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46 AN OPTIMAL PID CONTROLLER DESIGN FOR NONLINEAR OPTIMAL CONTROL PROBLEMS WITH CONTINUOUS STATE INEQUALITY CONSTRAINTS

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Abstract: In this paper, we consider an optimal PID control problem subject to continuous state inequality constraints. By applying the constraint transcription method, a local smoothing technique to these continuous state inequality constraint functions, we construct the corresponding smooth approximate functions. Then, by using the concept of the penalty function, these smooth approximate functions are appended to the cost function, forming a new cost function. Then, the constrained optimal PID control problem is approximated by a sequence of unconstrained optimal control problems. Each of which can be viewed and hence solved as an unconstrained nonlinear optimization problem. The gradient formula of the new appended cost function is derived, and a reliable computation algorithm is given.

Key words: Nonlinear optimal control; Continuous state inequality constraints; Constraint transcription; Local smoothing; Penalty function; PID control; Computational algorithm; Ship steering control.

1 INTRODUCTION

Steering control problem is always of great interest and well studied since it was introduced in 1920s. In this problem, it is required to design a controller to the steering of a system such that the tracking error between the heading angle of the system and a desired trajectory is as small as possible. PID control is considered to be one of the most reliable methods and has been widely used, in particular, for steering the heading angle of a ship in real practice. Adaptive control methods are also widely used, though not as widely as PID controllers. Adaptive controllers are harder to design. In particular, they may not be applicable when the output is required to move within a highly confined region.

In this paper, we consider a general nonlinear optimal control problem subject to continuous state inequality constraints, where the control is of the form of a PID controller. These continuous state inequality constraints arise due to the specifications on the rise time and the setting time and also due to the constraint imposed for avoiding overshoot. This constrained optimal control problem covers classical PID control problems as special cases, where their system dynamics are linear. It can be formulated as operating around a specified set point. Furthermore, in classical PID control problems, no hard constraints are allowed on the state of the system nor on the PID controller. The general constrained optimal PID control problem considered in this paper is also applicable to ship steering problems. In the ship steering control problem considered in this paper, the nonlinearity of its system dynamics, the practice requirements on the output (i.e., the rudder angle) and the controller output (i.e., dead band constraint), the specifications on the rise time and setting time and the constraint to avoid overshoot are taken explicitly into account. The constrained optimal control problem under a PID controller is an optimal parameter selection problem subject to continuous state inequality constraints as well as constraints on the control parameters. The constraint transcription method (Jennings L.S. (1990)) is first used to transform the continuous state inequality constraints into equality constraints in integral form. However, the integrands of these equality constraints in integral form are nonsmooth. Thus, a local smoothing technique (Teo K.L. (1997)) is applied to approximate these nonsmooth integrands by smooth functions. Then by using the concept of the penalty function, the integrals of these smooth approximate functions are appended to the cost function to form a new cost function. In this way, we obtain a sequence of optimal selection problems subject to constraints only on the control parameters. Each of which can be viewed as a nonlinear optimization problem and it is to be solved by using a gradient-based technique. For this, the gradient formula for the new appended cost function with respect to the control parameter vector is derived. Then, an efficient computational algorithm is proposed, and the optimal control software, MISER 3.3 (see Jennings L.S. (2004)), will be used.

2 PROBLEM STATEMENT

Consider a dynamical system:

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t), y(t), u(t)),\ t \in (0, T] & (2.1a) \\ \dfrac{dy(t)}{dt} = p(x(t)) & (2.1b) \\ x(0) = x^0 & (2.1c) \\ y(0) = y^0, & (2.1d) \end{cases}$$

where T is the terminal time, and $x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n,\ u = [u_1, \ldots, u_r]^\top \in \mathbb{R}^r,\ y = [y_1, \ldots, y_m]^\top \in \mathbb{R}^m,$ are, respectively, state, control and output vectors, while $f = [f_1, \ldots, f_n]^\top \in \mathbb{R}^n$ and $p = [p_1, \ldots, p_m]^T \in \mathbb{R}^m$ are, respectively, given continuously differentiable functions. x^0 and y^0 are initial conditions for x and y are given constant vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

We assume that the following conditions are satisfied.

(A1) Let $\mathbb{V}$ be a compact subset of $\mathbb{R}^r$. Then, there exists a constant C_1 such that

$$|f(x, u)| \leq C_1(1 + |x|)$$

for all $(x, u) \in \mathbb{R}^n \times \mathbb{V}$, where $|\cdot|$ denotes the usual Euclidean norm.

(A2) There exits a constant C_2 such that

$$|p(x)| \leq C_2(1 + |x|).$$

Remark 1. Suppose that the output equations are algebraic equations given below rather than the output system (2.1a) with the initial condition (2.1c).

$$y(t) = p(\hat{x}(t)) \tag{2.2}$$

where, without loss of generality,

$$\hat{x} = [x_1, \ldots, x_s]^T \tag{2.3}$$

with $s < n$. Furthermore, we assume that

$$\frac{d\hat{x}(t)}{dt} = q(x(t)) \tag{2.4}$$

where $q = [q_1, \ldots, q_s]^T$ is a continuously differentiable function. Then, it can be shown that

$$\frac{dy(t)}{dt} = \sum_{i=1}^{s} \frac{\partial p(\hat{x}(t))}{\partial x_i} \frac{dx_i(t)}{dt} = \sum_{i=1}^{s} \frac{\partial p(\hat{x}(t))}{\partial x_i} q(x(t)) \tag{2.5}$$

with initial condition

$$y(0) = p(\hat{x}(0)). \tag{2.6}$$

Thus, we see that the formulation of the output expressed in terms differential equations given by (2.1a) with initial condition (2.1c) is rather general. Certainly it covers the ship steering problem to be considered in the paper as a special case.

The control u is assumed to take the form of a PID controller given below.

$$u(t) = k_1(y(t) - r(t)) + k_2 \int_0^t (y(s) - r(s))ds + k_3 \frac{dy(t)}{dt}, \tag{2.7}$$

where $r(t)$ denotes a given reference input, which is a piecewise continuous function defined on $[0, T]$, while $k_i,\ i = 1, 2, 3$, are weighting coefficients for the proportional, integral and derivative terms of the PID controller, respectively.

We now specify the region within which the output trajectory is allowed to move. This region is defined in terms of the following continuous inequality constraints, which arise due to the specifications, such as the requirements on the rise time and the setting time and the constraints for avoiding overshoot.

$$g_i(t, y(t)) \geq 0,\ t \in [0, T],\ i = 1, \ldots, M. \tag{2.8}$$

For each $i = 1, \ldots, M$, the function g_i is continuously differentiable with respect to all its arguments.

The problem that is of interest to us may now be stated formally below. Given system (2.1), we wish to design a PID controller such that the output $y(t)$ of the corresponding closed loop system will move within the specified region and, at the same time, will tract a reference input as closely as possible. For this, we formulate a cost function stated as follows.

$$J(k) = \int_0^T \{\alpha_1(y(t) - r(t))^2 + \alpha_2[\frac{dy(t)}{dt}]^2 + \alpha_3[u(t)]^2\}dt, \tag{2.9}$$

where $\alpha_i,\ i = 1, 2, 3$, are the weighting parameters, and

$$k = [k_1, k_2, k_3]^T, \tag{2.10}$$

Define

$$z_1(t) = \int_0^t [y(s) - r(s)]ds. \tag{2.11}$$

Clearly, (2.11) is equivalent to

$$\begin{cases} \dfrac{dz_1(t)}{dt} = y(t) - r(t) & (2.12a) \\ z_1(0) = 0. & (2.12b) \end{cases}$$

Thus, system (2.1) with $u(t)$ chosen as a PID controller given by (2.7) can be written as:

$$\begin{cases} \dfrac{dx(t)}{dt} = \overline{f}(t, x(t), y(t), k) \ t \in [0, T] & (2.13a) \\ \dfrac{dy(t)}{dt} = p(x(t)) & (2.13b) \\ \dfrac{dz_1(t)}{dt} = y(t) - r(t), & (2.13c) \end{cases}$$

with initial conditions

$$\begin{cases} x(0) = x^0 & (2.14a) \\ y(0) = y^0 & (2.14b) \\ z_1(0) = 0, & (2.14c) \end{cases}$$

where

$$\overline{f}(t, x(t), y(t), k) = f(x(t), y(t), u(t)) \quad (2.15)$$

with $u(t)$ given by

$$u(t) = k_1(y(t) - r(t)) + k_2 z_1(t) + k_3 p(x(t)). \quad (2.16)$$

The specified region also remains the same as given by (2.8). The cost function (2.9) becomes:

$$\begin{aligned} \overline{J}(k) \quad = \quad & \int_0^T \{\alpha_1(y(t) - r(t))^2 + \alpha_2[p(x(t))]^2 \\ & + \alpha_3[k_1(y(t) - r(t)) + k_2 z_1(t) + k_3 p(x(t))]^2\} dt. \end{aligned} \quad (2.17)$$

The problem may now be stated as: Given the system (2.13) with initial condition (2.14), find a PID parameter vector k such that the cost function (2.17) is minimized subject to the continuous state inequality constraints (2.8). Let this problem be referred to as **Problem** (P).

Problem (P) is an optimal parameter selection problem.

3 CONSTRAINT APPROXIMATION

The constraint transcription technique which was first introduced in Jennings L.S. (1990) is now applied to the continuous state inequality constraints (2.8), leading to the following equivalent equality constraints:

$$h_i = \int_0^T \max\{g_i(t, y(t)), 0\} dt = 0, \ i = 1, ..., M. \quad (3.1)$$

However, the integrands appeared under the integration in (3.1) are nonsmooth. Thus, for each $i = 1, ..., M$, we shall approximate the nonsmooth function $\max\{g_i(t, x(t)), 0\}$ by a smooth function $\mathcal{L}_{i,\varepsilon}(t, y(t))$ given by

$$\mathcal{L}_{i,\varepsilon}(t,y(t)) = \begin{cases} 0, & \text{if} \quad g_i(t,y(t)) < -\varepsilon \\ (g_i(t,y(t)) + \varepsilon)^2 / 4\varepsilon, & \text{if} \; -\varepsilon \le g_i(t,y(t)) \le \varepsilon \\ g_i(t,y(t)), & \text{if} \quad g_i(t,y(t)) > \varepsilon, \end{cases} \tag{3.2}$$

where $\varepsilon > 0$ is an adjustable constant with small value. Then, for each $i = 1, ..., M$, we define

$$g_{i,\varepsilon}(t, y(t)) = \int_0^T \mathcal{L}_{i,\varepsilon}(t, y(t))dt. \tag{3.3}$$

We now use the concept of the penalty function to appended the functions $g_{i,\varepsilon}$ given by (3.3) to the cost function (2.17), forming a new cost function given below.

$$\overline{J}_{\varepsilon,\gamma}(k) = \int_0^T l_{\varepsilon,\gamma}(t, y(t), k)dt \tag{3.4}$$

where

$$\begin{aligned} l_{\varepsilon,\gamma}(t,x(t),y(t),k) \;\; = \;\; & \alpha_1(y(t) - r(t))^2 + \alpha_2[p(x(t))]^2 \\ & +\alpha_3[k_1(y(t) - r(t)) + k_2 z_1(t) + k_3 p(x(t))]^2 \\ & +\gamma \sum_{i=1}^{M} \int_0^T \mathcal{L}_{i,\varepsilon}(t, y(t))dt, \end{aligned} \tag{3.5}$$

and $\gamma > 0$ is a penalty parameter.

We may now state the following approximate problem for each $\varepsilon > 0$ and $\gamma > 0$, which is referred to as **Problem** $(P_{\varepsilon,\gamma})$. Given system (2.13) with initial condition (2.14), find a $k = [k_1, k_2, k_3]$ such that the cost function (3.5) is minimized.

The relationships between Problem $(P_{\varepsilon,\gamma})$ and Problem (P) are given in the following theorems. Their proofs are similar to those given for Theorem 2.1 and Theorem 2.2 in Teo K.L. (1993).

Theorem 3.1 *For any $\varepsilon > 0$, there exits a $\gamma(\varepsilon) > 0$ such that for all γ, $0 < \gamma < \gamma(\varepsilon)$, if $k^*_{\varepsilon,\gamma}$ is an optimal solution of Problem $(P_{\varepsilon,\gamma})$, then it satisfies the continuous state inequality constraints (2.8) of Problem (P).*

Theorem 3.2 *Let k^* and $k^*_{\varepsilon,\overline{\gamma}(\varepsilon)}$ be, respectively, optimal solutions of Problem (P) and Problem $(P_{\varepsilon,\gamma})$, where $\overline{\gamma}(\varepsilon)$ is chosen such that $k^*_{\varepsilon,\overline{\gamma}(\varepsilon)}$ satisfies the continuous state inequality constraints (2.8) of Problem (P). Then,*

$$\lim_{\varepsilon \to 0} \overline{J}(k^*_{\varepsilon,\overline{\gamma}(\varepsilon)}) = \overline{J}(k^*). \tag{3.6}$$

where $\overline{J}$ is defined by (2.17).

On the basis of Theorem 3.1 and Theorem 3.2, Problem (P) can be solved through solving a sequence of unconstrained optimal parameter selection problems $(P_{\varepsilon,\gamma})$. Each of these unconstrained optimal parameter selection problems can be solved as an unconstrained nonlinear optimization problem by using a gradient-based numerical method, such as any quasi-Newton method Teo K.L. (1991). Thus, the optimal control software, MISER 3.3, is applicable. Further details are given in the next section.

4 COMPUTATION METHOD

In this section, we will propose a reliable computational method for solving Problem (P) via solving a sequence of Problems $(P_{\varepsilon,\gamma})$, where for each $\varepsilon > 0$ and $\gamma > 0$, Problem $(P_{\varepsilon,\gamma})$ can be solved as an unconstrained nonlinear optimization problem. For doing this, it is required to provide, for each k, the value of the cost function $\overline{J}_{\varepsilon,\gamma}(k)$, as well as its gradient $\frac{\partial \overline{J}_{\varepsilon,\gamma}(k)}{\partial k}$. It is obvious that the value of the cost function $\overline{J}_{\varepsilon,\gamma}(k)$ can be readily obtained after system (2.13) with initial condition (2.14) corresponding to k is solved. For the gradient formula of the cost function $\overline{J}_{\varepsilon,\gamma}(k)$ corresponding to each k, we have the following theorem.

Theorem 4.1 *The gradient formula of the cost function $\overline{J}_{\varepsilon,\gamma}(k)$ with respect to k is given by*

$$\frac{\partial \overline{J}_{\varepsilon,\gamma}(k)}{\partial k} = \int_0^T \frac{\partial H_{\varepsilon,\gamma}(t, x(t), y(t), k, \lambda_{\varepsilon,\gamma}(t))}{\partial k} dt, \tag{4.1}$$

where $H_{\varepsilon,\gamma}(t, x, y, k, \lambda)$ is the Hamiltonian function given by

$$H_{\varepsilon,\gamma}(t, x, y, k, \lambda) = -l_{\varepsilon,\gamma}(t, x, y, k) + \lambda^T \overline{f}_i(t, x, y, k), \tag{4.2}$$

where the superscript T denotes the transpose, $l_{\varepsilon,\gamma}$ is as defined in (3.5) and $\lambda_{\varepsilon,\gamma}$ is the solution of following system of co-state differential equation

$$\frac{d\lambda(t)}{dt} = -\left[\frac{\partial H_{\varepsilon,\gamma}(t, x(t), y(t), k, \lambda(t))}{\partial x}, \ \frac{\partial H_{\varepsilon,\gamma}(t, x(t), y(t), k, \lambda(t))}{\partial y}\right]^T \tag{4.3a}$$

with the boundary condition

$$\lambda(T) = 0^T. \tag{4.3b}$$

Proof. The proof is similar to that given for Theorem 5.2.1 in Teo K.L. (1991).

For each $\varepsilon > 0$, $\gamma > 0$, Problem $(P_{\varepsilon,\gamma})$ can now be solved as an unconstrained nonlinear optimization problem using the gradient formula given in Theorem 4.1. Details are reported in the following as an algorithm.

Algorithm

1. Choose $\varepsilon > 0$, $\gamma > 0$ and k.

2. Solve Problem $(P_{\varepsilon,\gamma})$ by using the optimal control software, MISER 3.3, yielding $k^*_{\varepsilon,\gamma}$.

3. Check whether all the continuous state inequality constraints (2.8) are satisfied or not. If they are satisfied, go to Step 4. Otherwise, set $\gamma = 10\gamma$ and go to Step 2 with $k^*_{\varepsilon,\gamma}$ as the initial guess for the new optimization process.

4. If $\varepsilon > 10^{-7}$, set $\varepsilon = \varepsilon/10$ and go to Step 2, using $k^*_{\varepsilon,\gamma}$ as the initial guess for the new optimization process. Else we have a successful exit.

5 CONCLUSION

This paper considered an optimal PID control problem subject to continuous state inequality constraints. It was shown that, the problem can be solved via solving a sequence of unconstrained optimization problems. An efficient computational method was proposed.

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