\boldsymbol{P}_0^{+}(\boldsymbol{x}',\boldsymbol{y}')+\boldsymbol{M}(\boldsymbol{x}',\boldsymbol{y}',t)\right]^{+}$. However, calculation of the generalized inverse of a matrix is very complicated in one dimensional domain and is non-traceable for higher dimensional domains [3]. Thus, we approximate the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ in a finite dimensional subspace of the original infinite space. In this subspace, there is no need to calculate any generalized inverse. We start with an approximation of the system Green matrix $\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},t)$ by using a complete orthonormal set of N basis functions $\phi_i(\boldsymbol{x})$, $i=1,...,N$, i.e.,

$$\boldsymbol{G}(\boldsymbol{x},t;\boldsymbol{x}',t')=\boldsymbol{\Phi}^T(\boldsymbol{x})\boldsymbol{W}(t,t')\boldsymbol{\Phi}(\boldsymbol{x}'), \tag{20}$$

where $\boldsymbol{\Phi}(\boldsymbol{x})$ is an $n\times Nn$ matrix is given by

$$\boldsymbol{\Phi}(\boldsymbol{x})=\begin{bmatrix}\phi_1(\boldsymbol{x}) & \cdots & \phi_N(\boldsymbol{x}) & & 0 & \cdots & 0\\ 0 & \cdots & 0 & \ddots & \vdots & & \vdots\\ \vdots & & \vdots & & 0 & \cdots & 0\\ 0 & \cdots & 0 & & \phi_1(\boldsymbol{x}) & \cdots & \phi_N(\boldsymbol{x})\end{bmatrix}^T \tag{21}$$

and $\boldsymbol{W}(t,t')$ is an $Nn\times Nn$ symmetric matrix. The initial value $\boldsymbol{P}_0(\boldsymbol{x},\boldsymbol{y})$ is also approximated by

$$\boldsymbol{P}_0(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{\Phi}^T(\boldsymbol{x})\boldsymbol{W}_0\boldsymbol{\Phi}(\boldsymbol{y}), \tag{22}$$

where $\boldsymbol{W}_0$ is an $Nn \times Nn$ symmetric matrix. Now substituting (43) into (13) results in

$$\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{\Phi}^T(\boldsymbol{x})\boldsymbol{\Psi}(\boldsymbol{x}_i,\boldsymbol{\xi}_i,t)\boldsymbol{\Phi}(\boldsymbol{y}), \tag{23}$$

where we have used symmetry property of the Green function, and

$$\begin{aligned}\boldsymbol{\Psi}(\boldsymbol{x}_i,\boldsymbol{\xi}_i,t) = &\int_{t_0}^{t}\sum_{i,j=1}^{M_d}\boldsymbol{W}(t_0,\tau)\boldsymbol{\Phi}(\boldsymbol{\eta}_i)\bar{\boldsymbol{Q}}_d(\boldsymbol{\eta}_i,\boldsymbol{\alpha}_j,\tau)\boldsymbol{\Phi}^T(\boldsymbol{\alpha}_j)\boldsymbol{W}(t_0,\tau)d\tau\\ &+\int_{t_0}^{t}\sum_{i,j=1}^{M_b}\boldsymbol{W}(t_0,\tau)\boldsymbol{\Phi}(\boldsymbol{\xi}_i)\bar{\boldsymbol{Q}}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,\tau)\boldsymbol{\Phi}^T(\boldsymbol{\gamma}_j)\boldsymbol{W}(t_0,\tau)d\tau,\end{aligned} \tag{24}$$

and we have used the arguments $\boldsymbol{x}_i$ and $\boldsymbol{\xi}_i$ of $\boldsymbol{\Psi}(\boldsymbol{x}_i,\boldsymbol{\xi}_i,t)$ to denote $(\boldsymbol{\eta}_i,\boldsymbol{\alpha}_j)$ and $(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j)$ for a uniform definition of the measurement locations. Substituting (45) and (46) then into (12) and using the orthonormality property (i.e., $\int_D \boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{\Phi}^T(\boldsymbol{x})d\boldsymbol{x} = \boldsymbol{I}$) result in

$$\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{\Phi}^T(\boldsymbol{x})\boldsymbol{W}(t,t_0)\Big[\boldsymbol{W}_0^{-1}+\boldsymbol{\Psi}(\boldsymbol{x}_i,\boldsymbol{\xi}_i,t)\Big]^{-1}\boldsymbol{W}(t,t_0)\boldsymbol{\Phi}(\boldsymbol{y}), \tag{25}$$

which means that the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ has been approximated by N basis functions. It is seen from (48) and (47) that the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ depends on the measurement locations $\boldsymbol{x}_i$, $i = 1,...,M_d$ and $\boldsymbol{\xi}_i$, $i = 1,...,M_b$. Substituting (48) into the optimality index $I(t)$ in (42) gives

$$I(t) = \int_{D^2}\mathrm{tr}\Big(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{W}(t,t_0)\big[\boldsymbol{W}_0^{-1}+\boldsymbol{\Psi}(\boldsymbol{x}_i,\boldsymbol{\xi}_i,t)\big]^{-1}\boldsymbol{W}(t,t_0)\boldsymbol{\Phi}^T(\boldsymbol{y})\Big)d\boldsymbol{x}d\boldsymbol{y}, \tag{26}$$

which is to be minimized to result in optimal measurement locations $\boldsymbol{x}_i$, $i = 1,...,M_d$ and $\boldsymbol{\xi}_i$, $i = 1,...,M_b$.

3 Optimal actuator location design

3.1 Distributed Parameter Systems for Control

Let D be a open bounded set in Euclidean n-space E^n with piecewise smooth boundary S, and let t denote time defined on an interval $T = [t_0,t_f]$ with $t_f > t_0$. We consider a class of linear distributed parameter systems governed by the following linear PDE:

$$\begin{aligned}&\frac{\partial\boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t)+\sum_{i=1}^{M_d}\boldsymbol{B}_d(\boldsymbol{x}_i,t)\boldsymbol{u}_d(\boldsymbol{x}_i,t), \quad \forall\,\boldsymbol{x}\in D,\\ &\boldsymbol{\chi}(\boldsymbol{x},t_0) = \boldsymbol{\chi}_0(\boldsymbol{x}), \quad \forall\,\boldsymbol{x}\in D,\\ &\boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\chi}(\boldsymbol{\xi},t) = \sum_{i=1}^{M_b}\boldsymbol{B}_b(\boldsymbol{\xi}_i,t)\boldsymbol{u}_b(\boldsymbol{\xi}_i,t), \quad \forall\,\boldsymbol{\xi}\in S,\end{aligned} \tag{27}$$

defined for $t \in T$, where $\boldsymbol{x} = \mathrm{col}(x_1, ..., x_n) \in D$ is the n-dimensional spatial coordinate vector; $\boldsymbol{\chi}(\boldsymbol{x}, t) = \mathrm{col}(\chi_1(\boldsymbol{x}, t), ..., \chi_r(\boldsymbol{x}, t))$ is the r-dimensional vector state; $\boldsymbol{B}_d(\boldsymbol{x}_i, t)$ and $\boldsymbol{B}_b(\boldsymbol{\xi}_i, t)$ are $r \times k$ and $r \times l$ matrix functions, respectively; $\boldsymbol{u}_d(\boldsymbol{x}_i, t) \in \mathbb{U}_d$, $i \in \mathbb{M}_d$ with $\mathbb{M}_d$ being the set of all $i = 1, ..., M_d$, is the k-dimensional vector control input over the interior; $\boldsymbol{u}_b(\boldsymbol{\xi}_i, t) \in \mathbb{U}_b$, $i \in \mathbb{M}_b$ with $\mathbb{M}_b$ being the set of all $i = 1, ..., M_b$, is the l-dimensional vector control input distributed over the boundary. We assume that the permissible control spaces $\mathbb{U}_d$ and $\mathbb{U}_b$ are open, i.e., for each pair $\boldsymbol{u}_d \in \mathbb{U}_d$ and $\boldsymbol{u}_b \in \mathbb{U}_b$, there is a pair of small variations $\delta \boldsymbol{u}_d$ and $\delta \boldsymbol{u}_b$ such that $\boldsymbol{u}_d + \delta \boldsymbol{u}_d \in \mathbb{U}_d$ and $\boldsymbol{u}_b + \delta \boldsymbol{u}_b \in \mathbb{U}_b$. We make the following assumption on the operators $\boldsymbol{A}_{\boldsymbol{x}}$ and $\boldsymbol{\beta}_{\boldsymbol{\xi}}$, and the matrices $\boldsymbol{B}_{\boldsymbol{d}}(\boldsymbol{x}_i, t)$ and $\boldsymbol{B}_{\boldsymbol{b}}(\boldsymbol{\xi}_i, t)$.

Assumption 3.1

1) The matrix operators $\boldsymbol{A}_{\boldsymbol{x}}$ and $\boldsymbol{\beta}_{\boldsymbol{\xi}}$ are given by

$$\begin{aligned}
\boldsymbol{A}_{\boldsymbol{x}}[\bullet] &= \sum_{i,j=1}^{n} \boldsymbol{A}_{ij}(\boldsymbol{x}, t) \frac{\partial^2 [\bullet]}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \boldsymbol{B}_i(\boldsymbol{x}, t) \frac{\partial [\bullet]}{\partial x_i} + \boldsymbol{C}(\boldsymbol{x}, t)[\bullet], \\
\boldsymbol{\beta}_{\boldsymbol{\xi}}[\bullet] &= \sum_{j=1}^{n} \boldsymbol{A}_j(\boldsymbol{\xi}, t) \frac{\partial [\bullet]}{\partial x_j} + \boldsymbol{F}(\boldsymbol{\xi}, t)[\bullet],
\end{aligned} \tag{28}$$

where the $\boldsymbol{A}_{ij}(\boldsymbol{x}, t)$, $\boldsymbol{B}_i(\boldsymbol{x}, t)$, $\boldsymbol{C}(\boldsymbol{x}, t)$, and $\boldsymbol{F}(\boldsymbol{\xi}, t)$ are $r \times r$ matrices, which are assumed to be self-adjoint, and $\boldsymbol{A}_j(\boldsymbol{\xi}, t) = \sum_{i=1}^{n} \boldsymbol{A}_{ij}(\boldsymbol{\xi}, t) \cos(\boldsymbol{n}_{\boldsymbol{\xi}}, x_i)$, with $\boldsymbol{n}_{\boldsymbol{\xi}}$ being the outward normal to the boundary S at the point $\boldsymbol{\xi} \in S$, and $(\boldsymbol{n}_{\boldsymbol{\xi}}, x_i)$ being the angle between the outward normal $\boldsymbol{n}_{\boldsymbol{\xi}}$ and the x_i-axis. Furthermore, the matrix $\boldsymbol{A}_{ij}(\boldsymbol{x}, t)$ is symmetric, i.e., $\boldsymbol{A}_{ij}(\boldsymbol{x}, t) = \boldsymbol{A}_{ji}(\boldsymbol{x}, t)$.

2) There exist symmetric and positive definite matrices $\boldsymbol{R}_d(\boldsymbol{x}_i, \boldsymbol{y}_j, t)$, $(i, j) \in \mathbb{M}_d$, and $\boldsymbol{R}_b(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j, t)$, $(i, j) \in \mathbb{M}_b$, such that the matrices

$$\begin{aligned}
\bar{\boldsymbol{R}}_d(\boldsymbol{x}_i, \boldsymbol{y}_j, t) &= \boldsymbol{B}_d(\boldsymbol{x}_i, t) \boldsymbol{R}_d^{-1}(\boldsymbol{x}_i, \boldsymbol{y}_j, t) \boldsymbol{B}_d^T(\boldsymbol{y}_j, t), \\
\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j, t) &= \boldsymbol{B}_b(\boldsymbol{\xi}_i, t) \boldsymbol{R}_b^{-1}(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j, t) \boldsymbol{B}_b^T(\boldsymbol{\gamma}_j, t),
\end{aligned} \tag{29}$$

are bounded and symmetric and positive definite.

3) Let $\boldsymbol{A}_{\boldsymbol{x}}^*$ and $\boldsymbol{\beta}_{\boldsymbol{\xi}}^*$ be the adjoint operators of $\boldsymbol{A}_{\boldsymbol{x}}$ and $\boldsymbol{\beta}_{\boldsymbol{\xi}}$, i.e.,

$$\begin{aligned}
\boldsymbol{A}_{\boldsymbol{x}}^*[\bullet] &= \sum_{i,j=1}^{n} \frac{\partial^2 (\boldsymbol{A}_{ij}(\boldsymbol{x}, t)[\bullet])}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (\boldsymbol{B}_i(\boldsymbol{x}, t)[\bullet])}{\partial x_i} + \boldsymbol{C}(\boldsymbol{x}, t)[\bullet], \\
\boldsymbol{\beta}_{\boldsymbol{\xi}}^*[\bullet] &= \sum_{j=1}^{n} \boldsymbol{A}_j(\boldsymbol{\xi}, t) \frac{\partial [\bullet]}{\partial x_j} - \boldsymbol{K}(\boldsymbol{\xi}, t)[\bullet] + \boldsymbol{F}(\boldsymbol{\xi}, t)[\bullet],
\end{aligned} \tag{30}$$

with

$$\boldsymbol{K}(\boldsymbol{\xi}, t) = \sum_{i=1}^{n} \Big[\boldsymbol{B}_i(\boldsymbol{\xi}, t) - \sum_{j=1}^{n} \frac{\partial \boldsymbol{A}_{ij}(\boldsymbol{\xi}, t)}{\partial x_j} \Big] \cos(\boldsymbol{n}_{\boldsymbol{\xi}}, x_i). \tag{31}$$

In (30) and (31), we have used the notation $\frac{\partial[\bullet(\boldsymbol{\xi},t)]}{\partial x_j} = \frac{\partial[\bullet(\boldsymbol{x},t)]}{\partial x_j}\Big|_{\boldsymbol{x}=\boldsymbol{\xi}}$. *There exists a matrix operator* $\boldsymbol{L}_{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{y},t)$ *such that the system*

$$\begin{aligned}&\frac{\partial \boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t} = -\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t)\bar{\boldsymbol{A}}^*_{\boldsymbol{y}} - [\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}]^T\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t),\\ &\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t_0) = \boldsymbol{Z}_0(\boldsymbol{x},\boldsymbol{y}),\ \boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{Z}(\boldsymbol{\xi},\boldsymbol{y},t) = 0\end{aligned} \tag{32}$$

is exponentially stable at the origin, where

$$\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}[\bullet] = \boldsymbol{A}^*_{\boldsymbol{x}}[\bullet] + \boldsymbol{L}_{\boldsymbol{x}}[\bullet]. \tag{33}$$

Moreover, the matrix $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$ *defined by*

$$\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{L}_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{L}^T_{\boldsymbol{y}}, \tag{34}$$

is symmetric and positive definite for a symmetric and positive definite matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$.

3.2 Inverse optimal control design

3.2.1 Optimal control objective

Under Assumption 3.1 and assume that there are M_d controllers at points $\boldsymbol{x}_1,...,\boldsymbol{x}_{M_d}$ of the domain D and M_b controllers at points $\boldsymbol{\xi}_1,...,\boldsymbol{\xi}_{M_b}$ of the boundary S, the optimal control objective is to design admissible control pair $\boldsymbol{u}_d \in \mathbb{U}_d$ and $\boldsymbol{u}_b \in \mathbb{U}_b$,and their locations $\boldsymbol{x}_1,...,\boldsymbol{x}_{M_d}$ and $\boldsymbol{\xi}_1,...,\boldsymbol{\xi}_{M_b}$ so as to minimize the following cost functional:

$$J = \int_{t_0}^{t_f} L dt + J_f, \tag{35}$$

where

$$\begin{aligned}L =&\frac{1}{2}\int_{D^2}\big(\boldsymbol{\chi}(\boldsymbol{x},t),\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big)d\boldsymbol{x}d\boldsymbol{y} + \frac{1}{2}\sum_{i,j=1}^{Md}\Big[\big(\boldsymbol{u}_d(\boldsymbol{x}_i,t),\boldsymbol{R}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)\boldsymbol{u}_d(\boldsymbol{y}_j,t)\big)\Big]+\\ &\frac{1}{2}\sum_{i,j=1}^{Mb}\Big[\big(\boldsymbol{u}_b(\boldsymbol{\xi}_i,t),\boldsymbol{R}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,t)\boldsymbol{u}_b(\boldsymbol{\gamma}_j,t)\big)\Big],\end{aligned} \tag{36}$$

and

$$J_f =\frac{1}{2}\int_{D^2}\big(\boldsymbol{\chi}(\boldsymbol{x},t_f),\boldsymbol{Q}_f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\chi}(\boldsymbol{y},t)\big)d\boldsymbol{x}d\boldsymbol{y}. \tag{37}$$

In (36) and (37), $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$, $\boldsymbol{Q}_f(\boldsymbol{x},\boldsymbol{y})$, $\boldsymbol{R}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)$, and $\boldsymbol{R}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,t)$ are symmetric and positive definite matrices with $\boldsymbol{R}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)$, and $\boldsymbol{R}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,t)$ being defined in (29).

3.2.2 Inverse optimal control design

In [11], the inverse optimal control that minimizes the cost functional (35) is

$$\begin{aligned}
\boldsymbol{u}_d(\boldsymbol{x}_i,t) &= -\sum_{j=1}^{M_d} \boldsymbol{R}_d^{-1}(\boldsymbol{x}_i,\boldsymbol{x}_j,t)\boldsymbol{B}_d^T(\boldsymbol{x}_j,t)\int_D \boldsymbol{P}(\boldsymbol{x}_j,\boldsymbol{x},t)\boldsymbol{\chi}(\boldsymbol{x},t)d\boldsymbol{x},\\
\boldsymbol{u}_b(\boldsymbol{\xi}_i,t) &= -\sum_{j=1}^{M_b} \boldsymbol{R}_b^{-1}(\boldsymbol{\xi}_i,\boldsymbol{\xi}_j,t)\boldsymbol{B}_b^T(\boldsymbol{\xi}_j,t)\int_D \boldsymbol{P}(\boldsymbol{\xi}_j,\boldsymbol{x},t)\boldsymbol{\chi}(\boldsymbol{x},t)d\boldsymbol{x},
\end{aligned} \tag{38}$$

where the matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is given by

$$\begin{aligned}
&\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) = \bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t}), \quad \bar{t}=-t,\\
&\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t}) = \int_{D^2}\boldsymbol{G}(\boldsymbol{x},\bar{t};\boldsymbol{x'},-t_f)\Big[\boldsymbol{P}_f^+(\boldsymbol{x'},\boldsymbol{y'})+\boldsymbol{M}(\boldsymbol{x'},\boldsymbol{y'},\bar{t})\Big]^+\boldsymbol{G}^T(\boldsymbol{y},\bar{t};\boldsymbol{y'},-t_f)d\boldsymbol{x}'d\boldsymbol{y'},\\
&\boldsymbol{M}(\boldsymbol{x'},\boldsymbol{y'},\bar{t}) = \int_{-t_f}^{\bar{t}}\sum_{i,j=1}^{M_d}\boldsymbol{G}^T(\boldsymbol{\eta}_i,\tau;\boldsymbol{x'},-t_f)\bar{\boldsymbol{R}}_d(\boldsymbol{\eta}_i,\boldsymbol{\alpha}_j,\tau)\boldsymbol{G}(\boldsymbol{\alpha}_j,\tau;\boldsymbol{y'},-t_f)\\
&\qquad\qquad +\int_{-t_f}^{\bar{t}}\sum_{i,j=1}^{M_b}\boldsymbol{G}^T(\boldsymbol{\xi}_i,\tau;\boldsymbol{x'},-t_f)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,\tau)\boldsymbol{G}(\boldsymbol{\gamma}_j,\tau;\boldsymbol{y'},-t_f),
\end{aligned} \tag{39}$$

with the Green function being defined as

$$\frac{\partial \boldsymbol{G}(\boldsymbol{x},\bar{t};\boldsymbol{x}',\bar{t}')}{\partial \bar{t}} = \bar{\boldsymbol{A}}_{\boldsymbol{x}}^*\boldsymbol{G}(\boldsymbol{x},\bar{t};\boldsymbol{x}',\bar{t}'),\ \boldsymbol{G}(\boldsymbol{x},\bar{t}';\boldsymbol{x}',\bar{t}') = \boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}'),\ \boldsymbol{\beta}_{\boldsymbol{\xi}}^*\boldsymbol{G}(\boldsymbol{\xi},\bar{t};\boldsymbol{x}',\bar{t}') = 0. \tag{40}$$

3.3 Optimal control location design

3.3.1 Optimality index

Substituting the matrix $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$ given in (34) and the optimal controls $\boldsymbol{u}_d(\boldsymbol{x}_i,t)$ and $\boldsymbol{u}_b(\boldsymbol{\xi}_i,t)$ given in (38) into the cost functional L given in (36) yields

$$\begin{aligned}
L &= \frac{1}{2}\int_{D^2}\Big(\boldsymbol{\chi}(\boldsymbol{x},t),(\boldsymbol{L}_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)+\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{L}_{\boldsymbol{y}}^T)\boldsymbol{\chi}(\boldsymbol{y},t)\Big)d\boldsymbol{x}d\boldsymbol{y}\\
&+\frac{1}{2}\int_{D^2}\Bigg(\boldsymbol{\chi}(\boldsymbol{x},t),\sum_{i,j=1}^{Md}\Big[\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}_i,t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)\boldsymbol{P}(\boldsymbol{y}_j,\boldsymbol{y},t)\Big]\boldsymbol{\chi}(\boldsymbol{y},t)\Bigg)d\boldsymbol{x}d\boldsymbol{y}\\
&+\frac{1}{2}\int_{D^2}\Bigg(\boldsymbol{\chi}(\boldsymbol{x},t),\sum_{i,j=1}^{Mb}\Big[\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi}_i,t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,t)\boldsymbol{P}(\boldsymbol{\gamma}_j,\boldsymbol{y},t)\Big]\boldsymbol{\chi}(\boldsymbol{y},t)\Bigg)d\boldsymbol{x}d\boldsymbol{y},
\end{aligned} \tag{41}$$

which is a function of the control locations $\boldsymbol{x}_i$ and $\boldsymbol{\xi}_i$. An optimality index is to be chosen such that when it is minimized with respect to the control locations, the cost functional L given in (41) is minimized. From (39), we observe that the change in the matrix products $\bar{\boldsymbol{R}}_d(\boldsymbol{x}_i, \boldsymbol{y}_j, t)\boldsymbol{P}(\boldsymbol{y}_j, \boldsymbol{y}, t)$ and $\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j, t)\boldsymbol{P}(\boldsymbol{\gamma}_j, \boldsymbol{y}, t)$ is negligible with respect to the control locations $\boldsymbol{y}_j$ and $\boldsymbol{\gamma}_j$. This is due to the fact that the solution $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$, see (39), has the inverse of $\bar{\boldsymbol{R}}_d(\boldsymbol{x}_i, \boldsymbol{y}_j, t)$ and $\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j, t)$ as a factor. The same observation holds for the matrix products $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{x}_i, t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}_i, \boldsymbol{y}_j, t)$ and $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\xi}_i, t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j, t)$ with respect to the control locations $\boldsymbol{x}_i$ and $\boldsymbol{\xi}_i$. As a result, the cost functional is minimized when the matrix $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ is minimized with respect to the control locations. It is seen from (39) that the matrix $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ is a function of the control locations $\boldsymbol{x}_i$, $i = 1, ..., M_d$ over the interior D and $\boldsymbol{\xi}_i$, $i = 1, ..., M_b$ over the boundary S. In order to define optimal control locations, we need to choose an optimality index, i.e., a suitable measure of the matrix $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$. Since $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ is a square, symmetric and positive definite matrix, the magnitude of $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ is measured by its trace. Thus, the optimality index can be chosen as follows

$$I(t) = \int_{D^2} \mathrm{tr}(\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)) d\boldsymbol{x} d\boldsymbol{y}, \tag{42}$$

where tr(•) denotes the trace of •. Minimizing the function $I(t)$ will give optimal control locations.

3.3.2 Approximation of $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ in Finite Dimensional Subspace

For practical application, we need to compute the matrix $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ given by (39). This requires computation of the generalized inverse $\left[\boldsymbol{P}_f^+(\boldsymbol{x'}, \boldsymbol{y'}) + \boldsymbol{M}(\boldsymbol{x'}, \boldsymbol{y'}, \bar{t})\right]^+$. However, calculation of the generalized inverse of a matrix is very complicated in one dimensional domain and is non-traceable for higher dimensional domains [3]. Thus, we approximate the matrix $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ in a finite dimensional subspace of the original infinite space. In this subspace, there is no need to calculate any generalized inverse. We start with an approximation of the system Green matrix $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}, \bar{t})$, i.e., the solution of (40) by using a complete orthonormal set of N basis functions $\phi_i(\boldsymbol{x})$, $i = 1, ..., N$, i.e.,

$$\boldsymbol{G}(\boldsymbol{x}, \bar{t}; \boldsymbol{x'}, \bar{t}') = \boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{W}(\bar{t}, \bar{t}')\boldsymbol{\Phi}^T(\boldsymbol{x'}), \tag{43}$$

where $\boldsymbol{\Phi}(\boldsymbol{x})$ is an $n \times Nn$ matrix is given by

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \begin{bmatrix} \phi_1(\boldsymbol{x}) & \cdots & \phi_N(\boldsymbol{x}) & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \vdots & & \vdots \\ \vdots & & \vdots & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & & \phi_1(\boldsymbol{x}) & \cdots & \phi_N(\boldsymbol{x}) \end{bmatrix}^T \tag{44}$$

and $\boldsymbol{W}(\bar{t}, \bar{t}')$ is an $Nn \times Nn$ symmetric matrix. The initial value $\boldsymbol{P}_f(\boldsymbol{x}, \boldsymbol{y})$ is also approximated by

$$\boldsymbol{P}_f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{W}_f\boldsymbol{\Phi}^T(\boldsymbol{y}), \tag{45}$$

where $\boldsymbol{W}_f$ is an $Nn \times Nn$ symmetric matrix. Now substituting (43) into the third equation of (39) results in

$$\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \bar{t}) = \boldsymbol{\Phi}^T(\boldsymbol{x})\boldsymbol{\Psi}(\boldsymbol{x}_i, \boldsymbol{\xi}_i, \bar{t})\boldsymbol{\Phi}(\boldsymbol{y}), \tag{46}$$

where we have used symmetry property of the Green function, and

$$\begin{aligned}\boldsymbol{\Psi}(\boldsymbol{x}_i, \boldsymbol{\xi}_i, \bar{t}) = &\int_{-t_f}^{\bar{t}} \sum_{i,j=1}^{M_d} \boldsymbol{W}(\tau, -t_f)\boldsymbol{\Phi}(\boldsymbol{\eta}_i)\bar{\boldsymbol{R}}_d(\boldsymbol{\eta}_i, \boldsymbol{\alpha}_j, \tau)\boldsymbol{\Phi}^T(\boldsymbol{\alpha}_j)\boldsymbol{W}(\tau, -t_f)d\tau \\ &+ \int_{-t_f}^{\bar{t}} \sum_{i,j=1}^{M_b} \boldsymbol{W}(\tau, -t_f)\boldsymbol{\Phi}(\boldsymbol{\xi}_i)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j, \tau)\boldsymbol{\Phi}^T(\boldsymbol{\gamma}_j)\boldsymbol{W}(\tau, -t_f)d\tau,\end{aligned} \tag{47}$$

and we have used the arguments $\boldsymbol{x}_i$ and $\boldsymbol{\xi}_i$ of $\boldsymbol{\Psi}(\boldsymbol{x}_i, \boldsymbol{\xi}_i, \bar{t})$ to denote $(\boldsymbol{\eta}_i, \boldsymbol{\alpha}_j)$ and $(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j)$ for a uniform definition of the control locations. Substituting (45) and (46) then into (39) and using the orthonormality property (i.e., $\int_D \boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{\Phi}^T(\boldsymbol{x})d\boldsymbol{x} = \boldsymbol{I}$) result in

$$\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t) = \boldsymbol{\Phi}^T(\boldsymbol{x})\boldsymbol{W}(\bar{t}, t_f)\Big[\boldsymbol{W}_f^{-1} + \boldsymbol{\Psi}(\boldsymbol{x}_i, \boldsymbol{\xi}_i, \bar{t})\Big]^{-1}\boldsymbol{W}(\bar{t}, t_f)\boldsymbol{\Phi}(\boldsymbol{y}), \tag{48}$$

which means that the matrix $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ has been approximated by N basis functions. It is seen from (48) and (47) that the matrix $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ depends on the control locations $\boldsymbol{x}_i$, $i = 1, ..., M_d$ and $\boldsymbol{\xi}_i$, $i = 1, ..., M_b$. Substituting (48) into the optimality index $I(t)$ in (42) gives

$$I(t) = \int_{D^2} \mathrm{tr}\Big(\boldsymbol{\Phi}^T(\boldsymbol{x})\boldsymbol{W}(-t, t_0)\big[\boldsymbol{W}_0^{-1} + \boldsymbol{\Psi}(\boldsymbol{x}_i, \boldsymbol{\xi}_i, -t)\big]^{-1}\boldsymbol{W}(-t, t_0)\boldsymbol{\Phi}(\boldsymbol{y})\Big)d\boldsymbol{x}d\boldsymbol{y}, \tag{49}$$

where we have used $\bar{t} = -t$. Minimizing (49) gives optimal control locations $\boldsymbol{x}_i$, $i = 1, ..., M_d$ and $\boldsymbol{\xi}_i$, $i = 1, ..., M_b$.

3.3.3 Numerical Example

In this section, we present a numerical simulation to demonstrate the development of the optimal control allocations for the optimal control of a one-dimensional heat equation given by

$$\begin{aligned}&\frac{\partial \chi(x,t)}{\partial t} = a\frac{\partial^2 \chi(x,t)}{\partial x^2} + c\chi(x,t) + \sum_{i=1}^{M_d} u_d(x,t), \quad x \in (0,1), \\ &\chi(x,t_0) = \chi_0(x), \ \frac{\partial \chi(x,t)}{\partial x}\Big|_{x=0} = u_b(0,t), \quad \frac{\partial \chi(x,t)}{\partial x}\Big|_{x=1} = u_b(1,t)\end{aligned} \tag{50}$$

where a and c are constants. We first verify Assumption 3.1. It is seen that Assumption 3.1.1 holds. Assumption 3.1.2 also holds since $B_d(x_i, t) = 1$. We now verify Assumption

3.1.3. As such, from (50), we have $A_x[\bullet] = A_x^*[\bullet] = a\frac{\partial^2[\bullet]}{\partial x^2} + c[\bullet]$. Thus, we choose the operator $L_x[\bullet]$ as

$$L_x[\bullet] = (-\bar{a} - a)\frac{\partial^2[\bullet]}{\partial x^2} + (\bar{c} - c)[\bullet] \tag{51}$$

where $\bar{a}$ and $\bar{c}$ are positive constants satisfying

$$\bar{a} + a > 0, \;\; \bar{c} - c > 0. \tag{52}$$

According to (33), we have the operator $\bar{A}_x^*$ given by

$$\bar{A}_x^*[\bullet] = -\bar{a}\frac{\partial^2[\bullet]}{\partial x^2} + \bar{c}[\bullet]. \tag{53}$$

Thus, we have $z(x, y, t)$-system, see (32), given by

$$\begin{aligned} &\frac{\partial z(x,,y,t)}{\partial t} = \bar{a}\frac{\partial^2(x,t)}{\partial x^2} + \bar{a}\frac{\partial^2(x,t)}{\partial y^2} - 2\bar{c}z(x,y,t), \quad x \in (0,1), \\ &z(x,y,t_0) = z_0(x,y), \;\; \frac{\partial z(x,y,t)}{\partial x}\Big|_{x=0} = 0, \quad \frac{\partial z(x,,y,t)}{\partial x}\Big|_{x=1} = 0. \end{aligned} \tag{54}$$

The unique solution of (54) is

$$z(x,y,t) = \int_0^1 \int_0^1 G(x,t;x',t_0)z_0(x',y')G^T(y,t;y',t_0)dx'dy' \tag{55}$$

where the Green function $G(x, t; x', t_0)$ is the solution of

$$\begin{aligned} &\frac{\partial G(x,t;x',t_0)}{\partial t} = \bar{a}\frac{\partial^2 G(x,t;x',t_0)}{\partial x^2} - \bar{c}G(x,t;x',t_0), \\ &G(x,t_0;x',t_0) = \delta(x-x'), \;\; \frac{\partial G(x,t;x',t_0)}{\partial x}\Big|_{x=0} = 0, \;\; \frac{\partial G(x,t;x',t_0)}{\partial x}\Big|_{x=1} = 0. \end{aligned} \tag{56}$$

A calculation shows that the solution of (56) is

$$G(x,t;x',t_0) = e^{-\bar{c}(t-t_0)} + 2\sum_{n=1}^{\infty} e^{-(n^2\pi^2\bar{a}+\bar{c})(t-t_0)}\cos(n\pi x)\cos(n\pi x'), \tag{57}$$

which exponentially converges to zero as t tends to infinity. This in turn implies from (55) that the Green function $G(x, t; x', t_0)$ exponentially converges to zero. Hence, the first part of Assumption 3.1.3 has been verified. Next, we show that the matrix $Q(x, y, t)$ defined in (34) with L_x given by (51) is symmetric and positive definite. As such, the matrix $Q(x, y, t)$ is given y

$$Q(x,y,t) = -(\bar{a}+a)\frac{\partial^2 P(x,y,t)}{\partial x^2} - (\bar{a}+a)\frac{\partial^2 P(x,y,t)}{\partial y^2} - 2(\bar{c}-c)P(x,y,t), \tag{58}$$

where $P(x,y,t)$ is given in (39) with the Green function $G(x,t;x',t')$ being the solution of the following system

$$\begin{aligned}&\frac{\partial G(x,t;x',t')}{\partial t}=-\bar{a}\frac{\partial^2 G(x,t;x',t')}{\partial x^2}+\bar{c}G(x,t;x',t'),\\&G(x,t';x',t')=\delta(x-x'),\ \frac{\partial G(x,t;x',t')}{\partial x}\Big|_{x=0}=0,\ \frac{\partial G(x,t;x',t')}{\partial x}\Big|_{x=1}=0.\end{aligned}\tag{59}$$

A calculation shows that the solution of (59) is

$$G(x,t;x',t')=e^{\bar{c}(t-t')}+2\sum_{n=1}^{\infty}e^{(n^2\pi^2\bar{a}+\bar{c})(t-t')}\cos(n\pi x)\cos(n\pi x').\tag{60}$$

Substituting $G(x,t;x',t')$ given in (60) into $P(x,y,t)$ given in (39), then into (58) shows that the matrix $Q(x,y,t)$ is symmetric and positive definite as the constants $\bar{a}$ and $\bar{c}$ were chosen such that the conditions (52) hold. For numerical calculations, we assume that $a=0.5$, $c=0$, $P_f(x,y)=\delta(x-y)$,$r_d=1$, $R_b=1$, $M_d=2$, $t_0=0$, and $t_f=2s$. From (52), we choose $\bar{a}=0.1$ and $\bar{c}=0.5$. We used the Fletcher-Powell method [16] to obtain the optimal control locations: $x_1=x_2=0.5$, and indeed $x_{b1}=0$ and $x_{b2}=1$. The traces of the matrix $P(x,y,t)$ corresponding to various control locations are plotted in Fig. 1.

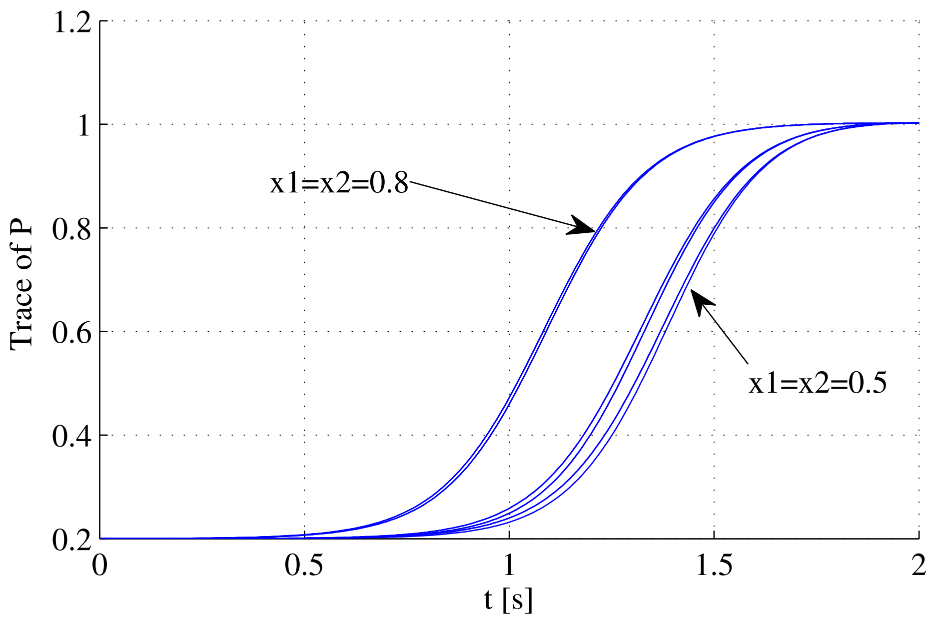

Figure 1: Traces of the matrix $P(x,y,t)$.

4 Conclusions

The paper presented a constructive method to determine optimal sensor and actuator locations for optimal design of filters/estimators and controllers over the interior and the boundary for a class of linear distributed parameter systems (DPSs). The most significant benefit of the inverse approach is that it avoids finding the symmetric and positive definite solution of the complicated Riccati PDE but the Bernoulli PDE instead, which is solved analytically in terms of the system Green function. This subsequently removes computational burden of the generalized inversion of matrices, which are required in computation of the solution of the Bernoulli PDE, by using a finite series of eigenfunctions to approximate the system Green function. The development in this paper can be further used for solving other filter design problems for distributed parameter systems and further improve performance of controlling practical distributed parameter systems as as marine riser systems in [12], [13], and [26].

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