

## On the blow-up phenomenon for a generalized Davey–Stewartson system

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The blow-up solutions of the Cauchy problem for a generalized Davey–Stewartson system, which models the wave propagation in a bulk medium made of an elastic material with coupled stresses, are investigated. The mass concentration is established for all the blow-up solutions of the system. The profile of the minimal blow-up solutions as  $t \rightarrow T$  (blow-up time) is discussed in detail in terms of the ground state.

*Keywords:* generalized Davey–Stewartson systems; mass concentration; minimal blow-up solutions; blow-up profile.

### 1. Introduction

The generalized Davey–Stewartson (GDS) system was introduced by Babaoğlu & Erbay (2004) to model the wave propagation in a bulk medium made of an elastic material with coupled stresses. In dimensionless form, the GDS system reads (see Babaoğlu & Erbay, 2004; Babaoğlu *et al.*, 2004)

$$\begin{cases} iu_t + \sigma \Delta u + \chi |u|^2 u + \gamma (\phi_x + \psi_y) u = 0, \\ \phi_{xx} + m_2 \phi_{yy} + n \psi_{xy} = (|u|^2)_x, \\ \lambda \psi_{xx} + m_1 \psi_{yy} + n \phi_{xy} = (|u|^2)_y, \end{cases} \quad (1.1)$$

where  $\chi, \gamma, m_1, m_2, n$  and  $\lambda$  are real constants and  $\sigma = \pm 1$ . The physical constants satisfy  $(m_2 - m_1)(\lambda - 1) = n^2$  with  $\lambda > 1$  and  $m_2 > m_1$ . Here,  $t$  is a non-dimensional time variable,  $x$  and  $y$  are spatial variables,  $u$  is the complex amplitude of the short transverse wave mode in the  $z$ -direction,  $\phi$  and  $\psi$  are the real long longitudinal and the long transverse wave modes in the  $x$ - and  $y$ -directions, respectively. Through the non-linear transformation  $\Phi_x = \phi_x + \psi_y - (1/m_1)|u|^2$ , system (1.1) reduces

to the original Davey–Stewartson (DS) system

$$\begin{cases} iu_t + \sigma(u_{xx} + u_{yy}) + \left(\chi + \frac{\gamma}{m_1}\right) |u|^2 u + \gamma \Phi_x u = 0, \\ \Phi_{xx} + m_1 \Phi_{yy} = \left(1 - \frac{1}{m_1}\right) (|u|^2)_x, \end{cases} \quad (1.2)$$

when  $n = 1 - \lambda = m_1 - m_2$ . System (1.2) is a model equation in the theory of shallow water waves (Davey & Stewartson, 1974).

In addition, we note that when  $\gamma = 0$  and  $\sigma = \chi = 1$ , systems (1.1) and (1.2) reduce to the cubic non-linear Schrödinger equation (NLS)

$$iu_t + \Delta u + \chi |u|^2 u = 0, \quad x \in \mathbb{R}^2, \quad (1.3)$$

which appears in non-linear optics. There is a large volume of the literature on the blow-up solutions and the standing wave of NLS-like equations including the DS system, and for details see, for example, Cazenave (2003), Cipelatti (1992), Cipelatti & Kavian (2001), Gan & Zhang (2008), Ghidaglia & Saut (1990), Guo & Wang (1999), Glassey (1977), Hmidi & Keraani (2005), Li *et al.* (2011), Merle (1993), Merle & Raphaël (2005), Merle & Tsutsumi (1990), Ogawa & Tsutsumi (1990), Ohta (1994), Papanicolaou *et al.* (1994), Sulem & Sulem (1999) and Weinstein (1983, 1986).

The GDS system was classified as elliptic–elliptic–elliptic when  $\sigma = 1$  and the physical parameters  $m_1, m_2$  and  $\lambda$  are all positive in Babaoğlu & Erbay (2004). In this paper, we investigate the Cauchy problem of the purely elliptic GDS system

$$\begin{cases} iu_t + \Delta u + \chi |u|^2 u + \gamma(\phi_x + \psi_y)u = 0, \\ \phi_{xx} + m_2 \phi_{yy} + n \psi_{xy} = (|u|^2)_x, \\ \lambda \psi_{xx} + m_1 \psi_{yy} + n \phi_{xy} = (|u|^2)_y, \\ u(0, x) = u_0(x), \end{cases} \quad (1.4)$$

where  $u = u(t, x): [0, T) \times \mathbb{R}^2 \rightarrow \mathbb{C}$ ,  $\phi = \phi(t, x), \psi = \psi(t, x): [0, T) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $0 < T \leq \infty$ ,  $i = \sqrt{-1}$ , and  $\Delta$  is the Laplace operator on  $\mathbb{R}^2$ .

Using  $\mathcal{F}(f) = \hat{f}$  as the Fourier transform in  $\mathbb{R}^2$  and solving the second and third equations in (1.4), we obtain

$$\hat{\phi} = \frac{i\xi_1}{\delta} (n\xi_2^2 - \lambda\xi_1^2 - m_1\xi_2^2) |\hat{u}|^2, \quad \hat{\psi} = \frac{i\xi_2}{\delta} (n\xi_1^2 - \xi_1^2 - m_2\xi_2^2) |\hat{u}|^2,$$

where  $\xi = (\xi_1, \xi_2)$  are the Fourier transform variables and  $\delta = (\lambda\xi_1^2 + m_2\xi_2^2)(\xi_1^2 + m_1\xi_2^2)$ . Computing the Fourier transform of  $(\phi_x + \psi_y)$  gives

$$\mathcal{F}(\phi_x + \psi_y) = \alpha(\xi) \widehat{|u|^2}, \quad (1.5)$$

where  $\alpha(\xi)$  has the form

$$\alpha(\xi) = \frac{\lambda\xi_1^4 + (1 + m_1 - 2n)\xi_1^2\xi_2^2 + m_2\xi_2^4}{(\lambda\xi_1^2 + m_2\xi_2^2)(\xi_1^2 + m_1\xi_2^2)}, \quad (1.6)$$

and has the following important properties (see Babaoğlu *et al.*, 2004):

- (A1) it is homogeneous of degree 0,

(A2)  $0 \leq \alpha(\xi) \leq \alpha_M = \max\{1, 1/m_1\}$ .

Now, we define the operator  $\mathcal{K}$  as

$$\mathcal{K}(v) = \mathcal{F}^{-1}(\alpha(\xi)\hat{v}), \tag{1.7}$$

from which system (1.4) can be reduced to the Cauchy problem

$$\begin{cases} iu_t + \Delta u + \chi|u|^2u + \gamma\mathcal{K}(|u|^2)u = 0, \\ u(0, x) = u_0(x). \end{cases} \tag{1.8}$$

Let us recall the main known facts about the Cauchy problem (1.8).

- (i) *Local well-posedness:* In view of the result of Babaoğlu *et al.* (2004), the Cauchy problem (1.8) is locally well-posed in the energy space  $H^1(\mathbb{R}^2)$ , that is, for any  $u_0 \in H^1(\mathbb{R}^2)$ , there exists a unique solution  $u(t, x)$  of the Cauchy problem (1.8) in  $C([0, T], H^1(\mathbb{R}^2))$  for some  $T \in (0, \infty)$  maximal existence time), either  $T = \infty$  or else  $T < \infty$  and  $\lim_{t \rightarrow T} \|u(t)\|_{H^1(\mathbb{R}^2)} = \infty$ . Moreover, the solution  $u(t)$  satisfies the conservation laws of mass ( $L^2$  norm) and energy

$$\int_{\mathbb{R}^2} |u(t)|^2 dx = \int_{\mathbb{R}^2} |u_0|^2 dx, \tag{1.9}$$

$$\mathcal{E}(u(t)) = \mathcal{E}(u_0), \tag{1.10}$$

for all  $t \in [0, T)$ , where

$$\mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{1}{4} \int_{\mathbb{R}^2} (\chi|u|^4 + \gamma|u|^2 \mathcal{K}(|u|^2)) dx. \tag{1.11}$$

- (ii) *Standing waves:* Due to the focusing effect of the non-linearity in (1.8), there exist standing wave solutions of the form

$$u(t, x) = e^{i\omega t} \phi(x),$$

where  $\omega > 0$  and  $\phi(x)$  is a non-trivial solution of the elliptic semi-linear problem

$$-\Delta\phi + \omega\phi - (\chi\phi^2 + \gamma\mathcal{K}(|\phi|^2))\phi = 0, \quad x \in \mathbb{R}^2. \tag{1.12}$$

It is shown in Eden & Erbay (2006) that (1.12) admits a non-trivial solution  $R_\omega$  which is the minimizer of the variational problem

$$d := \inf_{u \in H^1(\mathbb{R}^2)} J(u), \tag{1.13}$$

where

$$J(u) = \frac{\|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2}{\chi \|u\|_{L^4(\mathbb{R}^2)}^4 + \gamma \int_{\mathbb{R}^2} |u|^2 \mathcal{K}(|u|^2) dx}. \tag{1.14}$$

We note that such a solution is a ground state solution of (1.12).

- (iii) *Blow-up*: The existence of finite time blow-up solutions follows from the classical virial identity (see Babaoğlu *et al.*, 2004; Eden & Erbay, 2006; Eden *et al.*, 2008, 2009)

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 = 16\mathcal{E}(u_0),$$

which implies finite time blow-up for initial data  $u_0 \in \Sigma := H^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, |x|^2 dx)$ .

On the other hand, it is shown in Eden & Erbay (2006) that the solution of (1.8) is globally well-posed in  $H^1(\mathbb{R}^2)$  under the condition

$$\|u_0\|_{L^2(\mathbb{R}^2)} < N_c := \|R_\omega\|_{L^2(\mathbb{R}^2)}, \quad (1.15)$$

where  $R_\omega$  is the ground state of (1.12) and  $N_c$  is a universal constant (see (3.10)). In other words, Eden & Erbay (2006) established a necessary condition for the blow-up that the initial mass is larger than a critical values ( $\|u_0\|_{L^2(\mathbb{R}^2)} \geq N_c$ ). Moreover, the sharpness of this criterion follows from the existence of the pseudo-conformal symmetry (see Cipelatti & Kavian, 2001; Eden *et al.*, 2006; Ozawa, 1992). If  $u(t, x)$  solves (1.8), then so does  $[\mathcal{C}u](t, x)$ , which is defined by

$$[\mathcal{C}u](t, x) = (T - t)^{-1} \exp\left(\frac{-i|x|^2}{4(T - t)}\right) u\left(\frac{1}{T - t}, \frac{x}{T - t}\right),$$

where  $T > 0$ . By applying this transformation to the solitary wave  $u(t, x) = e^{it}R(x)$ , we obtain the blow-up solution with critical mass. More precisely, one has

$$\begin{aligned} [\mathcal{C}u](t, x) &= (T - t)^{-1} \exp\left(\frac{-i|x|^2}{4(T - t)}\right) u\left(\frac{1 + t}{T - t}, \frac{x}{T - t}\right) \\ &= (T - t)^{-1} \exp\left(\frac{-i|x|^2}{4(T - t)} + \frac{i}{T - t}\right) R\left(\frac{x}{T - t}\right). \end{aligned}$$

Simple calculation yields

$$\begin{aligned} \|\nabla \mathcal{C}u\|_{L^2(\mathbb{R}^2)}^2 &= \frac{\|\nabla R\|_{L^2(\mathbb{R}^2)}^2}{(T - t)^2} + \|xR\|_{L^2(\mathbb{R}^2)}^2 \longrightarrow \infty \quad \text{as } t \rightarrow T, \\ [\mathcal{C}u](0, x) &= \frac{1}{T} \exp\left(\frac{-i|x|^2}{4T} + \frac{i}{T}\right) R\left(\frac{x}{T}\right) \quad \text{and} \quad \|\mathcal{C}u\|_{L^2(\mathbb{R}^2)} = N_c := \|R_\omega\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Such a solution is often called minimal blow-up solution since the solution  $u(t, x)$  exists globally in time provided that  $\|u_0\|_{L^2(\mathbb{R}^2)} < N_c$ .

The above results show that the ground state solution  $R$  of (1.12) plays an important role in the study of the existence and non-existence of blow-up solutions. However, in the papers by Babaoğlu *et al.* (2004), Eden & Erbay (2006), Eden *et al.* (2008) and Eden *et al.* (2009), the proofs are based on a virial-type argument and provide no insight into the description of the singularity formation. It would be of considerable interest to extend the analysis of singularity formation for solutions to (1.8). The main aim of the present paper is to investigate the blow-up profile in terms of the ground state. More precisely, the mass concentration property is proved (Theorem 4.1). The limiting behaviour and the blow-up rate of the minimal blow-up solutions are established (Theorems 5.1 and 5.2).

The rest of the paper is organized as follows. In Section 2, some preliminaries are stated. In Section 3, the variational character of the ground state is investigated. In Section 4, the mass concentration is established based upon the concentration-compactness argument. In Section 5, the profiles and blow-up rate of minimal blow-up solutions are explored in terms of the ground state solution.

## 2. Preliminaries

Firstly, we recall some properties on the singular integral operator  $\mathcal{K}(\cdot)$ , which is defined by (1.7).

LEMMA 2.1 (Eden & Erbay, 2006) (i)  $\mathcal{K} : L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$  is bounded for all  $p \in (1, \infty)$  and  $\|\mathcal{K}(f)\|_{L^2(\mathbb{R}^2)}^2 \leq \alpha_M \|f\|_{L^2(\mathbb{R}^2)}^2$ .

(ii)  $\forall s \in \mathbb{R}, f \in H^s(\mathbb{R}^2)$ , then  $\mathcal{K}(f) \in H^s(\mathbb{R}^2)$ .

(iii) If  $f \in W^{m,p}(\mathbb{R}^2)$ , then  $\mathcal{K}(f) \in W^{m,p}(\mathbb{R}^2)$ , moreover

$$\partial_k \mathcal{K}(f) = \mathcal{K}(\partial_k f), \quad k = 1, 2.$$

(iv) The operator  $\mathcal{K}$  preserves the operations

- (1) translation:  $\mathcal{K}(\varphi(\cdot + y))(x) = \mathcal{K}(\varphi)(x + y), y \in \mathbb{R}^2$ .
- (2) dilatation:  $\mathcal{K}(\varphi(\lambda \cdot))(x) = \mathcal{K}(\varphi)(\lambda x), \lambda > 0$ .
- (3) conjugation:  $\overline{\mathcal{K}(\varphi)} = \mathcal{K}(\bar{\varphi})$ , where  $\bar{\varphi}$  is the complex conjugate of  $\varphi$ .

Making use of Lemma 2.1, we have the following remark.

REMARK 2.1 From the Parseval identity,

$$\int_{\mathbb{R}^2} f \cdot \bar{g} \, dx = \int_{\mathbb{R}^2} \mathcal{F}[f] \overline{\mathcal{F}[g]} \, d\xi, \quad d\xi = (2\pi)^{-2} \, dx,$$

and the definition of  $\mathcal{K}$ , we have

$$\int_{\mathbb{R}^2} |\varphi|^2 \mathcal{K}(|\varphi|^2) \, dx = \int_{\mathbb{R}^2} \mathcal{F}[|\varphi|^2] \alpha(\xi) \mathcal{F}[|\varphi|^2] \, dx = \int_{\mathbb{R}^2} \alpha(\xi) |\mathcal{F}[|\varphi|^2]|^2 \, d\xi.$$

Hence,

$$0 < \int_{\mathbb{R}^2} |\varphi|^2 \mathcal{K}(|\varphi|^2) \, dx \leq \alpha_M \int_{\mathbb{R}^2} |\varphi|^4 \, dx. \tag{2.1}$$

Now, we recall some compactness lemmas.

LEMMA 2.2 (Cazenave, 2003; Lions, 1984) If  $\mu > 0$ , and  $u_k$  is a bounded sequence of  $H^1(\mathbb{R}^2)$  such that

$$\int_{\mathbb{R}^2} |u_k|^2 \, dx = \mu,$$

then there exists a subsequence  $u_{k_j}$ , for which one of the following properties holds.

*Compactness:* There exists a sequence  $y_j$  in  $\mathbb{R}^2$ . For any  $\varepsilon > 0$ , there exists  $r < \infty$  such that

$$\int_{|x-y_j| \leq r} |u_{k_j}(x)|^2 dx \geq \mu - \varepsilon.$$

*Vanishing:* For all  $r < \infty$ , it holds that

$$\limsup_{j \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{|x-y| \leq r} |u_{k_j}(x)|^2 dx = 0.$$

*Dichotomy:* There exists a constant  $\alpha \in (0, \mu)$  and two sequences  $u_j^1, u_j^2 \subset H^1(\mathbb{R}^2)$ , with compact and disjoint supports, such that

$$\begin{aligned} \|u_j^1\|_{L^2(\mathbb{R}^2)} &\rightarrow \alpha, \\ \|u_j^2\|_{L^2(\mathbb{R}^2)} &\rightarrow (\mu - \alpha), \\ \|u_{k_j} - u_j^1 - u_j^2\|_{H^1(\mathbb{R}^2)} &\rightarrow 0, \\ \|u_{k_j} - u_j^1 - u_j^2\|_{L^p} &\rightarrow 0 \quad \text{for } 2 \leq p < \infty, \\ \text{distance}(\text{supp } u_j^1, \text{supp } u_j^2) &\rightarrow \infty. \end{aligned}$$

LEMMA 2.3 (Brezis & Lieb, 1984) Let  $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ ,  $\|\nabla f\|_{L^2(\mathbb{R}^2)} \leq C$  and  $\mu(|f| > \varepsilon) \geq \delta > 0$ . Then, for some constant  $\alpha = \alpha(C, \delta, \varepsilon)$ , there exists  $y \in \mathbb{R}^2$  such that

$$\mu\left(B(0, 1) \cap \left\{|f(x+y)| > \frac{\varepsilon}{2}\right\}\right) > \alpha.$$

LEMMA 2.4 (Lieb, 1983) Let  $1 < p < \infty$  and let  $f_j$  be a uniformly bounded sequence of functions in  $W^{1,p}$  such that  $\mu(|f_j| > \eta) \geq C$  for some positive constants  $C$  and  $\eta$ . Then there exists a sequence  $y_j \in \mathbb{R}^2$  such that

$$f_j(\cdot + y_j) \rightharpoonup f \neq 0 \quad \text{weakly in } W^{1,p}(\mathbb{R}^2).$$

LEMMA 2.5 Let  $v_n \in H^1(\mathbb{R}^2)$  such that

$$\int_{\mathbb{R}^2} |v_n|^2 dx \leq c_1, \quad \int_{\mathbb{R}^2} |\nabla v_n|^2 dx \leq c_2, \quad \int_{\mathbb{R}^2} \chi |v_n|^4 + \gamma |v_n|^2 \mathcal{K}(|v_n|^2) dx \geq c_3.$$

Then there exists a positive constant  $c_4 = c_4(c_1, c_2, c_3)$  and a sequence  $\{x_n\}$  such that

$$\int_{|x-x_n| < 1} |v_n|^2 dx > c_4. \tag{2.2}$$

*Proof.* It follows from (2.1) that

$$(\chi + |\gamma|\alpha_M) \int_{\mathbb{R}^2} |v_n|^4 dx \geq \int_{\mathbb{R}^2} \chi |v_n|^4 + \gamma |v_n|^2 \mathcal{K}(|v_n|^2) dx \geq c_3.$$

Using the Sobolev inequality yields

$$\int_{\mathbb{R}^2} |v_n|^6 dx \leq c \|v_n\|_{H^1(\mathbb{R}^2)}^6 \leq c(c_1 + c_2)^3 := c_5.$$

Choosing  $\varepsilon = \min\{\sqrt{c_3/4(\chi + |\gamma|\alpha_M)c_1}, \sqrt{c_3/4(\chi + |\gamma|\alpha_M)c_5}\}$ , we have

$$\begin{aligned} \frac{c_3}{\chi + |\gamma|\alpha_M} &\leq \int_{\mathbb{R}^2} |v_n|^4 dx \\ &= \int_{|v_n| \leq \varepsilon} |v_n|^4 dx + \int_{\varepsilon \leq |v_n| \leq 1/\varepsilon} |v_n|^4 dx + \int_{|v_n| \geq 1/\varepsilon} |v_n|^4 dx \\ &\leq \frac{c_3}{4(\chi + |\gamma|\alpha_M)c_1} \int_{|v_n| \leq \varepsilon} |v_n|^2 dx + \int_{\varepsilon \leq |v_n| \leq 1/\varepsilon} |v_n|^4 dx \\ &\quad + \frac{c_3}{4(\chi + |\gamma|\alpha_M)c_5} \int_{|v_n| \geq 1/\varepsilon} |v_n|^6 dx \\ &\leq \frac{c_3}{2(\chi + |\gamma|\alpha_M)} + \int_{\varepsilon \leq |v_n| \leq 1/\varepsilon} |v_n|^4 dx \\ &\leq \frac{c_3}{2(\chi + |\gamma|\alpha_M)} + \left(\frac{1}{\varepsilon}\right)^4 \mu\{|v_n| \geq \varepsilon\}, \end{aligned}$$

where  $\mu$  is the Lebesgue measure. The above inequality implies that for all  $n$ , there exists a constant  $C > 0$  such that

$$\mu\{|v_n| \geq \varepsilon\} \geq C > 0. \tag{2.3}$$

From Lemma 2.3, there is a constant  $\alpha(C, \varepsilon)$  (independent of  $n$ ) and a subsequence  $y_n$  such that

$$\mu\left(\{|x| < 1\} \cap \left\{|v_n(x + y_n)| > \frac{\varepsilon}{2}\right\}\right) > \alpha.$$

Thus,

$$\int_{|x| \leq 1} |v_n(x + y_n)|^2 dx \geq \left(\frac{\varepsilon}{2}\right)^2 \alpha,$$

which implies (2.2) with  $c_4 = (\varepsilon/2)^2 \alpha$ . □

### 3. The variational characterization of the ground state

For (1.12), we define the function sets

$$\begin{aligned} \mathcal{X}_\omega &= \text{the set of solutions for (1.12)} \\ &= \{u \in H^1(\mathbb{R}^2) : S'_\omega(u) = 0, u \neq 0\}, \\ \mathcal{G}_\omega &= \text{the set of ground states for (1.12)} \\ &= \{v \in \mathcal{X}_\omega : S_\omega(v) \leq S_\omega(u), \text{ for all } u \in \mathcal{X}_\omega\}, \end{aligned}$$

where the action functional  $S_\omega$  is defined by

$$S_\omega(\phi) = \frac{1}{2} \|\nabla\phi\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \omega \|\phi\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{4} \int_{\mathbb{R}^2} (\chi|\phi|^4 + \gamma|\phi|^2 \mathcal{K}(|\phi|^2)) \, dx. \tag{3.1}$$

From [Eden & Erbay \(2006\)](#), we have two identities for the solutions of (1.12). For any  $\phi_\omega \in \mathcal{X}_\omega$ , one has

$$\|\nabla\phi_\omega\|_{L^2(\mathbb{R}^2)}^2 + \omega \|\phi_\omega\|_{L^2(\mathbb{R}^2)}^2 - \int_{\mathbb{R}^2} (\chi|\phi_\omega|^4 + \gamma|\phi_\omega|^2 \mathcal{K}(|\phi_\omega|^2)) \, dx = 0, \tag{3.2}$$

and the Pohozaev identity

$$\omega \|\phi_\omega\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{2} \int_{\mathbb{R}^2} (\chi|\phi_\omega|^4 + \gamma|\phi_\omega|^2 \mathcal{K}(|\phi_\omega|^2)) \, dx = 0. \tag{3.3}$$

For any  $\phi_\omega \in \mathcal{X}_\omega$ , equalities (3.2) and (3.3) imply that

$$\|\nabla\phi_\omega\|_{L^2(\mathbb{R}^2)}^2 = \omega \|\phi_\omega\|_{L^2(\mathbb{R}^2)}^2, \tag{3.4}$$

$$\int_{\mathbb{R}^2} (\chi|\phi_\omega|^4 + \gamma|\phi_\omega|^2 \mathcal{K}(|\phi_\omega|^2)) \, dx = 2\omega \|\phi_\omega\|_{L^2(\mathbb{R}^2)}^2. \tag{3.5}$$

It follows from (3.1), (3.4) and (3.5) that

$$\mathcal{E}(\phi_\omega) = 0, \quad \forall \phi_\omega \in \mathcal{X}_\omega, \tag{3.6}$$

$$J(\phi_\omega) = S_\omega(\phi_\omega) = \frac{\|\phi_\omega\|_{L^2(\mathbb{R}^2)}^2}{2}, \tag{3.7}$$

where  $\mathcal{E}(\varphi)$  and  $J(u)$  are defined by (1.11) and (1.14), respectively.

Subsequently, one has

$$R_\omega \in \mathcal{G}_\omega \Leftrightarrow \begin{cases} R_\omega \in \mathcal{X}_\omega, \\ \|R_\omega\|_{L^2(\mathbb{R}^2)} \leq \|v\|_{L^2(\mathbb{R}^2)}, \quad \forall v \in \mathcal{X}_\omega. \end{cases} \tag{3.8}$$

Moreover, for any ground solution  $R(x) \in \mathcal{G}_\omega|_{\omega=1}$ ,  $x_0 \in \mathbb{R}^2$ ,  $\gamma \in \mathbb{R}$ , direct computation yields

$$R_\omega(x) = \omega^{1/2} e^{i\gamma} R(\omega^{1/2}(x - x_0)) \in \mathcal{G}_\omega \tag{3.9}$$

and

$$\|R_\omega\|_{L^2(\mathbb{R}^2)} = \|R\|_{L^2(\mathbb{R}^2)} =: N_c. \tag{3.10}$$

**REMARK 3.1** For any ground state solution  $R_\omega \in \mathcal{G}_\omega$ , it follows from (3.4), (3.8) and (3.10) that the  $L^2$  norm of the ground ( $\|R_\omega\|_{L^2(\mathbb{R}^2)}$ ) is unique and independent of  $\omega$  and  $\|\nabla R_\omega\|_{L^2(\mathbb{R}^2)}$  is a positive constant depending only on  $\omega$ , although it is an open question whether uniqueness (modulo phase and translation) of the ground states to (1.12) holds, i.e. we have

$$\mathcal{G}_\omega = \{e^{i\gamma} R_\omega(\cdot - y) : \gamma \in \mathbb{R}, y \in \mathbb{R}^2\},$$

for some fixed  $R_\omega \in \mathcal{G}_\omega$ .



It follows from (2.1) and the Gagliardo–Nirenberg inequality that

$$\int_{\mathbb{R}^2} \chi |\varphi|^4 + \gamma |\varphi|^2 \mathcal{K}(|\varphi|^2) \, dx \leq (\chi + |\gamma| \alpha_M) \int_{\mathbb{R}^2} |\varphi|^4 \, dx \leq C \|\varphi\|_{L^2(\mathbb{R}^2)}^2 \|\nabla \varphi\|_{L^2(\mathbb{R}^2)}^2.$$

The following proposition gives the best constant  $C$ .

**PROPOSITION 3.1** Let  $\chi < \min\{-\gamma \alpha_M, 0\}$ , where  $\alpha_M$  is a constant appearing in (A2). For any  $u \in H^1(\mathbb{R}^2)$ , it holds that

$$\chi \int_{\mathbb{R}^2} |u|^4 \, dx + \gamma \int_{\mathbb{R}^2} |u|^2 \mathcal{K}(|u|^2) \leq C_{\text{opt}} \|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2, \tag{3.11}$$

with

$$C_{\text{opt}} = \frac{2}{N_c^2}, \tag{3.12}$$

where  $N_c$  is defined by (3.10).

Proposition 3.1 is an immediate result of the following lemma and remark.

**LEMMA 3.1 (Eden & Erbay, 2006)** Let  $\omega > 0$  and  $\chi < \min\{-\gamma \alpha_M, 0\}$ , where  $\alpha_M$  is a constant appearing in (A2). Then, for the minimizing problem (1.14), it holds that

$$d = \min_{u \in H^1(\mathbb{R}^2)} J(u) = J(R_\omega) = \frac{N_c^2}{2},$$

where  $N_c$  is a universal constant defined by (3.10).

**REMARK 3.2** We note that the solution  $R_\omega$  of (1.12) is a ground state solution if it solves the minimizing problem (1.13). Indeed, for any  $R_\omega \in \mathcal{X}_\omega$ , equality (3.7) holds true. It follows that  $R_\omega$  is a minimizer of (1.13) if and only if it satisfies the minimizing problem

$$S(R_\omega) = \min_{\phi_\omega \in \mathcal{X}_\omega} S_\omega(\phi_\omega). \tag{3.13}$$

Using Proposition 3.1, we obtain several lemmas, which are useful in the subsequent sections.

From the argument of Weinstein (1983), Proposition 3.1 implies immediately the following lemma.

**LEMMA 3.2** For any  $f \in H^1(\mathbb{R}^2)$ , it holds that

$$\left[ 1 - \left( \frac{\|f\|_{L^2(\mathbb{R}^2)}}{N_c} \right)^2 \right] \|\nabla f\|_{L^2(\mathbb{R}^2)}^2 \leq 2\mathcal{E}(f), \tag{3.14}$$

where  $N_c$  is defined by (3.10) and  $\mathcal{E}$  is defined by (1.11).

Proceeding as for the Banica (2004, Lemma 2.1), we establish the following lemma.

**LEMMA 3.3** Let  $\theta$  be a real-valued function on  $\mathbb{R}^2$  and  $v \in H^1(\mathbb{R}^2)$  with  $\|v\|_{L^2(\mathbb{R}^2)} \leq N_c := \|R_\omega\|_{L^2(\mathbb{R}^2)}$ . Then

$$\left| \Im \int_{\mathbb{R}^2} \bar{v}(x) \nabla v \nabla \theta(x) \, dx \right| \leq \left( 2\mathcal{E}(v) \int_{\mathbb{R}^2} |v(x)|^2 |\nabla \theta(x)|^2 \, dx \right)^{1/2}. \tag{3.15}$$

*Proof.* It follows from (3.14) and  $\|v\|_{L^2(\mathbb{R}^2)} \leq N_c$  that

$$\mathcal{E}(e^{i\tau\theta}v) \geq 0,$$

for all real number  $\tau$ . On the other hand, one has

$$\mathcal{E}(e^{i\tau\theta}v) = \tau^2 \int_{\mathbb{R}^2} |v|^2 |\nabla\theta|^2 dx - \tau \int_{\mathbb{R}^2} \Im(v\nabla\bar{v})\nabla\theta dx + \mathcal{E}(v).$$

Thus, the discriminant of the equation for  $\tau$  must be negative or null and the desired inequality follows.  $\square$

Define the variational problem

$$I(\alpha) \equiv \min\{\mathcal{E}(f) \mid f \in H^1(\mathbb{R}^2), \|f\|_{L^2(\mathbb{R}^2)} = \alpha\}. \quad (3.16)$$

For  $I(\alpha)$ , we have the following lemma, whose proof relies heavily on the techniques presented in Weinstein (1986).

LEMMA 3.4 (a)  $I(\alpha) = 0$ , or  $I(\alpha) = -\infty$ .

(b) If  $\alpha < N_c$ , then  $I(\alpha) = 0$ , and any minimizing sequence converges to zero weakly in  $H^1(\mathbb{R}^2)$ .

*Proof.* (a) Let  $\lambda \in \mathbb{R} \setminus \{0\}$  and  $f^\lambda(x) = \lambda f(\lambda x)$ . Then for any  $f \in H^1(\mathbb{R}^2)$ , it holds that  $\|f^\lambda\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)}$  and  $\mathcal{E}(f^\lambda) = \lambda^2 \mathcal{E}(f)$ . This implies that  $I(\alpha) = 0$ , or  $I(\alpha) = -\infty$ .

Part (b) follows from (a) and Lemma 3.2.  $\square$

Using Remark 3.1, we can investigate the variational characterization of the ground state for (1.12). Let us define the set  $\mathcal{G}$  by

$$\mathcal{G} = \bigcup_{\omega \in \mathbb{R}^+} \mathcal{G}_\omega,$$

and a constrained minimization problem by

$$\mathcal{I}(N_c) := \inf\{\mathcal{E}(f) \mid f \in H^1(\mathbb{R}^2), \|f\|_{L^2(\mathbb{R}^2)} = N_c\}, \quad (3.17)$$

where the functional  $\mathcal{E}$  is defined by (1.11).

PROPOSITION 3.2 The following statements are equivalent:

- (i)  $u \in \mathcal{G}$  with  $\mathcal{G} = \bigcup_{\omega \in \mathbb{R}^+} \mathcal{G}_\omega$ .
- (ii) There exists  $R \in \mathcal{G}_1 := \mathcal{G}_\omega|_{\omega=1}$  such that  $u = \omega^{1/2} e^{i\gamma} R(\omega^{1/2}(x - x_0))$ .
- (iii)  $\mathcal{E}(u) = 0$  and  $\|u\|_{L^2(\mathbb{R}^2)} = N_c := \|R_\omega\|_{L^2(\mathbb{R}^2)}$ .
- (iv)  $u$  solves the minimization problem (3.17).

*Proof.* **Step 1:** (i)  $\Leftrightarrow$  (ii).

It follows from (3.8) and (3.10) that

$$u \in \mathcal{G}_\omega \Leftrightarrow \begin{cases} u \in \mathcal{X}_\omega, \\ \|u\|_{L^2(\mathbb{R}^2)} = N_c \end{cases} \tag{3.18}$$

and

$$u \in \mathcal{G}_\omega \Leftrightarrow \text{there exists } R \in \mathcal{G}_1 \text{ such that } u = \omega^{1/2} e^{i\gamma} R(\omega^{1/2}(x - x_0)). \tag{3.19}$$

**Step 2:** (ii)  $\Rightarrow$  (iii).

If (ii) holds true, direct computation yields  $\|u\|_{L^2(\mathbb{R}^2)} = N_c$ . Using (3.6), one obtains

$$\mathcal{E}(u) = \omega \mathcal{E}(R) = 0.$$

**Step 3:** (iii)  $\Rightarrow$  (iv).

It follows from Lemma 3.2 that

$$\mathcal{E}(f) \geq 0, \quad \text{if } \|f\|_{L^2(\mathbb{R}^2)} \leq N_c. \tag{3.20}$$

Hence,  $\mathcal{I}(N_c) = 0$ .

**Step 4:** (iv)  $\Rightarrow$  (i).

If  $u$  is a minimizer of the variational problem of (3.17), it satisfies the Euler–Lagrange equation (1.12). So  $u \in \mathcal{X}_\omega$  for some  $\omega > 0$ . Moreover, from (3.17) and (3.18), we know  $u \in \mathcal{G}_\omega \subset \mathcal{G}$ .  $\square$

#### 4. Mass concentration of the blow-up solution

In this section, we study the mass concentration of the blow-up solutions for the Cauchy problem (1.8) in  $\mathbb{R}^2$ .

**THEOREM 4.1** Suppose that  $\chi < \min\{-\gamma\alpha_M, 0\}$ , where  $\alpha_M$  is a constant appearing in (A2). Let  $u(t, x)$  be the solution of the Cauchy problem (1.8) which blows up in finite time  $T$ . Then, there is a function  $t \rightarrow x(t)$  such that for any  $r > 0$ ,

$$\liminf_{t \rightarrow T} \|u(t)\|_{L^2(B(x(t), r))} \geq N_c := \|R_\omega\|_{L^2(\mathbb{R}^2)}.$$

As a direct consequence, we have the following corollary, which is also an important result presented in [Eden & Erbay \(2006\)](#).

**COROLLARY 4.1** If  $\|u_0\|_{L^2(\mathbb{R}^2)} < N_c$ , then the solution  $u(t)$  of the Cauchy problem (1.8) exists globally.

Theorem 4.1 follows from the following proposition.

**PROPOSITION 4.1** Let  $\{u_n\}$  satisfy  $\|u_n\|_{L^2(\mathbb{R}^2)} \leq c_1$ ,  $\mathcal{E}(u_n) \leq c_2$  and  $\|\nabla u_n\|_{L^2(\mathbb{R}^2)} \rightarrow \infty$  as  $n \rightarrow \infty$ . Then, for all  $r > 0$ , there exists  $\{x_n\}$  such that

$$\liminf_{n \rightarrow \infty} \frac{\|u_n\|_{L^2(B(x_n, r))}}{N_c} \geq 1.$$

*Proof.* We prove it by contradiction.

Suppose there are  $r_0 > 0$ ,  $\gamma_0 > 0$  and a sequence  $\{u_n\}$  such that

$$\sup_{x \in \mathbb{R}^2} \int_{|x-y| < r_0} |u_n(y)|^2 dy \leq N_c - \gamma_0.$$

Considering the scaling

$$U_n(x) = \lambda_n^{-1} u_n(\lambda_n^{-1} x),$$

with  $\lambda_n = \|\nabla u_n\|_{L^2(\mathbb{R}^2)}$ , we have

$$\|U_n\|_{L^2(\mathbb{R}^2)} = \|u_n\|_{L^2(\mathbb{R}^2)} \leq c_1, \tag{4.1}$$

$$\|\nabla U_n\|_{L^2(\mathbb{R}^2)} = 1, \tag{4.2}$$

$$\liminf_{n \rightarrow \infty} \mathcal{E}(U_n) = \liminf_{n \rightarrow \infty} \frac{\mathcal{E}(u_n)}{\lambda_n^2} = 0$$

and

$$\sup_{x \in \mathbb{R}^2} \int_{|x-y| < r} |U_n(x)|^2 dx \leq N_c - \gamma_0, \quad 0 < r < \lambda_n r_0.$$

Extracting a subsequence  $U_n$ , we have

$$\int_{\mathbb{R}^2} (\chi |U_n|^4 + \gamma |U_n|^2 \mathcal{K}(|U_n|^2)) dx \rightarrow 2 \quad \text{for large } n \tag{4.3}$$

and

$$\liminf_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^2} \int_{|x-y| < r} |U_n(y)|^2 dy \leq N_c - \gamma_0, \quad \forall r > 0. \tag{4.4}$$

From (4.1–4.3), Lemmas 2.2 and 2.5, we have the dichotomy

$$U_n = U_n^1 + \tilde{U}_n^1,$$

such that, for a sequence  $\{x_n^1\}$  and some  $\varphi_1 \in H^1(\mathbb{R}^2)$ ,

$$U_n^1(x_n^1 + \cdot) \rightharpoonup \varphi_1 \quad \text{weakly in } H^1(\mathbb{R}^2), \text{ locally (strongly) in } L^4(\mathbb{R}^2) \quad \text{and} \quad L^2(\mathbb{R}^2)$$

$$\int_{|x-y| < 1} |U_n^1(x)|^2 dx \geq \gamma_1,$$

where  $\gamma_1$  is a positive constant depending only on  $c_1$ .

On the other hand, from (4.4), for all  $r > 0$  we have

$$\liminf_{n \rightarrow \infty} \int_{|x-x_n^1| < r} |U_n^1(x)|^2 \leq N_c - \gamma_0.$$

By usual the technique of concentration compactness method, we can find a suitable choice of  $U_n^1$  such that

$$\gamma_1 \leq \|\varphi_1\|_{L^2(\mathbb{R}^2)}^2 = \lim_{n \rightarrow \infty} \|U_n^1\|_{L^2(\mathbb{R}^2)}^2 \leq N_c - \gamma_0 \tag{4.5}$$

and

$$\|U_n^1\|_{L^2(\mathbb{R}^2)}^2 + \|\tilde{U}_n^1\|_{L^2(\mathbb{R}^2)}^2 - \|U_n\|_{L^2(\mathbb{R}^2)}^2 \rightarrow 0. \tag{4.6}$$

Using Lemma 3.2, we have

$$\mathcal{E}(\varphi_1) > 0. \tag{4.7}$$

On the other hand,

$$\begin{aligned} \mathcal{E}(\varphi_1) + \liminf_{n \rightarrow \infty} \mathcal{E}(\tilde{U}_n^1) &\leq \liminf_{n \rightarrow \infty} (\mathcal{E}(U_n^1) + \mathcal{E}(\tilde{U}_n^1)) \\ &\leq \liminf_{n \rightarrow \infty} \mathcal{E}(U_n) = 0. \end{aligned} \tag{4.8}$$

Therefore, by Lemma 3.2 and (4.7), it holds that

$$\liminf_{n \rightarrow \infty} \mathcal{E}(\tilde{U}_n^1) \leq -\mathcal{E}(\varphi_1) < 0.$$

Extracting a subsequence (still labelled by  $\tilde{U}_n^1$ ), we have

$$\|\tilde{U}_n^1\|_{L^2(\mathbb{R}^2)}^2 \rightarrow c_1^1 \leq c_1 - \gamma_1 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \mathcal{E}(\tilde{U}_n^1) < 0. \tag{4.9}$$

Redefining the subsequence

$$\lambda_n = \|\nabla \tilde{U}_n^1\|_{L^2(\mathbb{R}^2)} \quad \text{and} \quad U_n(x) = \lambda_n^{-1} \tilde{U}_n^1(\lambda_n^{-1}x),$$

and extracting a subsequence, we have

$$\|U_n\|_{L^2(\mathbb{R}^2)} \rightarrow c_1^1 \leq c_1 - \gamma_1$$

and

$$\liminf_{n \rightarrow \infty} \left( \sup_{x \in \mathbb{R}^2} \int_{|x-y| < r} |U_n(y)|^2 \right) \leq N_c^2 - \gamma_0, \quad \forall r > 0.$$

Iterating the same procedure, we obtain

$$U_n = U_n^2 + \tilde{U}_n^2,$$

where, for some  $\{x_n^2\}$ ,  $U_n^2$  satisfies

$$\int_{|x-x_n^2| < 1} |U_n^2(y)|^2 \geq \gamma_1.$$

Defining  $p$  as the number such that

$$-p\gamma_1 + c_1 < N_c^2,$$

and applying the same procedure at most  $p$  times, we can find  $j \leq p$  and a function  $U_n^j$  such that for large  $n$

$$\mathcal{E}(\tilde{U}_n^j) < 0 \quad \text{and} \quad \|\tilde{U}_n^j\|_{L^2(\mathbb{R}^2)}^2 < N_c^2.$$

This contradicts Lemma 3.2. □

### 5. Profile of blow-up solutions with minimal mass in $\mathbb{R}^2$

In this section, we investigate the limit profile of the blow-up solution with minimal mass ( $\|u\|_{L^2(\mathbb{R}^2)} = N_c$ ).

Following Weinstein (1986), we have the following proposition.

**PROPOSITION 5.1** Suppose that  $\chi < \min\{-\gamma\alpha_M, 0\}$ , where  $\alpha_M$  is a constant appearing in (A2). Let  $u(t)$  be a solution of the Cauchy problem (1.8) in  $C([0, T], H^1(\mathbb{R}^2))$  such that  $u(t)$  blows up in finite time  $T$ :  $\lim_{t \rightarrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = \infty$ . Set  $\lambda(t) = \|\nabla R\|_{L^2(\mathbb{R}^2)} / \|\nabla u(t)\|_{L^2(\mathbb{R}^2)}$  ( $R \in \mathcal{G}_1$ ) and  $(S_\lambda u)(x, t) = \lambda u(\lambda x, t)$ . If

$$\|u_0\|_{L^2(\mathbb{R}^2)} = N_c := \|R_\omega\|_{L^2(\mathbb{R}^2)},$$

then there are functions  $y(t) \in \mathbb{R}^2$ ,  $\gamma(t) \in \mathbb{R}$  and a ground state  $R \in \mathcal{G}_1$  such that

$$S_{\lambda(t)} u(\cdot + y(t), t) e^{i\gamma(t)} \longrightarrow R(\cdot) \quad \text{in } H^1(\mathbb{R}^2), \quad (t \rightarrow T).$$

*Proof.* We need only show that for any sequence  $t_k \rightarrow T$ , there are subsequences  $t_{k_j}$  and sequence  $\gamma_{k_j}$  such that

$$S_{\lambda(t_{k_j})} u(\cdot + y_{k_j}, t_{k_j}) e^{i\gamma(t_{k_j})} \longrightarrow R(\cdot) \quad \text{in } H^1(\mathbb{R}^2).$$

Letting  $t_k \rightarrow T$ , we choose  $\lambda_k = \lambda(t_k)$  to satisfy

$$\|\nabla S_{\lambda_k} u(\cdot + y_k, t_k)\|_{L^2(\mathbb{R}^2)} = \lambda_k \|\nabla u(\cdot + y_k, t_k)\|_{L^2(\mathbb{R}^2)} = \|\nabla R\|_{L^2(\mathbb{R}^2)}. \tag{5.1}$$

Setting  $u_k \equiv S_{\lambda_k} u(\cdot + y_k, t_k)$  and noticing that  $u(t_k)$  blows up as  $t_k \rightarrow T$ ,  $\lambda_k \rightarrow 0$  and

$$\|u_k\|_{L^2(\mathbb{R}^2)} = \|u(t_k)\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)}, \tag{5.2}$$

we get that  $u_k$  is uniformly bounded in  $H^1(\mathbb{R}^2)$ . Therefore, there is a weakly convergent subsequence  $u_{k_j}$ .

Noticing

$$\mathcal{E}(u_{k_j}) = \lambda_{k_j}^2 \mathcal{E}(u(t_{k_j})) = \lambda_{k_j}^2 \mathcal{E}(u_0) \rightarrow 0, \quad j \rightarrow \infty, \tag{5.3}$$

and the assumption  $\|u_0\|_{L^2(\mathbb{R}^2)} = N_c$ , by (5.2), (5.3) and (3.14), we get that  $u_k$  is a minimizing sequence for the variational problem (3.17).

Now, we study the convergence of the sequence  $u_{k_j}$ . Since the minimizing sequence  $u_k$  is bounded in  $H^1(\mathbb{R}^2)$  with  $\|u_k\|_{L^2(\mathbb{R}^2)} = N_c$ , then there exists a subsequence  $u_{k_j}$ , for which either compactness or vanishing or dichotomy occurs (Lemma 2.2). In order to achieve the compactness, let us prove that the *vanishing* and *dichotomy* cannot occur.

*Vanishing* occurs when

$$\limsup_{j \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{|x-y| \leq M} |u_{k_j}(x)|^2 dx = 0, \quad \forall M < \infty.$$

Lemma 2.5 with (5.1–5.3) implies that *vanishing* cannot occur.

If *dichotomy* occurs, there exist a constant  $\alpha \in (0, N_c)$  and two sequences  $\varphi_j^1 \in H^1(\mathbb{R}^2)$  and  $\varphi_j^2 \in H^1(\mathbb{R}^2)$  of compact support satisfying the following property: for all  $\varepsilon > 0$ , there exists  $j_0 > 0$  such that for  $j > j_0$

$$\|\varphi_j^1\|_{L^2(\mathbb{R}^2)} - \alpha \leq \varepsilon, \quad \|\varphi_j^2\|_{L^2(\mathbb{R}^2)} - (N_c - \alpha) \leq \varepsilon, \tag{5.4}$$

$$\|u_{k_j} - \varphi_j^1 - \varphi_j^2\|_{H^1(\mathbb{R}^2)} \leq \varepsilon,$$

$$\|u_{k_j} - \varphi_j^1 - \varphi_j^2\|_{L^p} \leq \varepsilon \quad \text{for } 2 \leq p < \infty,$$

$$\text{distance}(\text{supp } \varphi_j^1, \text{supp } \varphi_j^2) \rightarrow \infty. \tag{5.5}$$

Then, using the argument in the treatment of *vanishing*, we show that there exist a  $\theta > 0$  and a  $\nu > 0$  such that for all  $j$

$$0 < \nu < \mu\{\theta < |\varphi_j^1|\}. \tag{5.6}$$

Since  $\varphi_j^1$  is a bounded sequence in  $H^1(\mathbb{R}^2)$  satisfying (5.6), we get by Lemma 2.4, that there are a subsequence  $\varphi_{j_r}^1$  and a sequence  $y_r$  such that

$$\varphi_{j_r}^1(\cdot + y_r) \rightharpoonup \varphi \neq 0 \quad \text{in } H^1(\mathbb{R}^2). \tag{5.7}$$

Using (5.4) and (5.5) gives rise to

$$\begin{aligned} 0 = I(N_c) &\geq \liminf_{r \rightarrow \infty} \mathcal{E}(\varphi_{j_r}^1) + \liminf_{r \rightarrow \infty} \mathcal{E}(\varphi_{j_r}^2) \\ &\geq \liminf_{r \rightarrow \infty} \mathcal{E}(\varphi_{j_r}^1) + I(N_c - \alpha \pm \varepsilon) - \beta(\varepsilon) \\ &= \liminf_{r \rightarrow \infty} \mathcal{E}(\varphi_{j_r}^1) - \beta(\varepsilon), \end{aligned}$$

where  $\beta(\varepsilon) \rightarrow 0(\varepsilon \rightarrow 0)$ . Since  $\varepsilon$  is arbitrary, we have

$$\liminf_{r \rightarrow \infty} \mathcal{E}(\varphi_{j_r}^1) \leq 0.$$

Using  $\|\varphi_{j_r}^1\|_{L^2(\mathbb{R}^2)} < N_c$  and Lemma 3.2, we obtain

$$\liminf_{r \rightarrow \infty} \mathcal{E}(\varphi_{j_r}^1) \geq 0.$$

Thus, for any fixed  $n^*$ , it holds that

$$\begin{aligned} 0 = I(N_c) &\geq \liminf_{r \rightarrow \infty} \mathcal{E}(\varphi_{j_r}^1) = \sup_n \inf_{r \geq n} \mathcal{E}(\varphi_{j_r}^1) \\ &\geq \inf_{r \geq n^*} \mathcal{E}(\varphi_{j_r}^1). \end{aligned}$$

We then extract a minimizing subsequence, renamed by  $\varphi_{j_r}^1$ , such that  $\lim_{r \rightarrow \infty} \mathcal{E}(\varphi_{j_r}^1) = 0$ . Using Lemma 3.4 yields

$$\varphi_{j_r}^1 \rightharpoonup 0,$$

which contradicts (5.7).

Hence, the only remaining possibility *compactness* occurs: there exists a sequence  $y_j$  in  $\mathbb{R}^2$  such that

$$\forall \varepsilon > 0, \quad \exists M < \infty, \quad \int_{|x-y_j| \leq M} |u_{k_j}|^2 dx \geq N_c^2 - \varepsilon. \tag{5.8}$$

By (5.8), we obtain

$$\|R\|_{L^2}^2 - \varepsilon \leq \int_{|x-y_j| \leq M} |u_{k_j}|^2 dx \leq \int_{\mathbb{R}^2} |u_{k_j}|^2 dx \leq \|R\|_{L^2}^2. \tag{5.9}$$

For  $u_{k_j}(\cdot + y_j)$  being bounded in  $H^1(\mathbb{R}^2)$ , there exist  $\psi \in H^1(\mathbb{R}^2)$  and a subsequence of  $u_{k_j}$ , denoted by  $u_{k_j}$ , such that

$$u_{k_j}(\cdot + y_j) \rightharpoonup \psi \quad \text{in } H^1(\mathbb{R}^2).$$

Given  $M > 0$ , the embedding  $H^1(\mathbb{R}^2) \hookrightarrow L^2(\{|x| \leq r\})$  is compact and

$$\int_{|x| \leq r} |\psi|^2 dx = \lim_{j \rightarrow \infty} \int_{|x-x_m| \leq r} |u_{k_j}|^2 dx.$$

Making use of (5.9) derives

$$\int_{\mathbb{R}^2} |\psi|^2 dx \geq N_c^2 - \varepsilon,$$

for every  $\varepsilon > 0$  and

$$\int_{\mathbb{R}^2} |\psi|^2 dx = N_c^2.$$

It follows that

$$u_{k_j}(\cdot + y_j) \rightarrow \psi \quad \text{in } L^2(\mathbb{R}^2).$$

Applying the Gagliardo–Nirenberg inequality (3.11) gives rise to

$$u_{k_j}(\cdot + y_j) \rightarrow \psi \quad \text{in } L^4(\mathbb{R}^2).$$

To show  $u_{k_j} \rightarrow \psi$  in  $H^1(\mathbb{R}^2)$ , we only need to show that  $\|\nabla \psi\|_{L^2(\mathbb{R}^2)} = \|\nabla R\|_{L^2(\mathbb{R}^2)}$ .

Using (5.1) and (5.3), we have

$$\begin{aligned} 0 &= \lim_{t \rightarrow T} \mathcal{E}(\psi_{u_{k_j}}) \\ &= \frac{1}{2} \|\nabla R\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{4} \lim_{t \rightarrow T} \int_{\mathbb{R}^2} |u_{k_j}|^4 dx \\ &= \frac{1}{2} \|\nabla R\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{4} \lim_{t \rightarrow T} \int_{\mathbb{R}^2} |\psi|^4 dx. \end{aligned}$$

Hence, inequality  $\|\nabla \psi\|_{L^2(\mathbb{R}^2)} < \|\nabla R\|_{L^2(\mathbb{R}^2)}$  derives  $\mathcal{E}(\psi) < 0$ . This is impossible from Lemma 3.2 and the fact  $\psi \neq 0$ .



Since  $\psi$  is a minimizer of (3.17), it satisfies the Euler–Lagrange equation (1.12). Making use of  $\|\nabla|\psi|\|_{L^2(\mathbb{R}^2)} \leq \|\nabla\psi\|_{L^2(\mathbb{R}^2)}$ , we know that  $|\psi|$  is also a minimizer of the variational problem (3.17). Thus, it is a non-negative solution of (1.12). It follows from  $\|\psi\|_{L^2(\mathbb{R}^2)} = N_c$ ,  $\|\nabla\psi\|_{L^2(\mathbb{R}^2)} = \|\nabla R\|_{L^2(\mathbb{R}^2)}$  and Proposition 3.2 that

$$\psi = R(\cdot + y) e^{i\gamma},$$

for some  $R \in \mathcal{G}_1$ ,  $y \in \mathbb{R}^2$  and  $\gamma \in \mathbb{R}$ . □

Let  $u(t, x)$  be a solution of the Cauchy problem (1.8) and  $I(t) = \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 dx$ , we have (see Babaoğlu *et al.*, 2004 for detail)

$$\frac{d^2 I}{dt^2} = 16\mathcal{E}(u_0). \tag{5.10}$$

As a direct result of (5.10), we obtain the following lemma.

**LEMMA 5.1** If the solution  $u(t, x)$  of the Cauchy problem (1.8) blows up at finite time  $T$ . Then there is a constant  $c_0$  such that

$$\int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 dx \leq c_0, \quad \text{for all } t \in [0, T).$$

**THEOREM 5.1** Let  $u(t)$  be a solution of the Cauchy problem (1.8) in  $C([0, T), H^1(\mathbb{R}^2))$  such that  $u(t)$  blows up in finite time  $T : \lim_{t \rightarrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = \infty$ . If  $\|u_0\|_{L^2(\mathbb{R}^2)} = N_c$ , then there are  $R \in \mathcal{G}_1$  and  $x_0 \in \mathbb{R}^2$  such that

$$u(t, x) \rightarrow N_c^2 \delta_{x_0} \tag{5.11}$$

in the sense of distribution as  $t \rightarrow T$ . Here  $N_c$  is defined by (3.10).

*Proof.* It follows from Proposition 5.1 that

$$\lambda^2(t) |u(t, \lambda(t)(x + x(t)))|^2 \rightarrow |R(x)|^2 \quad \text{in } L^1(\mathbb{R}^2) \text{ as } t \rightarrow T$$

and

$$|u(t, x + x(t))|^2 \rightarrow N_c^2 \delta_{x=0} \quad \text{as } t \rightarrow T. \tag{5.12}$$

Using Lemma 5.1 derives

$$\limsup_{t \rightarrow T} |x(t)| \leq \frac{\sqrt{c_0}}{N_c}.$$

For a constant  $r_0 > 0$ , we have

$$\forall t \in [0, T), \quad |x(t)| \leq r_0 \tag{5.13}$$

and

$$\begin{aligned} \int_{B(0,r)} |u(t, x)|^2 x dx &= \int_{B(0,r)} |u(t, x)|^2 (x - x(t)) dx + \int_{B(0,r)} |u(t, x)|^2 x(t) dx \\ &= \int_{B(-x(t),R)} |u(t, y + x(t))|^2 y dy + \int_{B(-x(t),R)} |u(t, y + x(t))|^2 x(t) dy. \end{aligned}$$

From (5.13), for arbitrary  $r > r_0$ , there is a  $\delta > 0$  such that  $B(0, \delta) \subset B(-x(t), r)$ . Formula (5.12) implies

$$\int_{B(0,r)} |u(t, x)|^2 x dx - \int_{\mathbb{R}^2} |R(x)|^2 x(t) dx = 0.$$

On the other hand, it derives from Lemma 5.1 that

$$\int_{|x|>r} |u(t, x)|^2 x \, dx \leq \frac{c_0}{r}.$$

Thus,

$$\lim_{t \rightarrow T} \left\{ \int_{\mathbb{R}^2} |u(t, x)|^2 x \, dx - \int_{\mathbb{R}^2} |R(x)|^2 \, dx \right\} = 0. \tag{5.14}$$

By Lemma 3.3, we obtain

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{R}^2} |u(t, x)|^2 x \, dx \right| &= \left| 2\Im \int_{\mathbb{R}^2} \bar{u}(t, x) \nabla u(t, x) \, dx \right| \\ &= \left| 2\Im \sum_{j=1}^N \int_{\mathbb{R}^2} \bar{u}(t, x) \nabla u(t, x) \cdot \nabla \theta_j(x) \, dx \right| \\ &\leq 2 \sum_{j=1}^N \left( 2\mathcal{E}(u(t)) \int_{\mathbb{R}^2} |u(t, x)|^2 |\nabla \theta_j(x)|^2 \, dx \right)^{1/2} \\ &\leq C, \end{aligned}$$

where  $\theta_j(x) = x_j$ . This shows that there is  $x_0 \in \mathbb{R}^2$  such that

$$\lim_{t \rightarrow T} \int_{\mathbb{R}^2} |u(t, x)|^2 x \, dx = - \left( \int_{\mathbb{R}^2} |R(x)|^2 \, dx \right) x_0. \tag{5.15}$$

It follows from (5.14) and (5.15) that  $x(t) \rightarrow -x_0$  as  $t \rightarrow T$ . Then we have

$$|u(t, x)| \rightarrow N_c^2 \delta_{x=-x_0}. \quad \square$$

The following theorem gives the lower bound for the blow-up rate of the minimal blow-up solutions.

**THEOREM 5.2** Let  $u(t)$  be a solution of the Cauchy problem (1.8) in  $C([0, T], H^1(\mathbb{R}^2))$  such that  $u(t)$  blows up in finite time  $T : \lim_{t \rightarrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = \infty$ . If  $\|u_0\|_{L^2(\mathbb{R}^2)} = N_c := \|R_\omega\|_{L^2(\mathbb{R}^2)}$ , then there exists a constant  $C > 0$  such that

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \geq \frac{C}{T-t}, \quad \forall t \in [0, T).$$

*Proof.* Let  $h(x) \in C_0^\infty(\mathbb{R}^2)$  be a non-negative radial function such that

$$h(x) = h(|x|) = |x|^2, \quad \text{if } |x| < 1 \quad \text{and} \quad |h(x)|^2 \leq ch(x).$$

For  $A > 0$ , we define  $h_A(x) = A^2 h(x/A)$  and  $g_A(t) = \int_{\mathbb{R}^2} h_A(x - x_0) |u(t, x)|^2 \, dx$  with  $x_0$  defined by (5.11).

Using Lemma 3.3, for every  $t \in [0, T)$ , we have

$$\begin{aligned} \left| \frac{d}{dt} g_A(t) \right| &= \left| 2\Im \sum_{j=1}^N \int_{\mathbb{R}^2} |\bar{u}(t, x)| \nabla u(t, x) \cdot \nabla h_A(x - x_0) dx \right| \\ &\leq 2\mathcal{E}(u_0)^{1/2} \left( \int_{\mathbb{R}^2} |u(t, x)|^2 \nabla h_A(x - x_0) dx \right)^{1/2} \\ &\leq C(u_0) \sqrt{g_A(t)}, \end{aligned} \quad (5.16)$$

which implies

$$\left| \frac{d}{dt} \sqrt{g_A(t)} \right| \leq C(u_0).$$

Integrating both sides gives rise to

$$|\sqrt{g_A(t)} - \sqrt{g_A(t')}| \leq C(u_0)|t' - t|.$$

We note that (5.11) implies

$$g_p(t') \rightarrow N_c h_A(0) = 0 \quad \text{as } t' \rightarrow T.$$

Therefore, letting  $t' \rightarrow T$ , we have

$$g_A(t) \leq C(u_0)(T - t)^2.$$

Now fix  $t \in [0, T)$  and let  $A$  go to infinity, it holds that

$$\int_{\mathbb{R}^2} |x - x_0|^2 |u(t, x)|^2 dx \leq C(T - t)^2.$$

Then the uncertainty principle

$$\left( \int_{\mathbb{R}^2} |u(t, x)|^2 dx \right)^2 \leq \left( \int_{\mathbb{R}^2} |x - x_0|^2 |u(t, x)|^2 dx \right) \left( \int_{\mathbb{R}^2} |\nabla u(t, x)|^2 dx \right)$$

gives us a lower bound of the blow-up rate

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \geq \frac{C(u_0)}{C(T - t)}.$$

□

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