Volatility estimation from short time series of stock prices*

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Abstract

We consider estimation of the historical volatility of stock prices. It is assumed that the stock prices are represented as time series formed as samples of the solution of a stochastic differential equation with random and time varying parameters; these parameters are not observable directly and have unknown evolution law. The price samples are available with limited frequency only. In this setting, the estimation has to be based on short time series, and the estimation error can be significant. We suggest some supplements to the existing non-parametric methods of volatility estimation. Two modifications of the standard summation formula for the volatility are derived. In addition, a linear transformation eliminating the appreciation rate and preserving the volatility is suggested.

Key words: econometrics, short time series, volatility estimation, non-parametric estimation.

JEL classification: C14, C15, C58

Mathematical Subject Classification (2010): 91G70

1 Introduction

In this paper, estimation of historical volatility is considered for financial time series generated by stock prices. This estimation is important because it is a necessary step for the volatility forecast which is crucial for pricing of financial derivatives and for optimal portfolio selection. Usually, volatility forecasting is based on a model for the volatility evolution. These models have to be selected and matched based on the estimated past historical volatility. The

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methods of estimation and forecast of volatility were intensively studied; see, e.g., Mandelbrot (1963), Hull and White (1987), Clark (1973), Merton (1980), Andersen and Bollerslev (1998), Elliott *et al* (1998), Andersen *et al* (2001), Fouque *et al* (2000), Frey and Runggaldier (2001), Malliavin and Mancino (2002), Andersen *al* (2003), Barndorff-Nielsen *al* (2003), Aït Sahalia and Mykland (2004), Zhang *et al* (2005), Cvitanic *et al* (2006), Aït Sahalia and Yu (2009).

The present paper revisits the problem of estimation of historical volatility. We consider the so-called diffusion model, where the prices can be described as samples of the continuous time solution of a stochastic differential equation. We consider a nonparametric setting where the parameters of this equation (including the volatility) are not assumed to be constant, and their evolution law is not assumed to be known. In this setting, it is unreasonable to use long-memory data, since the volatility is changing in time. Therefore, the older historical data are not relevant and only recent observations should be used. In addition, we assume the data frequency is limited. This means that only short time series of prices are available. Under these assumptions, the estimation error can be significant. We suggest some modifications that may help to reduce the estimation error for this model. First, we suggest two modifications of the standard formula for volatility based on some special features of the Ito processes used for the diffusion model of stock price evolution. In addition, we suggest a linear transformation eliminating the appreciation rate and preserving the volatility. In some cases, it can help to reduce the impact of the presence of a time variable and unknown drift caused by the appreciation rate of the stock prices.

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2 The model

Consider a risky asset (stock, foreign currency unit, etc.) with time series of the prices $S_1, S_2, S_3, \ldots$, for example, daily prices.

We consider the so-called diffusion model for stock prices. For this model, $S_k = S(t_k)$, where $S(t)$ is a continuous time random process such that

$$dS(t) = S(t)[a(t)dt + \sigma(t)dw(t)]. \qquad (1)$$

Here $w(t)$ is a Wiener process, $a(t)$ is the appreciation rate, $\sigma(t)$ is the volatility, $t > 0$. We assume that a and σ are some scalar random processes such that $(a(t), \sigma(t))$ is independent from $w(\tau) - w(\theta)$ for all θ, τ such that $\theta > \tau \geq t$. We assume that the process $(a(t), \sigma(t))$

belongs to $L_2(0,T)$ with probability 1 (i.e., $\int_0^T [a(s)^2 + \sigma(s)^2]ds < +\infty$ with probability 1), for a given $T > 0$.

This model has many financial applications, including pricing of derivatives and optimal portfolio selection. Usually, practical implementation of the methods based on this model requires to estimate (a, σ) from the historical data. For constant a and σ, satisfactory estimates can be obtained from sufficiently large samples. For financial models, estimation of a is challenging since the trend for financial time series is usually relatively small and unstable. Estimation of σ gives more robust results. This paper studies estimation of $\sigma(t)$ only. Some results and references for the estimation of the appreciation rate can be found in Dokuchaev (2005) and Dokuchaev (2002), Ch.9, p.128.

In the continuous time setting, the process $\sigma(t)$ is always adapted to the filtration generated by the historical prices $S(s)$, $s \le t$. This implies that, in theory, $\sigma(t)$ can be estimated without error from the observation of the continuous path on the time interval $[t-\varepsilon, t]$ for an arbitrarily small $\varepsilon > 0$. In practice, only finite time series of the prices observed with limited frequency are available. This generates the error in matching the statistical estimates with the value of $\sigma(t)$ in the continuous time model. The problem of reducing this error is the main focus of this paper.

3 The estimation based on discrete time series

We consider estimates random and time variable volatility at time t based on statistics collected at time $[t - \Delta t, t]$, where $\Delta t > 0$ is given.

The volatility process is usually characterized by the process

$$v(t) = \frac{1}{\Delta t} \int_{t-\Delta t}^{t} \sigma(s)^2 ds.$$

In fact, the process $v(t)$ is the one that is usually estimated from the historical data, rather than the underlying process $\sigma(t)$ itself. Up to the end of this paper, we consider estimation of the process $v(t)$.

Our goal is to construct an estimate $v(t)$ from available samples $S(t_k)$, where $t_k \in [t-\Delta t, t]$, $k = m_0, m_0 + 1, ..., m$. We assume that the time points t_k are equally spaced with sampling interval $\delta = t_k - t_{k-1}$. Furthermore, we assume that $t_{m_0} = t - \Delta t$ and $t_m = t$. This means that $\Delta t = (m - m_0)\delta$.

3.1 The traditional estimate

The traditional estimate of $v(t)$ is represented by the sample variance of the series $\delta^{-1/2}\log(S(t_k)/\log(S(t_{k-1}))$. This traditional estimate can be calculated as

$$\widehat{v}(t_m) = \frac{1}{\Delta t}\sum_{k=m_0+1}^{m}(A_m - Z_k)^2, \tag{2}$$

where $\Delta t = (m - m_0)\delta$,

$$A_m = \frac{1}{m - m_0}\sum_{k=m_0+1}^{m} Z_k,$$

and where

$$Z_k = \log S(t_k) - \log S(t_{k-1}).$$

(See, e.g., estimate (9.1) in Dokuchaev (2007)). We suggest below two modifications of this estimate. These modifications are based on the assumptions that the underlying time series are generated by model (1) and on the properties of the continuous time Ito processes.

3.2 An alternative estimate

The following lemma is known (see, e.g., Remark 1.1 in Dokuchaev (2002) and Proposition 7.1 from Dokuchaev (2007)).

Lemma 3.1 *Model (1) implies that the value* $v(t) = \frac{1}{\Delta t}\int_{t-\Delta t}^{t}\sigma(s)^2 ds$ *can be found explicitly as*

$$v(t) = \frac{2}{\Delta t}\left(\int_{t-\Delta t}^{t}\frac{dS(s)}{S(s)} - \log S(t) + \log S(t - \Delta t)\right). \tag{3}$$

Proof. It is well known that any solution of equation (1) is such that

$$S(t) = S(t - \Delta t)\exp\left(\int_{t-\Delta t}^{t} a(s)ds - \frac{1}{2}\int_{t-\Delta t}^{t}\sigma(s)^2 ds + \int_{t-\Delta t}^{t}\sigma(s)dw(s)\right).$$

It follows that

$$\log S(t) - \log S(t - \Delta t) = \int_{t-\Delta t}^{t} a(s)ds - \frac{1}{2}\int_{t-\Delta t}^{t}\sigma(s)^2 ds + \int_{t-\Delta t}^{t}\sigma(s)dw(s).$$

In addition,

$$\int_{t-\Delta t}^{t}\frac{dS(t)}{S(t)} = \int_{t-\Delta t}^{t}[a(s)ds + \sigma(s)dw(s)].$$

Hence

$$\frac{1}{2}\int_{t-\Delta t}^{t}\sigma(s)^2 ds = \int_{t-\Delta t}^{t}\frac{dS(t)}{S(t)} - \log S(t) + \log S(t - \Delta t). \tag{4}$$

Then equation (4) follows. □

The numerical implementation of (3)

Unfortunately, the second integral in (4) cannot be calculated without error but rather has to be estimated using available prices $S(t_k)$.

Let

$$\xi_k = \frac{S(t_k) - S(t_{k-1})}{S(t_{k-1})}. \tag{5}$$

For $t = t_m$ and $t - \Delta t = t_{m_0}$, we have to use approximation

$$\int_{t-\Delta t}^{t} \frac{dS(s)}{S(s)} \sim \sum_{k=m_0+1}^{m} \xi_k. \tag{6}$$

Formula (3) leads to the following estimate of $v(t)$:

$$\widehat{v}(t) = \frac{2}{\Delta t}\left(\sum_{k=m_0+1}^{m} \xi_k - \log S(t_m) + \log S(t_{m_0})\right), \tag{7}$$

where $\Delta t = (m - m_0)\delta$.

3.3 Another alternative estimate

It appears that, for the diffusion model (1), the value $v(t)$ can be represented via different stochastic integrals. Let us give one more estimate of $v(t)$ that is different from both estimates (2) and (7).

Lemma 3.2 *Model (1) implies that*

$$\int_{t-\Delta t}^{t} \sigma(s)^2 ds = 2\log|X(t)|, \tag{8}$$

where $X(s)$ is a complex-valued process defined for $s \in [t - \Delta t, t]$ such that

$$\begin{aligned} &dX(s) = iX(s)\frac{dS(s)}{S(s)}, \quad s \in (t-\Delta t, t), \\ &X(t-\Delta t) = 1. \end{aligned} \tag{9}$$

Here $i = \sqrt{-1}$ is the imaginary unit.

Proof. By the Ito formula, the solution $X(t)$ of (9) is

$$\begin{aligned} X(t) &= X(t-\Delta t)\exp\left(i\int_{t-\Delta t}^{t} a(s)ds - \frac{i^2}{2}\int_{t-\Delta t}^{t}\sigma(s)^2 ds + i\int_{t-\Delta t}^{t}\sigma(s)dw(s)\right) \\ &= \exp\left(i\int_{t-\Delta t}^{t} a(s)ds + \frac{1}{2}\int_{t-\Delta t}^{t}\sigma(s)^2 ds + i\int_{t-\Delta t}^{t}\sigma(s)dw(s)\right). \end{aligned}$$

Hence

$$|X(t)| = \exp\left(\frac{1}{2}\int_{t-\Delta t}^{t}\sigma(s)^2 ds\right).$$

Then (8) follows. □

The numerical implementation of (8)

The numerical implementation of Lemma 3.1 and (8) suggests to estimate the value $X(t)$. Again, this value cannot be calculated without error but rather has to be estimated using available prices $S(t_k)$.

The time discretization of (9) leads to the stochastic difference equation

$$\begin{aligned} &X(t_k) - X(t_{k-1}) = iX(t_{k-1})\xi_k, \quad k \ge m_0 + 1, \\ &X(t_{m_0}) = 1. \end{aligned}$$

where ξ_k are defined by (5). This equation can be rewritten as

$$\begin{aligned} &X(t_k) = X(t_{k-1})(1 + i\xi_k), \quad k \ge m_0 + 1, \\ &X(t_{m_0}) = 1. \end{aligned}$$

Hence

$$X(t_m) = \prod_{k=m_0+1}^{m} (1 + i\xi_k).$$

Clearly,

$$|X(t_m)| = \prod_{k=m_0+1}^{m} |1 + i\xi_k| = \prod_{k=m_0+1}^{m} (1 + \xi_k^2)^{1/2},$$

and

$$\log|X(t_m)| = \sum_{k=m_0+1}^{m} \log[(1 + \xi_k^2)^{1/2}] = \frac{1}{2} \sum_{k=m_0+1}^{m} \log(1 + \xi_k^2).$$

Therefore, formula (8) leads to the following estimate of $v(t)$:

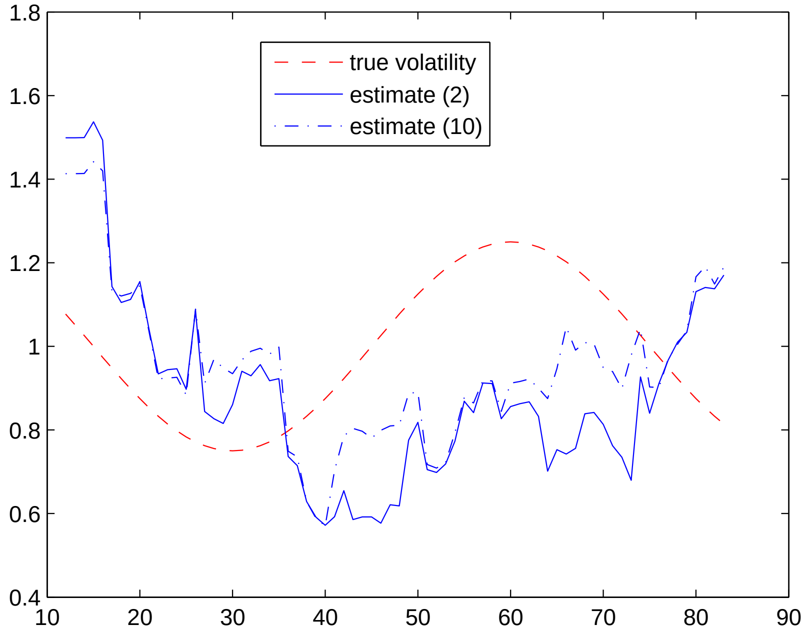
$$\widehat{v}(t) = \frac{1}{\Delta t} \sum_{k=m_0+1}^{m} \log(1 + \xi_k^2), \tag{10}$$

where $\Delta t = (m - m_0)\delta$.

Note that estimate (2) represents the sample variance of the series $\delta^{-1/2} \log(S(t_k)/\log(S(t_{k-1})$ and is not directly associated with model (1). Estimates (7) and (10) have a different origin; they were derived form the properties of model (1).

Figure 3.1 shows estimates (2) and (10) for applied for Monte-Carlo simulated prices with $\sigma(s) = 1 + 0.25\cos(s\pi/(3\Delta t))$ and with $a(s) \equiv 0.5$, in the setting described in Section 6 below. This figure demonstrates that these estimates produce close but still different results.

Figure 3.1: $---$: values of $\sigma(t)$; $—$: values of $\widehat{v}(t)$ defined by estimate (2); $-\cdot-\cdot-$: values of $\widehat{v}(t)$ defined by estimate (10). This figure shows that these estimates produce close but still different results.



4 Reducing the impact of the appreciation rate

Since only short time series $S(t_k)$ are observable, it is not possible to separate the impact of the noise $\sigma(t)dw(t)$ from the impact of the random and time variable input $a(t)dt$ defined by the appreciation rate process.

Let $\gamma(t)$ be an adapted process, and let

$$\widehat{S}(t) = S(0) + \int_0^t \gamma(s)\widehat{S}(s)S(s)^{-1}dS(s).$$

It follows from the definitions that $\widehat{S}(t)$ is the solution of the equation

$$\begin{aligned} &d\widehat{S}(t) = \gamma(t)\widehat{S}(t)S(t)^{-1}dS(t), \quad t > 0, \\ &\widehat{S}(0) = S(0), \end{aligned}$$

i.e.,

$$d\widehat{S}(t) = \widehat{S}(t)[\widehat{a}(t)dt + \widehat{\sigma}(t)dw(t)],$$

where

$$\widehat{a}(t) = \gamma(t)a(t), \quad \widehat{\sigma}(t) = \gamma(t)\sigma(t).$$

Lemma 4.1 *There exists a sequence of the processes $\gamma(t) = \gamma_j(t)$ such that $|\gamma_j(t)| \equiv 1$ for all j and that*

$$\int_0^T \gamma_j(t)f(t)dt \to 0 \quad \textit{as} \quad j \to +\infty \quad \textit{for any} \quad f(\cdot) \in L_2(0,T).$$

Proof. It suffices to take piecewise constant functions $\gamma_j(t) = (-1)^{k(j,t)}$, where $k(i,t) = 1$ if $t \in [2mT/j, (2m+1)T/j)$, $k(j,t) = -1$ if $t \in [(2m+1)T/j, (2m+2)T/j)$, $m = 0, 1, 2, ...$ Clearly, the required limit holds for all $f_j \in C(0,T)$, and the set $C(0,T)$ is dense in $L_2(0,T)$. Since $\|\gamma_j\|_{L_2(0,T)} = \text{const}$, it follows that $\gamma_j \to 0$ as $j \to +\infty$ weakly in $L_2(0,T)$. This completes the proof of Lemma 4.1. $\square$

Let us consider the sequence $\{\gamma(\cdot)\} = \{\gamma_j(\cdot)\}$ from the proof of Lemma 4.1 and the corresponding processes $\widehat{S}(t) = \widehat{S}_j(t)$, $\widehat{a}(t) = \widehat{a}_j(t) = \gamma_j(t)a(t)$, and $\widehat{\sigma}(t) = \widehat{\sigma}_j(t) = \gamma_j(t)\sigma(t)$. Since

$$\widehat{S}(t) = \widehat{S}(0) + \int_0^t \gamma_j(s)a(s)\widehat{S}(s)ds + \int_0^t \gamma_j(s)\sigma(s)\widehat{S}(s)dw(s),$$

we have that $\widehat{S}(t) = S(0)$ and

$$\widehat{S}(t) = \widehat{S}(0)\exp\left(\int_0^t \widehat{a}(s)ds - \frac{1}{2}\int_0^t \widehat{\gamma}(s)^2\sigma(s)^2 ds + \int_0^t \widehat{\sigma}(s)dw(s)\right)$$
$$= S(0)\exp\left(\int_0^t \gamma_j(s)a(s)ds - \frac{1}{2}\int_0^t \gamma_j(s)^2\sigma(s)^2 ds + \int_0^t \gamma_j(s)\sigma(s)dw(s)\right).$$

By the choice of $\{\gamma_j\}$, we have that $\sigma(t)^2 = \widehat{\sigma}(t)^2$ and

$$\int_0^t \widehat{a}(s)ds = \int_0^t \gamma_j(s)a(s)ds \to 0 \quad \text{as} \quad j \to +\infty \quad \text{a.s.}.$$

Therefore, the processes $\widehat{S}(t) = \widehat{S}_j(t)$ can be interpreted as processes with vanishing appreciation rate as $i \to +\infty$.

Clearly,

$$\frac{1}{\Delta t}\int_{t-\Delta t}^t \sigma(s)^2 ds = \frac{1}{\Delta t}\int_{t-\Delta t}^t \widehat{\sigma}(s)^2 ds$$

Therefore, the estimate of

$$\frac{1}{\Delta t}\int_{t-\Delta t}^t \sigma(s)^2 ds$$

can be obtained via calculating the similar value for the process $\widehat{S}(t) = \widehat{S}_j(t)$ for which the impact of the appreciation rate $a(t)$ is eliminated in the limit case where $j \to +\infty$.

We call $\widehat{S}(t)$ the process with eliminated appreciation rate. In fact, the process $\widehat{S}(t)$ converges to a martingale.

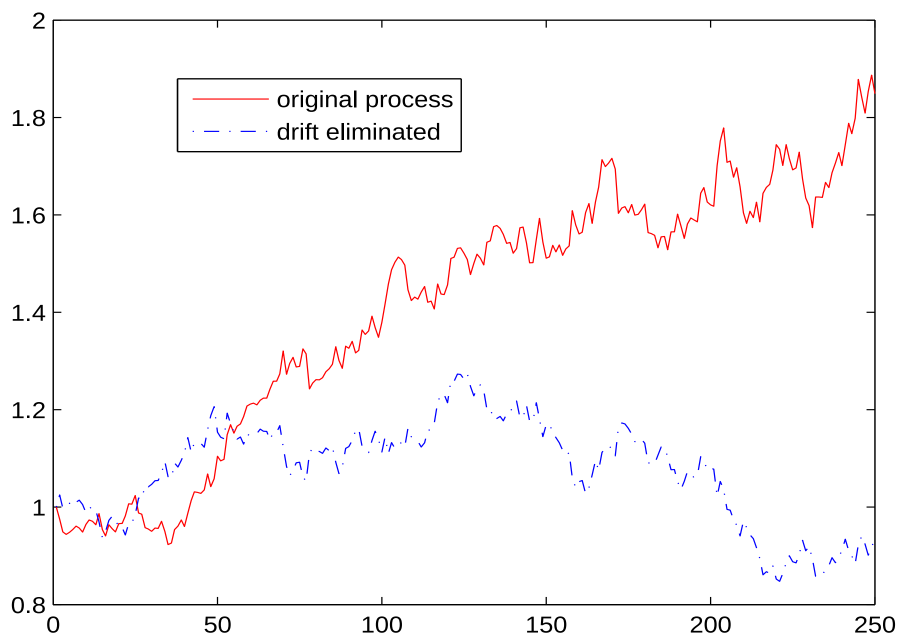
In practical calculations, the processes $\widehat{S}(t) = \widehat{S}_j(t)$ and $\gamma(t) = \gamma_j(t)$ are represented by discrete time processes; it is natural to select $\gamma_j(t_k) = (-1)^k$.

Figure 4.1 shows an example of the simulated processes $S(t)$ and $\widehat{S}(t)$ with $\gamma(t) = \gamma_j(t)$ defined as in the proof of Lemma 4.1, with the price parameters defined as

$$a(t) \equiv 0.5, \quad \sigma(t) \equiv 0.3, \quad t \in [0, 1],$$

and with $\delta = t_k - t_{k+1} = 0.004$. Here t_k are the times where the prices were observed; the same times are used as the points of discontinuity for $\gamma(t)$. This sample represents daily prices; the plot shows evolution on one year time horizon. It can be noted that the impact of the appreciation rate elimination barely seen from the local dynamics, since the volatility dominates the appreciation rate.

Figure 4.1: Drift elimination: $- - --$: the original process $S(t)$ with appreciation rate; $- \cdot - \cdot -$: the process $\widehat{S}(t)$ with eliminated appreciation rate, for the sample of daily prices. These processes have the same volatility.



5 The algorithm

Assume that the series of historical prices $S(t_k)$ is available, and that this is the series of data of some sufficient frequency, to justify the use of the continuous time diffusion model (1). We suggests the following procedure to estimate $v(t) = \frac{1}{\Delta t}\int_{t-\Delta t}^{t} \sigma(s)^2 ds$.

(i) Apply the appreciation rate eliminating procedure described above with $\gamma(t_k) = (-1)^k$. Let $\widehat{S}(t_k)$ be the corresponding process with eliminated appreciation rate.

(ii) Estimate the volatility using the series $\widehat{S}(t_k)$ and one of equations (2), (7), or (10).

The nature of the diffusion model (1) is such that a precise estimate of the volatility is achievable for the high frequency data only; the error increases if the frequency decreases. Therefore, it is preferable to use the data of the highest available frequency.

6 Some experiments

Monte-Carlo simulation

In our experiments, we used Monte-Carlo simulation of the time series for $S(t_k)$ evolving as

$$S(t_{k+1}) = S(t_k) + a(t_k)S(t_k)\delta + \sigma S(t_k)\sqrt{\delta}\,\eta_{k+1},$$

with mutually independent random η_k from the standard normal distribution $N(0,1)$, and with fixed

$$\sigma = 0.3, \qquad \delta = t_k - t_{k+1} = 0.004.$$

This choice of δ corresponds to the time series of the daily prices. In the experiments, we considered only the cases of short series with $m - m_0 = 10$. Note that the selection of the constant volatility in the experiments described above does not undermine the purpose to study processes with time variable volatility, since only short time series are used. Our choice of the short memory of $m - m_0 = 10$ periods corresponds to the case when the volatility is not expected to remain the same for longer than two weeks (or ten business days).

For $100,000$ Monte-Carlo trials, we simulated a sequence of 250 daily prices for every Monte-Carlo trial. For every sequence of prices, we considered subsequences of 10 consequent daily prices, for 240 possible initial times $t - \Delta t$. We estimated the volatility using these subsequences.

To compare different methods, we estimate the expected error

$$\mathbf{E}\left|\sigma - \widehat{v}(t)^{1/2}\right|.$$

More precisely, we estimate the sample mean error $\mathbb{E}e$, where

$$e = \left|\sigma - \widehat{v}(t_m)^{1/2}\right|.$$

Here $\mathbb{E}$ denotes the sample mean over all Monte-Carlo simulation trials and over all subsequences $\{S(t_k)\}$, $k = m_0, m_0+1, ..., m_0+10$ of the sequences of the simulated stock prices, for all possible m_0. The total size of these samples was above 1,000,000. We found that enlarging the sample does not improve the results. Actually, the experiments with samples $10,000$ Monte-Carlo trials produced the same results. Table 6.1 shows the values $\mathbb{E}e$ in the experiments described above for different choices of $a(s)$, $s \in [t-\Delta t, t]$.

Table 6.1: The mean error $\mathbb{E}e$ for estimates (2), (7), and (10) for Monte-Carlo simulated prices

$a(s)$	$\mathbb{E}e$ for (2)	$\mathbb{E}e$ for (7)	$\mathbb{E}e$ for (10)
$a(s) \equiv 0.5$; without drift elimination	0.0584477	0.0546515	0.05482599
$a(s) \equiv 0.5$; with drift elimination	0.0581887	0.0548541	0.05477097
$a(s) = 6\sin(2\pi(S(s) - S(s-\tau(s))))$; without drift elimination	0.06096515	0.06715185	0.06704281
$a(s) = 6\sin(2\pi(S(s) - S(s-\tau(s))))$; with drift elimination	0.05993648	0.0596709	0.05960616

Here $\tau(s) = 0.04\lfloor s/0.04\rfloor$; we denote by $\lfloor s\rfloor$ the integer part of s.

Further, we estimated the standard deviation σ_e of the error $e = \left|\sigma - \widehat{v}(t_m)^{1/2}\right|$ as the following:

(i) the error ε_t was calculated for $t - \Delta$ for $t = 11, ..., 250$;

(ii) the sample $\bar{e}$ was formed from 100,000 values $\frac{1}{240}\sum_{t=11}^{250}\varepsilon_t$ obtained in 100,000 Monte-Carlo trials.

(iii) σ_ε was calculated as the standard deviation of the series $\bar{e}$.

Table 6.2 shows the values σ_e in these experiments.

Table 6.2: σ_e for estimates (2), (7), and (10) for Monte-Carlo simulated prices

$a(s)$	σ_e for (2)	σ_e for (7)	σ_e for (10)
$a(s) \equiv 0.5$; without drift elimination	0.00675680407	0.0065980581	0.00660535275
$a(s) \equiv 0.5$; with drift elimination	0.00673944223	0.0066648496	0.00663868046
$a(s) = 6\sin(2\pi(S(s) - S(t-\tau)))$; without drift elimination	0.00720963109	0.011522603	0.0117100558
$a(s) = 6\sin(2\pi(S(s) - S(t-\tau)))$; with drift elimination	0.00724451753	0.00788851706	0.0078621931

The values $\mathbb{E}e$ and σ_e obtained in the experiments are very stable; the results are practically the same for much less Monte-Carlo trials.

Note that the appreciation rate elimination does not take effect in a single term k under the sums in (2),(7), and (10). However, it can make the error less systematic after mixing all $m-m_0$ terms in the sum. This explains some improvement achieved with the drift elimination for time dependent and random a in the experiment described above.

To achieve some effect from appreciation rate elimination in a single term k in the sum in (2), (7), or (10), the following modification of the algorithm described above can be used:

- Select $\nu \in \{1, 2, 3, ..\}$ and form the new sequence $\widehat{S}(\widehat{t_k})$ of prices, where $\widehat{t_k} = \nu t_k$.
- Estimate the volatility using the series $\widehat{S}(\widehat{t_k})$ and equation (2), (7), or (10).

It can be also noted that some minor improvement of the performance was observed for an estimate constructed as the mean of estimates (2),(7), and (10).

Experiment with historical prices

We have carried out some experiments for the time series representing the returns for the historical stock prices. Using daily price data from 1984 to 2009 for 19 American and Australian stocks (Citibank, Coca Cola, IBM, AMC, ANZ, LEI, LLC, LLN, MAY, MLG, MMF, MWB, MIM, NAB, NBH, NCM, NCP, NFM and NPC), we generated samples of price data for one synthetic price process $S(t_k)$. In fact, the full 25 years of data was not available for all the stocks; the total number of the prices in the sample was 69,948.

For the historical prices, the "true" volatility process is not available. Moreover, it cannot be even presumed with certainty that model (1) is suitable for particular prices samples.

Therefore, we cannot estimate the "error" in this experiment. So far, we will demonstrate only that different estimation rules produce close enough but still different distributions of random estimates.

We estimated the value of

$$\mathbf{E}\left(\frac{1}{\Delta t}\int_{t-\Delta t}^{t}\sigma(s)^2 ds\right)^{1/2}.$$

More precisely, we estimated the corresponding sample mean

$$\bar{\sigma} = \mathbb{E}\left[\widehat{v}(t_m)^{1/2}\right],$$

with estimates $\widehat{v}_k$ obtained accordingly to the different rules described above. We considered again short series of consisting of 10 daily prices, with $t-\Delta t = t_{m_0}$, and $t = t_m$, $m - m_0 = 10$. The sample mean $\mathbb{E}$ used here represents the averaging over all possible initial times t_m and and over different stocks; the total number of short time series was 66,590. The results of this experiment are presented in Table 6.3.

Table 6.3: $\bar{\sigma}$ for estimates (2), (7), and (10) for the large set of historical prices

	$\bar{\sigma}$ for (2)	$\bar{\sigma}$ for (7)	$\bar{\sigma}$ for (10)
Without drift elimination	0.2449	0.2516	0.2511
With drift elimination	0.2446	0.2454	0.2511

It can be seen from this table that the average values of the volatility calculated over a large number of short time series are close for different estimates. For a smaller number of short series, this effect is less noticeable. For instance, for the similar experiment with 50 prices for NAB and with the averaging over 29 short time series gives the results presented in Table 6.4.

Table 6.4: $\bar{\sigma}$ for estimates (2), (7), and (10) for a smaller set of historical prices

	$\bar{\sigma}$ for (2)	$\bar{\sigma}$ for (7)	$\bar{\sigma}$ for (10)
Without drift elimination	0.1387	0.1570	0.1569
With drift elimination	0.1549	0.1570	0.1569

Note that, in the experiment described by Table 6.4, the traditional estimate gives the value that is about 10% less than the values for other estimates.

We found already that, for Monte-Carlo simulation of the series generated by Ito equations, the different estimates, with or without appreciation rate elimination, produce different estimates for the same model. For historical prices, we observed again that the different estimates produced different results. Unlike the case of the Monte-Carlo simulation, we cannot tell which estimate produces a smaller error, since the true volatility is unknown. This leads to the conclusion that it could be beneficial to use and compare different method simultaneously for the estimation of the volatility from the historical prices.

7 Discussion and conclusion

We summarize our observations as the following.

(i) In some cases (not always), appreciation rate elimination reduces the estimation error. It may happen with estimate (7)-(10) as well as with estimate (2).

(ii) In some cases (not always), estimates (7)-(10) give lesser error than the mainstream estimate (2). It may happen with or without appreciation rate elimination.

The gain was modest but quite systematic and robust with respect to the changes of the parameters. For example, we observed that estimates (7)-(10) give 7% less error than the mainstream estimate (2) for experiments with constant a without appreciation rate elimination.

We have not determined yet the exact classification of models that is more appropriate for one or other method; we leave it for the future research. At the moment, we can state that even the fact of the existence of some alternative estimation methods that may reduce error in some cases is quite significant and calls to use these methods as a supplement to existing methods. It can lead to improvement of preciseness of volatility estimates and, therefore, can be useful for financial applications.

The significance of the preciseness of the volatility estimation can be illustrated as the following. For instance, assume that some volatility estimate is applied for option pricing as a parameter for the Black-Scholes formula. Consider, for example, calculation of a call option price with the exercise time $T = 1$, the initial stock price $S(0) = 1$, the risk-free short-term rate 0.03, and with the strike price 1.2. The option price calculated for constant volatility $\sigma = 0.4$ is 0.1016, and the option price calculated for constant volatility $1.05\sigma = 0.42$ is 0.1095.

This means that the 5% error for volatility estimate gives 7% error for the option price which is quite significant.

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