THEORY PROBAB. APPL.
Vol. 59, No. 1, pp. 136–144

Translated from Russian Journal

ON DIFFERENTIATION OF FUNCTIONALS CONTAINING THE FIRST EXIT OF A DIFFUSION PROCESS FROM A DOMAIN*

S. A. GUSEV† AND N. G. DOKUCHAEV‡

(*Translated by the authors*)

Abstract. One of the problems arising in the differentiation of functionals of random diffusion processes in domains with absorbing boundaries is to compute parametric derivatives for the functionals containing the first exit time τ from the domain for the underlying diffusion process. Earlier work [S. A. Gusev, *Numer. Anal. Appl.*, 1 (2008), pp. 314–331] proposed a method for solving this problem under some condition of existence of mean square derivatives for τ with respect to the parameter; this condition was restrictive and difficult to verify. In this paper, we show that this condition can be waived under some mild assumptions.

Key words. diffusion process, the first exit time, absorbing boundary, differentiation with respect to the parameters

DOI. 10.1137/S0040585X97986965

1. Introduction.

Differentiation of functionals of random processes with respect to parameters is commonly used for solving stochastic optimization problems. These problems are complicated for functionals containing the first exit time τ of the random process from a domain. It is known, for example, that the first exit time for a smooth process is not continuous with respect to the small deviations of parameters. On the other hand, there is some pathwise regularity of the first exit times of random diffusion processes [1], [2]. However, this regularity is insufficient to ensure regularity and existence for the corresponding derivatives of τ.

For diffusion processes, the differentiation of the functionals containing τ is possible in principle through the differentiation of the solutions of the Kolmogorov equations with respect to corresponding parameters. However, practical calculation of these derivatives is difficult.

Currently, there are not many works on the problem of differentiation of a functional containing the first exit time from the domain with respect to parameters. On the other hand, calculation of these derivatives is required in a variety of applications. For example, calculation of parametric derivatives of the solutions of parabolic equations arises in optimal selection of these parameters via gradient methods. For this aim, the probabilistic representation and the differentiation of stochastic processes containing τ can be used. Further, calculation of parametric derivatives for functionals of diffusion processes is used in financial mathematics. For example, evaluation of parametric derivatives of the option prices was studied in [3], [4] using the Malliavin stochastic calculus. Technically, the functionals considered in these papers were of a different type than those considered below in functional (2), but the task can be reduced to the calculation of (2) for problems of barrier-type options pricing.

In [5], a representation of the derivatives of functionals containing τ was suggested. This representation was based on Itô's formula applied to the function that vanishes on the boundary along with its first derivatives. This result was obtained under the condition that

*Received by the editors March 12, 2012; revised December 11, 2013. This work was supported by Russian Leading Science Schools Programme grant 5111.2014.1, Russian Foundation for Basic Research grant 14-01-00340-a, and Australian Research Council grant DP120100928.
http://www.siam.org/journals/tvp/59-1/T98696.html

†Institute of Computational Mathematics and Mathematical Geophysics, Siberian Branch of Russian Academy of Science, Novosibirsk, Russia (sag@osmf.sscc.ru).

‡Department of Mathematics and Statistics, Curtin University, Perth, Australia (n.dokuchaev@curtin.edu.au).

there exist mean square derivatives of τ with respect to the parameters. This condition is difficult to verify; moreover, we do not know of any examples where this condition is satisfied.

We show in this paper that the formula obtained in [5] is valid without this restrictive condition if coefficients of the equation are sufficiently smooth.

2. Problem statement. We assume that we are given a connected bounded domain $G \subset \mathbf{R}^d$ with some regular enough boundary ∂G, a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and a nondecreasing sequence of σ-algebras $\mathcal{F}_t \subseteq \mathcal{F}$, $t \geqq 0$. We are given a d-dimensional Wiener process W that is progressively measurable with respect to $\mathcal{F}_t$; the difference $W_s - W_t$ is independent on σ-algebra $\mathcal{F}_t$ for $s > t$.

Let $U \subset \mathbf{R}^m$ be an open set. For $x \in G$ and $t \in [0,T)$, consider a d-dimensional random process $X_s = X_s(\theta)$, which depends on a vector parameter $\theta \in U$ and is described by the stochastic differential equation (SDE)

$$X_s(\theta) = x + \int_t^s a(v, X_v(\theta), \theta)\, dv + \int_t^s \sigma(v, X_v(\theta), \theta)\, dW_v, \tag{1}$$

with measurable functions $a\colon\ [0,\infty) \times \mathbf{R}^d \times U \to \mathbf{R}^d$ and $\sigma\colon\ [0,\infty) \times \mathbf{R}^d \times U \to \mathbf{R}^{d\times d}$. We assume that the coefficients of (1) satisfy the following condition.

(A) The functions a and σ are bounded. There exists a constant $\mathcal{K}$ such that, for all $\theta \in U$, $v \geqq 0$, $x, y \in \mathbf{R}^d$, $i, j \in \{1, \dots, d\}$, the following holds:

$$|a_i(v,x,\theta) - a_i(v,y,\theta)| + |\sigma_{ij}(v,x,\theta) - \sigma_{ij}(v,y,\theta)| \leqq \mathcal{K}|x-y|,$$

where a_i and σ_{ij} are components of the vector a and matrix σ. These assumptions on a, σ ensure that, for any $\theta \in U$, there exists an $\mathcal{F}_s$-measurable process X_s such that (1) holds for all $s \geqq 0$ with probability 1 (see, e.g., [6]).

Symbol $E_{t,x}$ will denote the expectation relative to the probability measure $P_{t,x}$ that corresponds to the process outgoing at the time t from the point x. The definition of $P_{t,x}$ can be found, for example, in [7, p. 381].

In applications, it is common to consider expectations of the form

$$u(t,x,\theta) = E_{t,x}\left[\varphi(X_T(\theta), \theta)\chi_{\tau > T} + \int_t^{T\wedge\tau} f(v, X_v(\theta), \theta)\, dv\right], \tag{2}$$

where $\tau = \inf\{v\colon\ v > t,\ X_v \notin G\}$ is the first exit time of the process X out of the domain G, and χ_A in the indicator function of a set A.

Let $Q_T = (0,T) \times G$. It is known that, under some mild restrictions on φ and f, the value of (2) at $(t,x) \in Q_T$ coincides with the solution of the following boundary value problem for parabolic equation:

$$Lu + f(t,x,\theta) = 0, \qquad t \in (0,T), \quad (t,x) \in Q_T, \tag{3}$$

$$u(T,x,\theta) = \varphi(x,\theta), \qquad x \in G, \tag{4}$$

$$u(t,x,\theta) = 0, \qquad x \in \partial G. \tag{5}$$

The operator $L = L(\theta)$ in (3) is defined as

$$L \equiv \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i,j=1}^d b_{ij}(t,x,\theta)\partial^2_{x_i,x_j} + \sum_{i=1}^d a_i(t,x,\theta)\partial_{x_i}, \tag{6}$$

where b_{ij} are the components of the matrix $B \equiv \sigma\sigma^*$.

The functional defined by (2) contains τ. In this paper, we study the possibility of differentiating the value (2) with respect to θ. To do this, we need to make the following assumptions in addition to assumption (A).

(B) The matrix $B(t,x,\theta) = \{b_{ij}(t,x,\theta)\}$ is uniformly nondegenerate, i.e.,

$$B(t,x,\theta) \geqq \alpha_0 I$$

for some $\alpha_0 > 0$.

(C) The boundary ∂G is C^4-smooth, and the derivatives

$$\frac{\partial a}{\partial x},\ \frac{\partial^2 a}{\partial x^2},\ \frac{\partial a}{\partial \theta_i},\ \frac{\partial^2 a}{\partial x\,\partial \theta_i},\ \frac{\partial \sigma}{\partial x},\ \frac{\partial^2 \sigma}{\partial x^2},\ \frac{\partial \sigma}{\partial \theta_i},\ \frac{\partial^2 \sigma}{\partial x\,\partial \theta_i},\ \frac{\partial a}{\partial t},\ \frac{\partial \sigma}{\partial t}$$

are continuous and bounded in $[0,\infty) \times \mathbf{R}^d \times U$.

(D) For all $(x,\theta) \in Q \times U$, there exist derivatives $\partial\varphi/\partial x$ and $\partial\varphi/\partial\theta$.

(E) For all $\theta \in U$, the function f is continuous on $[0,T] \times \overline{Q}_T$, and it has continuous derivatives $\partial f/\partial x$ and $\partial f/\partial\theta$ for all $(t,x,\theta) \in Q_T \times U$.

In (E) and in what follows, $\bar{D}$ means the closure of a set D.

We denote by G_δ the set of all points G located on the distance greater than $\delta > 0$ from ∂G. In [8], some existence theorems for boundary problems for parabolic equations are given. The type of a functional space containing solutions depends on particular specifications of the boundary value problem. For the purposes of this paper, we need a solution that has continuous derivatives $\partial u/\partial x$ at the points of all nonempty sets $(0, T-\delta) \times G_\delta$. For problem (2)–(5), the required property is ensured by [8, Chap. III, Theorem 4.2; Chap. IV, Theorems 5.2 and 9.1] given that free terms of problem (2)–(5) satisfy the conditions listed in these theorems for existence of the solutions.

3. The main result: The calculation of $\boldsymbol{\partial u/\partial\theta}$. Throughout the paper we assume for simplicity that θ is scalar and $U \subset \mathbf{R}$ is an interval. Extension on the case of vector valued θ is straightforward.

The formal differentiation of (2) gives

$$\begin{aligned}\frac{\partial u}{\partial\theta}(t,x) = E_{t,x}\bigg[&\left(\frac{\partial\varphi}{\partial x}(T,X_T,\theta)Z_T + \frac{\partial\varphi}{\partial\theta}(T,X_T,\theta)\right)\chi_{\tau>T} \\ &+ \int_t^{T\wedge\tau}\left(\frac{\partial f}{\partial x}(v,X_v,\theta)Z_v + \frac{\partial f}{\partial\theta}(v,X_v,\theta)\right)dv\bigg] + \Phi(\theta),\end{aligned} \tag{7}$$

where

$$\Phi(\theta) := \lim_{\Delta\theta\to 0} E_{t,x}\left(\frac{\tau(\theta+\Delta\theta)-\tau(\theta)}{\Delta\theta}\, f(\tau,X_\tau)\chi_{\tau<T}\right)$$

(if the limit exists). The process

$$Z_s(\theta) = \int_t^s\left(\frac{\partial a}{\partial x}Z_v(\theta) + \frac{\partial a}{\partial\theta}\right)dv + \int_t^s\left(\frac{\partial\sigma}{\partial x}Z_v(\theta) + \frac{\partial\sigma}{\partial\theta}\right)dW(v) \tag{8}$$

is the mean-square derivative of $\partial X_\cdot/\partial\theta$ with respect to the parameter of the solution of (1). It is known (see, e.g., [9]) that the assumptions listed above ensure the existence of the mean-square derivative $Z_s(\theta) = \partial X_\cdot/\partial\theta$ that can be obtained as the solution of (1) and (8).

To determine the first term in (7), we need to know the process $(X_\cdot(\theta), Z_\cdot(\theta))$, which is Markov, since system (1), (8) satisfies conditions for existence of a strong solution. For a numerical application of the formula obtained below for $\partial u/\partial\theta$, the pair $(X_\cdot(\theta), Z_\cdot(\theta))$ can be obtained using numerical methods [5].

To calculate the second term in (7), we have to consider dependence of τ on θ. Here the difficulties arise, as was mentioned in the introduction.

Therefore, the key problem in finding the derivative $\partial u/\partial\theta$ by formula (7) is calculation of $\Phi(\theta)$. Following [5], we propose the formula to determine $\Phi(\theta)$, obtained on the basis of transition to the limit as $\Delta\theta \to 0$. In the present paper, we will not use the value $\partial\tau/\partial\theta$.

To prove the formula for $\Phi(\theta)$ given below, we use the estimate [2] for the expectation of the difference of the first exit times of two diffusion processes described by (1). In [2, Theorem. 2.3], it was shown that, for two different sets of coefficients $a^{(1)}$, $\sigma^{(1)}$ and $a^{(2)}$, $\sigma^{(2)}$ in SDE (1), given that these coefficients are bounded along with their derivatives with respect to x on the infinite cylinder $Q_\infty \equiv (0,\infty) \times G$, the corresponding processes and their first exit times are such that

$$E_{t,x}\left[\frac{1}{\lambda}\Big(\exp\left(\lambda|\tau_1-\tau_2|\right)-1\Big)\right] \leqq \max_{k=1,2}\sup_{(t,x)\in Q_\infty}\left|\frac{dv_k}{dx}\right| E_{t,x}\left|X^{(1)}_{\tau_1\wedge\tau_2}-X^{(2)}_{\tau_1\wedge\tau_2}\right|. \tag{9}$$

Here $\lambda > 0$ is some constant defined by d, G, $\sup_{Q_\infty} a^{(k)}$ $(k=1,2)$, and α_0. For the purpose of our paper, the coefficients $a^{(k)}$, $\sigma^{(k)}$ $(k=1,2)$ represent the coefficients of (1) for different values of $\theta=\theta_k$, i.e., $a^{(k)}(t,x)\equiv a(t,x,\theta_k)$, $\sigma^{(k)}(t,x)\equiv\sigma(t,x,\theta_k)$. In (9), $X^{(k)}_\cdot$ is the random process obtained from SDE (1), where a and σ are replaced by $a^{(k)}$ and $\sigma^{(k)}$, and τ_k is the first exit time for $X^{(k)}_\cdot$ from G, and v_k denotes the solution of the boundary value problem in Q_∞

$$L^{(k)}v_k+\lambda v_k+1=0, \tag{10}$$

$$v_k(t,x)\big|_{x\in\partial G}=0, \tag{11}$$

$$\operatorname*{ess\,sup}_{t>0}\|v_k(t,\cdot)\|_{L_2(G)}<\infty, \tag{12}$$

where $L^{(k)}$ is operator (6), where the coefficients for the first and second order derivatives with respect to x are components of the vectors $a^{(k)}$ and matrices $B^{(k)}\equiv\sigma^{(k)}(\sigma^{(k)})^\top$, respectively. By Theorem 2.1 from [2], $\sup_{x,t,\theta_k}|\partial v_k/\partial x|<+\infty$.

For the value under the expectation on the left-hand side of (9), we have, obviously, that

$$|\tau_1-\tau_2|^p \leqq p!\,\lambda^{-p}\Big(\exp\left(\lambda|\tau_1-\tau_2|\right)-1\Big) \quad \text{for} \quad p=1,2,\dots. \tag{13}$$

We will use inequalities (9), (13) below, with $X^{(1)}_\cdot$, $X^{(2)}_\cdot$ replaced with the processes $X_\cdot(\theta+\Delta\theta)$, $X_\cdot(\theta)$, and with the value $\max_{k=1,2}\sup_{(t,x)\in Q_\infty}|dv_k/dx|$ replaced with the value $\sup_{(t,x,\theta)\in Q_\infty\times U}|dv/dx|$, where v is the solution of problem (10)–(12) with operator (6).

Lemma 1. *For any integer* $p\geqq 1$,

$$E_{t,x}|\tau(\theta+\Delta\theta)-\tau(\theta)|^p\to 0 \quad \textit{as} \quad \Delta\theta\to 0. \tag{14}$$

Proof. We observe that the solution of SDE (1) is continuous in mean-square in θ. We denote $\widetilde{\tau}(\theta,\Delta\theta)=\tau(\theta)\wedge\tau(\theta+\Delta\theta)$. Applying (9) and (13) to $X_\cdot(\theta+\Delta\theta)$ and $X_\cdot(\theta)$, we obtain the estimate of the expectation for $|\tau(\theta+\Delta\theta)-\tau(\theta)|^p$,

$$E_{t,x}|\tau(\theta+\Delta\theta)-\tau(\theta)|^p \leqq C(p)E_{t,x}|X_{\tilde\tau(\theta,\Delta\theta)}(\theta+\Delta\theta)-X_{\tilde\tau(\theta,\Delta\theta)}(\theta)|, \tag{15}$$

where $C(p)=p!\,\lambda^{-p}\sup_{(t,x,\theta)\in Q_\infty\times U}|dv/dx|$.

Since the process $X_\cdot(\theta)$ is continuous in θ in mean-square, we have that

$$E_{t,x}|X_{\tilde\tau(\theta,\Delta\theta)}(\theta+\Delta\theta)-X_{\tilde\tau(\theta,\Delta\theta)}(\theta)|\to 0 \quad \text{as} \quad \Delta\theta\to 0.$$

This completes the proof of the lemma.

Lemma 2. *There exists a constant* $K>0$ *such that, as* $\Delta\theta\to 0$,

$$E_{t,x}\left|\frac{\tau(\theta+\Delta\theta)-\tau(\theta)}{\Delta\theta}\right|<K. \tag{16}$$

Proof. We know that the solution of SDE (1) is differentiable in mean-square in θ. Let us divide both parts of inequality (15), where $p=1$, by $\Delta\theta$,

$$E_{t,x}\left|\frac{\tau(\theta+\Delta\theta)-\tau(\theta)}{\Delta\theta}\right| \leqq C(1)E_{t,x}\left|\frac{\Delta X_{\tilde\tau(\theta,\Delta\theta)}(\theta)}{\Delta\theta}\right|, \tag{17}$$

where $\Delta X_{\tilde\tau(\theta,\Delta\theta)}(\theta)=X_{\tilde\tau(\theta,\Delta\theta)}(\theta+\Delta\theta)-X_{\tilde\tau(\theta,\Delta\theta)}(\theta)$. Further, applying the Cauchy–Bunyakovsky inequality and the mean-square differentiability of $X_\cdot$ with respect to θ, we obtain that, as $\Delta\theta \to 0$,

$$E_{t,x}\left|\frac{\Delta X_{\tilde{\tau}(\theta,\Delta\theta)}(\theta)}{\Delta\theta}\right| \leqq \left[E_{t,x}\left(\frac{\Delta X_{\tilde{\tau}(\theta,\Delta\theta)}(\theta)}{\Delta\theta}\right)^2\right]^{1/2}$$
$$\leqq \sup_{0\leqq v\leqq T}\left[E_{t,x}\chi_{v\leqq\tau\wedge T}Z_v^2(\theta)\right]^{1/2}, \tag{18}$$

where $Z.$ is the mean-square derivative of X with respect to θ. The value $E_{t,x}Z_v^2(\theta)$ is bounded, since the conditions of Theorem 4 from [9, p. 48] holds for (8). This completes the proof of the lemma.

For an arbitrarily selected function $r(x,\theta)$ such that $r \in C^1(\mathbf{R}^{d+1} \to \mathbf{R})$, we will use the notation

$$\frac{d}{d\theta}r(X.(\theta),\theta) = \frac{\partial r}{\partial x}\frac{\partial X.(\theta)}{\partial\theta} + \frac{\partial r}{\partial\theta},$$

where $\partial X.(\theta)/\partial\theta$ is the mean-square derivative.

The following theorem presents our main result.

THEOREM 1. *Assume that the coefficients of* (1) *and functions* φ *and* f *satisfy conditions* (A)–(E). *Then formula* (9) *holds. The limit* $\Phi(\theta)$ *exists and can be represented as*

$$\Phi(\theta) = -E_{t,x}\left[\chi_{\tau<T}\frac{f(\tau,X_\tau)}{(Lg)_\tau}\left(\int_t^\tau \frac{d}{d\theta}(Lg)_v\,dv + \int_t^\tau \frac{d}{d\theta}\sum_{l,j}\left(\frac{\partial g}{\partial x_l}\sigma_{lj}\right)_v dW_j(v)\right)\right] \tag{19}$$

for any $g \in C^4(\mathbf{R}^d \to \mathbf{R})$ *that vanishes on* ∂G *along with its first partial derivatives, and such that the values* Lg *are nonzero on* ∂G *for all* $\theta \in U$.

Remark. For any bounded domain G in $\mathbf{R}^d$ with C^d-smooth boundary ∂G, $d \geqq 2$, one can construct a C^d-smooth function $g(x)$ that vanishes on ∂G along with the first partial derivatives, and such that its second derivatives at the border do not vanish. Such a function can be obtained, for example, as an integer positive power of the distance to the boundary ∂G, which is defined in a neighborhood of ∂G as follows: $\rho(x) = \min_{y\in\partial G}|x-y|$. Typically function $\rho(x)$ is defined in such a way that it is positive for the points inside the domain and negative for the points outside the domain. The function $\rho(x)$ can be extended to $\mathbf{R}^d$ with its smoothness preserved. Construction of functions of this type is described in [10]. Respectively, a function g with the required properties exists under our assumption that the boundary of C^4-smooth.

Proof of Theorem 4. For $\psi \in C^1(\mathbf{R}\times\mathbf{R}^d \to \mathbf{R})$, we denote

$$R_j^\psi(v,X_v,\theta) = \sum_l \frac{\partial\psi(X_v(\theta))}{\partial x_l}\sigma_{lj}(v,X_v(\theta),\theta).$$

Since g vanishes on the boundary, we obtain from the Itô's formula that

$$0 = g(x) + \int_t^{\tau(\theta)} Lg(v,X_v(\theta),\theta)\,dv + \int_t^{\tau(\theta)}\sum_j R_j^g(v,X_v(\theta),\theta)\,dW_j(v). \tag{20}$$

Similarly to the proof of Lemma 1, we use the notation $\widetilde{\tau}(\theta,\Delta\theta) = \tau(\theta)\wedge\tau(\theta+\Delta\theta)$. Applying (20) for θ and $\theta + \Delta\theta$, we obtain

$$0 = \frac{1}{\Delta\theta}\Bigg[\int_t^{\tilde\tau(\theta,\Delta\theta)} \Big(Lg(v, X_v(\theta+\Delta\theta), \theta+\Delta\theta) - Lg(v, X_v(\theta), \theta)\Big)\, dv + \int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta+\Delta\theta)} Lg(v, X_v(\theta+\Delta\theta), \theta+\Delta\theta)\, dv - \int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta)} Lg(v, X_v(\theta), \theta)\, dv + \int_t^{\tilde\tau(\theta,\Delta\theta)} \sum_j \Big(R_j^g(v, X_v(\theta+\Delta\theta), \theta+\Delta\theta) - R_j^g(v, X_v(\theta), \theta)\Big)\, dW_j(v) + \int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta+\Delta\theta)} \sum_j R_j^g(v, X_v(\theta+\Delta\theta), \theta+\Delta\theta)\, dW_j(v) - \int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta)} \sum_j R_j^g(v, X_v(\theta), \theta)\, dW_j(v)\Bigg]. \tag{21}$$

Let us multiply (21) by $\chi_{\tau<T} f(\tau, X_\tau)/(Lg)_\tau$, where τ is defined for the parameter value θ. Let us consider the limit of the expectation of the resulting inequality as $\Delta\theta \to 0$.

Let us show that

$$\lim_{\Delta\theta\to 0} E_{t,x}\Bigg[\chi_{\tau(\theta)<T}\, \frac{f(\tau, X_\tau)}{(Lg)_\tau} \times \int_t^{\tilde\tau(\theta,\Delta\theta)} \frac{Lg(v, X_v(\theta+\Delta\theta), \theta+\Delta\theta) - Lg(v, X_v(\theta), \theta)}{\Delta\theta} dv\Bigg] = E_{t,x}\Bigg[\chi_{\tau(\theta)<T}\, \frac{f(\tau, X_\tau)}{(Lg)_\tau} \int_t^{\tau(\theta)} \frac{d}{d\theta}\, Lg(v, X_v(\theta), \theta)\, dv\Bigg]. \tag{22}$$

For this aim, let us state the following inequality, using the fact that the value Lg is separated from zero on ∂G and $\chi_{\tau(\theta)<T} \frac{f(\tau,X_\tau)}{(Lg)_\tau}$ is a.e. bounded:

$$\Bigg| E_{t,x}\Bigg[\chi_{\tau(\theta)<T}\, \frac{f(\tau, X_\tau)}{(Lg)_\tau} \times \Bigg(\int_t^{\tilde\tau(\theta,\Delta\theta)} \frac{Lg(v, X_v(\theta+\Delta\theta), \theta+\Delta\theta) - Lg(v, X_v(\theta), \theta)}{\Delta\theta}\, dv - \int_t^{\tau(\theta)} \frac{d}{d\theta}\, Lg(v, X_v(\theta), \theta)\, dv\Bigg)\Bigg]\Bigg| \leqq C \int_t^T \Bigg| E_{t,x}\Bigg[\chi_{v<\tilde\tau(\theta,\Delta\theta)} \Bigg(\frac{Lg(v, X_v(\theta+\Delta\theta), \theta+\Delta\theta) - Lg(v, X_v(\theta), \theta)}{\Delta\theta} - \frac{d}{d\theta}\, Lg(v, X_v(\theta), \theta)\Bigg)\Bigg]\Bigg|\, dv + C\Bigg| E_{t,x}\Bigg[\int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta)} \frac{d}{d\theta}\, Lg(v, X_v(\theta), \theta)\, dv\Bigg]\Bigg|, \tag{23}$$

where $C = \sup_{(v,x,\theta)\in[0,T]\times\partial G\times U} |f(v,x,\theta)/(Lg)(v,x,\theta)|$.

Since the derivatives of Lg with respect to x and θ are continuous and bounded in $\overline{G}$, it can be shown that, as $\Delta\theta \to 0$,

$$\frac{Lg(v, X_v(\theta+\Delta\theta), \theta+\Delta\theta) - Lg(v, X_v(\theta), \theta)}{\Delta\theta} \to \frac{d}{d\theta}\, Lg(v, X_v(\theta), \theta)$$

in probability. Hence the first term on the right-hand side of (23) converges to zero as $\Delta\theta \to 0$.

In addition, continuity and boundedness of the derivatives of Lg in $\overline{G}$ imply that the value $(d/d\theta)Lg(v, X_v(\theta), \theta)$ is bounded. Hence the second term on the right-hand side of (23) converges to zero as $\Delta\theta \to 0$, by Lemma 1:

$$\left| E_{t,x}\left[\int_{\tilde{\tau}(\theta,\Delta\theta)}^{\tau(\theta)} \frac{d}{d\theta}\, Lg(v, X_v(\theta), \theta)\, dv\right]\right| \leqq C_1 E_{t,x}|\tau(\theta) - \widetilde{\tau}(\theta, \Delta\theta)|$$

$$\leqq C_1 E_{t,x}\big|\Delta\tau\big| \to 0 \quad \text{as} \quad \Delta\theta \to 0, \tag{24}$$

where

$$C_1 = \sup_{(v,x,\theta)\in[0,T]\times\partial G\times U} \left|\frac{d}{d\theta}\, Lg(v, X_v(\theta), \theta)\right|, \quad \Delta\tau = \tau(\theta + \Delta\theta) - \tau(\theta).$$

Let us prove that, as $\Delta\theta \to 0$,

$$E_{t,x}\left[\chi_{\tau(\theta)<T}\, \frac{f(\tau, X_\tau)}{\Delta\theta(Lg)_\tau}\left(\int_{\tilde{\tau}(\theta,\Delta\theta)}^{\tau(\theta+\Delta\theta)} Lg(v, X_v(\theta + \Delta\theta), \theta + \Delta\theta)\, dv - \int_{\tilde{\tau}(\theta,\Delta\theta)}^{\tau(\theta)} Lg(v, X_v(\theta), \theta) dv\right)\right]$$

$$- E_{t,x}\left[\chi_{\tau(\theta)<T} f(\tau, X_\tau)\frac{\Delta\tau}{\Delta\theta}\right] \to 0. \tag{25}$$

The structure of the integrals in (25) implies that

$$\int_{\tilde{\tau}(\theta,\Delta\theta)}^{\tau(\theta+\Delta\theta)} Lg(v, X_v(\theta + \Delta\theta), \theta + \Delta\theta)\, dv - \int_{\tilde{\tau}(\theta,\Delta\theta)}^{\tau(\theta)} Lg(v, X_v(\theta), \theta)\, dv$$

$$= \int_{\tau(\theta)}^{\tau(\theta+\Delta\theta)} Lg(v, X_v(\theta + \eta\Delta\theta), \theta + \eta\Delta\theta)\, dv, \tag{26}$$

where

$$\eta = \begin{cases} 1 & \text{if } \tau(\theta) < \tau(\theta + \Delta\theta), \\ 0 & \text{if } \tau(\theta + \Delta\theta) < \tau(\theta). \end{cases}$$

Let us denote $C_2 = \sup_{(v,x,\theta)\in[0,T]\times\partial G\times U} |f(v, x, \theta)/(Lg)_\tau|$. Applying the Itô's formula to $Lg(v, X_v(\theta + \eta\Delta\theta), \theta + \eta\Delta\theta)$, and taking into account (26), we derive the estimate for the value presented in (25):

$$\left| E_{t,x}\left[\chi_{\tau(\theta)<T}\, \frac{f(\tau, X_\tau)}{\Delta\theta(Lg)_\tau} \int_{\tau(\theta)}^{\tau(\theta+\Delta\theta)} \big(Lg(v, X_v(\theta + \eta\Delta\theta), \theta + \eta\Delta\theta) - (Lg)_\tau\big)\, dv\right]\right|$$

$$\leqq C_2 E_{t,x}\left[\chi_{(\tau(\theta)<T)\&(\eta=1)}\, \frac{\Delta\tau}{\Delta\theta}\big|Lg(\tau, X_\tau(\theta + \Delta\theta), \theta + \Delta\theta) - Lg(\tau, X_\tau(\theta), \theta)\big|\right]$$

$$+ C_2\left| E_{t,x}\left[\chi_{\tau(\theta)<T}\, \frac{1}{\Delta\theta} \int_{\tau(\theta)}^{\tau(\theta+\Delta\theta)} dv \int_{\tau(\theta)}^{v} L^2 g(s, X_s(\theta + \eta\Delta\theta), \theta + \eta\Delta\theta)\, ds\right]\right|$$

$$+ C_2\left| E_{t,x}\left[\chi_{\tau(\theta)<T}\, \frac{1}{\Delta\theta} \int_{\tau(\theta)}^{\tau(\theta+\Delta\theta)} dv \times \int_{\tau(\theta)}^{v} \sum_j R_j^{Lg}(s, X_s(\theta + \eta\Delta\theta), \theta + \eta\Delta\theta)\, dW_j(s)\right]\right|. \tag{27}$$

Clearly, the second and third terms on the right-hand side of (27) converge to zero as $\Delta\theta \to 0$. Let us consider $E_{t,x}[\chi_{(\tau(\theta)<T)\&(\eta=1)}(\Delta\tau/\Delta\theta)|\Delta Lg|]$, where $\Delta Lg := Lg(\tau, X_\tau(\theta + \Delta\theta), \theta + \Delta\theta) - Lg(\tau, X_\tau(\theta), \theta)$. As was mentioned above, $\Delta Lg \to 0$ in probability as $\Delta\theta \to 0$. Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Let $\delta > 0$ be such that $\mathbf{P}\{|\Delta Lg| > \varepsilon_1\} < \varepsilon_2$ if $\Delta\theta < \delta$. In this case, for $\Delta\theta < \delta$,

$$E_{t,x}\left[\chi_{(\tau(\theta)<T)\&(\eta=1)}\, \frac{\Delta\tau}{\Delta\theta}\, |\Delta Lg|\right] < 2KM\varepsilon_2 + K\varepsilon_1,$$

where K is the constant from (16), and $M = \sup_{[0,T]\times\partial G\times U} Lg$. Therefore, (25) is proved.

Let us consider stochastic integrals with respect to the Wiener process in (21). Let us show that

$$\lim_{\Delta\theta\to 0} E_{t,x}\Bigg[\chi_{\tau(\theta)<T}\,\frac{f(\tau,X_\tau)}{(Lg)_\tau} \times \int_t^{\tilde\tau(\theta,\Delta\theta)} \sum_j \frac{R_j^g(v,X_v(\theta+\Delta\theta),\theta+\Delta\theta)-R_j^g(v,X_v(\theta),\theta)}{\Delta\theta}\,dW_j(v)\Bigg]$$
$$= E_{t,x}\Bigg[\chi_{\tau(\theta)<T}\,\frac{f(\tau,X_\tau)}{(Lg)_\tau}\int_t^{\tau(\theta)}\sum_j \frac{d}{d\theta}\big(R_j^g(v,X_v(\theta),\theta)\big)\,dW_j(v)\Bigg]. \tag{28}$$

For this, we consider the following inequalities:

$$\Bigg|E_{t,x}\Bigg[\chi_{\tau(\theta)<T}\,\frac{f(\tau,X_\tau)}{(Lg)_\tau} \times\Bigg(\int_t^T \chi_{v\leqq\tilde\tau(\theta,\Delta\theta)}\sum_j\Bigg(\frac{R_j^g(v,X_v(\theta+\Delta\theta),\theta+\Delta\theta)-R_j^g(v,X_v(\theta),\theta)}{\Delta\theta} - \frac{d}{d\theta}R_j^g(v,X_v(\theta),\theta)\Bigg)\,dW_j(v)\Bigg)$$
$$- \int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta)}\sum_j \frac{d}{d\theta}\left(R_j^g(v,X_v(\theta),\theta)\right)dW_j(v)\Bigg]\Bigg|$$
$$\leqq C_2\int_t^T\Bigg(\sum_j E_{t,x}\Bigg(\frac{R_j^g(v,X_v(\theta+\Delta\theta),\theta+\Delta\theta)-R_j^g(v,X_v(\theta),\theta)}{\Delta\theta} - \frac{d}{d\theta}R_j^g(v,X_v(\theta),\theta)\Bigg)^2\Bigg)^{1/2}dv$$
$$+ C_2E_{t,x}\Bigg|\int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta)}\sum_j\frac{d}{d\theta}\left(R_j^g(v,X_v(\theta),\theta)\right)dW_j(v)\Bigg|. \tag{29}$$

The function $R_j^g(v,X_v(\theta+\Delta\theta),\theta+\Delta\theta)$ and its derivatives with respect to x and θ are bounded in $\overline{G}$. Hence, it can be shown that

$$\frac{R_j^g(v,X_v(\theta+\Delta\theta),\theta+\Delta\theta)-R_j^g(v,X_v(\theta),\theta)}{\Delta\theta}\to\frac{d}{d\theta}R_j^g(v,X_v(\theta),\theta)$$

in probability as $\Delta\theta\to 0$. It follows that the first term on the right-hand side of (29) converges to zero as $\Delta\theta\to 0$. Since the function $(d/d\theta)R_j^g(\cdot,X.,\theta)$ is bounded, it follows that the second term on the right-hand side of (29) also converges to zero as $\Delta\theta\to 0$.

To complete the proof of the theorem, it suffices to show that

$$\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E_{t,x}\chi_{\tau(\theta)<T}\frac{f(\tau,X_\tau)}{(Lg)_\tau}\Bigg[\int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta+\Delta\theta)}\sum_j R_j^g(v,X_v(\theta+\Delta\theta),\theta+\Delta\theta)\,dW_j(v)$$
$$-\int_{\tilde\tau(\theta,\Delta\theta)}^{\tau(\theta)}\sum_j R_j^g(v,X_v(\theta),\theta)\,dW_j(v)\Bigg]=0. \tag{30}$$

The multiplier after $E_{t,x}$ was missing. The proof of (30) follows from the fact that the function R_j^g vanishes on the boundary of the intervals $[\widetilde\tau,\tau(\theta)]$ and $[\widetilde\tau,\tau(\theta+\Delta\theta)]$. Applying the Itô's formula to this function under the integrals in (30), we obtain that (30) holds. This completes the proof of Theorem 1.

Corollary 1. *Theorem* 1 *allows us to obtain a formula for* $\partial u/\partial\theta$ *that does not contain* $\partial\tau/\partial\theta$:

$$\frac{\partial u}{\partial\theta}(t,x) = E_{t,x}\bigg[\bigg(\frac{\partial\varphi}{\partial x}(X_T,\theta)Z_T + \frac{\partial\varphi}{\partial\theta}(X_T,\theta)\bigg)\chi_{\tau>T} + \int_t^{T\wedge\tau}\bigg(\frac{\partial f}{\partial x}(v,X_v)Z_v + \frac{\partial f}{\partial\theta}(v,X_v)\bigg)\,dv - \chi_{\tau<T}\,\frac{f(\tau,X_\tau)}{(Lg)_\tau}\bigg(\int_t^\tau \frac{d}{d\theta}(Lg)_v\,dv + \int_t^\tau \frac{d}{d\theta}\sum_{l,j}\bigg(\frac{\partial g}{\partial x_l}\,\sigma_{lj}\bigg)_v dW_j(v)\bigg)\bigg].$$

REFERENCES

[1] N. Dokuchaev, *On first exit times for homogeneous diffusion processes*, Theory Probab. Appl., 31 (1987), pp. 497–498.

[2] N. Dokuchaev, *Estimates for distances between first exit times via parabolic equations in unbounded cylinders*, Probab. Theory Related Fields, 129 (2004), pp. 290–314.

[3] E. Fournié, J. M. Lasry, J. Lebuchoux, P. L. Lions, and N. Touzi, *Applications of Malliavin calculus to Monte Carlo methods in finance*, Finance Stoch., 3 (1999), pp. 391–412.

[4] M. Montero and A. Kohatsu-Higa, *Malliavin calculus applied to finance*, Phys. A, 320 (2003), pp. 548–570.

[5] S. A. Gusev, *Estimation of derivatives with respect to parameters of a functional of a diffusion process moving in a domain with absorbing boundary*, Numer. Anal. Appl., 1 (2008), pp. 314–331.

[6] K. Itô, *On a stochastic integral equation*, Proc. Japan Acad., 22 (1946), pp. 32–35.

[7] I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, Nauka, Moscow, 1977 (in Russian).

[8] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltzeva, *Linear and Quasi-linear Equations of the Parabolic Type*, Nauka, Moscow, 1967 (in Russian).

[9] I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations*, Nukova Dumka, Kiev, 1968 (in Russian).

[10] G. E. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, Singapore, 2005.