Multi-Hop Non-Regenerative MIMO Relays – QoS Considerations

Yue Rong, *Member, IEEE*

***Abstract*—For non-regenerative multi-hop multiple-input multiple-output (MIMO) relay communication systems, the optimal source precoding matrix and the optimal relay amplifying matrices have been recently established for a broad class of objective functions subjecting to the transmission power constraint at each node. However, existing works do not consider any quality-of-service (QoS) constraints, which are important in practical communication systems. In this paper, we derive the optimal source and relay matrices of a multi-hop MIMO relay system that guarantee the predetermined QoS criteria be attained with the minimal total transmission power. In particular, we consider two types of receivers at the destination node: the linear minimal mean-squared error (MMSE) receiver and the nonlinear decision feedback equalizer (DFE) based on the MMSE criterion. We show that for both types of receivers, the solution to the original optimization problem can be upper-bounded by using a successive geometric programming (GP) approach and lower-bounded by utilizing a dual decomposition technique. Simulation results show that both bounds are tight, and to obtain the same QoS, the MIMO relay system using the nonlinear MMSE-DFE receiver requires substantially less total transmission power than the linear MMSE receiver-based system.**

***Index Terms*—MIMO relay, QoS, multi-hop relay, linear non-regenerative relay, MMSE, majorization.**

I. Introduction

As an efficient solution for wireless backhaul networks, non-regenerative multiple-input multiple-output (MIMO) relay communication systems recently have attracted much research interest [1]-[13]. For a two-hop MIMO relay system, the optimal relay amplifying matrix is obtained in [1]-[4] to maximize the mutual information (MI) between source and destination. In [5]-[7], optimal algorithms are developed to minimize the mean-squared error (MSE) of the signal waveform estimation at the destination.

The works of [1]-[7] have been generalized by [8], where a unified framework is established for two-hop linear non-regenerative MIMO relay systems with a broad class of objective functions. The framework in [8] has been further extended to multi-hop non-regenerative MIMO relay systems with arbitrary number of hops [9]. Both [8] and [9] consider a linear minimal mean-squared error (MMSE) receiver at the destination node. Recently, it has been shown in [10] that by using a nonlinear decision feedback equalizer (DFE) based on the MMSE criterion (referred to as the MMSE-DFE receiver) at the destination node, the system bit-error-rate (BER) performance can be significantly improved. Other recent works on multi-hop non-regenerative MIMO relay systems are found in [11]-[13].

Yue Rong is with the Department of Electrical and Computer Engineering, Curtin University, Bentley, WA 6102, Australia. E-mail: y.rong@curtin.edu.au.

The aim of [1]-[13] is to optimize a given objective function, subjecting to the transmission power constraint at each node. However, the quality-of-service (QoS) constraints are not addressed by [1]-[13]. Note that in practical communication systems, QoS criteria are very important. In this paper, we derive the optimal source and relay matrices of a multi-hop MIMO relay system which guarantee that the predetermined QoS criteria be attained with the minimal total transmission power. Based on the strong link between the diagonal elements of the MMSE matrix and most commonly used MIMO communication system objective functions [8]-[10], the QoS criteria are set up as the upper-bound of the MSE of each data stream. Moreover, we consider two types of receivers at the destination node: the linear MMSE receiver and the nonlinear MMSE-DFE receiver. For both receiver schemes, we show that the optimal source precoding matrix and the optimal relay amplifying matrices have a similar structure. In fact, the optimal source precoding matrix is the product of three matrices: the right singular matrix of the first-hop channel, a diagonal power loading matrix, and a (semi)-unitary matrix. The optimal amplifying matrix at any relay node is the product of the right singular matrix of its direct forward channel, a diagonal power loading matrix, and the Hermitian transpose of the left singular matrix of the direct backward channel. Compared with the linear MMSE receiver, an advantage of using the nonlinear MMSE-DFE receiver is that there is no constraint on the number of data streams. Note that the DFE receiver is also well-known as the successive interference cancellation (SIC) receiver [14].

After the optimal structure of the source and relay matrices is determined, the relay design problems boil down to optimal power loading problems with QoS constraints, which are nonconvex for both types of receivers and the globally optimal solution is difficult to obtain with a reasonable computational complexity. We show that the solution to the original optimization problems can be upper-bounded by using a successive geometric programming (GP) approach and lower-bounded by utilizing a dual decomposition technique. A theoretical analysis of the tightness of the upper and lower bounds is very difficult. Instead, we resort to numerical simulation to study the tightness of the bounds. Interestingly, we find that the upper and lower bounds are very close to each other. For practical applications the successive GP approach is preferred, since it has a lower computational complexity than the dual decomposition technique. Simulation results also demonstrate that to obtain the same QoS, the MIMO relay system using the nonlinear MMSE-DFE receiver requires substantially less total transmission power than the linear MMSE receiver-based system.

We would like to mention that the optimal source matrix design for a single-hop (point-to-point) MIMO system under QoS constraints is addressed in [15] for linear MMSE receiver, and in [16] for nonlinear MMSE-DFE receiver. Both [15] and [16] are summarized in [17]. Our paper generalizes the results in [15]-[17] from single-hop MIMO channels to multi-hop non-regenerative MIMO relay communication systems with any number of hops. Note that due to the introduction of multiple relay nodes, a rigorous proof of the theorems for multi-hop MIMO relay system is much more challenging than that for the single-hop MIMO channel, and is one contribution of this paper. The generalization from a single-hop MIMO system to multi-hop MIMO relay systems is significant.

The rest of this paper is organized as follows. In Section II, we introduce the model of a multi-hop linear non-regenerative MIMO relay communication system. The structures of the optimal source and relay matrices are shown in Section III for systems using the linear MMSE receiver and the non-linear MMSE-DFE receiver, respectively. In Section IV, we show some numerical examples. Conclusions are drawn in Section V.

II. System Model

We consider a wireless communication system with one source node, one destination node, and $L-1$ relay nodes ($L \geq 2$). We assume that due to the propagation path-loss, the signals transmitted by the ith node can only be received by its direct forward node, i.e., the $(i+1)$-th node. Thus, signals transmitted by the source node pass through L hops until they reach the destination node. We also assume that the number of antennas at each node is N_i, $i = 1, \cdots, L+1$, and the number of source symbols in each transmission is N_b. Like [1]-[10], a linear non-regenerative relay matrix is used at each relay.

The $N_1 \times 1$ signal vector transmitted by the source node is

$$\mathbf{s}_1 = \mathbf{F}_1 \mathbf{b} \tag{1}$$

where $\mathbf{b}$ is the $N_b \times 1$ source symbol vector, and $\mathbf{F}_1$ is the $N_1 \times N_b$ source precoding matrix. We assume that $\mathrm{E}[\mathbf{b}\mathbf{b}^H] = \mathbf{I}_{N_b}$, where $\mathrm{E}[\cdot]$ stands for the statistical expectation, $(\cdot)^H$ denotes the Hermitian transpose, and $\mathbf{I}_n$ is an $n \times n$ identity matrix. The $N_i \times 1$ signal vector received at the ith node is written as

$$\mathbf{y}_i = \mathbf{H}_{i-1}\mathbf{s}_{i-1} + \mathbf{v}_i, \qquad i = 2, \cdots, L+1 \tag{2}$$

where $\mathbf{H}_{i-1}$ is the $N_i \times N_{i-1}$ quasi-static MIMO fading channel matrix between the ith and the $(i-1)$-th nodes, i.e., the $(i-1)$-th hop, $\mathbf{v}_i$ is the $N_i \times 1$ independent and identically distributed (i.i.d.) additive white Gaussian noise (AWGN) vector at the ith node, and $\mathbf{s}_{i-1}$ is the $N_{i-1} \times 1$ signal vector transmitted by the $(i-1)$-th node. We assume that the noises are complex circularly symmetric with zero mean and unit variance.

The input-output relationship at node i is given by

$$\mathbf{s}_i = \mathbf{F}_i \mathbf{y}_i, \qquad i = 2, \cdots, L \tag{3}$$

where $\mathbf{F}_i$ is the $N_i \times N_i$ amplifying matrix at node i. Combining (1)-(3), we obtain the received signal vector at the destination node (the $(L+1)$-th node) as

$$\mathbf{y}_{L+1} = \bar{\mathbf{H}}\mathbf{b} + \bar{\mathbf{v}} \tag{4}$$

where $\bar{\mathbf{H}}$ and $\bar{\mathbf{v}}$ are the equivalent MIMO channel matrix and the noise vector, and given respectively by

$$\bar{\mathbf{H}} = \mathbf{H}_L\mathbf{F}_L \cdots \mathbf{H}_1\mathbf{F}_1 = \bigotimes_{i=L}^{1}(\mathbf{H}_i\mathbf{F}_i)$$
$$\bar{\mathbf{v}} = \mathbf{H}_L\mathbf{F}_L \cdots \mathbf{H}_2\mathbf{F}_2\mathbf{v}_2 + \cdots + \mathbf{H}_L\mathbf{F}_L\mathbf{v}_L + \mathbf{v}_{L+1} = \sum_{l=2}^{L}\Big(\bigotimes_{i=L}^{l}(\mathbf{H}_i\mathbf{F}_i)\mathbf{v}_l\Big) + \mathbf{v}_{L+1}.$$

Here for matrices $\mathbf{A}_i$, $\bigotimes_{i=l}^{k}(\mathbf{A}_i) \triangleq \mathbf{A}_l \cdots \mathbf{A}_k$.

From (1), we know that the power of the signals transmitted by the source node is $\mathrm{tr}(\mathbf{F}_1\mathbf{F}_1^H)$, where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. Based on (2) and (3), the power of the signal transmitted by the relay node i, $i = 2, \cdots, L$, is given by

$$\begin{aligned}&\mathrm{tr}\big(\mathrm{E}\big[\mathbf{s}_i\mathbf{s}_i^H\big]\big)\\ &= \mathrm{tr}\big(\mathbf{F}_i\mathrm{E}\big[\mathbf{y}_i\mathbf{y}_i^H\big]\mathbf{F}_i^H\big)\\ &= \mathrm{tr}\Bigg(\mathbf{F}_i\Bigg(\sum_{l=1}^{i-1}\Big(\bigotimes_{k=i-1}^{l}(\mathbf{H}_k\mathbf{F}_k)\bigotimes_{k=l}^{i-1}\big(\mathbf{F}_k^H\mathbf{H}_k^H\big)\Big) + \mathbf{I}_{N_i}\Bigg)\mathbf{F}_i^H\Bigg)\\ &\quad i = 2, \cdots, L.\end{aligned} \tag{5}$$

III. Optimal Source and Relay Matrices with QoS Constraints

In this section, we design non-regenerative multi-hop MIMO relay systems that meet the QoS requirements with the minimal (weighted) total transmission power. Based on the strong link between the diagonal elements of the MMSE matrix and most commonly used MIMO communication system objective functions such as the system BER and the source-destination MI [8]-[10], [15]-[17], the QoS criteria are set up as the upper-bound of MSE of each data stream. In particular, we derive the optimal source precoding matrix and the optimal relay amplifying matrices for destinations with the linear MMSE receiver, and the nonlinear MMSE-DFE receiver, respectively.

A. Linear MMSE receiver at the destination

Using a linear MMSE receiver, the estimated signal vector is

$$\hat{\mathbf{b}} = \mathbf{W}^H\mathbf{y}_{L+1} \tag{6}$$

where $\mathbf{W}$ is the $N_{L+1} \times N_b$ weight matrix of the linear MMSE receiver given by [18]

$$\mathbf{W} = (\bar{\mathbf{H}}\bar{\mathbf{H}}^H + \mathbf{C}_{\bar{v}})^{-1}\bar{\mathbf{H}}. \tag{7}$$

Here $(\cdot)^{-1}$ denotes the matrix inversion, and $\mathbf{C}_{\bar{v}} = \mathrm{E}[\bar{\mathbf{v}}\bar{\mathbf{v}}^H]$ is the noise covariance matrix

$$\mathbf{C}_{\bar{v}} = \sum_{l=2}^{L}\Big(\bigotimes_{i=L}^{l}(\mathbf{H}_i\mathbf{F}_i)\bigotimes_{i=l}^{L}(\mathbf{F}_i^H\mathbf{H}_i^H)\Big) + \mathbf{I}_{N_{L+1}}. \tag{8}$$

Using (4) and (6)-(8), the MMSE matrix denoted as $\mathbf{M}(\{\mathbf{F}_i\}) = \mathrm{E}\Big[(\hat{\mathbf{b}}-\mathbf{b})(\hat{\mathbf{b}}-\mathbf{b})^H\Big]$, is given by [9]

$$\begin{aligned}\mathbf{M}(\{\mathbf{F}_i\}) &= \left(\mathbf{I}_{N_b} + \bar{\mathbf{H}}^H \mathbf{C}_{\bar{v}}^{-1}\bar{\mathbf{H}}\right)^{-1} \\ &= \Bigg[\mathbf{I}_{N_b} + \bigotimes_{i=1}^{L}(\mathbf{F}_i^H\mathbf{H}_i^H)\Bigg(\sum_{l=2}^{L}\Big(\bigotimes_{i=L}^{l}(\mathbf{H}_i\mathbf{F}_i) \\ &\qquad \bigotimes_{i=l}^{L}(\mathbf{F}_i^H\mathbf{H}_i^H)\Big)+\mathbf{I}_{N_{L+1}}\Bigg)^{-1}\bigotimes_{i=L}^{1}(\mathbf{H}_i\mathbf{F}_i)\Bigg]^{-1} \end{aligned} \quad (9)$$

where $\{\mathbf{F}_i\} \triangleq \{\mathbf{F}_i, i=1,\cdots,L\}$.

The QoS-constrained optimization problem for MIMO relay systems using the linear MMSE receiver at the destination node can be written as

$$\min_{\{\mathbf{F}_i\}} \quad c_1\mathrm{tr}\big(\mathbf{F}_1\mathbf{F}_1^H\big) + \sum_{i=2}^{L} c_i \mathrm{tr}\Bigg(\mathbf{F}_i\Bigg(\sum_{l=1}^{i-1}\Big(\bigotimes_{k=i-1}^{l}(\mathbf{H}_k\mathbf{F}_k) \bigotimes_{k=l}^{i-1}\big(\mathbf{F}_k^H\mathbf{H}_k^H\big)\Big)+\mathbf{I}_{N_i}\Bigg)\mathbf{F}_i^H\Bigg) \quad (10)$$

$$\text{s.t.} \quad \mathbf{d}[\mathbf{M}(\{\mathbf{F}_i\})] \leq \mathbf{q} \quad (11)$$

where the objective function (10) is the weighted total transmission power consumed by the source node and all relay nodes in the relay network with $c_i > 0$, $i = 1,\cdots,L$, as weighting coefficients, $\mathbf{d}[\mathbf{A}]$ is a column vector containing all main diagonal elements of $\mathbf{A}$, and $\mathbf{q} = [q_1, q_2, \cdots, q_{N_b}]^T$ is the QoS requirement vector measured in terms of the MSE of each data stream that must be satisfied. A larger c_i over c_j, $j = 1,\cdots,L, j \neq i$, should be assigned to the ith node, if saving the power of the ith node over that of other nodes is desired. In the following, we focus on the objective function (10) with $c_i = 1$, $i = 1,\cdots,L$. Nonetheless, the algorithms developed in this paper can be straightforwardly extended to the case of general weighting coefficients. Note that due to the strong link between $\mathbf{d}[\mathbf{M}(\{\mathbf{F}_i\})]$ and the system BER and the source-destination MI [8]-[10], the QoS constraints in (11) can be equivalently represented as the BER and/or the MI constraint in each data stream. Obviously, from (9) we can see that any meaningful $\mathbf{q}$ should satisfy $0 < q_i < 1$, $i = 1,\cdots,N_b$. Without loss of generality, we assume that the elements of $\mathbf{q}$ are arranged in an *increasing* order. The following definition from [19] is required in order to solve the problem (10), (11).

DEFINITION 1 [19, 1.A.1, 1.A.2]: Consider any two real-valued $N \times 1$ vectors $\mathbf{x}, \mathbf{y}$, let $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[N]}$, $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[N]}$ denote the elements of $\mathbf{x}$ and $\mathbf{y}$ sorted in decreasing order, respectively, and $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(N)}$, $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(N)}$ denote the elements of $\mathbf{x}$ and $\mathbf{y}$ sorted in increasing order, respectively. We say that $\mathbf{x}$ is additively majorized by $\mathbf{y}$, denoted as $\mathbf{x} \prec_+ \mathbf{y}$, if $\sum_{i=1}^{n} x_{[i]} \leq \sum_{i=1}^{n} y_{[i]}$, for $n = 1,\cdots,N-1$, and $\sum_{i=1}^{N} x_{[i]} = \sum_{i=1}^{N} y_{[i]}$. We say that $\mathbf{x}$ is weakly additively submajorized by $\mathbf{y}$, denoted as $\mathbf{x} \prec_{+(\mathrm{w})} \mathbf{y}$, if $\sum_{i=1}^{n} x_{[i]} \leq \sum_{i=1}^{n} y_{[i]}$, for $n = 1,\cdots,N$. We say that $\mathbf{x}$ is weakly additively supermajorized by $\mathbf{y}$, denoted as $\mathbf{x} \prec^{+(\mathrm{w})} \mathbf{y}$, if $\sum_{i=k}^{N} x_{[i]} \geq \sum_{i=k}^{N} y_{[i]}$, or $\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}$, for $k = 1,\cdots,N$.

Let us write the singular value decomposition (SVD) of $\mathbf{H}_i$ as

$$\mathbf{H}_i = \mathbf{U}_i\mathbf{\Sigma}_i\mathbf{V}_i^H, \qquad i = 1,\cdots,L \quad (12)$$

where the dimensions of $\mathbf{U}_i$, $\mathbf{\Sigma}_i$, $\mathbf{V}_i$ are $N_{i+1} \times N_{i+1}$, $N_{i+1} \times N_i$, $N_i \times N_i$, respectively. We assume that the main diagonal elements of $\mathbf{\Sigma}_i$, $i = 1,\cdots,L$, are arranged in the *decreasing* order. Let us introduce $R_h \triangleq \min(\mathrm{rank}(\mathbf{H}_1), \mathrm{rank}(\mathbf{H}_2), \cdots, \mathrm{rank}(\mathbf{H}_L))$, where $\mathrm{rank}(\cdot)$ denotes the rank of a matrix. The following theorem establishes the structure of the optimal source precoding matrix and relay amplifying matrices when the linear MMSE receiver is applied at the destination node.

THEOREM 1: Assuming $N_b \leq R_h$ and $\mathrm{rank}(\mathbf{F}_i) = N_b, i = 1,\cdots,L$, for the linear non-regenerative multi-hop MIMO relay design problem (10), (11), the optimal source and relay matrices $\mathbf{F}_i$, $i = 1,\cdots,L$, are given by

$$\mathbf{F}_1 = \mathbf{V}_{1,1}\mathbf{\Lambda}_1\mathbf{U}_{F_1}^H, \quad \mathbf{F}_i = \mathbf{V}_{i,1}\mathbf{\Lambda}_i\mathbf{U}_{i-1,1}^H, \quad i = 2,\cdots,L \quad (13)$$

where $\mathbf{\Lambda}_i$, $i = 1,\cdots,L$, are $N_b \times N_b$ diagonal matrices, $\mathbf{U}_{F_1}$ is an $N_b \times N_b$ unitary matrix such that $[\mathbf{M}(\{\mathbf{F}_i\})]_{k,k} = q_k$, $k = 1,\cdots,N_b$, and $\mathbf{U}_{i,1}$ and $\mathbf{V}_{i,1}$ contain the leftmost N_b vectors of $\mathbf{U}_i$ and $\mathbf{V}_i$, respectively. Here for a matrix $\mathbf{A}$, $[\mathbf{A}]_{k,k}$ stands for the kth main diagonal element of $\mathbf{A}$.

PROOF: See Appendix A. □

The assumption of $N_b \leq R_h$ is motivated by the fact that using a linear receiver at the destination, the maximal number of independent data streams that can be sent from source to destination for any given $\{\mathbf{F}_i\}$ is no more than R_h. Moreover, the assumption of $\mathrm{rank}(\mathbf{F}_i) = N_b, i = 1,\cdots,L$, is sufficient to allow N_b independent data streams to be sent from source to destination.

From Theorem 1 we find that the optimal source precoding matrix, the optimal relay amplifying matrices, and the MMSE receiver matrix jointly diagonalize the multi-hop MIMO relay channel $\bar{\mathbf{H}}$ *after a rotation* $\mathbf{U}_{F_1}$ of the source precoding matrix. Substituting (13) back into (5), the transmission power at the source and relay nodes can be respectively written as

$$\mathrm{tr}\big(\mathbf{F}_1\mathbf{F}_1^H\big) = \sum_{k=1}^{N_b}\lambda_{1,k}^2 \quad (14)$$

$$\mathrm{tr}\Bigg(\mathbf{F}_i\Bigg(\sum_{l=1}^{i-1}\Big(\bigotimes_{k=i-1}^{l}(\mathbf{H}_k\mathbf{F}_k)\bigotimes_{k=l}^{i-1}\big(\mathbf{F}_k^H\mathbf{H}_k^H\big)\Big)+\mathbf{I}_{N_i}\Bigg)\mathbf{F}_i^H\Bigg) = \sum_{k=1}^{N_b}\lambda_{i,k}^2\left(\sum_{j=1}^{i-1}\prod_{l=j}^{i-1}\lambda_{l,k}^2\sigma_{l,k}^2+1\right), \; i = 2,\cdots,L \quad (15)$$

where $\lambda_{i,k}$ and $\sigma_{i,k}$, $i = 1,\cdots,L, k = 1,\cdots,N_b$, are the kth main diagonal elements of $\mathbf{\Lambda}_i$ and $\mathbf{\Sigma}_i$, respectively. Using (14), (15), and (93) in Appendix A, the optimal power loading parameters $\boldsymbol{\lambda} \triangleq [\lambda_{1,1},\cdots,\lambda_{L,N_b}]^T$ can be obtained by solving the following problem

$$\min_{\boldsymbol{\lambda}} \quad \sum_{k=1}^{N_b}\lambda_{1,k}^2+\sum_{i=2}^{L}\sum_{k=1}^{N_b}\lambda_{i,k}^2\left(\sum_{j=1}^{i-1}\prod_{l=j}^{i-1}\lambda_{l,k}^2\sigma_{l,k}^2+1\right) \tag{16}$$

$$\text{s.t.} \quad \mathbf{q}\prec^{+(\mathrm{w})}\left\{\left(1+\frac{\prod_{i=1}^{L}\sigma_{i,k}^2\lambda_{i,k}^2}{1+\sum_{i=2}^{L}\prod_{l=i}^{L}\sigma_{l,k}^2\lambda_{l,k}^2}\right)^{-1}\right\}_{k=1,\cdots,N_b} \tag{17}$$

$$\lambda_{i,k}>0, \qquad i=1,\cdots,L, \quad k=1,\cdots,N_b \tag{18}$$

where in (17) for a scalar x, $\{x_k\}_{k=1,\cdots,N} \triangleq [x_1,x_2,\cdots,x_N]^T$. Here the objective function (16) is obtained from (14) and (15), while the constraint (17) is obtained from (93) and (100) in Appendix A. Note that due to the constraint (18), the rank of $\mathbf{F}_i$, $i=1,\cdots,L$, is guaranteed to be N_b.

To simplify notations, let us introduce the following variable substitutions for $k=1,\cdots,N_b$

$$x_{1,k}\triangleq\lambda_{1,k}^2, \quad a_{i,k}\triangleq\sigma_{i,k}^2, \qquad i=1,\cdots,L \tag{19}$$

$$x_{i,k}\triangleq\lambda_{i,k}^2(a_{i-1,k}x_{i-1,k}+1), \quad i=2,\cdots,L\,. \tag{20}$$

Applying (19), (20) to (16)-(18) and expanding (17) using Definition 1, the optimization problem (16)-(18) can be equivalently written as

$$\min_{\mathbf{x}} \quad \sum_{i=1}^{L}\sum_{k=1}^{N_b}x_{i,k} \tag{21}$$

$$\text{s.t.} \quad \sum_{k=1}^{j}\left(1-\prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}}{1+a_{i,k}x_{i,k}}\right)\leq\sum_{k=1}^{j}q_k, \quad j=1,\cdots,N_b \tag{22}$$

$$x_{i,k}>0, \quad i=1,\cdots,L, \quad k=1,\cdots,N_b \tag{23}$$

where $\mathbf{x}\triangleq[x_{1,1},\cdots,x_{L,N_b}]^T$. It can be shown that the Hessian matrix of $\sum_{k=1}^{j}\left(1-\prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}}{1+a_{i,k}x_{i,k}}\right)$ is indefinite for $L\geq 2$ and $x_{i,k}>0$, $i=1,\cdots,L$, $k=1,\cdots,N_b$. Thus, the constraints in (22) are nonconvex and difficult to handle for $L\geq 2$. Consequently, the problem (21)-(23) is a nonconvex optimization problem and the globally optimal solution is difficult to obtain. In the following, we provide two numerical methods to solve the problem (21)-(23). The first method is based on a successive application of geometric programming (GP) [20]-[22]. While the second method utilizes the dual decomposition technique [23].

For the first method, let us introduce an auxiliary variable vector $\mathbf{z}\triangleq[z_1,\cdots,z_{N_b}]^T$ with $z_k\leq\prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}}{1+a_{i,k}x_{i,k}}$, $k=1,\cdots,N_b$. Then the problem (21)-(23) can be equivalently written as

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{i=1}^{L}\sum_{k=1}^{N_b}x_{i,k} \tag{24}$$

$$\text{s.t.} \quad \sum_{k=1}^{j}z_k\geq\sum_{k=1}^{j}(1-q_k), \qquad j=1,\cdots,N_b \tag{25}$$

$$z_k\prod_{i=1}^{L}\frac{1+a_{i,k}x_{i,k}}{a_{i,k}x_{i,k}}\leq 1, \qquad k=1,\cdots,N_b \tag{26}$$

$$x_{i,k}>0, \quad i=1,\cdots,L, \quad k=1,\cdots,N_b\,. \tag{27}$$

The objective function (24) is a posynomial. The constraints in (26) are posynomial upper-bound constraints [20]. However, constraints in (25) make the problem (24)-(27) a signomial programming (SP) problem, which is very difficult to solve [22]. If the constraints in (25) can also be converted to posynomial upper-bound constraints, then the problem (24)-(27) becomes a GP problem. Towards this end, we apply the geometric inequality to the left-hand side of (25) such that

$$\sum_{k=1}^{j}z_k\geq\prod_{k=1}^{j}\left(\frac{z_k}{\alpha_{j,k}}\right)^{\alpha_{j,k}} \tag{28}$$

where $\alpha_{j,k}>0, k=1,\cdots,j, j=1,\cdots,N_b$, and $\sum_{k=1}^{j}\alpha_{j,k}=1, j=1,\cdots,N_b$. Note that an approach similar to (28) has been applied to solve the power control problem in multiuser communication [22]. The major difference between [22] and the optimization problem (24)-(27) is that the power control problem in [22] may have infeasible constraints, while the problem (24)-(27) is always feasible. Substituting (25) with the inequalities $\prod_{k=1}^{j}\left(\frac{z_k}{\alpha_{j,k}}\right)^{\alpha_{j,k}}\geq\sum_{k=1}^{j}(1-q_k)$, we have

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{i=1}^{L}\sum_{k=1}^{N_b}x_{i,k} \tag{29}$$

$$\text{s.t.} \quad \beta_j\prod_{k=1}^{j}z_k^{-\alpha_{j,k}}\leq 1, \qquad j=1,\cdots,N_b \tag{30}$$

$$z_k\prod_{i=1}^{L}(1+a_{i,k}^{-1}x_{i,k}^{-1})\leq 1, \quad k=1,\cdots,N_b \tag{31}$$

$$x_{i,k}>0, \quad i=1,\cdots,L, \quad k=1,\cdots,N_b \tag{32}$$

where

$$\beta_j\triangleq\sum_{k=1}^{j}(1-q_k)\prod_{k=1}^{j}\alpha_{j,k}^{\alpha_{j,k}}\,. \tag{33}$$

The problem (29)-(32) is a GP problem in standard form, which can be converted to a convex optimization problem and efficiently solved by the interior-point method [20].

The procedure of applying the successive GP approach to solve the problem (21)-(23) is summarized in Table I. Here ε is a small positive number close to zero and the superscript (n) denotes the number of iterations.

TABLE I
PROCEDURE OF APPLYING THE SUCCESSIVE GP APPROACH TO SOLVE THE PROBLEM (21)-(23).

1) Initialize the algorithm at a feasible $\mathbf{x}^{(0)}$; Set $n=0$.
2) Compute $z_k^{(n)}=\prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}^{(n)}}{1+a_{i,k}x_{i,k}^{(n)}}$, $k=1,\cdots N_b$, $\alpha_{j,k}^{(n)}=\frac{z_k^{(n)}}{\sum_{k=1}^{j}z_k^{(n)}}$ and $\beta_j^{(n)}$ in (33), $j=1,\cdots,N_b$.
   Obtain $\mathbf{x}^{(n+1)}$ by solving the standard GP problem (29)-(32).
3) If $\max_{i,k}\left|x_{i,k}^{(n+1)}-x_{i,k}^{(n)}\right|\leq\varepsilon$, then end.
   Otherwise, let $n:=n+1$ and go to step 2).

Using (26) with $z_k=1-q_k, k=1,\cdots,N_b$, a feasible $\mathbf{x}^{(0)}$ can be obtained as $x_{i,k}^{(0)}=\left[a_{i,k}((1-q_k)^{-\frac{1}{L}}-1)\right]^{-1}$, $i=1,\cdots,L, k=1,\cdots,N_b$. We use the MOSEK convex optimization MATLAB toolbox [24] to solve the problem (29)-(32). Note that since the constraints in (30) are stricter than those in (25), the solution to the problem (24)-(27) is upper-bounded by that of the problem (29)-(32) before the convergence of the successive GP procedure. The tightness of this upper-bound depends on the coefficients $\alpha_{j,k}$, $k = 1, \cdots, j$, $j = 1, \cdots, N_b$. By using the successive GP approach, $\alpha_{j,k}$ are adaptively chosen to match the value of z_k, $k = 1, \cdots, N_b$, at each iteration. Moreover, when the successive GP procedure converges, i.e., when $\mathbf{x}^{(n)} \doteq \mathbf{x}^{(n+1)}$, we have $\sum_{k=1}^{j} z_k^{(n+1)} \doteq \prod_{k=1}^{j} \left(z_k^{(n)} / \alpha_{j,k}^{(n)} \right)^{\alpha_{j,k}^{(n)}}$. Thus, the problem (24)-(27) and the problem (29)-(32) are equivalent at the convergence point of the successive GP procedure, since at the convergence point, the constraint (25) is identical to the constraint (30). A rigorous analysis of the convergence of the successive GP algorithm is not available. However, we observed in the numerical simulations that a monotonic convergence of the successive GP algorithm is always achieved. The successive GP approach is also known as iterative monomial approximation in geometric programming literature [21], and its convergence has been observed for solving the power control problem in multiuser communication [22].

Now let us have a closer look at the problem (21)-(23). Interestingly, the variables $x_{i,k}$ are coupled only through the summations in (21) and (22). Such structure facilitates the application of the dual decomposition technique. First, the Lagrangian function associated with the problem (21), (22) can be written as

$$\begin{aligned} &\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}) \\ &= \sum_{i=1}^{L} \sum_{k=1}^{N_b} x_{i,k} + \sum_{j=1}^{N_b} \mu_j \left(\sum_{k=1}^{j} \left(1 - \prod_{i=1}^{L} \frac{a_{i,k} x_{i,k}}{1 + a_{i,k} x_{i,k}} - q_k \right) \right) \\ &= \sum_{k=1}^{N_b} \left[\sum_{i=1}^{L} x_{i,k} + \left(\sum_{j=k}^{N_b} \mu_j \right) \left(1 - \prod_{i=1}^{L} \frac{a_{i,k} x_{i,k}}{1 + a_{i,k} x_{i,k}} - q_k \right) \right] \\ &\triangleq \sum_{k=1}^{N_b} \mathcal{L}_k(\mathbf{x}_k, \tilde{\mu}_k) + \eta \end{aligned} \tag{34}$$

where $\mu_j \geq 0, j = 1, \cdots, N_b$, are the Lagrangian multipliers, $\boldsymbol{\mu} \triangleq [\mu_1, \cdots, \mu_{N_b}]^T$, $\tilde{\mu}_k \triangleq \sum_{j=k}^{N_b} \mu_j$, $\eta \triangleq \sum_{k=1}^{N_b} \tilde{\mu}_k (1 - q_k)$, $\mathbf{x}_k \triangleq [x_{1,k}, \cdots, x_{L,k}]^T$, and

$$\mathcal{L}_k(\mathbf{x}_k, \tilde{\mu}_k) \triangleq \sum_{i=1}^{L} x_{i,k} - \tilde{\mu}_k \prod_{i=1}^{L} \frac{a_{i,k} x_{i,k}}{1 + a_{i,k} x_{i,k}}, \ k = 1, \cdots, N_b \, . \tag{35}$$

The dual function [20] associated with the original problem (21)-(23) is given by $g(\boldsymbol{\mu}) = \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\mu})$. Now the optimization of $\mathbf{x}$ is carried out in two levels. At the lower level, we solve for $\mathbf{x}_k$, $k = 1, \cdots, N_b$, from the following decoupled subproblem with given $\tilde{\mu}_k$.

$$\min_{\mathbf{x}_k} \ \mathcal{L}_k(\mathbf{x}_k, \tilde{\mu}_k) \qquad \text{s.t.} \ \ x_{i,k} > 0, \quad i = 1, \cdots, L. \tag{36}$$

The problem (36) does not have a closed-form solution. We should resort to numerical methods such as the projected gradient algorithm [25] to solve it. The projected gradient method starts at an initial point $x_{i,k}^{(0)}$. At the nth iteration, $x_{i,k}^{(n)}$ is updated as $x_{i,k}^{(n+1)} = x_{i,k}^{(n)} + \delta_n \big(\tilde{x}_{i,k}^{(n)} - x_{i,k}^{(n)} \big)$, where $\delta_n \in (0, 1]$ is a step size and $\tilde{x}_{i,k}^{(n)} = \left[x_{i,k}^{(n)} - s_n \nabla_i \mathcal{L}_k(\mathbf{x}_k^{(n)}, \tilde{\mu}_k) \right]^+$. Here $[x]^+ \triangleq \max(x, \epsilon)$, s_n is a positive scalar, ϵ is a small positive number close to zero, and $\nabla_i \mathcal{L}_k(\mathbf{x}_k^{(n)}, \tilde{\mu}_k)$ denotes the gradient with respect to $x_{i,k}^{(n)}$. At the higher level, we update the dual variable $\boldsymbol{\mu}$ by solving the master dual problem

$$\max_{\boldsymbol{\mu}} \ \ g(\boldsymbol{\mu}) \qquad \text{s.t.} \ \ \mu_i \geq 0, \quad i = 1, \cdots, N_b. \tag{37}$$

The problem (37) can be solved by the sub-gradient method [23].

The procedure of applying the dual decomposition technique to solve the problem (21)-(23) is summarized in Table II. Note that the constraints in (22) are absorbed into the Lagrangian function (34), and when this algorithm converges, (22) is always satisfied. Since the dual decomposition method essentially solves the dual optimization problem, the result we obtain is a lower-bound of the original problem (21)-(23). Moreover, since the dual problem is always convex, the convergence of the dual decomposition algorithm is guaranteed.

TABLE II
PROCEDURE OF APPLYING THE DUAL DECOMPOSITION APPROACH TO SOLVE THE PROBLEM (21)-(23).

1) Initialize the algorithm at a feasible $\boldsymbol{\mu}^{(0)}$; Set $n = 0$.
2) Solve the subproblems (36) $k = 1, \cdots, N_b$, using $\boldsymbol{\mu}^{(n)}$, to obtain $\mathbf{x}^{(n)}$.
3) Solve the master problem (37) with $\mathbf{x}^{(n)}$ to obtain $\boldsymbol{\mu}^{(n+1)}$.
4) If $\max_i \left| \mu_i^{(n+1)} - \mu_i^{(n)} \right| \leq \varepsilon$, then end.
   Otherwise, let $n := n + 1$ and go to step 2).

Both the successive GP and the dual decomposition algorithms can be applied in wireless backhaul communications. We assume that the source node has the channel state information (CSI) knowledge of $\mathbf{H}_1$, the destination node knows $\mathbf{W}$, and the ith node, $i = 2, \cdots, L$, knows the CSI of its backward channel $\mathbf{H}_{i-1}$ and its forward channel $\mathbf{H}_i$. In practice, the backward CSI can be obtained through standard training methods [26]. The forward CSI required at the ith node ($\mathbf{H}_i$) is exactly the backward CSI at the $(i+1)$-th node, and thus can be obtained by a feedback from the $(i+1)$-th node. It is shown that at high training signal-to-noise ratio (SNR), the required CSI can be estimated with a high precision [26]. The iteration computations can be carried out at any node depending on the capability of all nodes. The selected node first collects the information on $\mathrm{rank}(\mathbf{H}_i)$ and $\sigma_{i,k}$ from the $(i+1)$-th node, $i = 1, \cdots, L$. Then it determines N_b ($N_b \leq R_h$) and performs the optimization and computes $\mathbf{U}_{F_1}$. Finally, it sends the optimal $\lambda_{i,k}$, $k = 1, \cdots, N_b$, to the ith node, $i = 1, \cdots, L$, and $\mathbf{U}_{F_1}$ to the source node. At the ith node, after the optimal $\lambda_{i,k}$, $k = 1, \cdots, N_b$, are received, the optimal matrix $\mathbf{F}_i$ is assembled using (13). Substituting (13) back into (7), we have $\mathbf{W} = \mathbf{U}_{L,1} \mathbf{D}_s \mathbf{U}_{F_1}^H$, where $\mathbf{D}_s$ is a diagonal matrix with the diagonal elements given by $[\mathbf{D}_s]_{k,k} = \left(\sum_{l=1}^{L} \prod_{i=l}^{L} \lambda_{i,k}^2 \sigma_{i,k}^2 + 1 \right)^{-1} \prod_{i=1}^{L} \lambda_{i,k} \sigma_{i,k}$, $k = 1, \cdots, N_b$. Thus, at the destination node, $\mathbf{U}_{L,1}$ is estimated through channel training, while $\mathbf{D}_s$ and $\mathbf{U}_{F_1}$ are forwarded by the selected node.

Note that since $\mathcal{L}_k(\mathbf{x}_k, \tilde{\mu}_k)$ in (35) is non-convex with respect to $x_{i,k}$, $i = 1, \cdots, L$, solving the subproblems (36) has a higher computational complexity than solving the GP problem (29)-(32), which can be converted to a convex optimization problem and efficiently solved by the interior-point method. Thus, in practice, the successive GP approach is preferred. The contribution of the dual decomposition method is that it establishes a lower-bound for the globally optimal solution to the original optimization problem (21)-(23). It is observed in Section IV through numerical simulations that this lower-bound is very close to the upper-bound obtained by the successive GP approach. Thus, the successive GP approach can be used with confidence in practice.

A suboptimal MIMO relay scheme can be developed by assuming a diagonal $\bar{\mathbf{H}}$, i.e., $\mathbf{F}_i$, $i = 1, \cdots, L$, are given by (13) with $\mathbf{U}_{F_1} = \mathbf{I}_{N_b}$. Thus, in this scheme the QoS constraint (11) is equivalent to

$$\left(1 + \frac{\prod_{i=1}^{L} \sigma_{i,k}^2 \lambda_{i,k}^2}{1 + \sum_{i=2}^{L} \prod_{l=i}^{L} \sigma_{l,k}^2 \lambda_{l,k}^2}\right)^{-1} \leq q_k, \qquad k = 1, \cdots, N_b .$$

Using the variable substitutions in (19), (20), the power loading problem for the suboptimal scheme is written as

$$\min_{\mathbf{x}} \quad \sum_{i=1}^{L} \sum_{k=1}^{N_b} x_{i,k} \tag{38}$$

$$\text{s.t.} \quad 1 - \prod_{i=1}^{L} \frac{a_{i,k} x_{i,k}}{1 + a_{i,k} x_{i,k}} \leq q_k, \quad k = 1, \cdots, N_b \tag{39}$$

$$x_{i,k} > 0, \qquad i = 1, \cdots, L, \quad k = 1, \cdots, N_b . \tag{40}$$

Note that constraints in (39) are equivalent to $\prod_{i=1}^{L} \frac{1 + a_{i,k} x_{i,k}}{a_{i,k} x_{i,k}} \leq \frac{1}{1 - q_k}$, $k = 1, \cdots, N_b$. Thus the problem (38)-(40) can be converted to

$$\min_{\mathbf{x}} \quad \sum_{i=1}^{L} \sum_{k=1}^{N_b} x_{i,k} \tag{41}$$

$$\text{s.t.} \quad \prod_{i=1}^{L} \left(1 + a_{i,k}^{-1} x_{i,k}^{-1}\right) \leq \frac{1}{1 - q_k}, \;\; k = 1, \cdots, N_b \tag{42}$$

$$x_{i,k} > 0, \qquad i = 1, \cdots, L, \quad k = 1, \cdots, N_b . \tag{43}$$

Since the objective function (41) is a posynomial and (42) are posynomial upper-bound inequality constraints, the problem (41)-(43) is a GP problem in standard form [20]. We use the MOSEK GP optimization MATLAB toolbox [24] to solve the problem (41)-(43).

It can be seen that the problem (38)-(40) has a smaller feasible region than that of the problem (21)-(23). Thus, we expect that the suboptimal scheme has a worse performance than the optimal approach in (21)-(23). However, the computational complexity of solving (38)-(40) is in the same order of one iteration of the successive GP approach in Table I. Such performance-complexity tradeoff is very useful for practical MIMO relay systems.

B. Nonlinear MMSE-DFE receiver at the destination

Using a nonlinear MMSE-DFE receiver and assuming that there is no error propagation, the estimated signal can be represented as [14]

$$\hat{\mathbf{b}} = \mathbf{W}^H \mathbf{y}_{L+1} - \mathbf{B}\mathbf{b} = \left(\mathbf{W}^H \bar{\mathbf{H}} - \mathbf{B}\right)\mathbf{b} + \mathbf{W}^H \bar{\mathbf{v}}$$

where $\mathbf{W}$ and $\mathbf{B}$ are the feed-forward and feedback matrix of the DFE receiver, respectively. In practice, the error propagation of the DFE receiver can be minimized by detecting the substream with the smallest error probability first. To minimize the MSE, the optimal $\mathbf{B}$ is given by $\mathbf{B} = \mathcal{U}\left[\mathbf{W}^H \bar{\mathbf{H}}\right]$, where $\mathcal{U}[\mathbf{A}]$ denotes the strictly upper-triangular part of $\mathbf{A}$. Let us introduce the following QR decomposition [27]

$$\mathbf{G} \triangleq \begin{bmatrix} \mathbf{C}_{\bar{v}}^{-\frac{1}{2}} \bar{\mathbf{H}} \\ \mathbf{I}_{N_b} \end{bmatrix} = \mathbf{Q}\mathbf{R} = \begin{bmatrix} \bar{\mathbf{Q}} \\ \underline{\mathbf{Q}} \end{bmatrix} \mathbf{R} \tag{44}$$

where $\mathbf{R}$ is an $N_b \times N_b$ upper-triangular matrix, $\mathbf{Q}$ is an $(N_b + N_{L+1}) \times N_b$ semi-unitary matrix with $\mathbf{Q}^H \mathbf{Q} = \mathbf{I}_{N_b}$, $\bar{\mathbf{Q}}$ is a matrix containing the first N_{L+1} rows of $\mathbf{Q}$, and $\underline{\mathbf{Q}}$ contains the last N_b rows of $\mathbf{Q}$. It has been shown in [10], [17] that $\mathbf{W}$, $\mathbf{B}$, and the MMSE matrix $\mathbf{M}$ can be represented as

$$\mathbf{W} = \mathbf{C}_{\bar{v}}^{-\frac{1}{2}} \bar{\mathbf{Q}} \mathbf{D}_R^{-1}, \qquad \mathbf{B} = \mathbf{D}_R^{-1}\mathbf{R} - \mathbf{I}_{N_b}, \qquad \mathbf{M} = \mathbf{D}_R^{-2}$$

where $\mathbf{D}_R$ is a matrix taking the diagonal elements of $\mathbf{R}$ as the main diagonal and zero elsewhere.

The QoS-constrained optimization problem for MIMO relay systems using the nonlinear MMSE-DFE receiver at the destination node can be written as

$$\min_{\{\mathbf{F}_i\}} \quad c_1 \mathrm{tr}\left(\mathbf{F}_1 \mathbf{F}_1^H\right) + \sum_{i=2}^{L} c_i \mathrm{tr}\left(\mathbf{F}_i \left(\sum_{l=1}^{i-1} \left(\bigotimes_{k=i-1}^{l} (\mathbf{H}_k \mathbf{F}_k) \bigotimes_{k=l}^{i-1} \left(\mathbf{F}_k^H \mathbf{H}_k^H\right)\right) + \mathbf{I}_{N_i}\right) \mathbf{F}_i^H\right) \tag{45}$$

$$\text{s.t.} \quad \begin{bmatrix} \mathbf{C}_{\bar{v}}^{-\frac{1}{2}} \bar{\mathbf{H}} \\ \mathbf{I}_{N_b} \end{bmatrix} = \mathbf{Q}\mathbf{R} \tag{46}$$

$$\mathbf{d}[\mathbf{R}] \geq \mathbf{q}^{-\frac{1}{2}} \tag{47}$$

where $\mathbf{q}^{-\frac{1}{2}} \triangleq \left[q_1^{-\frac{1}{2}}, \cdots, q_{N_b}^{-\frac{1}{2}}\right]^T$. In the following, we focus on the objective function (45) with $c_i = 1$, $i = 1, \cdots, L$. Nonetheless, the algorithms developed later can be straightforwardly extended to the case of general weighting coefficients.

Let us introduce the following definition that will be used to solve the problem (45)-(47).

DEFINITION 2 [17, p33]: For any two real-valued $N \times 1$ vectors $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x}$ is multiplicatively majorized by $\mathbf{y}$, denoted as $\mathbf{x} \prec_{\times} \mathbf{y}$, if $\prod_{i=1}^{n} x_{[i]} \leq \prod_{i=1}^{n} y_{[i]}$, for $n = 1, \cdots, N-1$, and $\prod_{i=1}^{N} x_{[i]} = \prod_{i=1}^{N} y_{[i]}$. We say that $\mathbf{x}$ is weakly multiplicatively submajorized by $\mathbf{y}$, denoted as $\mathbf{x} \prec_{\times(\mathrm{w})} \mathbf{y}$, if $\prod_{i=1}^{n} x_{[i]} \leq \prod_{i=1}^{n} y_{[i]}$, for $n = 1, \cdots, N$.

The following theorem establishes the structure of the optimal source precoding matrix and relay amplifying matrices when the nonlinear MMSE-DFE receiver is used at the destination node.

THEOREM 2: For the non-regenerative multi-hop MIMO relay design problem (45)-(47), assuming that $\mathrm{rank}(\mathbf{F}_i) = M \triangleq \min(N_b, R_h)$, $i = 1, \cdots, L$, the optimal source and relay matrices $\mathbf{F}_i$, $i = 1, \cdots, L$, are given by

$$\mathbf{F}_1 = \mathbf{V}_{1,1} \mathbf{\Delta}_1 \mathbf{V}_{F_1}^H, \quad \mathbf{F}_i = \mathbf{V}_{i,1} \mathbf{\Delta}_i \mathbf{U}_{i-1,1}^H, \quad i = 2, \cdots, L \tag{48}$$

where $\mathbf{\Delta}_i = \text{diag}(\delta_{i,1}, \delta_{i,2}, \cdots, \delta_{i,M})$ are $M \times M$ diagonal matrices, and $\mathbf{V}_{F_1}$ is an $N_b \times M$ semi-unitary matrix ($\mathbf{V}_{F_1}^H \mathbf{V}_{F_1} = \mathbf{I}_M$) such that the QR decomposition in (46) holds. In particular, $\boldsymbol{\delta} \triangleq [\delta_{1,1}, \cdots, \delta_{L,M}]^T$ is obtained by solving the following optimization problem

$$\min_{\boldsymbol{\delta}} \ \sum_{k=1}^{M} \delta_{1,k}^2 + \sum_{i=2}^{L} \sum_{k=1}^{M} \delta_{i,k}^2 \left(\sum_{j=1}^{i-1} \prod_{l=j}^{i-1} \delta_{l,k}^2 \sigma_{l,k}^2 + 1 \right) \tag{49}$$

$$\text{s.t.} \ \{q_k^{-1}\}_{k=1,\cdots,N_b} \prec_{\times(\text{w})} \left[\left\{ 1 + \frac{\prod_{i=1}^{L} \sigma_{i,k}^2 \delta_{i,k}^2}{1 + \sum_{i=2}^{L} \prod_{l=i}^{L} \sigma_{l,k}^2 \delta_{l,k}^2} \right\}_{k=1,\cdots,M}^T, \mathbf{1}_{N_b - M} \right]^T \tag{50}$$

$$\delta_{i,k} > 0, \qquad i = 1, \cdots, L, \quad k = 1, \cdots, M \tag{51}$$

where $\mathbf{1}_n$ denotes a $1 \times n$ vector with all 1 elements.

PROOF: See Appendix B. □

The motivation of the assumption $\text{rank}(\mathbf{F}_i) = M$ is to avoid any transmission power loss at each node. In practice, the node selected for performing the optimization first collects the information on $\text{rank}(\mathbf{H}_i)$, $i = 1, \cdots, L$. Then it determines $M = \min(N_b, R_h)$ and performs the optimization by solving the problem (49)-(51). Due to the constraint (51), the rank of $\mathbf{F}_i$, $i = 1, \cdots, L$, is guaranteed to be M. Comparing (48) with (13), we find that the optimal source and relay matrices have a similar structure for both the linear MMSE and the nonlinear MMSE-DFE receivers. The major differences are the power loading matrices $\mathbf{\Lambda}_i$ and $\mathbf{\Delta}_i$, $i = 1, \cdots, L$, and the rotation matrices at the source node $\mathbf{U}_{F_1}$ and $\mathbf{V}_{F_1}$. An intuitive explanation is that the structure of the relay matrices in (13) and (48) minimizes the total transmission power, while together with the relay matrices, the rotation matrix at the source node is used to guarantee the QoS constraints for different type of receivers. Moreover, as proved in [9] and [10], the source and relay matrices structure in (13) and (48) is optimal for most commonly used MIMO design criteria. Thus, we expect that the optimal source and relay matrices structure derived in this paper should be applicable in scenarios where the objective function is not the total transmission power (such as maximizing the source-destination mutual information subjecting to the QoS constraint on the MSE of each data stream). A rigorous proof of the optimal structure in such scenarios is an important future research topic.

Interestingly, when the nonlinear MMSE-DFE receiver is used at the destination, there is no constraint on N_b. In fact, as can be seen from Appendix B, N_b can be greater than R_h. Since the elements of $\mathbf{q}$ are arranged in an increasing order, the elements of $\mathbf{q}^{-1}$ are sorted in a decreasing order. Using the variable substitutions in (19), (20), and expanding (50) using Definition 2, the problem (49)-(51) can be equivalently rewritten as

$$\min_{\mathbf{x}} \ \sum_{i=1}^{L} \sum_{k=1}^{M} x_{i,k} \tag{52}$$

$$\text{s.t.} \ \prod_{k=1}^{j} q_k^{-1} \le \prod_{k=1}^{j} \left(1 - \prod_{i=1}^{L} \frac{a_{i,k} x_{i,k}}{1 + a_{i,k} x_{i,k}} \right)^{-1}, \quad j = 1, \cdots, M-1 \tag{53}$$

$$\prod_{k=1}^{N_b} q_k^{-1} \le \prod_{k=1}^{M} \left(1 - \prod_{i=1}^{L} \frac{a_{i,k} x_{i,k}}{1 + a_{i,k} x_{i,k}} \right)^{-1} \tag{54}$$

$$x_{i,k} > 0, \quad i = 1, \cdots, L, \quad k = 1, \cdots, M \tag{55}$$

where the constraint (54) is obtained since $q_k^{-1} > 1$, $k = 1, \cdots, N_b$, and thus $\max_{j=1,\cdots,N_b} \prod_{k=1}^{j} q_k^{-1} = \prod_{k=1}^{N_b} q_k^{-1}$.

Similar to Section III-A, the problem (52)-(55) can be solved by the successive GP approach and the dual decomposition method. We first discuss the successive GP approach. By introducing the auxiliary variables $z_k \le \prod_{i=1}^{L} \frac{a_{i,k} x_{i,k}}{1 + a_{i,k} x_{i,k}}$, $k = 1, \cdots M$, the problem (52)-(55) is equivalent to

$$\min_{\mathbf{x}, \mathbf{z}} \ \sum_{i=1}^{L} \sum_{k=1}^{M} x_{i,k} \tag{56}$$

$$\text{s.t.} \ \prod_{k=1}^{j} (1 - z_k) \le \prod_{k=1}^{j} q_k, \quad j = 1, \cdots, M-1 \tag{57}$$

$$\prod_{k=1}^{M} (1 - z_k) \le \prod_{k=1}^{N_b} q_k \tag{58}$$

$$z_k \prod_{i=1}^{L} \frac{1 + a_{i,k} x_{i,k}}{a_{i,k} x_{i,k}} \le 1, \qquad k = 1, \cdots, M \tag{59}$$

$$x_{i,k} > 0, \quad i = 1, \cdots, L, \quad k = 1, \cdots, M\,. \tag{60}$$

Obviously, constraints in (57) and (58) are not yet posynomial upper-bound constraints. Let us introduce an auxiliary variable vector $\mathbf{h} \triangleq [h_1, \cdots, h_M]^T$, with $1 - z_k \le h_k$, $k = 1, \cdots, M$, the constraints in (57) and (58) can be equivalently written as

$$\prod_{k=1}^{j} h_k \le \prod_{k=1}^{j} q_k, \qquad j = 1, \cdots, M-1 \tag{61}$$

$$\prod_{k=1}^{M} h_k \le \prod_{k=1}^{N_b} q_k \tag{62}$$

$$z_k + h_k \ge 1, \qquad k = 1, \cdots, M. \tag{63}$$

To convert the constraints in (63) to posynomial upper-bound constraint, we apply the following geometric inequality $z_k + h_k \ge (z_k/\gamma_k)^{\gamma_k} (h_k/\theta_k)^{\theta_k}$, where $\gamma_k > 0$, $\theta_k > 0$, and $\gamma_k + \theta_k = 1$. Now a tightened version of the problem (56)-(60) is given by

$$\min_{\mathbf{x},\mathbf{z},\mathbf{h}} \quad \sum_{i=1}^{L}\sum_{k=1}^{M} x_{i,k} \tag{64}$$

$$\text{s.t.} \quad \prod_{k=1}^{j} h_k q_k^{-1} \le 1, \qquad j = 1,\cdots,M-1 \tag{65}$$

$$\prod_{k=1}^{M} h_k \prod_{k=1}^{N_b} q_k^{-1} \le 1 \tag{66}$$

$$\gamma_k^{\gamma_k}\theta_k^{\theta_k} z_k^{-\gamma_k} h_k^{-\theta_k} \le 1, \qquad k = 1,\cdots,M \tag{67}$$

$$z_k \prod_{i=1}^{L}(1 + a_{i,k}^{-1}x_{i,k}^{-1}) \le 1, \quad k = 1,\cdots,M \tag{68}$$

$$x_{i,k} > 0, \quad i = 1,\cdots,L, \quad k = 1,\cdots,M\,. \tag{69}$$

The problem (64)-(69) is a GP problem in standard form. In a similar fashion to Section III-A, the power loading problem (52)-(55) can be solved by a successive GP approach, where in each iteration, the GP problem (64)-(69) is solved. The steps are summarized in Table III.

TABLE III
PROCEDURE OF APPLYING THE SUCCESSIVE GP APPROACH TO SOLVE THE PROBLEM (52)-(55).

1) Initialize the algorithm at a feasible $\mathbf{x}^{(0)}$; Set $n = 0$.
2) Compute $z_k^{(n)} = \prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}^{(n)}}{1+a_{i,k}x_{i,k}^{(n)}}$, $h_k^{(n)} = 1 - z_k^{(n)}$, $\gamma_k^{(n)} = z_k^{(n)}$, $\theta_k^{(n)} = h_k^{(n)}$, $k = 1,\cdots,M$.
   Obtain $\mathbf{x}^{(n+1)}$ by solving the standard GP problem (64)-(69).
3) If $\max_{i,k}\left|x_{i,k}^{(n+1)} - x_{i,k}^{(n)}\right| \le \varepsilon$, then end.
   Otherwise, let $n := n + 1$ and go to step 2).

To solve the problem (52)-(55) using the dual decomposition technique, we first apply the $\log$ operation to both sides of the constraints in (53) and (54). The problem (52)-(55) is now equivalent to

$$\min_{\mathbf{x}} \quad \sum_{i=1}^{L}\sum_{k=1}^{M} x_{i,k} \tag{70}$$

$$\text{s.t.} \quad \sum_{k=1}^{j}\log\left(1 - \prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}}{1+a_{i,k}x_{i,k}}\right) \le \sum_{k=1}^{j}\log q_k, \quad j = 1,\cdots,M-1 \tag{71}$$

$$\sum_{k=1}^{M}\log\left(1 - \prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}}{1+a_{i,k}x_{i,k}}\right) \le \sum_{k=1}^{N_b}\log q_k \tag{72}$$

$$x_{i,k} > 0, \quad i = 1,\cdots,L, \quad k = 1,\cdots,M\,. \tag{73}$$

The Lagrangian function associated with the problem (70)-(72) is

$$\begin{aligned}
&\mathcal{J}(\mathbf{x},\boldsymbol{\nu}) \\
&= \sum_{i=1}^{L}\sum_{k=1}^{M} x_{i,k} \\
&\quad + \sum_{j=1}^{M-1}\nu_j\left(\sum_{k=1}^{j}\left(\log\left(1 - \prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}}{1+a_{i,k}x_{i,k}}\right) - \log q_k\right)\right) \\
&\quad + \nu_M\left(\sum_{k=1}^{M}\log\left(1 - \prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}}{1+a_{i,k}x_{i,k}}\right) - \sum_{k=1}^{N_b}\log q_k\right) \\
&\triangleq \sum_{k=1}^{M}\mathcal{J}_k(\mathbf{x}_k,\tilde{\nu}_k) + \xi
\end{aligned}$$

where $\nu_j \ge 0, j = 1,\cdots,M$, are the Lagrangian multipliers, $\boldsymbol{\nu} \triangleq [\nu_1,\cdots,\nu_M]^T$, $\tilde{\nu}_k \triangleq \sum_{j=k}^{M}\nu_j$, $k = 1,\cdots,M$, $\xi \triangleq -\sum_{k=1}^{M}\tilde{\nu}_k\log q_k - \nu_M\sum_{k=M+1}^{N_b}\log q_k$, and

$$\mathcal{J}_k(\mathbf{x}_k,\tilde{\nu}_k) \triangleq \sum_{i=1}^{L}x_{i,k} + \tilde{\nu}_k\log\left(1 - \prod_{i=1}^{L}\frac{a_{i,k}x_{i,k}}{1+a_{i,k}x_{i,k}}\right) \quad k = 1,\cdots,M\,. \tag{74}$$

Now the decoupled subproblem with given $\tilde{\nu}_k$ is

$$\min_{\mathbf{x}_k} \quad \mathcal{J}_k(\mathbf{x}_k,\tilde{\nu}_k) \qquad \text{s.t.} \;\; x_{i,k} > 0, \quad i = 1,\cdots,L. \tag{75}$$

The master dual problem is given by

$$\max_{\boldsymbol{\nu}} \quad f(\boldsymbol{\nu}) \qquad \text{s.t.} \;\; \nu_i \ge 0, \quad i = 1,\cdots,M. \tag{76}$$

where $f(\boldsymbol{\nu}) = \min_{\mathbf{x}}\mathcal{J}(\mathbf{x},\boldsymbol{\nu})$. The procedure of applying the dual decomposition approach to solve the problem (52)-(55) is listed in Table IV.

TABLE IV
PROCEDURE OF APPLYING THE DUAL DECOMPOSITION APPROACH TO SOLVE THE PROBLEM (52)-(55).

1) Initialize the algorithm at a feasible $\boldsymbol{\nu}^{(0)}$; Set $n = 0$.
2) Solve the subproblems (75) $k = 1,\cdots,M$, using $\boldsymbol{\nu}^{(n)}$, to obtain $\mathbf{x}^{(n)}$.
3) Solve the master problem (76) with $\mathbf{x}^{(n)}$ to obtain $\boldsymbol{\nu}^{(n+1)}$.
4) If $\max_i\left|\nu_i^{(n+1)} - \nu_i^{(n)}\right| \le \varepsilon$, then end.
   Otherwise, let $n := n + 1$ and go to step 2).

It will be shown in Section IV that although the successive GP and the dual decomposition approaches provide an upper-bound and a lower-bound of the problem (52)-(55), respectively, their performance are almost identical. Note that since $\mathcal{J}_k(\mathbf{x}_k,\tilde{\nu}_k)$ in (74) is non-convex with respect to $x_{i,k}$, $i = 1,\cdots,L$, solving the subproblems (75) has a higher computational complexity than solving the GP problem (64)-(69). Thus, in practice, the successive GP approach should be used.

IV. NUMERICAL EXAMPLES

In this section, we study the performance of the proposed algorithms. In the simulations, all channel matrices have i.i.d. complex Gaussian entries with zero-mean and variances $1/N_i$ for $\mathbf{H}_i$, $i = 1,\cdots,L$. All simulation results are averaged over 1000 independent channel realizations. Unless mentioned explicitly, we set $c_i = 1$, $i = 1, \cdots, L$.

In the first example, we compare the performance of the successive GP approach and the dual decomposition technique. We simulate a 2-hop ($L = 2$) relay system with $N_i = N = 3$, $i = 1, 2, 3$. The MSE requirement at each data stream is set to be identical, i.e., $q_i = q$, $i = 1, 2, 3$. Table V shows the performance of both approaches in terms of total transmission power versus MSE for the linear MMSE receiver and the nonlinear MMSE-DFE receiver. It can be seen that the dual decomposition technique is only slightly better than the successive GP approach. Since the former approach provides a lower-bound and the latter approach establishes an upper-bound for the system performance, the results in Table V indicate that both bounds are tight. Thus, either of the approaches can be applied to solve the original optimization problem. In the following examples, for clarity, we only show the performance of the successive GP approach.

In the second example, we compare the performance of MIMO relay systems with the linear MMSE receiver, MIMO relay systems using the nonlinear MMSE-DFE receiver, and MIMO relay systems with the suboptimal scheme (38)-(40). We choose $N = 3$, $L = 2$, and identical MSE requirement at all streams. Fig. 1 shows the total transmission power required by three systems versus MSE. From Fig. 1 we find that the system using the nonlinear MMSE-DFE receiver requires much less total transmission power than that using the linear MMSE receiver, especially at low MSEs.

In the third example, we simulate a 3-hop ($L = 3$) MIMO relay system with $N = 3$ and $q_i = q, i = 1, 2, 3$. From Fig. 2 we see that compared with $L = 2$, the total power required by all three systems increases. This is expected, since for non-regenerative relay systems, noises at all relay nodes are amplified and superimposed at the destination node. Thus, in order to achieve the same MSE, a three-hop relay system requires more total transmission power than a two-hop system. We also observe from Fig. 2 that the relay system using the nonlinear MMSE-DFE receiver has the best performance in terms of the required total transmission power. For this example, the optimal power allocation (dB) among the first three nodes (the source node and two relay nodes) is listed in Table VI. It can be seen from Table VI that for both the linear MMSE and the nonlinear MMSE-DFE receivers, each node gets approximately the same amount of power.

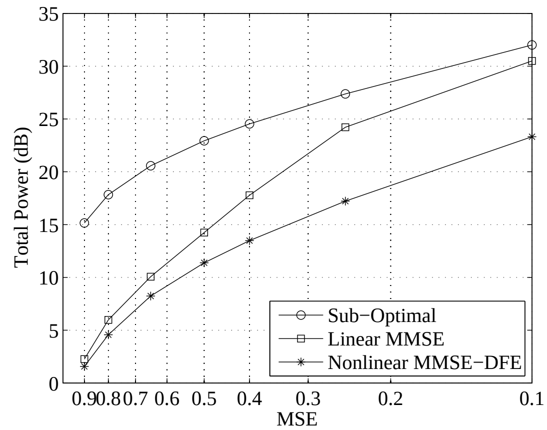

Fig. 1. Example 2: Total power versus MSE (q); $N = 3$, $L = 2$, $q_1 = q_2 = q_3 = q$.

TABLE V

EXAMPLE 1: COMPARISON OF THE SUCCESSIVE GP AND THE DUAL DECOMPOSITION APPROACHES; $N = 3$, $L = 2$.

MSE (q)	0.9	0.5	0.1	0.05	0.01	0.005	0.001
Successive GP (linear MMSE)	1.0050	14.2241	30.5001	33.1698	40.5201	43.5733	60.6046
Dual decomposition (linear MMSE)	1.0049	14.2106	30.4996	32.9231	40.1781	43.1298	60.1036
Successive GP (nonlinear MMSE-DFE)	0.5094	11.3751	23.4131	27.1305	34.8870	38.0352	55.4729
Dual decomposition (nonlinear MMSE-DFE)	0.5093	11.3708	23.4101	26.9713	34.6257	37.8921	55.1331

TABLE VI

EXAMPLE 3: POWER ALLOCATION AMONG THE FIRST THREE NODES; $N = 3$, $L = 3$, $c_1 = c_2 = c_3 = 1$.

MSE (q)	0.9	0.8	0.65	0.5	0.4	0.25	0.1
Node 1, linear MMSE	0.9648	5.3437	9.1789	13.1696	17.1009	22.0624	27.3032
Node 2, linear MMSE	0.8909	5.2972	9.1538	13.1849	17.0505	21.6592	26.8234
Node 3, linear MMSE	1.0426	5.4596	9.3404	13.3707	17.2554	22.3598	27.7923
Node 1, nonlinear MMSE-DFE	-0.0028	3.2947	7.3386	10.4884	12.5157	16.1756	22.3452
Node 2, nonlinear MMSE-DFE	-0.0743	3.2212	7.2715	10.4546	12.4796	16.1474	22.3426
Node 3, nonlinear MMSE-DFE	0.0790	3.3848	7.4595	10.6315	12.6684	16.3411	22.5354

TABLE VII

EXAMPLE 3: POWER ALLOCATION AMONG THE FIRST THREE NODES; $N = 3$, $L = 3$, $c_1 = 10$, $c_2 = 1$, $c_3 = 0.1$.

MSE (q)	0.9	0.8	0.65	0.5	0.4	0.25	0.1
Node 1, linear MMSE	-3.9825	0.8589	5.1619	9.4453	13.0070	18.9301	24.9992
Node 2, linear MMSE	2.3116	7.1846	10.9855	15.0621	18.9840	23.4610	28.4922
Node 3, linear MMSE	7.7768	12.5774	16.1578	20.0893	24.1123	29.9148	35.5158
Node 1, nonlinear MMSE-DFE	-4.5812	-0.8445	3.1894	6.6697	8.8369	12.5907	18.7554
Node 2, nonlinear MMSE-DFE	1.7211	5.0720	8.8818	12.3932	14.4077	17.8788	24.1208
Node 3, nonlinear MMSE-DFE	7.1740	10.2333	14.1305	17.4519	19.3862	22.8985	29.0905

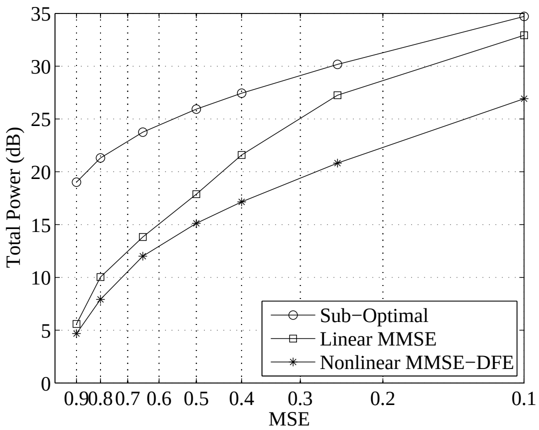

Fig. 2. Example 3: Total power versus MSE (q); $N = 3$, $L = 3$, $q_1 = q_2 = q_3 = q$.

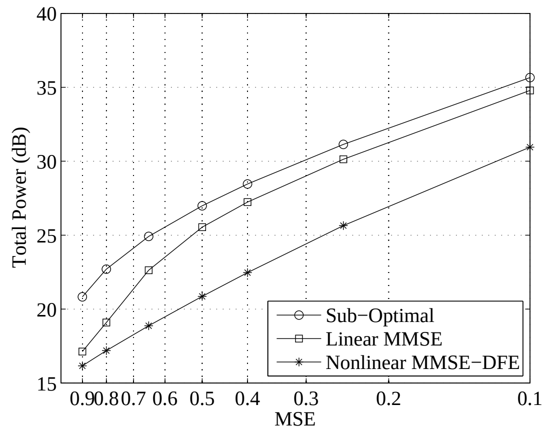

Fig. 3. Example 4: Total power versus MSE (q); $N = 3$, $L = 3$, $q_1 = q/4, q_2 = q/2, q_3 = q$.

To study the effects of unequal weighting coefficients to the power allocation among different nodes, we simulate the same three-hop MIMO relay system with $c_1 = 10$, $c_2 = 1$, and $c_3 = 0.1$ in the objective functions (10) and (45). The power (dB) consumed by the first three nodes is listed in Table VII. Compared with Table VI, it can be seen from Table VII that the node with a larger coefficient (i.e., node 1) consumes less transmission power, while the node having a smaller coefficient (i.e., node 3) requires more transmission power.

A 3-hop MIMO relay system with different QoS requirement at each stream is simulated in our fourth example. We set $N = 3$ and $q_1 = q/4, q_2 = q/2, q_3 = q$. The total power required by three systems is displayed in Fig. 3. It can be seen that due to the stricter MSE constraints for the first two streams, all systems require more power than those in the third example. Similar to previous examples, the system using the nonlinear MMSE-DFE receiver requires the least amount of power. We also observe from Fig. 3 that the gap of the required power between the system with the linear MMSE receiver and the suboptimal scheme becomes smaller when the streams have different QoS requirement. This is due to the fact that the amount of performance gap between two algorithms depends on the difference in the feasible regions of the optimization problems (38)-(40) and (21)-(23) for two algorithms. When all data streams have identical QoS constraints, such difference is quite big. The feasible region difference becomes smaller when each data stream has a different QoS constraint as in this example[1].

From Figs. 1-3, we find that the system using the MMSE-DFE receiver requires less total transmission power than the system with the linear MMSE receiver. A similar performance difference can be expected if the QoS constraints are imposed upon the rate/BER of each data stream. The reason is that as we mentioned, the rate/BER of each stream can be directly represented as a function of the MSE of each stream, and thus, the rate/BER constraint at each stream can be equivalently converted to the MSE constraint at each stream. However, if the QoS constraint is imposed upon the *sum-rate* of all data streams, both systems require the same amount of total transmission power. This is due to the fact that when the sum-rate is used as the design metric, the optimal source precoding matrix, the optimal relay amplifying matrices, and the optimal feed-forward matrix $\mathbf{W}$ of the nonlinear MMSE-DFE receiver jointly diagonalize the multi-hop MIMO relay channel, and thus, the linear MMSE and the nonlinear MMSE-DFE receivers have exactly the same performance.

From Figs. 1-3, we also see that both the system using the linear MMSE receiver and the system with the nonlinear MMSE-DFE receiver tremendously outperform the suboptimal scheme. However, the suboptimal scheme has a smaller computational complexity than the former two schemes. Such performance-complexity tradeoffs provide flexibility in practical MIMO relay systems.

V. Conclusions

We derived the optimal structure of source and relay matrices for multi-hop MIMO relay systems with QoS constraints using the linear MMSE receiver and the nonlinear MMSE-DFE receiver at the destination node, respectively. The successive GP approach and the dual decomposition technique were used to solve the optimization problem. We found that at the same MSE level, the MIMO relay system using the nonlinear MMSE-DFE receiver requires much less total transmission power than the system with the linear MMSE receiver.

Appendix A
Proof of Theorem 1

The following four lemmas are required to prove Theorem 1.

Lemma 1 [19, 9.H.2]: For m $N \times N$ complex matrices $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_m$, let $\mathbf{B} = \bigotimes_{i=1}^{m} \mathbf{A}_i$, then $\boldsymbol{\sigma}_b \prec_{+(\mathrm{w})} (\boldsymbol{\sigma}_{a_1} \odot \boldsymbol{\sigma}_{a_2} \odot \cdots \odot \boldsymbol{\sigma}_{a_m})$, where $\boldsymbol{\sigma}_b$, and $\boldsymbol{\sigma}_{a_i}$, $i = 1, \cdots, m$, denote $N \times 1$ vectors containing the singular values of $\mathbf{B}$ and $\mathbf{A}_i$

[1] As an intuitive explanation, for any $q > 0$, the difference between two regions $R_1 = \{(x_1, x_2) | \, 0 < x_1 \leq q, \, 0 < x_2, \, x_1 + x_2 \leq 2q\}$ and $R_2 = \{(x_1, x_2) | \, 0 < x_1 \leq q, \, 0 < x_2 \leq q\}$ is bigger than that of two regions $S_1 = \{(x_1, x_2) | \, 0 < x_1 \leq q/2, \, 0 < x_2, \, x_1 + x_2 \leq 3q/2\}$ and $S_2 = \{(x_1, x_2) | \, 0 < x_1 \leq q/2, \, 0 < x_2 \leq q\}$.

arranged in the same order, respectively, and $\odot$ denotes the Schur (element-wise) product of two vectors.

LEMMA 2 [19, 9.B.1]: For a Hermitian matrix $\mathbf{A}$ with the vector of its main diagonal elements $\mathbf{d}[\mathbf{A}]$ and the vector of its eigenvalues $\boldsymbol{\lambda}[\mathbf{A}]$, it follows that $\mathbf{d}[\mathbf{A}] \prec_{+} \boldsymbol{\lambda}[\mathbf{A}]$.

LEMMA 3 [19, 5.A.9.a]: For two vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x} \prec^{+(\mathrm{w})} \mathbf{y}$, there exists a vector $\mathbf{u}$ such that $\mathbf{u} \leq \mathbf{x}$ and $\mathbf{u} \prec_{+} \mathbf{y}$.

LEMMA 4 [19, 9.H.1.h]: For two $N \times N$ positive semidefinite matrices $\mathbf{A}$ and $\mathbf{B}$ with eigenvalues $\lambda_{a,i}$ and $\lambda_{b,i}$, $i = 1, \cdots, N$, arranged in the same order, respectively, it follows that $\mathrm{tr}(\mathbf{AB}) \geq \sum_{i=1}^{N} \lambda_{a,i}\lambda_{b,N+1-i}$.

Now we set out to prove Theorem 1 by first considering the constraint (11). Let us define

$$\mathbf{A}_1 = \mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^H\mathbf{H}_1^H \tag{77}$$

$$\mathbf{A}_i = \mathbf{H}_i\mathbf{F}_i(\mathbf{A}_{i-1} + \mathbf{I}_{N_i})\mathbf{F}_i^H\mathbf{H}_i^H, \qquad i = 2, \cdots, L \tag{78}$$

and write $\mathbf{A}_i = \mathbf{U}_{A_i}\boldsymbol{\Lambda}_{A_i}\mathbf{U}_{A_i}^H, i = 1, \cdots, L$, as the eigen-decomposition of $\mathbf{A}_i$, where $\boldsymbol{\Lambda}_{A_i}$ is an $N_b \times N_b$ diagonal matrix containing all nonzero eigenvalues of $\mathbf{A}_i$ sorted in the *decreasing* order for all i, and $\mathbf{U}_{A_i}$ is the associated $N_{i+1} \times N_b$ matrix of eigenvectors. From (77) and (78), we have

$$\mathbf{H}_1\mathbf{F}_1 = \mathbf{U}_{A_1}\boldsymbol{\Lambda}_{A_1}^{\frac{1}{2}}\mathbf{U}_0 \tag{79}$$

$$\mathbf{H}_i\mathbf{F}_i = \mathbf{U}_{A_i}\boldsymbol{\Lambda}_{A_i}^{\frac{1}{2}}\mathbf{S}_i(\mathbf{A}_{i-1} + \mathbf{I}_{N_i})^{-\frac{1}{2}}, \quad i = 2, \cdots, L \tag{80}$$

where $\mathbf{U}_0$ is an $N_b \times N_b$ unitary matrix, $\mathbf{S}_i$, $i = 2, \cdots, L$, are $N_b \times N_i$ semi-unitary matrices with $\mathbf{S}_i\mathbf{S}_i^H = \mathbf{I}_{N_b}$. It will be seen that the objective function (10) is invariant to $\mathbf{U}_0$ and $\mathbf{S}_i$, $i = 2, \cdots, L$. Substituting (79) and (80) into

$$\mathbf{X}(\{\mathbf{F}_i\}) \triangleq \bar{\mathbf{H}}^H\left(\bar{\mathbf{H}}\bar{\mathbf{H}}^H + \mathbf{C}_{\bar{v}}\right)^{-1}\bar{\mathbf{H}} \tag{81}$$

we have

$$\begin{aligned}
&\mathbf{X}(\{\mathbf{F}_i\}) \\
&= \mathbf{U}_0^H\boldsymbol{\Lambda}_{A_1}^{\frac{1}{2}}\mathbf{U}_{A_1}^H\bigotimes_{i=2}^{L}\left((\mathbf{A}_{i-1} + \mathbf{I}_{N_i})^{-\frac{1}{2}}\mathbf{S}_i^H\boldsymbol{\Lambda}_{A_i}^{\frac{1}{2}}\mathbf{U}_{A_i}^H\right) \\
&\quad \times(\mathbf{A}_L + \mathbf{I}_{N_{L+1}})^{-1}\bigotimes_{i=L}^{2}\left(\mathbf{U}_{A_i}\boldsymbol{\Lambda}_{A_i}^{\frac{1}{2}}\mathbf{S}_i(\mathbf{A}_{i-1} + \mathbf{I}_{N_i})^{-\frac{1}{2}}\right) \\
&\quad \times\mathbf{U}_{A_1}\boldsymbol{\Lambda}_{A_1}^{\frac{1}{2}}\mathbf{U}_0 \\
&= \mathbf{U}_0^H\boldsymbol{\Lambda}_{A_1}^{\frac{1}{2}}\mathbf{U}_{A_1}^H\bigotimes_{i=2}^{L}\left(\mathbf{U}_{A_{i-1}}(\boldsymbol{\Lambda}_{A_{i-1}} + \mathbf{I}_{N_b})^{-\frac{1}{2}}\mathbf{U}_{A_{i-1}}^H\mathbf{S}_i^H\right. \\
&\quad \left.\times\boldsymbol{\Lambda}_{A_i}^{\frac{1}{2}}\mathbf{U}_{A_i}^H\right)\mathbf{U}_{A_L}(\boldsymbol{\Lambda}_{A_L} + \mathbf{I}_{N_b})^{-1}\mathbf{U}_{A_L}^H\bigotimes_{i=L}^{2}\left(\mathbf{U}_{A_i}\boldsymbol{\Lambda}_{A_i}^{\frac{1}{2}}\right. \\
&\quad \left.\times\mathbf{S}_i\mathbf{U}_{A_{i-1}}(\boldsymbol{\Lambda}_{A_{i-1}} + \mathbf{I}_{N_b})^{-\frac{1}{2}}\mathbf{U}_{A_{i-1}}^H\right)\mathbf{U}_{A_1}\boldsymbol{\Lambda}_{A_1}^{\frac{1}{2}}\mathbf{U}_0\,.
\end{aligned} \tag{82}$$

Applying Lemma 1 to $\mathbf{X}(\{\mathbf{F}_i\})$ in (82), we have

$$\boldsymbol{\lambda}[\mathbf{X}(\{\mathbf{F}_i\})] \prec_{+(\mathrm{w})} \mathbf{d}\big[\tilde{\mathbf{X}}\big] \tag{83}$$

where $\tilde{\mathbf{X}}$ is a diagonal matrix given by

$$\begin{aligned}
\tilde{\mathbf{X}} &\triangleq \boldsymbol{\Lambda}_{A_1}^{\frac{1}{2}}\bigotimes_{i=2}^{L}\left((\boldsymbol{\Lambda}_{A_{i-1}} + \mathbf{I}_{N_b})^{-\frac{1}{2}}\boldsymbol{\Lambda}_{A_i}^{\frac{1}{2}}\right)(\boldsymbol{\Lambda}_{A_L} + \mathbf{I}_{N_b})^{-1} \\
&\quad \bigotimes_{i=L}^{2}\left(\boldsymbol{\Lambda}_{A_i}^{\frac{1}{2}}(\boldsymbol{\Lambda}_{A_{i-1}} + \mathbf{I}_{N_b})^{-\frac{1}{2}}\right)\boldsymbol{\Lambda}_{A_1}^{\frac{1}{2}} \\
&= \bigotimes_{i=1}^{L}(\boldsymbol{\Lambda}_{A_i}(\boldsymbol{\Lambda}_{A_i} + \mathbf{I}_{N_b})^{-1})\,.
\end{aligned} \tag{84}$$

Applying the matrix inversion lemma to (9), the MMSE matrix $\mathbf{M}(\{\mathbf{F}_i\})$ can be written as

$$\mathbf{M}(\{\mathbf{F}_i\}) = \mathbf{I}_{N_b} - \bar{\mathbf{H}}^H\left(\bar{\mathbf{H}}\bar{\mathbf{H}}^H + \mathbf{C}_{\bar{v}}\right)^{-1}\bar{\mathbf{H}} = \mathbf{I}_{N_b} - \mathbf{X}(\{\mathbf{F}_i\})\,. \tag{85}$$

From (83) and (85) we obtain

$$\boldsymbol{\lambda}[\mathbf{M}(\{\mathbf{F}_i\})] = \boldsymbol{\lambda}[\mathbf{I}_{N_b} - \mathbf{X}(\{\mathbf{F}_i\})] \prec^{+(\mathrm{w})} \mathbf{d}\big[\mathbf{I}_{N_b} - \tilde{\mathbf{X}}\big]\,. \tag{86}$$

In (86), $\boldsymbol{\lambda}[\mathbf{I}_{N_b} - \mathbf{X}(\{\mathbf{F}_i\})] = \mathbf{d}\big[\mathbf{I}_{N_b} - \tilde{\mathbf{X}}\big]$ is obtained at $\mathbf{S}_i = \boldsymbol{\Phi}\mathbf{U}_{A_{i-1}}^H$, $i = 2, \cdots, L$, where $\boldsymbol{\Phi}$ stands for an arbitrary $N_b \times N_b$ diagonal matrix with unit-norm main diagonal elements, i.e., $|[\boldsymbol{\Phi}]_{i,i}| = 1, [\boldsymbol{\Phi}]_{i,j} = 0, i,j = 1, \cdots, N_b, i \neq j$. Without affecting $\boldsymbol{\lambda}[\mathbf{M}(\{\mathbf{F}_i\})]$, we choose $\mathbf{S}_i = \mathbf{U}_{A_{i-1}}^H$, $i = 2, \cdots, L$.

For any given $\{\mathbf{F}_i\}$, there exists an $N_b \times N_b$ unitary matrix $\mathbf{U}_{F_1}$ such that $\mathbf{U}_{F_1}^H\mathbf{X}(\{\mathbf{F}_i\})\mathbf{U}_{F_1}$ is diagonal. In other words, there exists $\tilde{\mathbf{F}}_1 = \mathbf{F}_1\mathbf{U}_{F_1}$ such that the rotated MMSE matrix $\mathbf{M}\big(\tilde{\mathbf{F}}_1, \mathbf{F}_2, \cdots, \mathbf{F}_L\big)$ is diagonal. Using $\tilde{\mathbf{F}}_1$ and $\mathbf{U}_{F_1}$, the original MMSE matrix $\mathbf{M}(\{\mathbf{F}_i\})$ is equal to $\mathbf{U}_{F_1}\mathbf{M}\big(\tilde{\mathbf{F}}_1, \mathbf{F}_2, \cdots, \mathbf{F}_L\big)\mathbf{U}_{F_1}^H$. The objective function (10) is same for $\tilde{\mathbf{F}}_1$ and $\mathbf{F}_1$, and can be written as

$$\begin{aligned}
P &= \mathrm{tr}\big(\tilde{\mathbf{F}}_1\tilde{\mathbf{F}}_1^H\big) + \mathrm{tr}\big(\mathbf{F}_2(\mathbf{H}_1\tilde{\mathbf{F}}_1\tilde{\mathbf{F}}_1^H\mathbf{H}_1^H + \mathbf{I}_{N_2})\mathbf{F}_2^H\big) \\
&\quad + \sum_{i=3}^{L}\mathrm{tr}\left(\mathbf{F}_i\left(\sum_{l=2}^{i-1}\Big(\bigotimes_{k=i-1}^{l}(\mathbf{H}_k\mathbf{F}_k)\mathbf{H}_1\tilde{\mathbf{F}}_1\tilde{\mathbf{F}}_1^H\mathbf{H}_1^H\right.\right. \\
&\quad \left.\left.\bigotimes_{k=l}^{i-1}\big(\mathbf{F}_k^H\mathbf{H}_k^H\big)\Big) + \mathbf{I}_{N_i}\right)\mathbf{F}_i^H\right)\,.
\end{aligned} \tag{87}$$

Now the problem (10)-(11) can be equivalently written as

$$\min_{\mathbf{U}_{F_1}, \tilde{\mathbf{F}}_1, \mathbf{F}_2, \cdots, \mathbf{F}_L} \quad P \tag{88}$$

$$\text{s.t.} \quad \mathbf{X}\big(\tilde{\mathbf{F}}_1, \mathbf{F}_2, \cdots, \mathbf{F}_L\big) \quad \text{is diagonal} \tag{89}$$

$$\mathbf{d}\Big[\mathbf{U}_{F_1}\mathbf{M}\big(\tilde{\mathbf{F}}_1, \mathbf{F}_2, \cdots, \mathbf{F}_L\big)\mathbf{U}_{F_1}^H\Big] \leq \mathbf{q}\,. \tag{90}$$

Note that the steps of (87)-(90) are also used in [15] for single-hop MIMO communication systems. From Lemma 2 we know that

$$\begin{aligned}
&\mathbf{d}\Big[\mathbf{U}_{F_1}\mathbf{M}\big(\tilde{\mathbf{F}}_1, \mathbf{F}_2, \cdots, \mathbf{F}_L\big)\mathbf{U}_{F_1}^H\Big] \\
&= \mathbf{d}[\mathbf{M}(\{\mathbf{F}_i\})] \prec_{+} \boldsymbol{\lambda}[\mathbf{M}(\{\mathbf{F}_i\})]\,.
\end{aligned} \tag{91}$$

Based on Lemma 3 and (90), (91), a matrix $\mathbf{U}_{F_1}$ satisfying the QoS constraint (90) can be found if and only if

$$\mathbf{q} \prec^{+(\mathrm{w})} \boldsymbol{\lambda}[\mathbf{M}(\{\mathbf{F}_i\})]\,. \tag{92}$$

Interestingly, combining (86) and (92), we find that for all $\{\mathbf{F}_i\}$ that satisfy (92), the following inequality also holds

$$\mathbf{q} \prec^{+(\mathrm{w})} \mathbf{d}\big[\mathbf{I}_{N_b} - \tilde{\mathbf{X}}\big]\,. \tag{93}$$

In other words, (93) has a relaxed feasible region than that of (92). Since (92) is equivalent to (90), we can replace the constraint (90) by (93) without increasing the value of the objective function (88).

Now we set out to consider the objective function (88). First, we introduce some notations: for $i = 1, \cdots, L$, $r_i \triangleq \mathrm{rank}(\mathbf{H}_i)$, $\mathbf{U}_i \triangleq [\mathbf{U}_{i,r_i}, \mathbf{U}_{i,\bar{r}_i}]$, where $\mathbf{U}_{i,r_i}$ and $\mathbf{U}_{i,\bar{r}_i}$ contain the left singular vectors of $\mathbf{H}_i$ associated with the nonzero and zero singular values of $\mathbf{H}_i$, respectively, $\mathbf{\Sigma}_{i,r_i}$ is a diagonal matrix containing the nonzero singular values of $\mathbf{H}_i$, $\mathbf{\Sigma}_{i,1}$ contains the largest N_b singular values of $\mathbf{H}_i$ sorted in the same order as the diagonal elements of $\mathbf{\Lambda}_{A_i}$. We also define $\hat{\mathbf{F}}_i \triangleq \mathbf{V}_i^H \mathbf{F}_i$, $i = 1, \cdots, L$. Substituting the SVD of $\mathbf{H}_1$ in (12) into (79) and left multiplying by $\mathbf{U}_1^H$ on both sides, we have

$$\begin{bmatrix} \mathbf{\Sigma}_{1,r_1} & \mathbf{0}_{r_1 \times (N_1 - r_1)} \\ \mathbf{0}_{(N_2 - r_1) \times r_1} & \mathbf{0}_{(N_2 - r_1) \times (N_1 - r_1)} \end{bmatrix} \hat{\mathbf{F}}_1 = \mathbf{U}_1^H \mathbf{U}_{A_1} \mathbf{\Lambda}_{A_1}^{\frac{1}{2}} \mathbf{U}_0 \,. \tag{94}$$

Obviously, (94) is true if and only if $\mathbf{U}_{1,\bar{r}_1}^H \mathbf{U}_{A_1} = \mathbf{0}_{(N_2 - r_1) \times N_b}$ and $\hat{\mathbf{F}}_1$ has the following minimum norm solution

$$\hat{\mathbf{F}}_1 = \begin{bmatrix} \mathbf{\Sigma}_{1,r_1}^{-1} & \mathbf{0}_{r_1 \times (N_1 - r_1)} \end{bmatrix}^T \mathbf{U}_{1,r_1}^H \mathbf{U}_{A_1} \mathbf{\Lambda}_{A_1}^{\frac{1}{2}} \mathbf{U}_0 \,. \tag{95}$$

Similar to (94) and (95), from (80), we have that for $i = 2, \cdots, L$, $\mathbf{U}_{i,\bar{r}_i}^H \mathbf{U}_{A_i} = \mathbf{0}_{(N_{i+1} - r_i) \times N_b}$ and

$$\hat{\mathbf{F}}_i = \begin{bmatrix} \mathbf{\Sigma}_{i,r_i}^{-1} & \mathbf{0}_{r_i \times (N_i - r_i)} \end{bmatrix}^T \times \mathbf{U}_{i,r_i}^H \mathbf{U}_{A_i} \mathbf{\Lambda}_{A_i}^{\frac{1}{2}} \mathbf{S}_i (\mathbf{A}_{i-1} + \mathbf{I}_{N_i})^{-\frac{1}{2}} \,. \tag{96}$$

Since $\tilde{\mathbf{F}}_1$, $\mathbf{F}_i$, $i = 2, \cdots, L$, and $\hat{\mathbf{F}}_i$, $i = 1, \cdots, L$, result in the same objective function (88), to determine $\mathbf{U}_{A_1}$ in (95) and $\mathbf{U}_{A_i}$ in (96), $i = 2, \cdots, L$, we substitute (95) and (96) into the objective function (88) and have

$$P = \sum_{i=1}^{L} \mathrm{tr}\big(\mathbf{\Sigma}_{i,r_i}^{-1} \mathbf{U}_{i,r_i}^H \mathbf{U}_{A_i} \mathbf{\Lambda}_{A_i} \mathbf{U}_{A_i}^H \mathbf{U}_{i,r_i} \mathbf{\Sigma}_{i,r_i}^{-1}\big) \tag{97}$$

We note that the transmission power (97) is invariant to $\mathbf{U}_0$ and $\mathbf{S}_i$, $i = 2, \cdots, L$. Using Lemma 4, we know that under $\mathrm{rank}(\mathbf{F}_i) = N_b$, (97) is minimized if and only if $\mathbf{U}_{A_i}^H \mathbf{U}_{i,r_i} = [\mathbf{\Phi}, \mathbf{0}_{N_b \times (r_i - N_b)}]$, $i = 1, \cdots, L$, and the minimum is $\sum_{i=1}^{L} \mathrm{tr}(\mathbf{\Lambda}_{A_i} \mathbf{\Sigma}_{i,1}^{-2})$. Without loss of generality, we choose $\mathbf{\Phi} = \mathbf{I}_{N_b}$. Therefore, we have $\mathbf{U}_{A_i} = \mathbf{U}_{i,1}$, $i = 1, \cdots, L$. From (95) we find that

$$\mathbf{F}_1 = \mathbf{V}_1 \Big[\mathbf{\Sigma}_{1,1}^{-1} \mathbf{\Lambda}_{A_1}^{\frac{1}{2}}, \mathbf{0}_{N_b \times (N_1 - N_b)}\Big]^T \mathbf{U}_0 = \mathbf{V}_{1,1} \mathbf{\Sigma}_{1,1}^{-1} \mathbf{\Lambda}_{A_1}^{\frac{1}{2}} \mathbf{U}_0 \,. \tag{98}$$

Note that $\mathbf{U}_0$ will be determined later. Together with $\mathbf{S}_i = \mathbf{U}_{A_{i-1}}^H$, we obtain from (96) that

$$\mathbf{F}_i = \mathbf{V}_{i,1} \mathbf{\Sigma}_{i,1}^{-1} \mathbf{\Lambda}_{A_i}^{\frac{1}{2}} (\mathbf{\Lambda}_{A_{i-1}} + \mathbf{I}_{N_b})^{-\frac{1}{2}} \mathbf{U}_{i-1,1}^H \quad i = 2, \cdots, L \,. \tag{99}$$

Thus, the optimal structure of $\mathbf{F}_i$, $i = 2, \cdots, L$, is given by (13) with $\mathbf{\Lambda}_i = \mathbf{\Sigma}_{i,1}^{-1} \mathbf{\Lambda}_{A_i}^{\frac{1}{2}} (\mathbf{\Lambda}_{A_{i-1}} + \mathbf{I}_{N_b})^{-\frac{1}{2}}$.

Finally, substituting (98) and (99) back into (81), we have $\mathbf{X}(\{\mathbf{F}_i\}) = \mathbf{U}_0^H \tilde{\mathbf{X}} \mathbf{U}_0$. Thus we obtain $\mathbf{X}(\tilde{\mathbf{F}}_1, \mathbf{F}_2, \cdots, \mathbf{F}_L) = \mathbf{U}_{F_1}^H \mathbf{U}_0^H \tilde{\mathbf{X}} \mathbf{U}_0 \mathbf{U}_{F_1}$. In order for the constraint (89) to hold, $\mathbf{U}_0$ should be $\mathbf{U}_{F_1}^H$. From (98), we obtain $\mathbf{F}_1 = \mathbf{V}_{1,1} \mathbf{\Sigma}_{1,1}^{-1} \mathbf{\Lambda}_{A_1}^{\frac{1}{2}} \mathbf{U}_{F_1}^H$. Thus we have proved that the optimal structure of $\mathbf{F}_1$ is as in (13) with $\mathbf{\Lambda}_1 = \mathbf{\Sigma}_{1,1}^{-1} \mathbf{\Lambda}_{A_1}^{\frac{1}{2}}$. Consequently, using the optimal structure of $\mathbf{F}_i$, $i = 1, \cdots, L$, $\tilde{\mathbf{X}}$ in (84) can be represented as

$$\tilde{\mathbf{X}} = \mathbf{D}_h (\mathbf{D}_h^2 + \mathbf{D}_c + \mathbf{I}_{N_b})^{-1} \mathbf{D}_h \tag{100}$$

where $\mathbf{D}_h$ and $\mathbf{D}_c$ are $N_b \times N_b$ diagonal matrices with the kth diagonal elements $k = 1, \cdots, N_b$, given by $[\mathbf{D}_h]_{k,k} = \prod_{l=1}^{L} \lambda_{l,k} \sigma_{l,k}$ and $[\mathbf{D}_c]_{k,k} = \sum_{l=2}^{L} \prod_{i=l}^{L} \lambda_{i,k}^2 \sigma_{i,k}^2$, respectively.

Appendix B
Proof of Theorem 2

The following two lemmas are required to prove Theorem 2.

Lemma 5 [19, 9.H.1.b]: For m $N \times N$ complex matrices $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_m$, let $\mathbf{B} = \bigotimes_{i=1}^{m} \mathbf{A}_i$, then $\boldsymbol{\sigma}_b \prec_\times (\boldsymbol{\sigma}_{a_1} \odot \boldsymbol{\sigma}_{a_2} \odot \cdots \odot \boldsymbol{\sigma}_{a_m})$, where $\boldsymbol{\sigma}_b$, and $\boldsymbol{\sigma}_{a_i}$, $i = 1, \cdots, m$, are defined in Lemma 2.

Lemma 6 [10]: For two $N \times 1$ vectors $\mathbf{x}$ and $\mathbf{y}$, if $\mathbf{x} \prec_\times \mathbf{y}$, then $\{(1 - x_i)^{-1}\}_{i=1,\cdots,N} \prec_{\times(\mathrm{w})} \{(1 - y_i)^{-1}\}_{i=1,\cdots,N}$.

Let us introduce the SVD $\mathbf{F}_1 = \tilde{\mathbf{U}}_{F_1} \tilde{\mathbf{\Lambda}}_{F_1} \tilde{\mathbf{V}}_{F_1}^H$, where the dimensions of $\tilde{\mathbf{U}}_{F_1}$, $\tilde{\mathbf{\Lambda}}_{F_1}$, $\tilde{\mathbf{V}}_{F_1}$ are $N_1 \times N_1$, $N_1 \times N_b$, $N_b \times N_b$, respectively. We assume that the main diagonal elements of $\tilde{\mathbf{\Lambda}}_{F_1}$ are arranged in the decreasing order. Since $\mathrm{rank}(\mathbf{F}_1) = M$, we also have $\mathbf{F}_1 = \mathbf{U}_{F_1} \mathbf{\Lambda}_{F_1} \mathbf{V}_{F_1}^H$, where $\mathbf{U}_{F_1} = \big[\tilde{\mathbf{U}}_{F_1}\big]_{1:M}$, $\mathbf{V}_{F_1} = \big[\tilde{\mathbf{V}}_{F_1}\big]_{1:M}$, $\mathbf{\Lambda}_{F_1} = \big[\tilde{\mathbf{\Lambda}}_{F_1}\big]_{1:M,1:M}$. From (44) we have

$$\mathbf{G} = \begin{bmatrix} \mathbf{C}_{\bar{v}}^{-\frac{1}{2}} \tilde{\mathbf{H}} \tilde{\mathbf{U}}_{F_1} \tilde{\mathbf{\Lambda}}_{F_1} \tilde{\mathbf{V}}_{F_1}^H \\ \mathbf{I}_{N_b} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N_{L+1}} & \mathbf{0}_{N_{L+1} \times N_b} \\ \mathbf{0}_{N_b \times N_{L+1}} & \tilde{\mathbf{V}}_{F_1} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{\bar{v}}^{-\frac{1}{2}} \tilde{\mathbf{H}} \tilde{\mathbf{U}}_{F_1} \tilde{\mathbf{\Lambda}}_{F_1} \\ \mathbf{I}_{N_b} \end{bmatrix} \tilde{\mathbf{V}}_{F_1}^H$$

where $\tilde{\mathbf{H}} = \bigotimes_{i=L}^{2} (\mathbf{H}_i \mathbf{F}_i) \mathbf{H}_1$. Let us write the generalized triangular decomposition (GTD) [28] of $\mathbf{\Psi}$ as

$$\mathbf{\Psi} \triangleq \begin{bmatrix} \mathbf{C}_{\bar{v}}^{-\frac{1}{2}} \tilde{\mathbf{H}} \tilde{\mathbf{U}}_{F,1} \tilde{\mathbf{\Lambda}}_{F_1} \\ \mathbf{I}_{N_b} \end{bmatrix} = \mathbf{Q}_\Psi \mathbf{R} \mathbf{P}_\Psi^H \tag{101}$$

where $\mathbf{Q}_\Psi$ is an $(N_{L+1} + N_b) \times N_b$ semi-unitary matrix with $\mathbf{Q}_\Psi^H \mathbf{Q}_\Psi = \mathbf{I}_{N_b}$, and $\mathbf{P}_\Psi$ is an $N_b \times N_b$ unitary matrix. It can be shown from [28] that (101) holds *if and only if* $\mathbf{d}[\mathbf{D}_R] \prec_\times \boldsymbol{\sigma}_\Psi$, where $\boldsymbol{\sigma}_\Psi$ is a column vector containing singular values of $\mathbf{\Psi}$. Without affecting the power constraints, we take $\tilde{\mathbf{V}}_{F_1} = \mathbf{P}_\Psi^H$, or equivalently

$$\mathbf{V}_{F_1} = \big[\mathbf{P}_\Psi^H\big]_{1:M} \,. \tag{102}$$

Then we can write the QR decomposition of $\mathbf{G}$ as

$$\mathbf{G} = \mathbf{Q}\mathbf{R}, \qquad \mathbf{Q} = \begin{bmatrix} \mathbf{I}_{N_{L+1}} & \mathbf{0}_{N_{L+1} \times N_b} \\ \mathbf{0}_{N_b \times N_{L+1}} & \mathbf{P}_\Psi^H \end{bmatrix} \mathbf{Q}_\Psi \,. \tag{103}$$

Because $\mathbf{\Psi}$ and $\mathbf{G}$ have the same singular values, from (101) and (103) we know that the constraint (46) can be equivalently written as

$$\mathbf{d}[\mathbf{D}_R] \prec_\times \boldsymbol{\sigma}_G \,. \tag{104}$$

Applying Lemma 5 to (82) and taking into account that $\text{rank}(\mathbf{F}_i) = M$, $i = 1, \cdots, L$, we obtain that

$$\boldsymbol{\lambda}_X \prec_\times \left[\left(\mathbf{d}\left[\tilde{\mathbf{X}}\right]\right)^T, \mathbf{0}_{1\times(N_b-M)}\right]^T \tag{105}$$

where $\boldsymbol{\lambda}_X$ is a column vector containing all eigenvalues of $\mathbf{X}$. Applying Lemma 6 to (105), we have

$$\begin{aligned}&\{(1-\lambda_{X,i})^{-1}\}_{i=1,\cdots,N_b}\\ &\prec_{\times(\mathrm{w})}\left[\left\{(1-\tilde{x}_{i,i})^{-1}\right\}^T_{i=1,\cdots,M}, \mathbf{1}_{N_b-M}\right]^T\end{aligned} \tag{106}$$

where $\lambda_{X,i}$ denotes the ith eigenvalue of $\mathbf{X}$ and $\tilde{x}_{i,i}$ denotes the (i,i)-th element of $\tilde{\mathbf{X}}$. From (44), we can write

$$\begin{aligned}\mathbf{G}^H\mathbf{G} &= \mathbf{I}_{N_b} + \bar{\mathbf{H}}^H\mathbf{C}_{\bar{v}}^{-1}\bar{\mathbf{H}}\\ &= \left[\mathbf{I}_{N_b} - \bar{\mathbf{H}}^H\left(\bar{\mathbf{H}}^H\bar{\mathbf{H}} + \mathbf{C}_{\bar{v}}\right)^{-1}\bar{\mathbf{H}}\right]^{-1}\\ &= (\mathbf{I}_{N_b} - \mathbf{X})^{-1}\end{aligned} \tag{107}$$

where the matrix inversion lemma is applied to obtain the second equation. From (107), we find that $\left\{\sigma^2_{G,i}\right\}_{i=1,\cdots,N_b} = \left\{(1-\lambda_{X,i})^{-1}\right\}_{i=1,\cdots,N_b}$, where $\sigma_{G,i}$ is the ith singular value of $\mathbf{G}$. Using (106) we obtain $\left\{\sigma^2_{G,i}\right\}_{i=1,\cdots,N_b} \prec_{\times(\mathrm{w})} \left[\left\{(1-\tilde{x}_{i,i})^{-1}\right\}^T_{i=1,\cdots,M}, \mathbf{1}_{N_b-M}\right]^T$. Moreover, since (104) is equivalent to $\mathbf{d}\left[\mathbf{D}_R^2\right] \prec_\times \left\{\sigma^2_{G,i}\right\}_{i=1,\cdots,N_b}$, we have

$$\mathbf{d}\left[\mathbf{D}_R^2\right] \prec_{\times(\mathrm{w})}\left[\left\{(1-\tilde{x}_{i,i})^{-1}\right\}^T_{i=1,\cdots,M}, \mathbf{1}_{N_b-M}\right]^T. \tag{108}$$

We would like to mention that for all $\mathbf{D}_R$ and $\{\mathbf{F}_i\}$ that satisfy (104), inequality (108) also holds. In other words, (108) has a relaxed feasible region than that of (104). Since (104) is equivalent to (46), we can replace the constraint (46) by (108) without increasing the value of the objective function (45). Moreover, from (82) we see that $\boldsymbol{\lambda}_X = \left[\left(\mathbf{d}\left[\tilde{\mathbf{X}}\right]\right)^T, \mathbf{0}_{1\times(N_b-M)}\right]^T$ holds at $\mathbf{S}_i = \mathbf{\Phi}\mathbf{U}^H_{A_{i-1}}$, $i = 2, \cdots, L$. Without affecting the objective function (45), we choose $\mathbf{S}_i = \mathbf{U}^H_{A_{i-1}}$, $i = 2, \cdots, L$.

Now let us consider the objective function (45). In a way similar to (94), (95), (97), (98), by solving (79), we obtain the optimal $\mathbf{F}_1$ as

$$\mathbf{F}_1 = \mathbf{V}_1\left[\mathbf{\Sigma}^{-1}_{1,1}\mathbf{\Lambda}^{\frac{1}{2}}_{A_1}, \mathbf{0}_{M\times(N_1-M)}\right]^T\mathbf{U}_0 = \mathbf{V}_{1,1}\mathbf{\Sigma}^{-1}_{1,1}\mathbf{\Lambda}^{\frac{1}{2}}_{A_1}\mathbf{U}_0\,.$$

Note that $\mathbf{U}_0$ does not affect $\boldsymbol{\lambda}_X$. In fact, $\mathbf{U}_0$ should be chosen as $\mathbf{V}^H_{F_1}$ in (102) such that the QR decomposition of $\mathbf{G}$ in (103) holds. Thus $\mathbf{F}_1 = \mathbf{V}_{1,1}\mathbf{\Sigma}^{-1}_{1,1}\mathbf{\Lambda}^{\frac{1}{2}}_{A_1}\mathbf{V}^H_{F_1}$, and we have proved that the optimal structure of $\mathbf{F}_1$ is as in (48) with $\mathbf{\Delta}_1 = \mathbf{\Sigma}^{-1}_{1,1}\mathbf{\Lambda}^{\frac{1}{2}}_{A_1}$. In a way similar to (96), (97), (99), we obtain the optimal $\mathbf{F}_i$, $i = 2, \cdots, L$ as (99). Thus, we have proved that the optimal structure of $\mathbf{F}_i$ is as in (48) with $\mathbf{\Delta}_i = \mathbf{\Sigma}^{-1}_{i,1}\mathbf{\Lambda}^{\frac{1}{2}}_{A_i}(\mathbf{\Lambda}_{A_{i-1}} + \mathbf{I}_M)^{-\frac{1}{2}}$.

The constraint (47) is equivalent to $\mathbf{q}^{-1} \leq \mathbf{d}[\mathbf{D}_R^2]$, which indicates that $\mathbf{q}^{-1} \prec_{\times(\mathrm{w})} \mathbf{d}[\mathbf{D}_R^2]$. From (108) we have

$$\mathbf{q}^{-1} \prec_{\times(\mathrm{w})} \left[\left\{(1-\tilde{x}_{i,i})^{-1}\right\}^T_{i=1,\cdots,M}, \mathbf{1}_{N_b-M}\right]^T. \tag{109}$$

Finally, by applying (48) to $\tilde{x}_{i,i}$ in (109) we obtain (50).

References

[1] X. Tang and Y. Hua, "Optimal design of non-regenerative MIMO wireless relays," *IEEE Trans. Wireless Commun.*, vol. 6, pp. 1398-1407, Apr. 2007.

[2] Z. Fang, Y. Hua, and J. C. Koshy, "Joint source and relay optimization for a non-regenerative MIMO relay," in *Proc. IEEE Workshop on Sensor Array and Multi-Channel Signal Process.*, Waltham, WA, Jul. 2006, pp. 239-243.

[3] O. Muñoz-Medina, J. Vidal, and A. Agustín, "Linear transceiver design in nonregenerative relays with channel state information," *IEEE Trans. Signal Process.*, vol. 55, pp. 2593-2604, Jun. 2007.

[4] I. Hammerström and A. Wittneben, "Power allocation schemes for amplify-and-forward MIMO-OFDM relay links," *IEEE Trans. Wireless Commun.*, vol. 6, pp. 2798-2802, Aug. 2007.

[5] W. Guan and H. Luo, "Joint MMSE transceiver design in nonregenerative MIMO relay systems," *IEEE Commun. Lett.*, vol. 12, pp. 517-519, Jul. 2008.

[6] A. S. Behbahani, R. Merched, and A. M. Eltawil, "Optimizations of a MIMO relay network," *IEEE Trans. Signal Process.*, vol. 56, pp. 5062-5073, Oct. 2008.

[7] Y. Rong, "Linear non-regenerative multicarrier MIMO relay communications based on MMSE criterion," *IEEE Trans. Commun.*, vol. 58, pp. 1918-1923, Jul. 2010.

[8] Y. Rong, X. Tang, and Y. Hua, "A unified framework for optimizing linear non-regenerative multicarrier MIMO relay communication systems," *IEEE Trans. Signal Process.*, vol. 57, pp. 4837-4851, Dec. 2009.

[9] Y. Rong and Y. Hua, "Optimality of diagonalization of multi-hop MIMO relays," *IEEE Trans. Wireless Commun.*, vol. 8, pp. 6068-6077, Dec. 2009.

[10] Y. Rong, "Optimal linear non-regenerative multi-hop MIMO relays with MMSE-DFE receiver at the destination," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 2268-2279, Jul. 2010.

[11] S. Yeh and O. Lévêque, "Asymptotic capacity of multi-level amplify-and-forward relay networks," in *Proc. IEEE ISIT*, Nice, France, Jun. 2007, pp. 1436-1440.

[12] S. Yang and J.-C. Belfiore, "Diversity of MIMO multihop relay channels," submitted to *IEEE Trans. Inf. Theory*, [Online]. Avaliable: http://arxiv.org/abs/0708.0386v1.

[13] N. Fawaz, K. Zarifi, M. Debbah, and D. Gesbert, "Asymptotic capacity and optimal precoding strategy of multi-level precode & forward in correlated channels," in *Proc. IEEE ITW*, Porto, Portugal, May 2008, pp. 209-213.

[14] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge University Press, 2005.

[15] D. P. Palomar, M. A. Lagunas, and J. M. Cioffi, "Optimum linear joint transmit-receive processing for MIMO channels with QoS constraints," *IEEE Trans. Signal Process.*, vol. 52, pp. 1179-1197, May 2004.

[16] Y. Jiang, W. Hager, and J. Li, "Tunable channel decomposition for MIMO communications using channel state information," *IEEE Trans. Signal Process.*, vol. 54, pp. 4405-4418, Nov. 2006.

[17] D. P. Palomar and Y. Jiang, *MIMO Transceiver Design via Majorization Theory.* Now Publishers, 2007.

[18] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory.* Englewood Cilffs, NJ: Prentice Hall, 1993.

[19] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*. Academic Press, 1979.

[20] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U. K.: Cambridge University Press, 2004.

[21] C. S. Beightler and D. T. Philips, *Applied Geometric Programming*. Wiley, 1976.

[22] M. Chiang, *Geometric Programming for Communication Systems.* Now Publishers, 2005.

[23] D. P. Palomar and M. Chiang, "A tutorial on decomposition methods for network utility maximization," *IEEE J. Selet. Areas. Commun.*, vol. 24, pp. 1439-1451, Aug. 2006.

[24] The MOSEK optimization toolbox for MATLAB manual. Version 5.0. [Online]. Available: http://www.mosek.com.

[25] D. P. Bertsekas, *Nonlinear Programming*. 2nd. Edition, Athena Scientific, Belmont, 1995.

[26] P. Lioliou and M. Viberg, "Least-squares based channel estimation for MIMO relays," in *Proc. IEEE WSA 2008,* Darmstadt, Germany, Feb. 2008, pp. 90-95.

[27] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[28] Y. Jiang, W. Hager, and J. Li, "The generalized triangular decomposition," *Mathematics of Computation*, vol. 77, pp. 1037-1056, Apr. 2008.