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Research Article

Rapid Convergence of Solution for Hybrid System with Causal Operators

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We investigated the convergence of iterative sequences of approximate solutions to a class of periodic boundary value problem of hybrid system with causal operators and established two sequences of approximate solutions that converge to the solution of the problem with rate of order $k \geq 2$.

1. Introduction

Recently, the problem of qualitative theory of dynamic systems with causal operators has attracted much attention since such systems include several types of dynamic systems, such as ordinary differential equations, integrodifferential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral equations. Therefore, the study of the theory of causal systems becomes very important. This is because a single result involving causal operators covers interesting corresponding results from many categories of dynamic systems, thus avoiding duplication and monotony of repetition. For more details, we can refer to the monographs [1–9] and the references cited therein. Since it is difficult to find the solutions of differential equations with causal operators, we need to look for the approximate solutions. Quasilinearization combined with the technique of upper and lower solutions is an effective and fruitful technique for obtaining approximate solutions to a wide variety of nonlinear problems. The main advantages of the method are the practicality of finding successive approximations of the unknown solution as well as the quadratic convergence rate. Some recent results in the development of the method and its real-world applications can be found in [10–19].

Hybrid systems have also attracted much attention in recent years. Hybrid systems are dynamical systems that evolute continuously in time but have formatting changes, called modes, at a sequence of discrete times. Some recent works on hybrid systems are included in [20–26]. However, to our best knowledge, very few results have been achieved on hybrid system with causal operators; particularly methods for finding approximate solutions with rapid convergence are yet to be developed. Hence, the purpose of this paper is to develop the method of quasilinearization for the periodic boundary value problem of hybrid system with causal operators. We will prove that the problem has solutions which can be approximated via monotone sequences with rate of convergence of order $k \geq 2$.

2. Preliminaries

In this section, we present the following definition and lemma which will help to prove our main result.

Let $E = C(I, \mathbb{R})$, where $I = [0, T]$, $T > 0$ is an appropriate positive constant, and $Q \in C(E, E)$.

Definition 1 (see [2]). The operator Q is said to be a causal or nonanticipatory operator if the following property is satisfied: for each couple of elements x, y of E such that $x(s) = y(s)$ for $0 \leq s \leq t$, one also has $(Qx)(s) = (Qy)(s)$ for $0 \leq s \leq t$ with $t < T$, T being arbitrary.

Let the points $t_j \in I$ be fixed such that $t_0 = 0$, $t_{p+1} = T$ and $t_j < t_{j+1}$, $j = 0, 1, 2, \ldots, p$.

We consider the following periodic boundary value problem (PBVP) of hybrid system with causal operators:

$$u' = Q\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right)\right), \quad t \in \left(t_j, t_{j+1}\right], \; j = 0, 1, \ldots, p, \qquad (1)$$
$$u(0) = u(T),$$

where $Q \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a continuous causal operator, the functions $\Lambda_j : \mathbb{R} \to \mathbb{R}$ are increasing, and there exist constants $L_j > 0$ such that, for any points $t_j \in \mathbb{R}$ and $u(t_j) \le v(t_j)$, the following equalities or inequalities are satisfied:

$$\begin{gathered}\Lambda_j\left(u\left(t_j\right)\right) - \Lambda_j\left(v\left(t_j\right)\right) = \Lambda_j\left(u\left(t_j\right) - v\left(t_j\right)\right), \\ \left(\Lambda_j u\left(t_j\right)\right)^k = \Lambda_j u^k\left(t_j\right), \qquad (2) \\ \Lambda_j\left(u\left(t_j\right)\right) - \Lambda_j\left(v\left(t_j\right)\right) \le L_j\left(u\left(t_j\right) - v\left(t_j\right)\right),\end{gathered}$$

and if $m < 0$, then $\Lambda_j m < 0$; that is, $\Lambda_j(-a) = -\Lambda_j a$, in which $a > 0$, $j = 0, 1, \ldots, p$.

The function $\alpha(t) \in C^1(I, \mathbb{R})$ is called a lower solution of the PBVP (1) if the following inequalities are satisfied:

$$\alpha'(t) \le Q\left(t, \alpha(t), \Lambda_j\left(\alpha\left(t_j\right)\right)\right), \quad \text{for } t \in \left(t_j, t_{j+1}\right], \; j = 0, 1, \ldots, p, \qquad (3)$$
$$\alpha(0) \le \alpha(T).$$

Analogously, we can define an upper solution of the PBVP (1) by introducing the inequalities in (3) in opposite directions.

Let the functions $\alpha, \beta \in C^1(I, \mathbb{R})$ be such that $\alpha(t) \le \beta(t)$. Consider the sets

$$\Omega = \left\{u \in C(I, \mathbb{R}) : \alpha(t) \le u(t) \le \beta(t)\right\}. \qquad (4)$$

Similar to the proof of Theorem 3.2.1 in [2], we have the following lemma.

Lemma 2. *Let $v, w \in C(I, \mathbb{R})$ be lower and upper solutions of the PBVP (1) satisfying $v(t) \le w(t)$, $t \in I$. Suppose that the operator Q is bounded on Ω. Then, there exists a solution $x(t)$ of (1) in the closed set Ω, such that $v(t) \le x(t) \le w(t)$, $t \in I$.*

3. Main Result

Consider the Banach space $C(I)$ with the usual norm $\|\cdot\|_\infty$. For a given sequence $\{x_n\} \subset C(I)$, we say that $\{x_n\}$ converges to x with order of convergence k, if $\{x_n\}$ converges to x in $C(I)$ and there exist $n_0 \in N$ and $\lambda > 0$ such that

$$\left\|x_{n+1} - x\right\|_\infty \le \lambda \left\|x_n - x\right\|_\infty^k, \quad n \ge n_0; \; \lambda \text{ is a constant}. \qquad (5)$$

Theorem 3. *Let the following conditions hold:*

(H_1) *The functions $\alpha(t)$, $\beta(t)$ are lower and upper solutions to the PBVP (1) and $\alpha(t) \le \beta(t)$ for $t \in [0, T]$.*

(H_2) *There exist continuous functions $(t, u(t), \Lambda_j(u(t_j)))$, $(\partial^i Q/\partial(\Lambda_j(u_j))^i)(t, u(t), \Lambda_j(u(t_j)))$, and constants $M_i \ge 0$ and $N_i \ge 0$ such that*

$$\begin{gathered}\frac{\partial^i Q}{\partial u^i}\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right)\right) \ge -i! M_i, \\ \frac{\partial^i Q}{\partial\left(\Lambda_j\left(u_j\right)\right)^i}\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right)\right) \ge -i! N_i, \qquad (6) \\ i = 0, 1, \ldots, k.\end{gathered}$$

Then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the unique solution ψ of the PBVP (1), and the convergence is of order $k \ge 2$.

Proof. Firstly, we note that the condition (H_2) implies that the PBVP (1) has a unique solution $\psi(t)$ between $\alpha(t)$ and $\beta(t)$. To construct the sequence $\{\alpha_n\}$, for given

$$\begin{aligned}Q&\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right)\right) \\ &= \sum_{i=0}^{k-1} \frac{\partial^i Q}{\partial u^i}\left(t, v(t), \Lambda_j\left(v\left(t_j\right)\right)\right) \frac{(u-v)^i}{i!} \\ &+ \frac{\partial^k Q}{\partial u^k}\left(t, \xi(t), \Lambda_j\left(\xi\left(t_j\right)\right)\right) \frac{(u-v)^k}{k!} \\ &+ \sum_{i=0}^{k-1} \frac{\partial^i Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t, v(t), \Lambda_j\left(v\left(t_j\right)\right)\right) \\ &\cdot \frac{\left(\Lambda_j\left(u\left(t_j\right)\right) - \Lambda_j\left(v\left(t_j\right)\right)\right)^i}{i!} \qquad (7) \\ &+ \frac{\partial^k Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^k}\left(t, \xi(t), \Lambda_j\left(\xi\left(t_j\right)\right)\right) \\ &\cdot \frac{\left(\Lambda_j\left(u\left(t_j\right)\right) - \Lambda_j\left(v\left(t_j\right)\right)\right)^k}{k!},\end{aligned}$$

where $\xi(t) \in [v, u]$, $\alpha(t) \le v \le u \le \beta(t)$, define the following function:

$$\begin{aligned}g&\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right); v(t), \Lambda_j\left(v\left(t_j\right)\right)\right) \\ &= \sum_{i=0}^{k-1} \frac{\partial^i Q}{\partial u^i}\left(t, v(t), \Lambda_j\left(v\left(t_j\right)\right)\right) \frac{(u-v)^i}{i!} - M_k (u \\ &- v)^k + \sum_{i=0}^{k-1} \frac{\partial^i Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t, v(t), \Lambda_j\left(v\left(t_j\right)\right)\right)\end{aligned}$$

$$\cdot \frac{\left(\Lambda_j\left(u\left(t_j\right)\right)-\Lambda_j\left(v\left(t_j\right)\right)\right)^i}{i!} - N_k\left(\Lambda_j\left(u\left(t_j\right)\right) - \Lambda_j\left(v\left(t_j\right)\right)\right)^k, \tag{8}$$

in which the function $g \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Using (H_2), (7), and (8), we get

$$g\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right); v(t), \Lambda_j\left(v\left(t_j\right)\right)\right) \leq Q\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right)\right), \quad t \in I, \ v, u \in \Omega. \tag{9}$$

Now, consider the following boundary value problem:

$$u'(t) = g\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right); \alpha(t), \Lambda_j\left(\alpha\left(t_j\right)\right)\right), \quad t \in I, \quad u(0) = u(T). \tag{10}$$

It follows from (9) that

$$\beta'(t) \geq Q\left(t, \beta(t), \Lambda_j\left(\beta\left(t_j\right)\right)\right) \geq g\left(t, \beta(t), \Lambda_j\left(\beta\left(t_j\right)\right); \alpha(t), \Lambda_j\left(\alpha\left(t_j\right)\right)\right), \tag{11}$$
$$\beta(0) \geq \beta(T),$$

$$\alpha'(t) \leq Q\left(t, \alpha(t), \Lambda_j\left(\alpha\left(t_j\right)\right)\right) = g\left(t, \alpha(t), \Lambda_j\left(\alpha\left(t_j\right)\right); \alpha(t), \Lambda_j\left(\alpha\left(t_j\right)\right)\right), \tag{12}$$
$$\alpha(0) \leq \alpha(T).$$

That is, α and β are lower and upper solutions of (10), respectively.

Thus, using Lemma 2, we conclude that problem (10) has the unique solution α_1 and $\alpha_1 \in [\alpha, \beta]$.

Now, suppose that $\alpha_0 = \alpha \leq \alpha_1 \leq \cdots \alpha_n \leq \beta$, where α_n is the unique solution of

$$u'(t) = g\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right); \alpha_{n-1}(t), \Lambda_j\left(\alpha_{n-1}\left(t_j\right)\right)\right), \quad t \in I, \quad u(0) = u(T). \tag{13}$$

In this case, we have

$$\beta'(t) \geq Q\left(t, \beta(t), \Lambda_j\left(\beta\left(t_j\right)\right)\right) \geq g\left(t, \beta(t), \Lambda_j\left(\beta\left(t_j\right)\right); \alpha_n(t), \Lambda_j\left(\alpha_n\left(t_j\right)\right)\right),$$
$$\beta(0) \geq \beta(T),$$
$$\alpha_n'(t) = g\left(t, \alpha_n(t), \Lambda_j\left(\alpha_n\left(t_j\right)\right); \alpha_{n-1}(t), \Lambda_j\left(\alpha_{n-1}\left(t_j\right)\right)\right) \leq Q\left(t, \alpha_n(t), \Lambda_j\left(\alpha_n\left(t_j\right)\right)\right) = g\left(t, \alpha_n(t), \Lambda_j\left(\alpha_n\left(t_j\right)\right); \alpha_n(t), \Lambda_j\left(\alpha_n\left(t_j\right)\right)\right),$$
$$\alpha_n(0) = \alpha(T). \tag{14}$$

We conclude, using again Lemma 2, that there exists a unique solution $\alpha_{n+1} \in [\alpha_n, \beta]$ for

$$u'(t) = g\left(t, u(t), \Lambda_j\left(u\left(t_j\right)\right); \alpha_n(t), \Lambda_j\left(\alpha_n\left(t_j\right)\right)\right), \quad t \in I, \quad u(0) = u(T). \tag{15}$$

Thus, we know that $\{\alpha_n\}$ is a nondecreasing sequence and is bounded in $C^1(I)$. According to the standard arguments (see [12]), the Ascoli-Arzela Theorem guarantees the existence of a subsequence which converges uniformly to a continuous function $\psi \in [\alpha, \beta]$.

Since

$$\alpha_n(t) = u(0) + \int_0^t g\left(s, \alpha_n(s), \Lambda_j\left(\alpha_n\left(t_j\right)\right); \alpha_{n-1}(t), \Lambda_j\left(\alpha_{n-1}\left(t_j\right)\right)\right) ds \tag{16}$$

we have

$$\psi(t) = u(0) + \int_0^t g\left(s, \psi(s), \Lambda_j\left(\psi\left(t_j\right)\right); \psi(t), \Lambda_j\left(\psi\left(t_j\right)\right)\right) ds = u(0) + \int_0^t Q\left(s, \psi(s), \Lambda_j\left(\psi\left(t_j\right)\right)\right) ds, \tag{17}$$

and ψ is the unique solution of the PBVP (1) in $[\alpha, \beta]$.

Now, we prove that the convergence is of order k. For this purpose, using (7), we have

$$\psi'(t) = \sum_{i=0}^{k-1} \frac{\partial^i Q}{\partial u^i}\left(t, \alpha_n(t), \Lambda_j\left(\alpha_n\left(t_j\right)\right)\right) \cdot \frac{\left(\psi(t) - \alpha_n(t)\right)^i}{i!} + \frac{\partial^k Q}{\partial u^k}\left(t, \rho_n(t), \Lambda_j\left(\rho_n\left(t_j\right)\right)\right) \cdot \frac{\left(\psi(t) - \alpha_n(t)\right)^k}{k!}$$
$$+ \sum_{i=0}^{k-1} \frac{\partial^i Q}{\partial\left(\Lambda_j u\left(t_j\right)\right)^i}\left(t, \alpha_n(t), \Lambda_j\left(\alpha_n\left(t_j\right)\right)\right) \cdot \frac{\left(\Lambda_j\left(\psi\left(t_j\right)\right) - \Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)^i}{i!}$$

$$
\begin{aligned}
&+\frac{\partial^k Q}{\partial\left(\Lambda_j u\left(t_j\right)\right)^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right)\\
&\cdot\frac{\left(\Lambda_j\left(\psi\left(t_j\right)\right)-\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)^k}{k!},\\
&\psi(0)=\psi(T),\\
&\qquad\text{where } \rho_n\in[\alpha_n,\psi].
\end{aligned}
\tag{18}
$$

On the other hand, by (8) and (15), it is verified that, for $n\ge 0$,

$$
\begin{aligned}
\alpha'_{n+1}(t)&=g\left(t,\alpha_{n+1}(t),\Lambda_j\left(\alpha_{n+1}\left(t_j\right)\right);\alpha_n(t),\right.\\
\Lambda_j\left(\alpha_n\left(t_j\right)\right)\Big)&=\sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial u^i}\left(t,\alpha_n(t),\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)\\
&\cdot\frac{\left(\alpha_{n+1}(t)-\alpha_n(t)\right)^i}{i!}-M_k\left(\alpha_{n+1}(t)-\alpha_n(t)\right)^k\\
&+\sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,\alpha_n(t),\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)\\
&\cdot\frac{\left(\Lambda_j\left(\alpha_{n+1}\left(t_j\right)\right)-\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)^i}{i!}\\
&-N_k\left(\Lambda_j\left(\alpha_{n+1}\left(t_j\right)\right)-\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)^k,\\
\alpha_{n+1}(0)&=\alpha_{n+1}(T).
\end{aligned}
\tag{19}
$$

Let $e_{n+1}=\psi-\alpha_{n+1}$ and $a_n=\alpha_{n+1}-\alpha_n$; then $a_n^k(t)\le e_n^k(t)$, for all $n\in N$ and $t\in I$. Thus, we have

$$
\begin{aligned}
e'_{n+1}&=\sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial u^i}\left(t,\alpha_n(t),\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)\\
&\cdot\left[\frac{e_n^i(t)-a_n^i(t)}{i!}\right]+\frac{\partial^k Q}{\partial u^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right)\\
&\cdot\frac{e_n^k(t)}{k!}+M_k a_n^k(t)\\
&+\sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,\alpha_n(t),\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)\\
&\cdot\frac{\left(\Lambda_j\left(e_n\left(t_j\right)\right)\right)^i-\left(\Lambda_j\left(a_n\left(t_j\right)\right)\right)^i}{i!}\\
&+\frac{\partial^k Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right)\\
&\cdot\frac{\left(\Lambda_j\left(e_n\left(t_j\right)\right)\right)^k}{k!}+N_k\left(\Lambda_j\left(a_n\left(t_j\right)\right)\right)^k.
\end{aligned}
\tag{20}
$$

The continuity of $\partial^k Q/\partial u^k$ and $\partial^k Q/\partial(\Lambda_j(u(t_j)))^k$ in Ω implies that there exist $A_k>0$ and $B_k>0$ such that

$$
\begin{aligned}
&\frac{\partial^k Q}{\partial u^k}\left(t,u(t),\Lambda_j\left(u\left(t_j\right)\right)\right)\le k!A_k,\\
&\frac{\partial^k Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^k}\left(t,u(t),\Lambda_j\left(u\left(t_j\right)\right)\right)\le k!B_k,\\
&\qquad t,u\in\Omega.
\end{aligned}
\tag{21}
$$

Finally, as for all $E,F\in\mathbb{R}$, $E^i-F^i=(E-F)\sum_{j=0}^{i-1}E^{i-1-j}F^j$, we get that

$$
\begin{aligned}
&e'_{n+1}(t)-P_n(t)e_{n+1}(t)-H_n(t)\Lambda_j\left(e_{n+1}\left(t_j\right)\right)\\
&\quad\le C_k e_n^k(t)+D_k\Lambda_j\left(e_n^k\left(t_j\right)\right),\quad t\in I,
\end{aligned}
\tag{22}
$$

where $C_k=A_k+M_k>0$, $D_k=B_k+N_k>0$, and

$$
\begin{aligned}
P_n(t)&=\sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial u^i}\left(t,\alpha_n(t),\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)\\
&\cdot\left[i!\sum_{j=0}^{i-1}e_n^{i-1-j}(t)\,a_n^j(t)\right]\\
H_n(t)&=\sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial\left(\Lambda_j u\left(t_j\right)\right)^i}\left(t,\alpha_n(t),\Lambda_j\left(\alpha_n\left(t_j\right)\right)\right)\\
&\cdot\left[i!\sum_{j=0}^{i-1}\left(\Lambda_j\left(e_n\left(t_j\right)\right)\right)^{i-1-j}\left(\Lambda_j\left(a_n\left(t_j\right)\right)\right)^j\right].
\end{aligned}
\tag{23}
$$

Since $\{\alpha_n\}$ converges uniformly to ψ in I, (21) implies that there exists $n_0\in N$ and $P>0, H>0$, such that $P_n(t)\le -P<0$, $H_n(t)\le -H<0$, for $n>n_0$, and $t\in I$. Then, there exists a continuous function $\sigma_n\le 0$ on I such that

$$
\begin{aligned}
&e'_{n+1}(t)+Pe_{n+1}(t)+H\Lambda_j\left(e_{n+1}\left(t_j\right)\right)\\
&\quad=C_k e_n^k(t)+\sigma_n(t)+D_k\Lambda_j\left(e_n^k\left(t_j\right)\right)\\
&\quad\quad+\Lambda_j\left(\sigma_n\left(t_j\right)\right),\quad t\in I,\\
&e_{n+1}(0)=e_{n+1}(T),
\end{aligned}
\tag{24}
$$

or equivalently

$$
\begin{aligned}
e_{n+1}(t)&=\int_0^T G(t,s,P)\left[C_k e_n^k(t)+\sigma_n(t)\right]ds\\
&+\int_0^T G(t,s,H)\\
&\cdot\left[D_k\Lambda_j\left(e_n^k\left(t_j\right)\right)+\Lambda_j\left(\sigma_n\left(t_j\right)\right)\right]ds,
\end{aligned}
\tag{25}
$$

where G is the Green function associated with the following linear boundary value problem:

$$
\begin{aligned}
&u'+Pu+H\left(\Lambda_j\left(u\left(t_j\right)\right)\right)=\sigma(t)+\Lambda_j\left(\sigma\left(t_j\right)\right),\\
&u(0)=u(T).
\end{aligned}
\tag{26}
$$

From [4], we have that G is positive on $I \times I$, since the solution of problem (26) is given by

$$\begin{aligned} u(t) &= \int_0^T G(t,s,P)\,\sigma(s)\,ds \\ &\quad + \int_0^T G(t,s,H)\left(\Lambda_j\left(\sigma\left(s_j\right)\right)\right)ds, \end{aligned} \tag{27}$$

where $\int_0^T G(t,s,P)ds = 1/P$, $\int_0^T G(t,s,H)ds = 1/H$. We can thus conclude that, for any $t \in I$ and $n \geq n_0$,

$$\begin{aligned} 0 &\leq \psi(t) - \alpha_{n+1}(t) \\ &\leq \frac{C_k}{P}\max_{t\in I} e_n^k(t) + \frac{D_k}{H}\Lambda_j\left(e_n^k\left(t_j\right)\right) \\ &\leq \frac{C_k}{P}\max_{t\in I} e_n^k(t) + \frac{L_j D_k}{H} e_n^k\left(t_j\right) \\ &\leq \frac{C_k}{P}\max_{t\in I} e_n^k(t) + \frac{L_j D_k}{H}\left\|e_n^k(t)\right\|, \end{aligned} \tag{28}$$

where $e_n^k(t_j) \leq \|e_n^k(t)\| = \max\{|e_n^k(t)| : t \in [0,T]\}$. Hence,

$$\left\|\psi(t) - \alpha_{n+1}(t)\right\|_\infty \leq \lambda\left\|\psi - \alpha_n(t)\right\|_\infty^k, \tag{29}$$

for all $n \geq n_0$, and $\lambda = \max\{C_k/P, L_j D_k/H\} > 0$.

Similarly, to construct the sequence $\{\beta_n\}$, define the following function:

$$\begin{aligned} &h\left(t,u(t),\Lambda_j\left(u\left(t_j\right)\right);v(t),\Lambda_j\left(v\left(t_j\right)\right)\right) \\ &= \begin{cases} \displaystyle\sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial u^i}\left(t,v(t),\Lambda_j\left(v\left(t_j\right)\right)\right)\frac{(u-v)^i}{i!} - M_k(u-v)^k + \sum_{i=0}^{k-1}\frac{\partial^i Q}{\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,v(t),\Lambda_j\left(v\left(t_j\right)\right)\right)\frac{\left(\Lambda_j\left(u\left(t_j\right)\right)-\Lambda_j\left(v\left(t_j\right)\right)\right)^i}{i!} - N_k\left(\Lambda_j\left(u\left(t_j\right)\right)-\Lambda_j\left(v\left(t_j\right)\right)\right)^k, & \text{if } k \text{ is odd}, \\ \displaystyle\sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial u^i}\left(t,v(t),\Lambda_j\left(v\left(t_j\right)\right)\right)\frac{(u-v)^i}{i!} + A_k(u-v)^k + \sum_{i=0}^{k-1}\frac{\partial^i Q}{\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,v(t),\Lambda_j\left(v\left(t_j\right)\right)\right)\frac{\left(\Lambda_j\left(u\left(t_j\right)\right)-\Lambda_j\left(v\left(t_j\right)\right)\right)^i}{i!} + B_k\left(\Lambda_j\left(u\left(t_j\right)\right)-\Lambda_j\left(v\left(t_j\right)\right)\right)^k, & \text{if } k \text{ is even}, \end{cases} \end{aligned} \tag{30}$$

where the function $h \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and M_k, N_k, A_k, and B_k are nonnegative constants given by (6) and (21), respectively. Similar to the discussion of $g(t,u(t),\Lambda_j(u(t_j)); v(t), \Lambda_j(v(t_j)))$ above, we have

$$\begin{aligned} &h\left(t,u(t),\Lambda_j\left(u\left(t_j\right)\right);v(t),\Lambda_j\left(v\left(t_j\right)\right)\right) \\ &\quad \geq Q\left(t,u(t),\Lambda_j\left(u\left(t_j\right)\right)\right), \quad t \in I,\ u,v \in \Omega. \end{aligned} \tag{31}$$

Now, let $\beta_0 = \beta$; for $n \geq 1$, we define β_n by induction, as the unique solution of the following boundary value problem:

$$\begin{aligned} &u'(t) \\ &\quad = h\left(t,u(t),\Lambda_j\left(u\left(t_j\right)\right);\beta_{n-1}(t),\Lambda_j\left(\beta_{n-1}\left(t_j\right)\right)\right), \\ &\qquad t \in I, \\ &u(0) = u(T). \end{aligned} \tag{32}$$

We can obtain $\alpha \leq \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0 \leq \beta$. Similar to the discussion of $\{\alpha_n\}$, $\{\beta_n\}$ is a nonincreasing sequence and is bounded in $C^1(I)$. Then $\{\beta_n\}$ converges uniformly in $C(I)$ to the continuous function $\psi \in [\alpha,\beta]$. Since

$$\begin{aligned} \beta_n(t) &= u(0) + \int_0^t h\left(t,\beta_n(s),\Lambda_j\left(\beta_n\left(t_j\right)\right);\beta_{n-1}(s),\right. \\ &\quad \left.\Lambda_j\left(\beta_{n-1}\left(t_j\right)\right)\right)ds, \end{aligned} \tag{33}$$

we have

$$\begin{aligned} \psi(t) &= u(0) + \int_0^t h\left(s,\psi(s),\Lambda_j\left(\psi\left(t_j\right)\right);\psi(s),\right. \\ &\quad \left.\Lambda_j\left(\psi\left(t_j\right)\right)\right)ds = u(0) + \int_0^t Q\left(s,\psi(s),\right. \\ &\quad \left.\Lambda_j\left(\psi\left(t_j\right)\right)\right)ds. \end{aligned} \tag{34}$$

Therefore, ψ is the unique solution of the PBVP (1) in $[\alpha,\beta]$. Furthermore, we prove that the convergence is of order k. For this purpose, using (7), we have

$$\begin{aligned} \psi'(t) &= \sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial u^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right) \\ &\quad \cdot \frac{\left(\psi(t)-\beta_n(t)\right)^i}{i!} + \frac{\partial^k Q}{\partial u^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right) \\ &\quad \cdot \frac{\left(\psi(t)-\beta_n(t)\right)^k}{k!} \\ &\quad + \sum_{i=0}^{k-1}\frac{\partial^i Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right) \\ &\quad \cdot \frac{\left(\Lambda_j\left(\psi\left(t_j\right)\right)-\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)^i}{i!} \\ &\quad + \frac{\partial^k Q}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right) \end{aligned}$$

$$\cdot \frac{\left(\Lambda_j\left(\psi\left(t_j\right)\right)-\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)^k}{k!},$$
$$\psi(0)=\psi(T),$$
$$\rho_n\in[\psi,\beta_n]. \quad (35)$$

On the other hand, by (30) and (32), it is verified that, for $n\geq 0$, if k is odd, then

$$\begin{aligned}\beta_{n+1}'(t)&=h\left(t,\beta_{n+1}(t),\Lambda_j\left(\beta_{n+1}\left(t_j\right)\right);\beta_n(t),\right.\\ &\left.\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)=\sum_{i=0}^{k-1}\frac{\partial^iQ}{\partial u^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)\\ &\cdot\frac{\left(\beta_{n+1}(t)-\beta_n(t)\right)^i}{i!}-M_k\left(\beta_{n+1}(t)-\beta_n(t)\right)^k\\ &+\sum_{i=0}^{k-1}\frac{\partial^iQ}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)\\ &\cdot\frac{\left(\Lambda_j\left(\beta_{n+1}\left(t_j\right)\right)-\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)^i}{i!}\\ &-N_k\left(\Lambda_j\left(\beta_{n+1}\left(t_j\right)\right)-\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)^k,\end{aligned} \quad (36)$$

while if k is even, then

$$\begin{aligned}\beta_{n+1}'(t)&=\sum_{i=0}^{k-1}\frac{\partial^iQ}{\partial u^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)\\ &\cdot\frac{\left(\beta_{n+1}(t)-\beta_n(t)\right)^i}{i!}+A_k\left(\beta_{n+1}(t)-\beta_n(t)\right)^k\\ &+\sum_{i=0}^{k-1}\frac{\partial^iQ}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)\\ &\cdot\frac{\left(\Lambda_j\left(\beta_{n+1}\left(t_j\right)\right)-\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)^i}{i!}\\ &+B_k\left(\Lambda_j\left(\beta_{n+1}\left(t_j\right)\right)-\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)^k.\end{aligned} \quad (37)$$

Let $f_n=\psi-\beta_n$, and $b_n=\beta_{n+1}-\beta_n$. Then, we have that if k is odd, then

$$\begin{aligned}-f_{n+1}'&=\beta_{n+1}'-\psi'=\sum_{i=0}^{k-1}\frac{\partial^iQ}{\partial u^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)\\ &\cdot\frac{b_n^i(t)-f_n^i(t)}{i!}-\frac{\partial^kQ}{\partial u^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right)\\ &\cdot\frac{f_n^k(t)}{k!}-M_kb_n^k(t)\\ &+\sum_{i=0}^{k-1}\frac{\partial^iQ}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)\\ &\cdot\frac{\left(\Lambda_j\left(b_n\left(t_j\right)\right)\right)^i-\left(\Lambda_j\left(f_n\left(t_j\right)\right)\right)^i}{i!}\\ &-\frac{\partial^kQ}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right)\\ &\cdot\frac{\left(\Lambda_j\left(f_n\left(t_j\right)\right)\right)^k}{k!}-N_k\left(\Lambda_j\left(b_n\left(t_j\right)\right)\right)^k,\end{aligned} \quad (38)$$

while if k is even, then

$$\begin{aligned}-f_{n+1}'&=\beta_{n+1}'-\psi'=\sum_{i=0}^{k-1}\frac{\partial^iQ}{\partial u^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)\\ &\cdot\frac{b_n^i(t)-f_n^i(t)}{i!}-\frac{\partial^kQ}{\partial u^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right)\\ &\cdot\frac{f_n^k(t)}{k!}+A_kb_n^k(t)\\ &+\sum_{i=0}^{k-1}\frac{\partial^iQ}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^i}\left(t,\beta_n(t),\Lambda_j\left(\beta_n\left(t_j\right)\right)\right)\\ &\cdot\frac{\left(\Lambda_j\left(b_n\left(t_j\right)\right)\right)^i-\left(\Lambda_j\left(f_n\left(t_j\right)\right)\right)^i}{i!}\\ &-\frac{\partial^kQ}{\partial\left(\Lambda_j\left(u\left(t_j\right)\right)\right)^k}\left(t,\rho_n(t),\Lambda_j\left(\rho_n\left(t_j\right)\right)\right)\\ &\cdot\frac{\left(\Lambda_j\left(f_n\left(t_j\right)\right)\right)^k}{k!}+B_k\left(\Lambda_j\left(b_n\left(t_j\right)\right)\right)^k.\end{aligned} \quad (39)$$

Furthermore

$$\begin{aligned}(-1)^k\left(b_n^k(t)\right)&\leq(-1)^k\left(f_n^k(t)\right),\quad\text{if } k \text{ odd},\\ b_n^k(t)&\leq f_n^k(t),\quad\text{if } k \text{ even},\end{aligned} \quad (40)$$

for all $n\in N$ and $t\in I$. We can write that if k is odd, then

$$\begin{aligned}&-f_{n+1}'(t)-P_n(t)\left(-f_{n+1}(t)\right)\\ &\quad-H_n(t)\left(-\Lambda_j\left(f_{n+1}\left(t_j\right)\right)\right)\\ &\leq 2M_k\left(-f_n^k(t)\right)+2N_k\left(-\Lambda_j\left(f_n^k\left(t_j\right)\right)\right),\end{aligned} \quad (41)$$

while if k is even, then

$$\begin{aligned}&-f_{n+1}'(t)-P_n(t)\left(-f_{n+1}(t)\right)\\ &\quad-H_n(t)\left(-\Lambda_j\left(f_{n+1}\left(t_j\right)\right)\right)\\ &\leq C_kf_n^k(t)+D_k\Lambda_j\left(f_n^k\left(t_j\right)\right),\end{aligned} \quad (42)$$

where $t \in I$, $C_k = A_k + M_k > 0$, $D_k = B_k + N_k > 0$, and

$$P_n(t) = \sum_{i=0}^{k-1} \frac{\partial^i Q}{\partial u^i} \left(t, \beta_n(t), \Lambda_j\left(\beta_n\left(t_j\right)\right)\right) \cdot \left[i! \sum_{j=0}^{i-1} b_n^{i-1-j}(t) f_n^j(t) \right],$$
$$H_n(t) = \sum_{i=0}^{k-1} \frac{\partial^i Q}{\partial\left(\Lambda_j u\left(t_j\right)\right)^i} \left(t, \beta_n(t), \Lambda_j\left(\beta_n\left(t_j\right)\right)\right) \cdot \left[i! \sum_{j=0}^{i-1} \left(\Lambda_j\left(b_n\left(t_j\right)\right)\right)^{i-1-j} \left(\Lambda_j\left(f_n\left(t_j\right)\right)\right)^j \right]. \tag{43}$$

Since $\{\beta_n\}$ converges uniformly to ψ in I, (21) implies that there exist $n_0 \in N$ and $P > 0, H > 0$, such that $P_n(t) \le -P < 0$, $H_n(t) \le -H < 0$, for $n > n_0$ and $t \in I$. Thus, there exists a continuous function $\sigma_n \le 0$ on I such that if k is odd,

$$\begin{aligned} -f'_{n+1}(t) - P\left(-f_{n+1}(t)\right) - H\left(-\Lambda_j\left(f_{n+1}\left(t_j\right)\right)\right) &= 2M_k\left(-f_n^k(t)\right) + \sigma_n(t) + 2N_k\left(-\Lambda_j\left(f_n^k\left(t_j\right)\right)\right) \\ &\quad + \Lambda_j\left(\sigma_n\left(t_j\right)\right), \end{aligned} \tag{44}$$
$$f_{n+1}(0) = f_{n+1}(T);$$

if k is even,

$$\begin{aligned} -f'_{n+1}(t) - P\left(-f_{n+1}(t)\right) - H\left(-\Lambda_j\left(f_{n+1}\left(t_j\right)\right)\right) &= C_k f_n^k(t) + \sigma_n(t) + D_k \Lambda_j\left(b_n^k\left(t_j\right)\right) \\ &\quad + \Lambda_j\left(\sigma_n\left(t_j\right)\right), \end{aligned} \tag{45}$$
$$f_{n+1}(0) = f_{n+1}(T). \tag{46}$$

Or equivalently, if k is odd, then

$$\begin{aligned} -f'_{n+1}(t) &= \int_0^T G(t,s,P)\left[2M_k\left(-f_n^k(t)\right) + \sigma_n(t)\right] ds \\ &\quad + \int_0^T G(t,s,H) \cdot \left[2N_k\left(-\Lambda_j\left(f_n^k\left(t_j\right)\right)\right) + \Lambda_j\left(\sigma_n\left(t_j\right)\right)\right] ds, \end{aligned} \tag{47}$$

while if k is even, then

$$\begin{aligned} -f'_{n+1}(t) &= \int_0^T G(t,s,P)\left[C_k\left(f_n^k(t)\right) + \sigma_n(t)\right] ds \\ &\quad + \int_0^T G(t,s,H) \cdot \left[D_k\left(\Lambda_j\left(f_n^k\left(t_j\right)\right)\right) + \Lambda_j\left(\sigma_n\left(t_j\right)\right)\right] ds, \end{aligned} \tag{48}$$

where G is the same with the above.

We conclude that, for every $t \in I$ and $n \ge n_0$, if k is odd, then

$$\begin{aligned} 0 &\le \beta_{n+1}(t) - \psi(t) \\ &\le \frac{2M_k}{P} \max\left(-f_n^k(t)\right) + \frac{2N_k L_j}{H}\left(-f_n^k\left(t_j\right)\right) \\ &\le \frac{2M_k}{P} \max\left(-f_n^k(t)\right) + \frac{2N_k L_j}{H}\left\|-f_n^k(t)\right\|, \\ \left\|\psi(t) - \beta_{n+1}(t)\right\|_\infty &\le \lambda_1 \left\|\psi(t) - \beta_n(t)\right\|_\infty^k, \end{aligned} \tag{49}$$

for all $n \ge n_0$ and $\lambda_1 = \max\{2M_k/P, 2N_k L_j/H\}$, while if k is even, then

$$\begin{aligned} 0 &\le \beta_{n+1}(t) - \psi(t) \\ &\le \frac{C_k}{P} \max f_n^k(t) + \frac{D_k L_j}{H}\left(f_n^k\left(t_j\right)\right) \\ &\le \frac{C_k}{P} \max f_n^k(t) + \frac{D_k L_j}{H}\left\|f_n^k(t)\right\|, \end{aligned} \tag{50}$$

and hence

$$\left\|\psi(t) - \beta_{n+1}(t)\right\|_\infty \le \lambda_2 \left\|\psi(t) - \beta_n(t)\right\|_\infty^k \tag{51}$$

for all $n \ge n_0$ and $\lambda_2 = \max\{C_k/P, D_k L_j/H\}$.

The proof is complete. □

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors completed the paper together. All authors read and approved the final paper.

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References

[1] C. Corduneanu, *Functional Equations with Causal Operators*, Taylor & Francis, 2002.

[2] V. Lakshmikantham, S. Leela, Z. Drici, and F. A. Mcrae, *Theory of Causal Differential Equations*, World Scientific Press, Singapore, 2009.

[3] Z. Drici, F. A. McRae, and J. Vasundhara Devi, "Monotone iterative technique for periodic boundary value problems with causal operators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 6, pp. 1271–1277, 2006.

[4] Z. Drici, F. A. McRae, and J. V. Devi, "Differential equations with causal operators in a Banach space," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 62, no. 2, pp. 301–313, 2005.

[5] F. Geng, "Differential equations involving causal operators with nonlinear periodic boundary conditions," *Mathematical and Computer Modelling*, vol. 48, no. 5-6, pp. 859–866, 2008.

[6] T. Jankowski, "Boundary value problems with causal operators," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 68, no. 12, pp. 3625–3632, 2008.

[7] C. T. H. Baker, G. Bocharov, E. Parmuzin, and F. Rihan, "Some aspects of causal & neutral equations used in modelling," *Journal of Computational and Applied Mathematics*, vol. 229, no. 2, pp. 335–349, 2009.

[8] J. Jiang, C. F. Li, and H. T. Chen, "Existence of solutions for set differential equations involving causal operator with memory in Banach space," *Journal of Applied Mathematics and Computing*, vol. 41, no. 1-2, pp. 183–196, 2013.

[9] J. Jiang, D. Cao, and H. T. Chen, "The fixed point approach to the stability of fractional differential equations with causal operators," *Qualitative Theory of Dynamical Systems*, pp. 1–16, 2015.

[10] T. Jankowski, "Nonlinear boundary value problems for second order differential equations with causal operators," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1380–1392, 2007.

[11] V. Lakshmikantham and A. S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, vol. 440 of *Mathematics and Its Applications*, Kluwer Academic publishers, Dodrecht, The Netherlands, 1998.

[12] A. Cabada and J. J. Nieto, "Quasilinearization and rate of convergence for higher-order nonlinear periodic boundary-value problems," *Journal of Optimization Theory and Applications*, vol. 108, no. 1, pp. 97–107, 2001.

[13] T. Jankowski, "Quadratic approximation of solutions for differential equations with nonlinear boundary conditions," *Computers and Mathematics with Applications*, vol. 47, no. 10-11, pp. 1619–1626, 2004.

[14] Kamar, A. R. Abd-Ellateef, and Z. Drici, "Generalized quasilinearization method for systems of nonlinear differential equations with periodic boundary conditions," *Dynamics of Continuous, Discrete & Impulsive Systems Series A: Mathematical Analysis*, vol. 12, pp. 77–85, 2005.

[15] F. M. Atici and S. G. Topal, "The generalized quasilinearization method and three point boundary value problems on time scales," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 577–585, 2005.

[16] A. Buica, "Quasilinearization method for nonlinear elliptic boundary-value problems," *Journal of Optimization Theory and Applications*, vol. 124, no. 2, pp. 323–338, 2005.

[17] B. Ahmad, "A quasilinearization method for a class of integro-differential equations with mixed nonlinearities," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 997–1004, 2006.

[18] M. Kot and W. M. Schaffer, "Discrete-time growth-dispersal models," *Mathematical Biosciences*, vol. 80, no. 1, pp. 109–136, 1986.

[19] V. B. Mandelzweig and F. Tabakin, "Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs," *Computer Physics Communications*, vol. 141, no. 2, pp. 268–281, 2001.

[20] A. Nerode and W. Kohn, *Models in Hybrid Systems*, vol. 36 of *Lecture Notes in Computer Science*, Springer, Berlin, Germany, 1993.

[21] V. Lakshmikantham and X. Liu, "Impulsive hybrid systems and stability theory," *Dynamic Systems and Applications*, vol. 7, no. 1, pp. 1–9, 1998.

[22] L. M. Hall and S. G. Hristova, "Quasilinearization for the periodic boundary value problem for hybrid differential equation," *Central European Journal of Mathematics*, vol. 2, no. 2, pp. 250–259, 2004.

[23] V. Lakshmikantham, J. V. Devi, and A. S. Vatsala, "Stability in terms of two measures of hybrid systems with partially visible solutions," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 62, no. 8, pp. 1536–1543, 2005.

[24] T. G. Bhaskar, V. Lakshmikantham, and J. V. Devi, "Nonlinear variation of parameters formula for set differential equations in a metric space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5–7, pp. 735–744, 2005.

[25] V. Lakshmikantham and J. Vasundhara Devi, "Hybrid systems with time scales and impulses," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 65, no. 11, pp. 2147–2152, 2006.

[26] B. Ahmad and S. Sivasundaram, "The monotone iterative technique for impulsive hybrid set valued integro-differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 12, pp. 2260–2276, 2006.