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A FINITE DIFFERENCE METHOD FOR PRICING EUROPEAN AND AMERICAN OPTIONS UNDER A GEOMETRIC LÉVY PROCESS

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Abstract. In this paper we develop a numerical approach to a fractional-order differential Linear Complementarity Problem (LCP) arising in pricing European and American options under a geometric Lévy process. The LCP is first approximated by a nonlinear penalty fractional Black-Scholes (fBS) equation. We then propose a finite difference scheme for the penalty fBS equation. We show that both the continuous and the discretized fBS equations are uniquely solvable and establish the convergence of the numerical solution to the viscosity solution of the penalty fBS equation by proving the consistency, stability and monotonicity of the numerical scheme. We also show that the discretization has the 2nd-order truncation error in both the spatial and time mesh sizes. Numerical results are presented to demonstrate the accuracy and usefulness of the numerical method for pricing both European and American options under the geometric Lévy process.

1. Introduction.

Pricing financial options has attracted much attention from both mathematicians and financial engineers in the last decade. A financial option is a contract that gives its owner the right, not obligation, to buy (*call option*) or to sell (*put option*) a fixed quantity of a stock at a fixed price (*strike price*) on (European type) or before (American type) a given expiry date. In a complete market, Black & Scholes [2] demonstrated that the price of a European option on a stock, whose price follows geometric Brownian motion with constant drift and volatility, satisfies a second order partial differential equation, known as the Black-Scholes (BS) equation (or model), with proper boundary and terminal conditions. One major shortcoming of the BS model is that the Gaussian shocks used in the model underestimate the probability that stock prices exhibit large movements or jumps over small time steps as illustrated by empirical data in financial market. To overcome this problem, we assume that the underlying stock price S_t of an option follows, as proposed in [7], the following a geometric Lévy process

$$d(\ln S_t) = (r - v)dt + dL_t \tag{1.1}$$

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with the solution

$$S_T = S_t e^{(r-v)(T-t)+\int_t^T dL_u},$$

where T is a future known time, r is the risk-free interest rate, v a convexity adjustment so that the expected value of S_T becomes $\mathbb{E}[S_T] = e^{r(T-t)}S_t$, and dL_t is the increment of a Lévy process under the equivalent martingale measure (EMM). Boyarchenko and Levendorskii [4] proposed the use of a modified Lévy-stable (LS) (Lévy-α-stable) process to model the dynamics of securities. This modification introduces a damping effect in the tails of the LS distribution, which are known as KoBoL processes. Carr, Geman, Madan and Yor [7] proposed the CGMY process including both positive and negative jumps. In this paper, we are concerned with options based on finite moment log-stable (FMLS) processes. In [8], the authors show that a classical hedging portfolio can be substantially improved by employing 'non-local' or fractional differential operators. Since over a time step Δt, the stock price S_t can diffuse or jump to a value $S_{t+\Delta t}$ far away from S_t, the localized information becomes less relevant. The fractional derivative weighs information of the portfolio over a range of values of the underlying stock [8] rather than localized information. When the Brownian motion component is replaced by a Lévy process, the Black-Scholes equation becomes a partial integro-differential equation (PIDE). In [8], by Fourier transform, the PIDE is written as a fractional partial differential equation. In what follows, we refer it to as a fractional Black-Scholes (fBS) equation. Fractional partial differential equations (fPDEs), as generalizations of classical integer-order partial differential equations, are increasingly used to model problems in many areas such as finance and fluid flows. Fractional spatial derivatives are used to model anomalous diffusion or dispersion in which a particle spreads at a rate inconsistent with the classical Brownian motion. Since closed-form solutions to fPDEs of practical significance can rarely be found, various numerical techniques have been proposed for fPDEs. Fractional derivatives can be represented in different forms such as those of Riemann-Liouville (RL) and Grüwald-Letnikov (GL) [18]. Most existing discretization methods have been developed for fPDEs in GL form (cf., for example, [20, 21, 5, 17, 25]).

Unlike a European option whose value is determined by the fBS equation, the value of an American option under the Lévy process is governed by a linear differential complementarity problem involving the fBS operator. Various penalty methods have been developed for solving complementarity problems in both infinite and dimensions [24, 15, 12, 27, 13, 28, 16, 6]. In this work, we develop a numerical method for the fractional differential linear complementarity problem (LCP), or the variational inequality, arising from pricing American options under the Lévy process. We first approximate the LCP by a nonlinear fBS equation using the linear penalty approach used in [29, 1, 28]. We then develop a finite difference scheme based on a numerical quadrature rule for the spatial integral term and Crank-Nicolson time-stepping scheme for the penalized nonlinear fBS equation which contains the fBS governing European option valuation as a special case. The truncation error of this discretization is shown to be of 2nd-order in both space and time. We will show the solution to the discretized system converges to the exact viscosity solution of the penalized fBS equation by proving that the numerical scheme is consistent, stable and monotone. Numerical results will be presented to demonstrate the accuracy and usefulness of the numerical scheme using some model fPDEs and fBS equations.

The organization of this paper is as follows. In Section 2, we will first give a brief account of the continuous LCP governing the American option valuation and apply the penalty method to the LCP to yield a penalized nonlinear fBS equation. We then discuss briely the unique solvability of the penalized fBS equation. In Section 3, we develop a discretization scheme for the fractional derivative and a full discretization scheme for the penalized fBS equation. The consistency, stability, and monotonicity of the numerical method are proved in Section 4. In Section 5, we first use a model fPDE to demonstrate that our numerical method is 2nd-order accurate in both space and time. We then present numerical results on European and American call and put options to show that the method produces practically useful results.

2. **The continuous problem and its unique solvability.**

2.1. **The continuous problem.**

A time-dependent random variable X_t is a Lévy process, if and only if it has independent and stationary increments with the following log-characteristic function in Lévy-Khintchine representation

$$\ln \mathbb{E}[e^{\mathrm{i}\xi X_t}] := t\Psi(\xi) = mit\xi - \frac{1}{2}\sigma^2 t\xi^2 + t\int_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}\xi x} - 1 - \mathrm{i}\xi h(x))W(dx),$$

where $\mathrm{i} = \sqrt{-1}$, $m \in \mathbb{R}$ is the drift rate, $\sigma \geq 0$ is the (constant) volatility, $h(x)$ is a truncation function, W is the Lévy measure satisfying

$$\int_{\mathbb{R}} \min\{1, x^2\} W(dx) < \infty,$$

and $\Psi(\xi)$ is the characteristic exponent of the Lévy process which is a combination of a drift component, a Brownian motion component and a jump component. These three components are determined by the Lévy-Khintchine triplet (m, σ^2, W). If the Lévy measure is of the form $W(dx) = w(x)dx$, $w(x)$ is then called the Lévy density. For an LS process, the Lévy density is given by

$$w_{LS}(x) = \begin{cases} Dq|x|^{-1-\alpha} & \text{for} \qquad x < 0, \\ Dpx^{-1-\alpha} & \text{for} \qquad x > 0, \end{cases}$$

where $D > 0$, $p, q \in [-1, 1]$ and $p + q = 1$ satisfying $0 < \alpha \leq 2$. The characteristic exponent of the LS process is

$$\Psi_{LS}(\xi) = -\frac{\sigma^\alpha}{4\cos(\alpha\pi/2)}\left[(1-s)(\mathrm{i}\xi)^\alpha + (1+s)(-\mathrm{i}\xi)^\alpha\right] + \mathrm{i}m\xi.$$

The parameters α and σ are respectively the stability index and scaling parameter. The parameter $s := p - q$ is called the skewness parameter satisfying $-1 \leq s \leq 1$, and m is a location parameter. When $s = 1$ (resp. $s = -1$) the random variable X is maximally skewed to the left (resp. right). When $\alpha = 2$ and $s = 0$, it becomes Gaussian case. A particular feature of the FMLS process is that it only exhibits downwards jumps, while upwards movements have continuous paths. The characteristic exponent of the LS process with $s = -1$, is

$$\Psi_{FMLS}(\xi) = \frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right)(-\mathrm{i}\xi)^\alpha, \tag{2.1}$$

where $\nu := \frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right)$ is the convexity adjustment of the random walk.

For a given function $u(x)$, one form of the α-th derivative of u is

$$ {}_{x_0}D_x^\alpha u(x) = \frac{u(x_0)}{\Gamma(1-\alpha)(x-x_0)^\alpha} + \frac{u'(x_0)}{\Gamma(2-\alpha)(x-x_0)^{\alpha-1}} + \frac{1}{\Gamma(2-\alpha)} \int_{x_0}^{x} \frac{u''(\xi)}{(x-\xi)^{\alpha-1}} d\xi \tag{2.2} $$

for $x > x_0$, where x_0 is a given real number and $\Gamma(\cdot)$ denotes the Gamma function. When $u(x_0) = 0$ and $u'(x_0) = 0$, it reduces to the Caputo's representation of the fractional partial derivative.

It is shown in [8] that the value U of a European option written on a stock, whose price S follows (1.1) with $L_t = \Psi_{FMLS}$ defined in (2.1), is determined by the following fBS equation:

$$ \mathcal{L}U := -U_t + aU_x - b \cdot {}_{x_{\min}}D_x^\alpha U + rU = 0 \tag{2.3} $$

for $(x,t) \in I \times [0,T) := (x_{\min}, x_{\max}) \times [0,T)$ with the boundary and terminal conditions:

$$ U(x_{\min}, t) = U_0(t), \qquad U(x_{\max}, t) = U_1(t) \tag{2.4} $$

$$ U(x,T) = U^*(x), \tag{2.5} $$

satisfying the compatibility conditions $U_0(T) = U^*(x_{\min})$ and $U_1(T) = U^*(x_{\max})$, where $x = \ln S$, $x_{\min} << 0$ and $x_{\max} > 0$ are two constants representing the lower and upper bounds for x, and

$$ a = -r - \frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right), \qquad b = -\frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right). $$

In (2.3), r is the risk-free rate and $\alpha \in (1,2)$ is the order of fractional derivative, and $U_0(t)$, $U_1(t)$ and $U^*(x)$ are known functions. For vanilla options, $U^*(x) = \max\{e^x - K, 0\}$ for a call and $U^*(x) = \max\{K - e^x, 0\}$ for a put, where K denotes the strike price of the option.

Note that the original spatial solution domain is $(-\infty, \infty)$. In computation, we truncate this infinite domain by the lower and upper bounds $x_{\min}$ and $x_{\max}$, as done in (2.3)–(2.4). Thus, we assume that $x_{\min}$ and $x_{\max}$ are chosen such that $x_{\min} << 0$ and $e^{x_{\max}} >> K$. Note that the use of (2.2) requires $U(x_{\min}, t) = 0$ and $U_x(x_{\min}, t) = 0$ in order to avoid the singularity at $x_{\min}$. Both of these can be achieved, up to a truncation error, by transforming (2.3) into an fBS equation satisfying the homogeneous Dirichlet boundary condition.

Let $F(x,t)$ be the function defined by

$$ F(x,t) = \frac{U_1(t) - U_0(t)}{e^{x_{\max}} - e^{x_{\min}}}(e^x - e^{x_{\min}}) + U_0(t). $$

Clearly, $F(x,t)$ satisfies the boundary conditions (2.4) and (2.9) (up to a truncation error). Also, F is an exponential function of x and thus it is invariant under the 1st and α-th order differentiation operations with respect to x. Taking $\mathcal{L}F$ from both sides of (2.3) and introducing a new variable $V(x,t) = F(x) - U(x,t)$, we have

$$ \mathcal{L}V(x,t) = f(x,t), \tag{2.6} $$

where $f(x,t) = \mathcal{L}F$. The boundary and terminal conditions (2.4)–(2.5) then become

$$ V(x_{\min}, t) = 0 = V(x_{\max}, t), \quad t \in [0,T), \tag{2.7} $$

$$ V(x,T) = V^*(x) := F(x,T) - U^*(x), \quad x \in I. \tag{2.8} $$

From the definitions of u and F and $x = \ln S$, we have

$$\lim_{x_{\min}\to-\infty} V_x(x_{\min}, t) = -\lim_{x_{\min}\to-\infty}\left(\frac{Ke^{x_{\min}}}{e^{x_{\max}} - e^{x_{\min}}} + U_S(S(x_{\min}), t)e^{x_{\min}}\right) = 0, \tag{2.9}$$

since $U_S(S(x), t)$ is bounded on $(-\infty, x_{\max})$. Thus, $V_x(x_{\min}, t) \to 0$ exponentially as $x_{\min} \to -\infty$. Therefore, from (2.2), (2.7) and (2.9), we see that the fractional derivative involved in $\mathcal{L}V$ now becomes the following Caputo's form:

$$_{x_{\min}}D_x^{\alpha}V(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_{x_{\min}}^{x}\frac{V_{xx}(\xi,t)}{(x-\xi)^{\alpha-1}}d\xi, \tag{2.10}$$

up to a truncation error when $x_{\min} << 0$.

While the price of an European option is governed by (2.3)–(2.5), it is well known that the price of an American option is determined by the following a linear complementarity problem [23]:

$$\begin{aligned} \mathcal{L}U &\geq 0, & (2.11a)\\ U &\geq U^*, & (2.11b)\\ \mathcal{L}U\cdot(U-U^*) &= 0 & (2.11c)\end{aligned}$$

for $(x, t) \in I \times [0, T)$, with a set of boundary and terminal conditions of the form (2.4)–(2.5). Under the same transformation as for obtaining (2.6), it is easy to show that (2.11) can be written as

$$\begin{aligned} \mathcal{L}V &\leq f,\\ V &\leq V^*,\\ (\mathcal{L}V - f)\cdot(V - V^*) &= 0\end{aligned}$$

for $(x, t) \in I \times [0, T)$ satisfying the boundary and terminal conditions (2.7)–(2.8).

In [23], the authors proposed and analyzed a power penalty method for (2.12) for the case that the Black-Scholes operator is the 2nd order differential operator, i.e., $\alpha = 2$ in (2.3). The penalty method proposed in [23] is extended to (2.12) in [6]. In this work, we use the linear penalty method to solve (2.12) as used in [1, 28], i.e., we approximate (2.12) by the the following penalized fBS equation:

$$\mathcal{L}V_\lambda(x,t) + \lambda[V_\lambda(x,t) - V^*(x)]_+ = f(x,t), \quad (x,t) \in I \times (0,T), \tag{2.13}$$

with the boundary and initial conditions (2.7)–(2.8), where $\lambda > 1$ is a penalty constant and $[z]_+ = \max\{0, z\}$ for any function z. The convergence of V_λ to V for the case that $\alpha \in (1, 2)$ and $\lambda > 1$ is in [6]. In the present work, we will concentrate on the construction and the convergence of a discretization scheme for (2.13).

Note that (2.13) contains (2.6) as a special case when $\lambda = 0$. Therefore, in what follows, our discussion will be focused on (2.13) unless mentioned otherwise.

2.2. **The variational problem and its solvability.** We now consider the unique solvability of (2.13). First, we formulate it as a variational problem, and then we show that the variational problem has a unique solution. We start this discussion by introducing some function spaces.

For an open set $\Omega \subset \mathbb{R}$ and $1 \leq p \leq \infty$, we let $L^p(\Omega) = \left\{v : (\int_\Omega |v(x)|^p dx)^{1/p} < \infty\right\}$ denote the space of all p-power integrable functions on Ω equipped with the usual L^p-norm $\|\cdot\|_{L^p(\Omega)}$, and $(\cdot,\cdot)$ denote the usual inner product. For any $\gamma \in (0, 1]$, we let

$$H^\gamma(\mathbb{R}) := \left\{v : v \text{ and } {}_{-\infty}D_x^\gamma v \in L^2(\mathbb{R})\right\}.$$

$|\cdot|_\gamma$ and $\|\cdot\|_\gamma$ are two functionals defined respectively as

$$|u|_\gamma = \|_{-\infty}D_x^\gamma u\|_{L^2(\Omega)}, \quad \|u\|_\gamma = (\|u\|^2_{L^2(\Omega)} + \|_{-\infty}D_x^\gamma u\|^2_{L^2(\Omega)})^{1/2},$$

for any $u \in H^\gamma(\mathbb{R})$. Then it is easy to show that $|\cdot|_\gamma$ and $\|\cdot\|_\gamma$ are semi-norm and norm on $H^\gamma(\mathbb{R})$ respectively. It has been shown in [11] that $H^\gamma(\mathbb{R})$ equipped with $\|\cdot\|_\gamma$ is a Sobolev space.

Similarly to the above definition of fractional Sobolev space, we also define the Sobolev space of functions having a support on the finite interval $I = (x_{\min}, x_{\max})$ given by

$$H_0^\gamma(I) = \left\{v : v, ({}_{x_{\min}}D_x^\gamma u) \in L^2(I), v(x_{\min}) = v(x_{\max}) = 0\right\},$$

where ${}_{x_{\min}}D_x^\gamma u$ is defined in (2.2) with x_0 replaced with $x_{\min}$.

In what follows, we also use $\langle\cdot,\cdot\rangle$ to denote the duality paring between $H_0^\gamma(I)$ and its dual space $H_0^{-\gamma}(I)$. Using the notation defined above, we pose the following variational problem: with the boundary and initial conditions (2.7)–(2.8):

Problem 2.1. *Find $u_\lambda(t) \in H^{\alpha/2}(I)$ for $t \in [0,T)$ almost everywhere (a.e.) satisfying (2.8), such that, for all $v \in H^{\alpha/2}(I)$,*

$$\left\langle -\frac{\partial u_\lambda(t)}{\partial t}, v\right\rangle + A\left(u_\lambda(t), v; t\right) + \left(\lambda\left[u_\lambda(x,t) - V^*(x)\right]_+, v\right) = (f(t), v) \tag{2.14}$$

where $A(\cdot,\cdot;t)$ is a bilinear form defined by

$$A(u,v;t) = a\left\langle\frac{\partial u}{\partial x}, v\right\rangle + b\left\langle {}_{x_{\min}}D_x^{\alpha-1}u, \frac{\partial v}{\partial x}\right\rangle + r(u,v), \quad u,v \in H_0^{\alpha/2}(I).$$

Using integration by parts, it is easy to verify that Problem (2.1) is the variational problem of (2.13) with (2.6)–(2.8) (cf. [11]). It has also been shown in [11] that the bilinear form $A(\cdot,\cdot;t)$ is coercive and continuous, as given in the following lemma:

Lemma 2.1. *There exist positive constants C_1 and C_2 such that for any $v, w \in H_0^{\alpha/2}(I)$,*

$$A(v,v;t) \geq C_1\|v\|^2_{\alpha/2}$$
$$A(v,w;t) \leq C_2\|v\|_{\alpha/2}\|w\|_{\alpha/2}$$

for $t \in (0,T)$ a.e..

Using this lemma, we have the following theorem.

Theorem 2.1. *Problem 2.1 has a unique solution.*

The proof of this theorem is just a re-statement of that of Theorem 3.1 of [23] which shows that the nonlinear form on the RHS of (2.14), (i.e., the nonlinear operator on the RHS of (2.13)) is strongly monotone and continuous, based on Lemma 2.1. For brevity, we omit this discussion.

3. Discretization of (2.13) .

We now consider the discretization of the fractional partial differential equation (2.13). Several efficient and accurate discretization scheme have been proposed for linear, nonlinear and penalized 2nd-order Black-Scholes equations [26, 22, 1, 16, 14]. However, to our best knowledge, there is no essentially work on the numerical approximation of the penalized fBS equation. In this section, we propose a discretization method for (2.13).

3.1. **Discretization of the α-th derivative.** Let the interval $I = (x_{\min}, x_{\max})$ be divided into M sub-intervals with mesh nodes

$$x_i = x_{\min} + ih, \qquad i = 0, 1, ...M, \tag{3.1}$$

where $h = (x_{\max} - x_{\min})/M$. For clarity, we omit the variable t in this subsection. When $1 < \alpha < 2$, from (2.10) we have that, for any $i \in \{1, 2, ..., M\}$,

$$\begin{aligned} {}_{x_{\min}}D_x^\alpha V(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_{x_{\min}}^{x_i} \frac{V_{xx}(y)}{(x_i - y)^{\alpha-1}} dy \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{V_{xx}(y)}{(x_i - y)^{\alpha-1}} dy \\ &=: \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{i} I_k^i. \end{aligned} \tag{3.2}$$

To discretize I_k^i, we first rewrite it as

$$\begin{aligned} I_k^i &= \int_{x_{k-1}}^{x_k} \frac{V_{xx}(y) - V_{xx}(x_k)}{(x_i - y)^{\alpha-1}} dy + \int_{x_{k-1}}^{x_k} \frac{V_{xx}(x_k)}{(x_i - y)^{\alpha-1}} dy \\ &\approx V_{xxx}(x_k) \int_{x_{k-1}}^{x_k} \frac{y - x_k}{(x_i - y)^{\alpha-1}} dy + V_{xx}(x_k) \int_{x_{k-1}}^{x_k} \frac{1}{(x_i - y)^{\alpha-1}} dy. \end{aligned} \tag{3.3}$$

In the above, we used a truncated Taylor expansion for $V_{xx}(y) - V_{xx}(x_k)$. The derivatives $V_{xx}(x_k)$ and $V_{xxx}(x_k)$ are then approximated respectively by the following finite differences:

$$V_{xx}(x_k) \approx \delta_x^2 V_k := \frac{V_{k-1} - 2V_k + V_{k+1}}{h^2}, \tag{3.4}$$

$$V_{xxx}(x_k) \approx \delta_x^3 V_k := \frac{-V_{k-2} + 3V_{k-1} - 3V_k + V_{k+1}}{h^3}, \tag{3.5}$$

where V_i denotes an approximation to $V(x_i)$ for any feasible i. The two integrals on the RHS of (3.3) can be evaluated exactly, and using (3.1), it is easy to show that these integrals are given by

$$P_k^i := \frac{1}{h^{2-\alpha}} \int_{x_{k-1}}^{x_k} \frac{1}{(x_i - y)^{\alpha-1}} dy = \frac{(i-k+1)^{2-\alpha} - (i-k)^{2-\alpha}}{(2-\alpha)}, \tag{3.6}$$

$$Q_k^i := \frac{1}{h^{3-\alpha}} \int_{x_{k-1}}^{x_k} \frac{y - x_k}{(x_i - y)^{\alpha-1}} dy = \frac{(i-k+1)^{3-\alpha} - (i-k)^{3-\alpha}}{(2-\alpha)(3-\alpha)} - \frac{(i-k+1)^{2-\alpha}}{2-\alpha}. \tag{3.7}$$

Therefore, I_k^i can then be approximated by

$$I_k^i \approx h^{-\alpha} \left[P_k^i (V_{k-1} - 2V_k + V_{k+1}) + Q_k^i (-V_{k-2} + 3V_{k-1} - 3V_k + V_{k+1}) \right]. \tag{3.8}$$

Since P_k^i and Q_k^i are functions of $(i-k)$, we have $P_k^i = P_{k+1}^{i+1}$ and $Q_k^i = Q_{k+1}^{i+1}$ for all $k = 1, 2, ..., i$.

For P_k^i and Q_k^i, we have the following lemma:

Lemma 3.1. *For any $i = 1, 2, ..., M$, the sequences $\{P_k^i\}_{k=1}^i$ and $\{Q_k^i\}_{k=1}^i$ satisfy*

$$0 < P_1^i < P_2^i < ... < P_{i-1}^i < P_i^i,$$
$$Q_1^i < Q_2^i < ... < Q_{i-1}^i < Q_i^i < 0.$$

The proof is trivial and thus it is omitted.

Replacing I_k^i in (3.2) with the RHS of (3.8), we define the following approximation to the α-derivative at x_i:

$$_{x_{\min}}D_x^\alpha V(x_i) = \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{i} I_k^i \approx D_h^\alpha V := \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{i+1} g_k^i V_{i-k+1}, \tag{3.9}$$

where g_k^i's are given by

$$\begin{aligned}
g_0^i &= Q_i^i + P_i^i,\\
g_1^i &= Q_{i-1}^i - 3Q_i^i + P_{i-1}^i - 2P_i^i,\\
g_2^i &= Q_{i-2}^i - 3Q_{i-1}^i + 3Q_i^i + P_{i-2}^i - 2P_{i-1}^i + P_i^i,\\
g_k^i &= Q_{i-k}^i - 3Q_{i-k+1}^i + 3Q_{i-k+2}^i - Q_{i-k+3}^i + P_{i-k}^i - 2P_{i-k+1}^i + P_{i-k+2}^i,
\end{aligned}$$

for $k = 3, 4, ..., i+1$. Clearly, g_k^i is a weighted sum of Q_{i-k}^i, Q_{i-k+1}^i, Q_{i-k+2}^i, Q_{i-k+3}^i, P_{i-k}^i, P_{i-k+1}^i and P_{i-k+2}^i. Using (3.6) and (3.7) one can derive that

$$g_0^i = \frac{1}{(2-\alpha)(3-\alpha)}, \tag{3.10}$$

$$g_1^i = \frac{2^{3-\alpha}-4}{(2-\alpha)(3-\alpha)}, \tag{3.11}$$

$$g_2^i = \frac{3^{3-\alpha}-4\times 2^{3-\alpha}+6}{(2-\alpha)(3-\alpha)}, \tag{3.12}$$

$$g_k^i = \frac{1}{(2-\alpha)(3-\alpha)}[(k+1)^{3-\alpha} - 4k^{3-\alpha} + 6(k-1)^{3-\alpha} - 4(k-2)^{3-\alpha} + (k-3)^{3-\alpha}], \tag{3.13}$$

for any $k = 3, 4, ..., i+1$. From (3.10)–(3.13), we see that g_k^i's are independent of i, and so in what follows, we write g_k^i as g_k. The following lemmas establish some properties of g_k.

Lemma 3.2. *For any $\alpha \in (0,1)$, the coefficients $g_k(\alpha)$, $\quad k = 0, 1, ..., i+1$ satisfy:*

(1) $g_0 > 0$, $g_1 < 0$, and $g_k > 0$ for $k = 3, 4, 5, ..., i+1$,
(2) there exists an $\alpha^* \in (1,2)$ such that $g_2(\alpha) < 0$ when $\alpha \in (1, \alpha^*)$ and $g_2(\alpha) > 0$ when $\alpha \in (\alpha^*, 2)$, and
(3) $\sum_{k=0}^{i+1} g_k < 0$.

Proof. (1) From (3.10) and (3.12), it is easy to verify that $g_0 > 0$ and $g_1 < 0$ for any $\alpha \in [0,1]$.

Let us consider $\bar{g}_{k+1} = g_{k+1}(2-\alpha)(3-\alpha)$ for $k \geq 2$. From (3.13), we have

$$\begin{aligned}
\bar{g}_{k+1} &= \left[(k+2)^{3-\alpha} - 4(k+1)^{3-\alpha} + 6k^{3-\alpha} - 4(k-1)^{3-\alpha} + (k-2)^{3-\alpha}\right]\\
&= \left[(k+2)^{3-\alpha} - 3(k+1)^{3-\alpha} + 3k^{3-\alpha} - (k-1)^{3-\alpha}\right]\\
&- \left[(k+1)^{3-\alpha} - 3k^{3-\alpha} + 3(k-1)^{3-\alpha} - (k-2)^{3-\alpha}\right]\\
&=: f_1(k+1) - f_1(k).
\end{aligned}$$

To show $\bar{g}_{k+1} > 0$, it suffices to show $f_1(k)$ is strictly increasing, which is equivalent to showing that $f_1'(k) > 0$. Differentiating f_1 with respect to k gives

$$\begin{aligned} f_1'(k+1) &= (3-\alpha)\left[(k+2)^{2-\alpha} - 3(k+1)^{2-\alpha} + 3k^{2-\alpha} - (k-1)^{2-\alpha}\right] \\ &= (3-\alpha)\left\{[(k+2)^{2-\alpha} - 2(k+1)^{2-\alpha} + k^{2-\alpha}]\right. \\ &\left.-[(k+1)^{2-\alpha} - 2k^{2-\alpha} + (k-1)^{2-\alpha}]\right\} \\ &=: (3-\alpha)(f_2(k+1) - f_2(k)). \end{aligned}$$

We now show $f_2(k)$ is also strictly increasing. Differentiating $f_2(k)$, we have

$$\begin{aligned} f_2'(k) &= (2-\alpha)\left[(k+1)^{1-\alpha} - 2k^{1-\alpha} + (k-1)^{1-\alpha}\right] \\ &= (2-\alpha)\{\left[(k+1)^{1-\alpha} - k^{1-\alpha}\right] - \left[k^{1-\alpha} - (k-1)^{1-\alpha}\right]\} \\ &=: (2-\alpha)(f_3(k) - f_3(k-1)). \end{aligned}$$

It remains to prove f_3 is strictly increasing. Differentiating $f_3(k)$, we have

$$f_3'(k) = (1-\alpha)\left[(k+1)^{-\alpha} - k^{-\alpha}\right] > 0,$$

when $1 < \alpha < 2$. Therefore $\bar{g}_{k+1} > 0$ for $k \geq 2$, or, $g_k = \bar{g}_k/[(2-\alpha)(3-\alpha)] > 0$ for $k \geq 3$.

(2) The proof of this is trivial and we omit it.

(3) For the finite difference scheme in (3.9), the approximation of the α-th derivative of $V(x) = 1$ becomes exact, i.e.,

$$D_h^\alpha 1 = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{\infty} g_k = 0.$$

Therefore $\sum_{k=0}^{\infty} g_k = 0$. Since $g_k > 0, k \geq 3$, then we have $\sum_{k=i+2}^{\infty} g_k > 0$, so the partial sum $\sum_{k=0}^{i+1} g_k < 0$. □

3.2. **Full discretization of (2.13).** For a positive integer N, let $(0, T)$ be divided into N sub-intervals with the mesh points

$$t_j = (N-j)\Delta t, \qquad j = 0, 1, ..., N,$$

where $\Delta t = T/N$. Using the central differencing for the first derivative in space, Crank-Nicolson time stepping method and the scheme (3.9) for the α-th derivative, we construct the following discretization scheme for (2.3):

$$\begin{aligned} &\mathcal{L}_{h,\Delta t} V_i^j \qquad (3.14)\\ &:= \frac{V_i^{j+1} - V_i^j}{\Delta t} + \frac{1}{2}\left(a\frac{V_{i+1}^{j+1} - V_{i-1}^{j+1}}{2h} - \frac{b}{\Gamma(2-\alpha)h^\alpha}\sum_{k=0}^{i+1} g_k V_{i+1-k}^{j+1} + rV_i^{j+1} + d_i(V_i^{j+1})\right) \\ &+ \frac{1}{2}\left(a\frac{V_{i+1}^{j} - V_{i-1}^{j}}{2h} - \frac{b}{\Gamma(2-\alpha)h^\alpha}\sum_{k=0}^{i+1} g_k V_{i+1-k}^{j} + rV_i^{j} + d_i(V_i^{j})\right) = \frac{1}{2}(f_i^j + f_i^{j+1}), \qquad (3.15) \end{aligned}$$

with the boundary and terminal conditions:

$$V_0^{j+1} = 0 = V_M^{j+1}, \quad V_i^0 = V^*(ih), \tag{3.16}$$

for $i = 1, 2, ..., M-1$ and $j = 0, 2, ..., N-1$, where V_i^j denotes an approximation to $V(x_i, t_j)$, $f_i^k = f(x_i, t_k)$ for $k = j$ and $j+1$, and $d_i(V_i^j) = \lambda[V_i^j - V_i^*]_+$, V^* is defined in (2.8).

Let $\mu = -b\frac{\Delta t}{\Gamma(2-\alpha)h^\alpha}$ and $\eta = a\frac{\Delta t}{2h}$, we rewrite equation (3.15) as

$$
\begin{aligned}
&\left[1 + \frac{1}{2}\left(\mu g_1 + r\Delta t\right)\right] V_i^{j+1} + \frac{1}{2} d_i(V_i^{j+1})\Delta t \\
&\frac{1}{2}\left[(\eta + \mu g_0)V_{i+1}^{j+1} + \left(-\eta + \mu g_2\right) V_{i-1}^{j+1} + \mu \sum_{k=3}^{i+1} g_k V_{i-k+1}^{j+1}\right] \\
&= \left[1 - \frac{1}{2}\left(\mu g_1 + r\Delta t\right)\right] V_i^j - \frac{1}{2} d_i(V_i^{j+1})\Delta t \\
&-\frac{1}{2}\left[(\eta + \mu g_0)V_{i+1}^{j} + \left(-\eta + \mu g_2\right) V_{i-1}^{j} + \mu \sum_{k=3}^{i+1} g_k V_{i-k+1}^{j}\right] + \frac{\Delta t}{2}(f_i^{j+1} + f_i^j)
\end{aligned}
$$

for $j = 0, 1, ..., N-1$. This system can further be written as the following matrix equation:

$$
\left(I + \frac{1}{2}C\right)\vec{V}^{j+1} + \frac{1}{2}D(\vec{V}^{j+1}) = \left(I - \frac{1}{2}C\right)\vec{V}^{j} - \frac{1}{2}D(\vec{V}^{j}) + \frac{\Delta t}{2}\left(\vec{f}^{j+1} + \vec{f}^{j}\right), \tag{3.17}
$$

where $\vec{V}^k = (V_1^k, V_2^k, \cdots, V_{M-1}^k)^\top$, $\vec{f}^k = (f_1^k, f_2^k, ..., f_{M-1}^k)^\top$ and $D(\vec{V}^k) = \Delta t\lambda \text{diag}([V_1^k - V_1^*]_+, ..., [V_{M-1}^k - V_{M-1}^*]_+)$ for all feasible ks. The system matrix $C = (c_{ij})$ is an $(M-1)\times(M-1)$ matrix of the form

$$
C = G + B + r\Delta t I, \tag{3.18}
$$

where I denotes the $(M-1)\times(M-1)$ identity matrix,

$$
G = \mu \begin{bmatrix} g_1 & g_0 & 0 & 0 & \cdots & 0 \\ g_2 & g_1 & g_0 & 0 & \cdots & 0 \\ g_3 & g_2 & g_1 & g_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ g_{M-2} & g_{M-3} & \cdots & g_2 & g_1 & g_0 \\ g_{M-1} & g_{M-2} & \cdots & g_3 & g_2 & g_1 \end{bmatrix}, \quad B = \eta \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix}.
$$

Clearly, G and B arise respectively from the discretization of the α-th derivative and the first derivative $\partial V/\partial x$. It is trivial to show that the elements c_{ij} are given by

$$
c_{ij} = \begin{cases} \mu g_0 + \eta, & j = i+1 \\ \mu g_1 + r\Delta t, & j = i \\ \mu g_2 - \eta, & j = i-1 \\ \mu g_k, & j = i-k+1, \quad k = 3, 4, ..., i, \\ 0, & \text{otherwise.} \end{cases}
$$

We comment that C is a Toeplitz matrix as the elements in each diagonal of C is a constant. Also, (3.17) is a nonlinear system for $\vec{V}^{j+1}$ since the diagonal matrix D is a nonlinear and non-smooth function of $\vec{V}^{j+1}$. Note that D is monotonically increasing in $\vec{V}^{j+1}$, in practice a Newton's or non-smooth Newton's method can be used for solving (3.17) numerically.

4. **Consistency, stability and monotonicity of the scheme.** In this section, we show that the solution to (3.15) converges to the viscosity solution to (2.13) by proving that the numerical scheme proposed in the previous section is consistent, stable and monotone. We start this discussion with the following theorem:

Theorem 4.1. *The finite difference scheme for* (2.3)*, defined by* (3.15)*, is consistent, with a truncation error of order* $\mathcal{O}(\Delta t^2 + h^2)$*.*

Proof. In what follows, we let C denote a generic positive constant, independent of h and Δt. We first consider the truncation error in the approximation D_h^α to D^α at x_i for any $i = 1, 2, ..., M-1$. From (3.2) and (3.3) we have that, for any function $V(x)$ sufficiently smooth on I,

$$\begin{aligned}
{}_{x_0}D_x^\alpha V(x_i) &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{V''(y)}{(x_i-y)^{\alpha-1}} dy \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{i} \left[\int_{x_{k-1}}^{x_k} \frac{V''(y) - V''(x_k)}{(x_i-y)^{\alpha-1}} dy + \int_{x_{k-1}}^{x_k} \frac{V''(x_k)}{(x_i-y)^{\alpha-1}} dy \right] \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{i} \left[V'''(x_k) \int_{x_{k-1}}^{x_k} \frac{y-x_k}{(x_i-y)^{\alpha-1}} dy \right. \\
&\left. + V''(x_k) \int_{x_{k-1}}^{x_k} \frac{1}{(x_i-y)^{\alpha-1}} dy \right] + E_i,
\end{aligned}$$

where E_i denotes the following remainder in the approximation of $V''(y) - V''(x_{k-1})$ by a truncated Taylor's expansion:

$$E_i = \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{V^{(4)}(\xi_k)}{2} \frac{(y-x_k)^2}{(x_i-y)^{\alpha-1}} dy$$

for a $\xi_k \in (x_{k-1}, x_k)$. From this equality we have, for $i = 1, 2, ..., M-1$,

$$|E_i| \leq \frac{||V^{(4)}||_\infty}{2\Gamma(2-\alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{(y-x_k)^2}{(x_i-y)^{\alpha-1}} dy, \tag{4.1}$$

where $||V^{(4)}||_\infty$ denotes the maximum norm of $V^{(4)}$ on I. Since $x_i - x_k = (i-k)h$, integrating by parts gives

$$
\begin{aligned}
&\sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}\frac{(x_k-y)^2}{(x_i-y)^{\alpha-1}}dy\\
=&\sum_{k=1}^{i}\Bigg[\frac{(x_k-x_{k-1})^2(x_i-x_{k-1})^{2-\alpha}}{2-\alpha}-\frac{2(x_k-x_{k-1})(x_i-x_{k-1})^{3-\alpha}}{(2-\alpha)(3-\alpha)}\\
&\qquad-\frac{2(x_i-x_k)^{4-\alpha}-2(x_i-x_{k-1})^{4-\alpha}}{(2-\alpha)(3-\alpha)(4-\alpha)}\Bigg]\\
=&\frac{h^{4-\alpha}}{2-\alpha}\sum_{k=1}^{i}\left[(i-k+1)^{2-\alpha}-\frac{2(i-k+1)^{3-\alpha}}{3-\alpha}-\frac{2(i-k)^{4-\alpha}-2(i-k+1)^{4-\alpha}}{(3-\alpha)(4-\alpha)}\right]\\
=&\frac{2h^{4-\alpha}}{2-\alpha}\sum_{k=1}^{i}\left[\frac{k^{2-\alpha}}{2}-\frac{k^{3-\alpha}}{3-\alpha}+\frac{k^{4-\alpha}-(k-1)^{4-\alpha}}{(3-\alpha)(4-\alpha)}\right]\\
=:&2h^{4-\alpha}\sum_{k=1}^{i}S_k,
\end{aligned}
\tag{4.2}
$$

where the definition of S_k is obvious. Using the expansion

$$
\begin{aligned}
(k-1)^{4-\alpha}=&\,k^{4-\alpha}-(4-\alpha)k^{3-\alpha}+\frac{(4-\alpha)(3-\alpha)}{2!}k^{2-\alpha}-\frac{(4-\alpha)(3-\alpha)(2-\alpha)}{3!}k^{1-\alpha}\\
&+\frac{(4-\alpha)(3-\alpha)(2-\alpha)(1-\alpha)}{4!}k^{-\alpha}-\frac{(4-\alpha)(3-\alpha)(2-\alpha)(1-\alpha)(-\alpha)}{5!}k^{-\alpha-1}\\
&+\frac{(4-\alpha)(3-\alpha)(2-\alpha)(1-\alpha)(-\alpha)(-\alpha-1)}{6!}k^{-\alpha-2}+\dots,
\end{aligned}
$$

we can easily show that

$$
S_k=\frac{1}{3!}k^{1-\alpha}-\frac{(1-\alpha)}{4!}k^{-\alpha}+\frac{(1-\alpha)(-\alpha)}{5!}k^{-\alpha-1}-\frac{(1-\alpha)(-\alpha)(-\alpha-1)}{6!}k^{-\alpha-2}+\dots.
$$

Thus, $\sum_{k=1}^{i}S_k$ can be written as

$$
\begin{aligned}
\sum_{k=1}^{i}S_k=&\frac{1}{3!}\sum_{k=1}^{i}k^{1-\alpha}+\left[-\frac{(1-\alpha)}{4!}+\frac{(1-\alpha)(-\alpha)}{5!}-\frac{(1-\alpha)(-\alpha)(-\alpha-1)}{6!}\dots\right]\\
&+\sum_{k=2}^{i}\left[-\frac{(1-\alpha)}{4!}k^{-\alpha}+\frac{(1-\alpha)(-\alpha)}{5!}k^{-\alpha-1}-\frac{(1-\alpha)(-\alpha)(-\alpha-1)}{6!}k^{-\alpha-2}\dots\right].
\end{aligned}
\tag{4.3}
$$

The first term on the RHS of (4.3) has the following upper bound:

$$
\sum_{k=1}^{i}k^{1-\alpha}\le\int_0^i s^{1-\alpha}ds=\frac{i^{2-\alpha}}{2-\alpha}. \tag{4.4}
$$

For the Gamma function, when $0<a<1$ and n is a positive integer, we have

$$
\frac{\Gamma(n+a)}{\Gamma(n+1)}<n^{a-1}
$$

from which, it can be shown that

$$\frac{\Gamma(\alpha+j-1)}{\Gamma(4+j)} < \frac{j^{\alpha-2}}{(j+3)(j+2)(j+1)} < j^{\alpha-5}.$$

Using this inequality and the properties of the Gamma function, we have

$$\begin{aligned}
&-\frac{(1-\alpha)}{4!}+\frac{(1-\alpha)(-\alpha)}{5!}-\frac{(1-\alpha)(-\alpha)(-\alpha-1)}{6!}\\
=&\sum_{j=1}^{\infty}\frac{(\alpha-1)\alpha\cdots(\alpha+j-2)}{(3+j)!}=\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{\infty}\frac{\Gamma(\alpha+j-1)}{\Gamma(4+j)}\\
=&\frac{1}{\Gamma(\alpha-1)}\left(\frac{\Gamma(\alpha)}{\Gamma(5)}+\sum_{j=2}^{\infty}\frac{\Gamma(\alpha+j-1)}{\Gamma(4+j)}\right)<\frac{1}{\Gamma(\alpha-1)}\left(\frac{\Gamma(\alpha)}{\Gamma(5)}+\sum_{j=2}^{\infty}j^{5-\alpha}\right)\\
<&\frac{1}{\Gamma(\alpha-1)}\left(\frac{\Gamma(\alpha)}{\Gamma(5)}+\int_1^{\infty}j^{5-\alpha}dj\right)=\frac{1}{\Gamma(\alpha-1)}\left(\frac{\Gamma(\alpha)}{\Gamma(5)}+\frac{1}{4-\alpha}\right)<C. \qquad (4.5)
\end{aligned}$$

The third term on the RHS of (4.3) can be estimated as follows:

$$\sum_{k=2}^{i}\left[-\frac{(1-\alpha)}{4!}k^{-\alpha}+\frac{(1-\alpha)(-\alpha)}{5!}k^{-\alpha-1}-\frac{(1-\alpha)(-\alpha)(-\alpha-1)}{6!}k^{-\alpha-2}\dots\right] \qquad (4.6)$$

$$\begin{aligned}
\le& C\sum_{k=2}^{i}\left(k^{-\alpha}+k^{-\alpha-1}+k^{-\alpha-2}\dots\right)=C\sum_{k=2}^{i}k^{-\alpha}\frac{k}{k-1}\le C\sum_{k=2}^{i}k^{-\alpha}\\
\le& C\int_1^{i}j^{-\alpha}dj=C\frac{1}{\alpha-1}\left(1-i^{1-\alpha}\right)\le C.
\end{aligned}$$

Replacing the three terms on the RHS of (4.3) with their respective upper bounds (4.4)–(4.6) and combining the resulting estimate with (4.2) and (4.1), we have

$$\begin{aligned}
|E_i| &\le \frac{h^{4-\alpha}}{\Gamma(2-\alpha)}\|V^{(4)}\|_\infty\sum_{k=1}^{i}S_k\\
&\le \frac{1}{\Gamma(2-\alpha)}\|V^{(4)}\|_\infty h^{4-\alpha}\left(\frac{1}{3!}\frac{i^{2-\alpha}}{2-\alpha}+C\right)\\
&\le Ch^{4-\alpha}i^{2-\alpha}\\
&\le Ch^2, \qquad (4.7)
\end{aligned}$$

for any $i=1,2,...,M-1$, since $i<M=1/h$.

It is standard to verify that the finite difference operators in (3.4) and (3.5) satisfy

$$|V''(x_k)-\delta_x^2V(x_k)|\le Ch^2, \qquad |V'''(x_k)-\delta_x^3V(x_k)|\le Ch. \qquad (4.8)$$

From (4.7), (4.8) and (3.9), we see that the truncation error in the discretization of the α-th derivative is bounded by

$$\begin{aligned}
& \left|{}_{x_0}D_{x_i}^{\alpha}V(x_i) - D_h^{\alpha}V(x_i)\right| \\
\leq\ & C\left(h^2 + h\sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}\frac{y-x_{k-1}}{(x_i-y)^{\alpha-1}}dy + h^2\sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}\frac{1}{(x_i-y)^{\alpha-1}}dy\right) \\
\leq\ & Ch^2\left(1+\int_{x_0}^{x_i}\frac{1}{(x_i-y)^{\alpha-1}}dy\right) \\
\leq\ & Ch^2, \qquad (4.9)
\end{aligned}$$

as the last integral in the above expression exists. Finally, it is well-known that Crank-Nicolson time stepping scheme, the central differencing and the mid-point quadrature rule used in (3.15) are all at least 2nd-order accurate on uniform meshes. Combining this fact with (4.9), we have

$$|\mathcal{L}V(x_i) - \mathcal{L}_{h,\Delta t}V(x_i)| \leq C(h^2+\Delta t^2).$$

Therefore, the discretization is consistent. □

Theorem 4.2. *The finite difference scheme defined by* (3.15) *is unconditionally stable.*

Proof. Let us first consider the case that $\lambda = 0$ in (2.13), or $d_i = 0$ in (3.15) for all i.

We use the discrete Fourier transform to prove the stability of the Crank-Nicolson method. Using μ and η introduced in Subsection 3.2, we rewrite (3.15) as

$$\begin{aligned}
& V_i^{j+1} - V_i^j + \frac{1}{2}\left[\eta\left(V_{i+1}^{j+1} - V_{i-1}^{j+1}\right) + \mu\sum_{k=0}^{i+1}g_kV_{i-k+1}^{j+1} + r\Delta tV_i^{j+1}\right] \\
& +\frac{1}{2}\left[\eta\left(V_{i+1}^{j} - V_{i-1}^{j}\right) + \mu\sum_{k=0}^{i+1}g_kV_{i-k+1}^{j} + r\Delta tV_i^{j}\right] = \Delta t\bar{f}_i^j \qquad (4.10)
\end{aligned}$$

for $i = 1, 2, ..., M-1$ and any feasible j, where $\bar{f}_i^j = (f_i^j + f_i^{j+1})/2$. This system has the matrix form (3.17) and from the definition (3.18) we see that all the coefficient matrices in (3.17) are Toeplitz matrices. Thus, each of the terms in (3.17) can be written as convolution of a coefficient vector with a finite support, one of $(\cdots, 0, V_1^n, ..., V_{M-1}^n, 0, \cdots)$ and $(\cdots, 0, \bar{f}_1^j, ..., \bar{f}_{M-1}^j, 0, \cdots)$ for $n = j$ and $j+1$. Applying the discrete Fourier transform to the system, or equivalently replacing V_m^n and $\bar{f}_m^n$ in (4.10) with $W^n e^{m\xi h\mathrm{i}}$ and $\bar{F}^n e^{m\xi h\mathrm{i}}$ respectively for all admissible m and n, we have

$$\begin{aligned}
& (W^{j+1} - W^j)e^{i\xi h\mathrm{i}} + \frac{1}{2}e^{i\xi h\mathrm{i}}W^{j+1}\left[\eta(e^{\xi h\mathrm{i}} - e^{-\xi h\mathrm{i}}) + \mu\sum_{k=0}^{i+1}g_ke^{\xi(-k+1)h\mathrm{i}} + r\Delta t\right] \\
& +\frac{1}{2}e^{i\xi h\mathrm{i}}W^j\left[\eta(e^{\xi h\mathrm{i}} - e^{-\xi h\mathrm{i}}) + \mu\sum_{k=0}^{i+1}g_ke^{\xi(-k+1)h\mathrm{i}} + r\Delta t\right] = \Delta t\bar{F}^j,
\end{aligned}$$

where $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$, and W^n and $\bar{F}^n$ are respectively the discrete Fourier transform of V^n and $\bar{f}^n$ for $n = j$ and $j+1$. Dividing both sides by $e^{i\xi h\mathrm{i}}$ and rearranging the resulting equation, we get

$$\begin{aligned}
W^{j+1} &= W^j \frac{1-\frac{1}{2}\left[\eta(e^{\xi h \mathrm{i}}-e^{-\xi h \mathrm{i}})+\mu\sum_{k=0}^{i+1} g_k e^{\xi(-k+1)h\mathrm{i}}+r\Delta t\right]}{1+\frac{1}{2}\left[\eta(e^{\xi h \mathrm{i}}-e^{-\xi h \mathrm{i}})+\mu\sum_{k=0}^{i+1} g_k e^{\xi(-k+1)h\mathrm{i}}+r\Delta t\right]} \\
&\quad + \frac{\Delta t \bar{F}^j}{1+\frac{1}{2}\left[\eta(e^{\xi h \mathrm{i}}-e^{-\xi h \mathrm{i}})+\mu\sum_{k=0}^{i+1} g_k e^{\xi(-k+1)h\mathrm{i}}+r\Delta t\right]} \\
&= W^j \frac{2-(A+B\mathrm{i})}{2+(A+B\mathrm{i})} + \frac{2\Delta t \bar{F}^j}{2+(A+B\mathrm{i})} \\
&= W^j \frac{(2-(A-B\mathrm{i}))(2+(A-B\mathrm{i}))}{(2+A)^2+B^2} + \frac{2(2+A-B\mathrm{i})}{(2+A)^2+B^2}\Delta t \bar{F}^j,
\end{aligned}$$

where

$$A = \mu \sum_{k=0}^{i+1} g_k \cos\left((1-k)\xi h\right) + r\Delta t, \qquad B = \eta \sin(\xi h) + \mu \sum_{k=0}^{i+1} g_k \sin\left((1-k)\xi h\right).$$

Taking magnitudes on both sides of the above equation gives

$$\begin{aligned}
|W^{j+1}| &\leq \frac{|W^j|\sqrt{(2-A)^2+B^2}\sqrt{(2+A)^2+B^2}}{(2+A)^2+B^2} + |\bar{F}^j|\frac{2\Delta t}{\sqrt{(2+A)^2+B^2}} \\
&= |W^j|\sqrt{\frac{(2-A)^2+B^2}{(2+A)^2+B^2}} + |\bar{F}^j|\frac{2\Delta t}{\sqrt{(2+A)^2+B^2}} \qquad (4.11)
\end{aligned}$$

It is known that the α-th derivative of $\cos(x)$ is $\cos(x+\alpha\pi/2)$. Thus, using the consistency result in Theorem 4.1 we have, when $\xi \neq 0$,

$$\begin{aligned}
\frac{1}{\Gamma(2-\alpha)h^\alpha}\sum_{k=0}^{i+1} g_k \cos\left(\xi h(1-k)\right) &= \frac{1}{\Gamma(2-\alpha)h^\alpha}\sum_{k=0}^{i+1} g_k \cos\left(\xi h((i+1-k)-i)\right) \\
&= \frac{1}{\Gamma(2-\alpha)h^\alpha}\sum_{k=0}^{i+1} g_k \cos\left(\xi(x_{i+1-k}-x_i)\right) \\
&= \frac{1}{|\xi|^\alpha} D^\alpha \cos\left(\xi(x-x_i)\right)|_{x=x_i} + \mathcal{O}(h^2) \\
&= \frac{1}{|\xi|^\alpha}\cos\left(\xi(x-x_i)+\frac{\pi}{2}\alpha\right)|_{x=x_i} + \mathcal{O}(h^2) \\
&= \frac{1}{|\xi|^\alpha}\cos\left(\frac{\pi}{2}\alpha\right) + \mathcal{O}(h^2).
\end{aligned}$$

Using this estimate, from the definition of A we see that

$$A = -\frac{b\Delta t}{2|\xi|^\alpha}\cos\left(\frac{\pi}{2}\alpha\right) + r\Delta t + \mathcal{O}(h^2\Delta t) = \mathcal{O}(\Delta t), \qquad (4.12)$$

when h is sufficiently small. Similarly, when $\xi = 0$,

$$\frac{1}{\Gamma(2-\alpha)h^\alpha}\sum_{k=0}^{i+1} g_k \cos\left(\xi h(1-k)\right) = \frac{1}{\Gamma(2-\alpha)h^\alpha}\sum_{k=0}^{i+1} g_k = D^\alpha 1 + \mathcal{O}(h^2) = \mathcal{O}(h^2).$$

Thus (4.12) still holds for this case. Therefore, using (4.12), we have from (4.11) when Δt is sufficiently small,

$$\begin{aligned}|W^{j+1}| &\leq |W^j|\sqrt{\frac{(2-\mathcal{O}(\Delta t))^2+B^2}{(2+\mathcal{O}(\Delta t))^2+B^2}} + 2\Delta t\left|\bar{F}^j\right| \leq \left|W^j\right|(1+L\Delta t) + 2\Delta t\left|\bar{F}^j\right| \\ &\leq \left|W^{j-1}\right|(1+L\Delta t)^2 + 2\Delta t\left[\left|\bar{F}^{j-1}\right|(1+L\Delta t) + \left|\bar{F}^j\right|\right] \\ &\leq \cdots \leq \left|W^0\right|(1+L\Delta t)^{j+1} + 2\Delta t\sum_{k=0}^{j}\left|\bar{F}^k\right|(1+L\Delta t)^{j-k}.\end{aligned}$$

for a constant $L > 0$, independent of h and Δt. Note $(1+1/z)^z$ is monotonically increasing for $z > 0$ and $\lim_{z\to+\infty}(1+1/z)^z = e$. We have from the above estimate and $\Delta t = T/N$

$$\begin{aligned}\left|W^{j+1}\right| &\leq \left(\left|W^0\right| + 2\Delta t\sum_{k=0}^{j}\left|\bar{F}^k\right|\right)\left(1+\frac{LT}{N}\right)^N \leq e^{LT}\left(\left|v^0\right| + 2\Delta t\sum_{k=0}^{j}\left|\bar{F}^k\right|\right) \\ &\leq \bar{L}\left(\left|v^0\right| + \frac{1}{N}\sum_{k=0}^{j}\left|\bar{F}^k\right|\right),\end{aligned}$$

where $\bar{L}$ denotes a generic positive constant, independent of h and Δt. Using Cauchy-Schwarz inequality, we then have

$$\left|W^{j+1}\right|^2 \leq \bar{L}\left(\left|W^0\right|^2 + \frac{j}{N^2}\sum_{k=0}^{j}\left|\bar{F}^k\right|^2\right) \leq \bar{L}\left(\left|W^0\right|^2 + \frac{1}{N}\sum_{k=0}^{j}\left|\bar{F}^k\right|^2\right)$$

for any $j \leq N-1$. Note W^{j+1}, W^0 and $\bar{F}^k$ are all functions of $\xi \in [-\pi/h, \pi/h]$. For any function $u \in H_0^{\alpha/2}(x_{\min}, x_{\max})$, let $||u||_{0,h} = \left(h\sum_{i=1}^{M-1}|u_i|^2\right)^{1/2}$ denote the discrete L^2-norm of u. Then, using the properties of the discrete Fourier and its inverse transforms (particularly Parseval's equality) we have

$$\begin{aligned}||V^{j+1}||_{0,h}^2 &= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h}\left|v^{j+1}\right|^2 d\xi \leq \frac{\bar{L}}{2\pi}\left(\int_{-\pi/h}^{\pi/h}|W^0|^2 d\xi + \frac{1}{N}\sum_{k=0}^{j}\int_{-\pi/h}^{\pi/h}\left|\bar{F}^k\right|^2 d\xi\right) \\ &= \bar{L}\left(||V^0||_{0,h}^2 + \frac{1}{N}\sum_{k=0}^{j}||\bar{f}^k||_{0,h}^2\right) \leq \bar{L}\left(||V^0||_{0,h}^2 + ||f||_{L^\infty(I\times(0,T))}^2\right),\end{aligned}$$

from which we have (recall $\bar{L}$ is a generic positive constant)

$$||V^{j+1}||_{0,h} \leq \bar{L}\left(||V^0||_{0,h} + ||f||_{L^\infty(I\times(0,T))}\right).$$

Therefore, the numerical method is unconditionally is stable when $\lambda = 0$. In the case that $\lambda > 0$, from the definition of d_i we have that, for any feasible i and k,

$$d_i(V_i^k) = \lambda[V_i^k - V_i^*]_+ = \frac{\lambda}{2}\left[\mathrm{sign}(V_i^k - V_i^*) + 1\right](V_i^k - V_i^*) =: \lambda\rho_i^k(V_i^k - V_i^*),$$

where $\rho_i^k = 0$ or 1. Therefore, (3.15) can be written as

$$V_i^{j+1} - V_i^j + \frac{1}{2}\left[\eta\left(V_{i+1}^{j+1} - V_{i-1}^{j+1}\right) + \mu\sum_{k=0}^{i+1} g_k V_{i-k+1}^{j+1} + \left(r + \lambda\rho_i^{j+1}\right)\Delta t V_i^{j+1}\right]$$
$$+ \frac{1}{2}\left[\eta\left(V_{i+1}^{j} - V_{i-1}^{j}\right) + \mu\sum_{k=0}^{i+1} g_k V_{i-k+1}^{j} + \left(r + \lambda\rho_i^{j+1}\right)\Delta t V_i^{j}\right] = \Delta t\left(\bar{f}_i^j + \bar{\rho}_i^j\lambda V_i^*\right), \tag{4.13}$$

where $\bar{\rho}_i^j = (\rho_i^i + \rho_i^{j+1})/2 = 0$, $1/2$ or 1. Comparing (4.13) with (4.10) we see that (4.13) is in the same form as (4.10) with $r\Delta t$ replaced with $\Delta(r + \lambda\rho_i^k)$ for $k = j$ or $j+1$ and $\Delta\bar{f}_i^j$ with $\Delta t\left(\bar{f}_i^j + \bar{\rho}_i^j\lambda V_i^*\right)$. All of these terms are of the order $\mathcal{O}(\Delta t)$. Thus, following the same analysis presented above for $\lambda = 0$, we have that the scheme is also stable when $\lambda > 0$. Therefore, we have proved the theorem. □

We now show that the numerical scheme is monotone.

Theorem 4.3. *(Monotonicity) The discretization scheme established in* (3.15) *is monotone when* $\Delta t \leq \frac{2}{r}$.

Proof. We notation simplicity, we first consider the case that $\lambda = 0$. Let

$$F_i^{j+1}\left(V_i^{j+1}, V_{i+1}^{j+1}, V_{i-1}^{j+1}, \dots, V_0^{j+1}, V_i^j, V_{i+1}^j, V_{i-1}^j, \dots, V_0^j\right)$$
$$:= \left[1 + \frac{1}{2}(\mu g_1 + r\Delta t)\right]V_i^{j+1} + \frac{1}{2}(\eta + \mu g_0)V_{i+1}^{j+1} - \frac{1}{2}\left(\eta - \mu g_2\right)V_{i-1}^{j+1} + \frac{\mu}{2}\sum_{k=3}^{i+1} g_k V_{i-k+1}^{j+1}$$
$$- \left[1 - \frac{1}{2}(\mu g_1 + r\Delta t)\right]V_i^{j} + \frac{1}{2}(\eta + \mu g_0)V_{i+1}^{j} - \frac{1}{2}\left(\eta - \mu g_2\right)V_{i-1}^{j} + \frac{\mu}{2}\sum_{k=3}^{i+1} g_k V_{i-k+1}^{j}.$$

To prove that F_i^{j+1} is monotone, we first show $\left(\sum_{k=0}^{i+1} g_k\right) - \frac{1}{2}g_1 > 0$. Let $\beta = 3 - \alpha$. From (3.10)–(3.13) we have

$$\sum_{k=0}^{3} g_k - \frac{1}{2}g_1 = g_0\left[-1 + 3\times 2^\beta - 3\times 3^\beta + 4^\beta - \frac{1}{2}\times\left(2^\beta - 4\right)\right] = g_0 f(\beta),$$

where $f(\beta) := (1 + 2.5\times 2^\beta - 3\times 3^\beta + 4^\beta)$. It now suffices to show that $f(\beta) > 0$ for $\beta \in (1,2)$. Since $f(2) = 0$, we need only to show that $f(\beta)$ is strictly decreasing for $\beta \in (1,2)$. Differentiating f gives

$$f'(\beta) = 2.5\times\ln(2)\times 2^\beta - 3\times\ln(3)\times 3^\beta + \ln(4)\times 4^\beta.$$

It is easy to show, even graphically, that $f'(\beta) < 0$ for all $\beta \in [1,2]$. Therefore,

$$\sum_{k=0}^{3} g_k - \frac{1}{2}g_1 > 0$$

From Lemma 3.2, we have that $g_k > 0$ for $k > 3$. Thus, when $i \geq 2$,

$$\sum_{k=0}^{i+1} g_k - \frac{1}{2}g_1 \geq \sum_{k=0}^{3} g_k - \frac{1}{2}g_1 > 0$$

We now use the above result to prove the monotonicity of F_i^{j+1}. When $\Delta t \leq \frac{2}{r}$, we have from the definition of F_i^{j+1} that, for any $\varepsilon > 0$ and feasible i and j,

$$
\begin{aligned}
&F_i^{j+1}\left(V_i^{j+1}, V_{i+1}^{j+1}+\varepsilon, V_{i-1}^{j+1}+\varepsilon, \ldots, V_0^{j+1}+\varepsilon, V_i^j+\varepsilon, V_{i+1}^j+\varepsilon, V_{i-1}^j+\varepsilon, \ldots, V_0^j+\varepsilon\right) \\
=&F_i^{j+1}\left(V_i^{j+1}, V_{i+1}^{j+1}, V_{i-1}^{j+1}, \ldots, V_0^{j+1}, V_i^j, V_{i+1}^j, V_{i-1}^j, \ldots, V_0^j\right) \\
&-\left[1-\frac{1}{2}(\mu g_1+r\Delta t)\right]\varepsilon+(\eta+\mu g_0)\varepsilon-(\eta-\mu g_2)\,\varepsilon+\mu\sum_{k=3}^{i+1} g_k\varepsilon \\
\leq&F_i^{j+1}\left(V_i^{j+1}, V_i^j, V_{i+1}^{j+1}, V_{i-1}^{j+1}, \ldots, V_0^{j+1}\right)+\mu\left(\sum_{k=0}^{i+1} g_k-\frac{1}{2}g_1\right)\varepsilon-\left(1-\frac{1}{2}r\Delta t\right)\varepsilon \\
\leq&F_i^{j+1}\left(V_i^{j+1}, V_i^j, V_{i+1}^{j+1}, V_{i-1}^{j+1}, \ldots, V_0^{j+1}\right),
\end{aligned}
$$

since $\mu < 0$.
Furthermore, since $g_1 < 0$, we have

$$
\begin{aligned}
&F_i^{j+1}\left(V_i^{j+1}+\varepsilon, V_{i+1}^{j+1}, V_{i-1}^{j+1}, \ldots, V_0^{j+1}, V_i^j, V_{i+1}^j, V_{i-1}^j, \ldots, V_0^j\right) \\
&= F_i^{j+1}\left(V_i^{j+1}, V_{i+1}^{j+1}, V_{i-1}^{j+1}, \ldots, V_0^{j+1}, V_i^j, V_{i+1}^j, V_{i-1}^j, \ldots, V_0^j\right)+\left[1+\frac{1}{2}\left(\mu g_1+r\Delta t\right)\right]\varepsilon \\
&> F_i^{j+1}\left(V_i^{j+1}, V_{i+1}^{j+1}, V_{i-1}^{j+1}, \ldots, V_0^{j+1}, V_i^j, V_{i+1}^j, V_{i-1}^j, \ldots, V_0^j\right).
\end{aligned}
$$

When $\lambda > 0$, it is easy to see $d(V_i^k)$ is monotonically increasing in V_i^k for any feasible k. Therefore, both $d(V_i^j)$ and $d(V_i^{j+1})$ are monotone and thus the scheme is also monotone. □

Combining Theorems 4.1, 4.2 and 4.3 ,we have the following convergence result.

Theorem 4.4. *Let V be the viscosity solution to (2.6) − (2.8) and $V_{h,\Delta t}$ be the solution to (3.15) − (3.16). Then, $V_{h,\Delta t}$ converges to V as $(h, \Delta t) \to (0, 0)$.*

In fact, conventionally, Theorems 4.1 and 4.2 already imply the convergence of our numerical scheme. Barles and Souganidis showed in [3] that any finite difference scheme for a general nonlinear 2nd-order PDE which is locally consistent, stable and monotone generates a solution converging uniformly on a compact subset of $[0, T] \times \mathbb{R}$ to the unique viscosity solution of the PDE. In [10] and [9], Cont and Tankov extended this result to partial integro-differential equations (PIDEs). Since (2.3) is an PIDE, Theorem 4.4 is just a consequence of the results established in [3, 10, 9].

We also comment that though the theoretical results in this section have been established for Crank-Nicolson's time-stepping scheme, they hold true for a general two-level time-stepping scheme with a splitting parameter $\theta \in [0.5, 1]$. However, when $\theta \in (0.5, 1]$, the truncation error in Theorem 4.1 is of the order $\mathcal{O}(\Delta t + h^2)$ instead of $\mathcal{O}(\Delta t^2 + h^2)$. For brevity, we omit this discussion.

5. **Numerical Experiments.** In this section, we first use an example with a known exact solution to demonstrate the rate of convergence of our scheme. We then show the usefulness and practicality of the method by applying it to several European and American option pricing problems. All the computations have been performed in double precision under MATLAB environment.

Example 1. Fractional diffusion equation with non-homogeneous boundary conditions:

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - {}_{-5}D_x^{1.5}u(x,t) = e^{2x}(1-2\sqrt{2}t), \quad 0<t\le 1, \quad -5<x<1,\\
&u(-5,t) = e^{-10}t, \quad 0<t\le 1,\\
&u(1,t) = e^2 t, \quad 0<t\le 1,\\
&u(x,0) = 0, \quad -5<x<1.
\end{aligned}$$

The exact solution to the above problem is $u(x,t) = te^{2x}$. Note for this test we have $u(-5,t) \approx 0$ and $u_x(-5,t) = 0$, and so, we may straightforwardly apply our numerical scheme to this test without the transformation used in Section 2. This problem is solved using a sequence of meshes $h_k = \Delta t_k = \frac{1}{5}\times 2^{-k}$ for $k = 0,1,...,5$. For each k, the following discrete maximum norm is computed:

$$E_i = \max_{0<j<N}\max_{0<i<M}\left\{\left|u(x_i,t_j) - U_i^j\right|\right\},$$

where $U = (U_i^j)$ denotes the numerical solution. These computed errors, along with computed rates of convergence $\log_2(E_{k+1}/E_k)$, for $k = 0,1,...,5$ are listed in Table 5.1, from which we see that the rates of convergence of our method are of order $\mathcal{O}(\Delta t^2 + h^2)$, coinciding with the truncation error established in Theorem 4.1. For comparison, we have also solved the problem using a combination of the Crank-Nicolson time-stepping scheme and two popular existing spatial finite difference methods proposed respectively in [19] and [17]. These two existing methods are denoted as LG and L2 respectively. The computed errors E_k^{LG}'s and E_k^{L2}'s and the rates of convergence for LG and L2 are also listed in Table 5.1, from which it is clear that both of the existing methods are 1st-order accurate, one order lower than our method.

$h=\Delta t=\frac{1}{5\times 2^k}$	E_k^{LG}	$\log_2\frac{E_{k+1}^{LG}}{E_k^{LG}}$	E_k^{L2}	$\log_2\frac{E_{k+1}^{L2}}{E_k^{L2}}$	E_k	$\log_2\frac{E_{k+1}}{E_k}$
$k=0$	0.12244		0.17372		0.023095	
$k=1$	0.06201	0.98	0.09873	0.82	0.049195	2.23
$k=2$	0.03120	0.99	0.05368	0.88	0.011224	2.13
$k=3$	0.01565	1.00	0.02829	0.92	0.000280	2.00
$k=4$	0.00784	1.00	0.01463	0.95	0.000070	2.00
$k=5$	0.00392	1.00	0.00748	0.97	0.000017	2.00

TABLE 5.1. Computed rates of convergence of ours and two existing methods for Example 1

Example 2: European call option governed by (2.3) with $r = 0.05$, $\sigma = 0.25$, $a = 0.0384$, $b = 0.0884$, $x_{\min} = \ln 0.1$, $x_{\max} = \ln 100$, $T = 1$ and $K = 50$. The initial and boundary conditions are respectively:

$$U(x,T) = \max(e^x - K, 0), \quad U(x_{\min},t) = 0, \quad U(x_{\max},t) = e^{x_{\max}} - Ke^{-r(T-t)}.$$

To solve this problem, we choose a mesh with $h = 0.02$ and $\Delta t = 1/52$. The numerical solution from our method for $\alpha = 1.5$ is plotted in Figure 5.1 against t and the original independent variable $S = e^x$. From the figure we see that the method is numerically very stable.

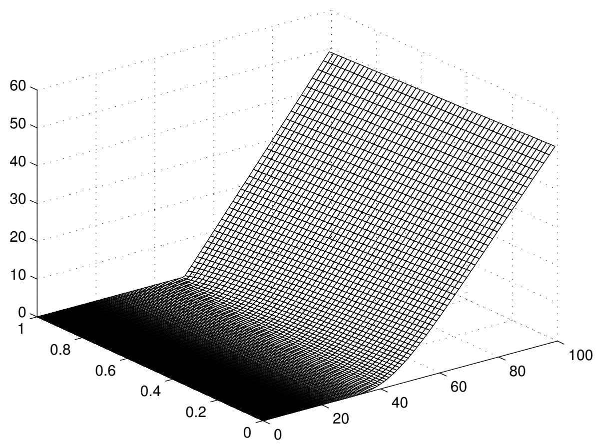

FIGURE 5.1. European call option value; $\alpha = 1.5$.

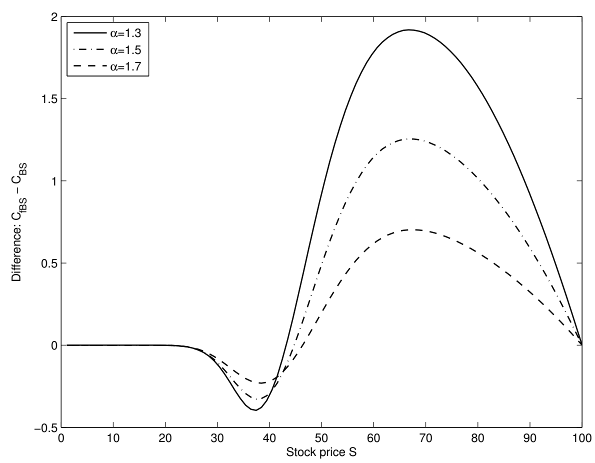

FIGURE 5.2. Difference between the European call option values from the fBS and BS equations.

To see the influence of α on the option price, we solve the problem for $\alpha = 1.3, 1.5, 1.7$ and 2, and plot in Figure 5.2, the difference between the values from the fBS equation (i.e., $\alpha < 2$) and the BS equation ($\alpha = 2$), denoted respectively $C_{fBS}(x, t)$ and $C_{BS}(x, t)$, at $t = 0$. From this figure we see that the call price increases as α decreases when S is greater than a critical value. This is likely because when α is close to 1, the solution to the fBS equation exhibits jump (or convection) nature, while when α is close to 2, it is of mainly diffusion nature. As a result, an option on a stock of jump nature is more expensive than one of diffusion nature, similar to the case that an option price is an increasing function of the volatility.

Example 3: European put option. The market parameters are the same as in Example 2, and the initial and boundary conditions are

$$U(x,T) = \max(K - e^x), \quad U(x_{\min}, t) = Ke^{-r(T-t)}, \quad U(x_{\max}, t) = 0.$$

The problem has been solved using the same mesh as that for Example 2, and the solution for $\alpha = 1.5$ is depicted in Figure 5.3. To gauge the influence of α on the value of the option, we have also solved the problem for $\alpha = 1.3, 1.5, 1.7$ and 2. The difference between the value $P_{fBS}(x,t)$ from the fBS model and the value $P_{BS}(x,t)$ from the BS model at $t = 0$ is depicted in Figure 5.4 for each of the chosen values of α. From the figure, we see that the differences are qualitatively identical to those in Figure 5.2. In fact, it can be easily shown using the *Put-Call Parity*: $C_{fBS} - C_{BS} = P_{fBS} - P_{BS}$. Our computation shows that

$$\max_i \left| \left[C_{fBS}(x_i, 0) - C_{BS}(x_i, 0)\right] - \left[P_{fBS}(x_i, 0) - P_{BS}(x_i, 0)\right] \right| = 0.0672,$$

indicating that the *Put-Call Parity* is satisfied by our numerical results from the fBS equation.

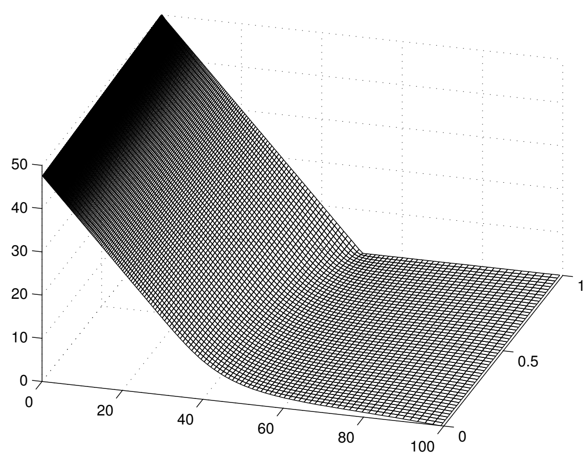

FIGURE 5.3. European put option value; $\alpha = 1.5$.

Example 4: American put option. The set of market and option parameters are the same as in Example 2. The boundary and initial conditions and the lower bound are given by

$$U(x,T) = U^*(x) = \max(K - e^x), \quad U(x_{\min}, t) = K, \quad U(x_{\max}, t) = 0.$$

This problem is solved by our numerical scheme on the uniform partition of the solution domain $(\ln(0.1), \ln(100)) \times (0, 1)$ in (x,t) with $M = 100$ and $N = 104$. The penalty parameter is chosen to be $\lambda = 10^{10}$. The difference between the computed value of this American option with $\alpha = 1.5$ and the lower bound U^* is plotted in, Figure 5.5. To see the difference between the American and European put options, we have also plot the difference between the computed European option value from Example 3 and U^* in Figure 5.5. From the figure, we see that the American option is more expensive than its European counterpart. Also, it can be seen that the value

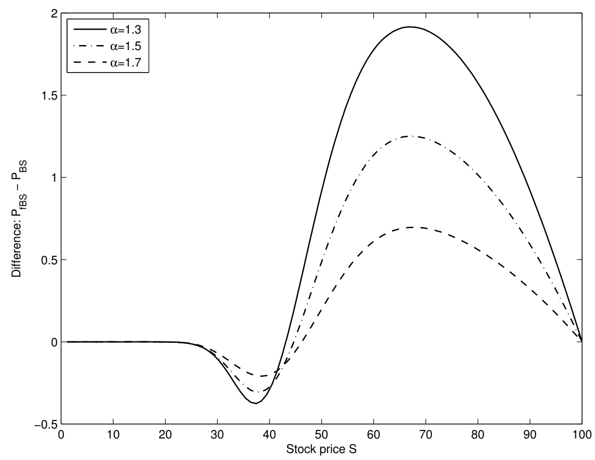

FIGURE 5.4. Difference between the European put option values from the fBS and BS equations.

of the American option is bounded below by U^*, while the value of the European option falls below U^* in a sub-region of the solution domain.

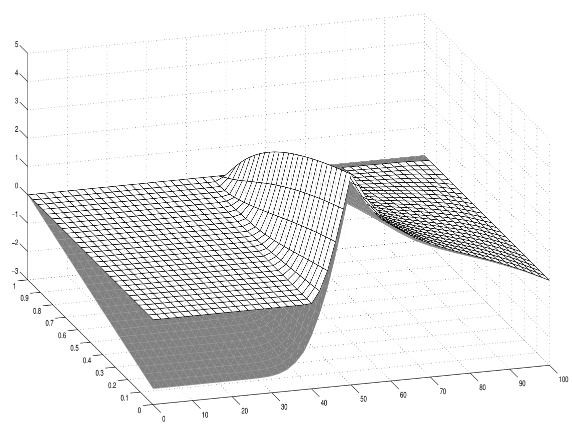

FIGURE 5.5. Differences between option prices and the lower bound: American option (upper), European option (lower).

Finally, we plot in Figure 5.6 the differences between the American put values from the fBS model and that from the BS model at the cross-section $t = 0$. From the figure we see that the value of the American option is a decreasing function of α when S is greater than a critical value, as observed in the Examples 2 and 3.

6. **Conclusion.** In this paper, we constructed and analyzed a novel 2nd-order finite difference method for the penalized fractional Black-Scholes equation governing

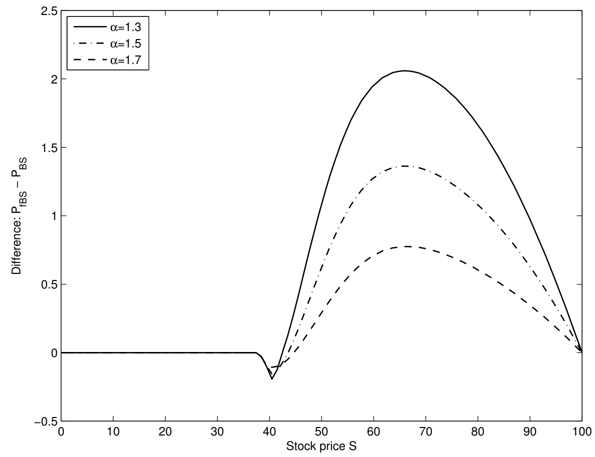

FIGURE 5.6. Differences between the American put option values from the fBS and BS equations.

European and American option pricing. We have proved the convergence of numerical method by showing that the method is consistent, stable and monotone. In particular, we have shown that the truncation error of the scheme is of 2nd-order as opposed to the 1st-order truncation errors for the existing numerical methods for the fBS equation. Numerical experiments have been carried out to verify the theoretical findings. The numerical results show that our method is 2nd-order accurate and gives practically useful and correct results when it is used for pricing European and American options.

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