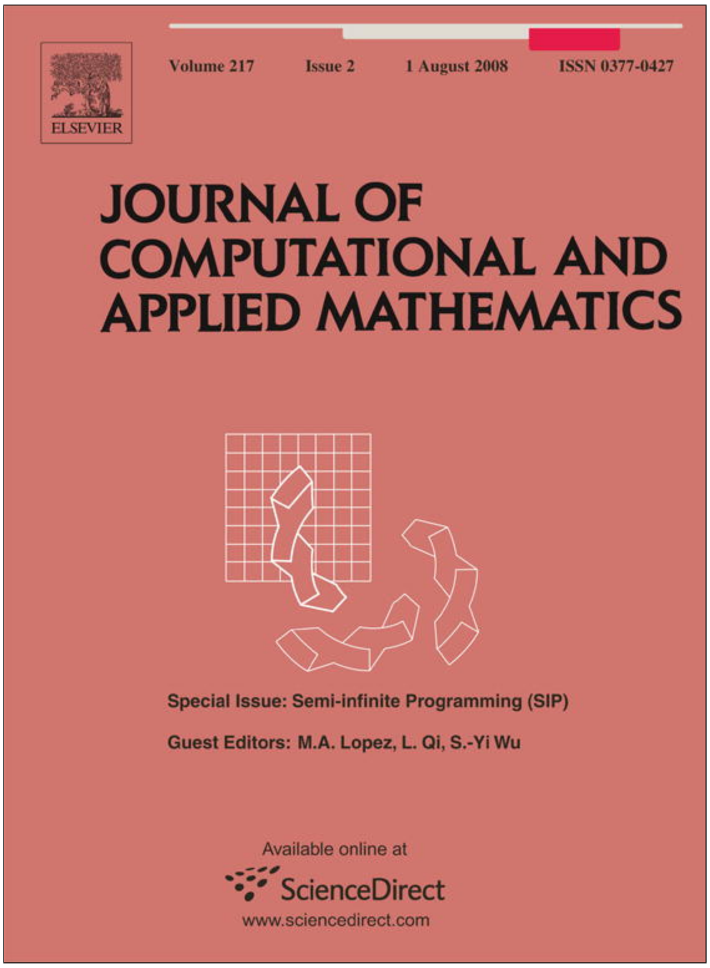
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Higher-order Mond–Weir duality for set-valued optimization☆

S.J. Li[a,*], K.L. Teo[b], X.Q. Yang[c]

[a] *College of Mathematics and Science, Chongqing University, Chongqing 400044, China*
[b] *Department of Mathematics and Statistics, Curtin University of Technology, G.P.O. Box U1987, Perth, WA 6845, Australia*
[c] *Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong*

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Abstract

In this paper, we introduce a higher-order Mond–Weir dual for a set-valued optimization problem by virtue of higher-order contingent derivatives and discuss their weak duality, strong duality and converse duality properties.

MSC: 90C29; 90C31; 65K10

Keywords: Set-valued optimization; Higher-order contingent derivative; Mond–Weir duality

1. Introduction

For various different types of convex minimization problems (for example, linear programming, convex programming and optimal control), there are associated maximization problems (called dual), involving different variables, which attain the same optimal value as the original problem (called primal). It is very important to discuss the relationship between primal problem and dual problem.

Recently, one finds that many optimization problems encountered in economics and other fields involve vector-valued (or set-valued) mappings as constraints and objectives. Then, optimization problems with vector-valued mappings (or set-valued mapping) have received much attention in recent years. Several authors have discussed duality properties of optimization problems with vector-valued mapping. In [16], Weir and Mond proved weak, strong and converse duality for weak minima of multiple objective optimization problems under different pseudo-convexity and quasiconvexity assumptions. In [8], Mishra et al. investigated a general Mond–Weir type of duality results in terms of right differentials of generalized d-type-I functions involved in the multiobjective programming problem. In [9], Preda and Koller introduced a Mond–Weir duality scheme for optimization problems involving set functions, i.e., defined on a measure space (with the variables being measurable sets), and also studied the Mond–Weir type of duality results under generalized pseudo-convexity and generalized quasiconvexity assumptions.

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* Corresponding author.
E-mail addresses: lisj@cqu.edu.cn (S.J. Li), K.L.Teo@curtin.edu.au (K.L. Teo), mayangxq@polyu.edu.hk (X.Q. Yang).

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There are also some investigations on duality properties of optimization problems with set-valued mappings. In [10], Sach and Craven obtained Wolfe-type and Mond–Weir-type duality theorems of set-valued optimization problems under the condition that set-valued mappings satisfy an invex property and by virtue of tangent derivative of set-valued mapping introduced in [2]. In [11], Sach et al. discussed Mond–Weir-type and Wolfe-type weak duality and strong duality results of set-valued optimization problems under the condition that set-valued mappings satisfy generalized invex properties and by virtue of the codifferential of set-valued mappings introduced in [1]. It should be mentioned that the Lagrangian duality for vector optimization with set-valued mappings in infinite dimensional spaces has been considered in [3–5,7,12]. The conjugate duality has been investigated in [15,13].

In this paper, we recall mth-order tangent sets and mth-order contingent derivative of set-valued mappings (see [2]) and some properties of higher-order derivatives for a S-convex set-valued mapping. Then, by virtue of the mth-order contingent derivative, we introduce a kind of higher-order Mond–Weir-type duality, which is a generalization of Mond–Weir duality for single-valued functions (see [16]). We establish weak duality, strong duality and converse duality results for optimization problems with set-valued mappings.

The rest of paper is organized as follows. In Section 2, we recall some basic definitions, the mth-order contingent set and the mth-order adjacent set. Then, we discuss their properties. In Section 3, we recall the mth-order contingent derivative and discuss its important properties. In Section 4, we introduce a kind of higher-order Mond–Weir duality for a set-valued optimization problem and study weak duality, strong duality and converse duality properties between this set-valued optimization problem and its higher-order Mond–Weir duality problem.

2. Mathematical preliminaries and higher-order tangent sets

Let X be a Banach space and Y and Z be two ordered Banach spaces, in which relations are defined by pointed closed convex cone S with $\operatorname{int} S \neq \emptyset$ and D with $\operatorname{int} D \neq \emptyset$, respectively. S^+ and D^+ are the polar cones of S and D, respectively. Suppose that $F : X \to 2^Y$ and $G : X \to 2^Z$ are two set-valued mappings. $A \subset X$, set $F(A)=\bigcup_{x\in A} F(x)$. For any $B \subset Y$ and $C \subset Y$, we assume that

$$B - C \geqslant \theta \iff y_2 - y_1 \in S \quad \forall y_2 \in B, y_1 \in C.$$

Definition 2.1. Let B be a set of Y and $y_0 \in B$.

(i) y_0 is said to be a weakly maximal point of B if there is no $y \in B$ such that $y - y_0 \in \operatorname{int} S$, and $\max_{\operatorname{int} S} B$ denotes the set of all weakly maximal points of B.
(ii) y_0 is said to be a weakly minimal point of B if there is no $y \in B$ such that $y - y_0 \in -\operatorname{int} S$, and $\min_{\operatorname{int} S} B$ denotes the set of all weakly minimal points of B.

Definition 2.2. F is called S-convex if

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + S \quad \forall x_1, x_2 \in X \text{ and } \lambda \in [0, 1].$$

Definition 2.3. F is called pseudo-Lipschitzian at (x_0, y_0), where $y_0 \in F(x_0)$, if there exist $M > 0$ and neighborhoods V of x_0 and W of y_0 such that

$$F(x_1) \cap W \subset F(x_2) + M\|x_1 - x_2\|B \quad \forall x_1, x_2 \in V.$$

Definition 2.4 (*Tanino [14]*). A compact base for S is a nonempty compact subset B of S with $\theta \notin B$ such that every $d \in S$, $d \neq \theta$, has a unique representation of the form αb, where $b \in B$ and $\alpha > 0$.

Let X be supplied with a distance d and K be a subset of X. We denote by

$$d(x, K) = \inf_{y\in K} d(x, y)$$

the distance from x to K, where we set $d(x, \emptyset) = +\infty$.

Definition 2.5. Let x belong to a subset K of X and $v_1, \ldots, v_{m-1}$ be elements of X. We say that the subset

$$T_K^{(m)}(x, v_1, \ldots, v_{m-1}) = \limsup_{h \to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}$$
$$= \left\{ y \in X \middle| \liminf_{h \to 0^+} d\left(y, \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}\right) = 0 \right\}$$

is the mth-order contingent set of K at $(x, v_1, \ldots, v_{m-1})$.

Definition 2.6. Let x belong to a subset K of X and $v_1, \ldots, v_{m-1}$ be elements of X. We say that the subset

$$T_K^{\flat(m)}(x, v_1, \ldots, v_{m-1}) = \liminf_{h \to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}$$
$$= \left\{ y \in X \; \lim_{h \to 0^+} d\left(y, \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}\right) = 0 \right\}$$

is the mth-order adjacent set of K at $(x, v_1, \ldots, v_{m-1})$.

Now we state some results of the mth-order contingent and adjacent sets, which have been obtained in [6].

Proposition 2.1. *If K is a convex subset and $v_1, \ldots, v_{m-1} \in K$, then*

$$T_K^{\flat(m)}(x_0, v_1 - x_0, \ldots, v_{m-1} - x_0) = T_K^{(m)}(x_0, v_1 - x_0, \ldots, v_{m-1} - x_0)$$
$$= \mathrm{cl}\left(\bigcup_{h>0} \frac{K - x_0 - h(v_1 - x_0) - \cdots - h^{m-1}(v_{m-1} - x_0)}{h^m}\right).$$

Proposition 2.2. *If K is convex, then $T_K^{\flat(m)}(x_0, v_1, \ldots, v_{m-1})$ is convex.*

Corollary 2.1. *If K is a convex subset and $v_1, \ldots, v_{m-1} \in K$, then sets $T_K^{(m)}(x_0, v_1 - x_0, \ldots, v_{m-1} - x_0)$ and $\mathrm{cl}(\bigcup_{h>0}(K - x_0 - h(v_1 - x_0) - \cdots - h^{m-1}(v_{m-1} - x_0))/h^m)$ are convex.*

3. Higher-order derivatives of set-valued mappings

In this section, we shall recall the definitions of the mth-order contingent derivative for set-valued mappings in [2]. Then, we shall investigate its properties under the condition that the set-valued mapping is S-convex.

Definition 3.1. Let X, Y be normed spaces and $F : X \to 2^Y$ be a set-valued map. The mth-order contingent derivative $D^{(m)}F(x, y, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ of F at $(x, y) \in \mathrm{Graph}(F)$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{cases} \mathrm{Graph}(D^{(m)}F(x, y, u_1, v_1, \ldots, u_{m-1}, v_{m-1})) \\ \quad = T_{\mathrm{Graph}(F)}^{(m)}(x, y, u_1, v_1, \ldots, u_{m-1}, v_{m-1}), \end{cases}$$

where $\mathrm{Graph}(H)$ denotes the graph of the set-valued mapping H, i.e., $\mathrm{Graph}(H) = \{(x, y) | y \in H(x), x \in \mathrm{Dom}(H)\}$.

We also define the S-directed mth-order contingent derivative $D_S^{(m)}F(x, y, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ of F at (x, y) for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ to be the mth-order contingent derivative of the set-valued mapping

$$F(x) + S = \{y + s | y \in F(x), s \in S\}$$

at (x, y) for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$. By Proposition 2.1, we have the following result.

Proposition 3.1. *Let F be S-convex on convex set $A \subset \mathrm{Dom}(F)$, $(x_0, y_0) \in \mathrm{Graph}(F)$ and let $u_1, \ldots, u_{m-1} \in A$ and $v_1 \in F(u_1) + S, \ldots, v_{m-1} \in F(u_{m-1}) + S$. Then, for any $x \in A$,*

$$y \in D_S^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x)$$

if and only if for any sequence $\{h_n\}$ with $h_n \to 0^+$ there exists sequence $\{(x_n, y_n)\}$ with $y_n \in F(x_n)$ such that

$$\frac{(x_n, y_n) - (x_0, y_0) - h_n(u_1 - x_0, v_1 - y_0) - \cdots - h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0)}{h_n^m} \to (x, y).$$

By similar proof method of Theorem 4.1 in [6], we have the following result.

Proposition 3.2. *Let F be S-convex on convex set $A \subset \mathrm{Dom}(F)$. Then, for all $x', x'' \in A$ and any $y' \in F(x')$,*

$$F(x'') - y' \subset D_S^{(m)} F(x', y', u_1 - x', v_1 - y', \ldots, u_{m-1} - x', v_{m-1} - y')(x'' - x'),$$

where $u_1, \ldots, u_{m-1} \in A$ and $v_1 \in F(u_1) + S, \ldots, v_{m-1} \in F(u_{m-1}) + S$.

Proposition 3.3. *Let F be S-convex on $\mathrm{Dom}(F)$, $(x_0, y_0) \in \mathrm{Graph}(F)$ and let $u_1, \ldots, u_{m-1} \in \mathrm{Dom}(F)$, $v_1 \in F(u_1) + S, \ldots, v_{m-1} \in F(u_{m-1}) + S$. Suppose that S has a compact base and that there exists an $\bar{x} \in \mathrm{conv}\{x_0, u_1, \ldots, u_{m-1}\}$ with $\bar{x} \in \mathrm{int}(\mathrm{Dom}(F))$. Suppose that there exists a pointed closed cone $\bar{S}$ such that $S \backslash \{\theta\} \subset \mathrm{int}\, \bar{S}$ and*

$$\mathrm{conv}\{(x_0, y_0), (u_1, v_1), \ldots, (u_{m-1}, v_{m-1})\} \cap \mathrm{int}(\mathrm{Graph}(F + \bar{S})) = \emptyset. \tag{1}$$

Then,

$$\begin{aligned} &D_S^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\ &\quad \subset D^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) + S \quad \forall x \in A. \end{aligned}$$

Proof. Let $y \in D_S^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x)$. Then, there exist sequences $\{(x_n, y_n)\} \subset \mathrm{Graph}(F)$, $\{h_n\} \subset R^+ \backslash \{0\}$ with $h_n \to 0^+$ and $\{d_n\} \subset S$ such that

$$\begin{aligned} &\frac{(x_n, y_n + d_n) - (x_0, y_0) - h_n(u_1 - x_0, v_1 - y_0) - \cdots - h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0)}{h_n^m} \\ &\quad \to (x, y). \end{aligned} \tag{2}$$

Let us consider two possible cases for sequence $\{d_n\}$.

Case 1: There exists n_0 such that $d_n = \theta$, for $n \geqslant n_0$. By the definition of higher-order contingent derivative, we have

$$y \in D^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x).$$

Case 2: There exists a subsequence without loss of generality we still write as d_n such that $d_n \neq \theta$, for all n.

Now, we confirm that the sequence $\{\|d_n\|/h_n^m\}$ is bounded. Indeed, suppose that sequence $\{\|d_n\|/h_n^m\}$ is unbounded. Without loss of generality, we assume that $\|d_n\|/h_n^m \to +\infty$. Since S has a compact base, let

$$d_n/\|d_n\| \to d' \in S \backslash \{\theta\}. \tag{3}$$

It follows from the S-convexity of F on $\mathrm{Dom}(F)$ that $\mathrm{Graph}(F + \bar{S})$ is a convex set. By (1) and a standard separation theorem of convex sets, there exists a nonzero vector $(\lambda, \mu) \in X \times Y$ such that

$$\langle \lambda, \tilde{x} \rangle + \langle \mu, \tilde{y} \rangle \geqslant \langle \lambda, x \rangle + \langle \mu, y \rangle \tag{4}$$

for any $(\tilde{x}, \tilde{y}) \in \mathrm{conv}\{(x_0, y_0), (u_1, v_1), \ldots, (u_{m-1}, v_{m-1})\}$ and $(x, y) \in \mathrm{Graph}(F + \bar{S})$. Since there exists an $\bar{x} \in \mathrm{conv}\{x_0, u_1, \ldots, u_{m-1}\}$ with $\bar{x} \in \mathrm{int}(\mathrm{Dom}(F))$, $\mu \neq 0$. Take an arbitrary $s \in \bar{S}$. It follows from (4) and $(x_0, y_0 + s) \in \mathrm{Graph}(F + \bar{S})$ that

$$\langle \mu, s \rangle \leqslant 0.$$

This implies that

$$\mu \in (\bar{S})^{-}\backslash\{\theta\}. \tag{5}$$

From (1), (3) and (5), we have

$$d' \in S\backslash\{\theta\} \subset \operatorname{int} \bar{S}$$

and so

$$\langle \mu, d' \rangle < 0. \tag{6}$$

It follows from (2) that

$$\frac{x_n - x_0 - h_n(u_1 - x_0) - \cdots - h_n^{m-1}(u_{m-1} - x_0)}{\|d_n\|} \to \theta \tag{7}$$

and

$$\frac{y_n - y_0 - h_n(v_1 - y_0) - \cdots - h_n^{m-1}(v_{m-1} - y_0)}{\|d_n\|} \to -d'. \tag{8}$$

Obviously, when n is large enough, we have

$$\begin{aligned}&(x_0, y_0) + h_n(u_1 - x_0, v_1 - y_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0)\\ &\quad \in \operatorname{conv}\{(x_0, y_0), (u_1, v_1), \ldots, (u_{m-1}, v_{m-1})\}.\end{aligned} \tag{9}$$

It follows from (4) and (7)–(9) that

$$\langle \mu, d' \rangle \geqslant 0$$

which contradicts (6).

Thus, sequence $\{\|d_n\|/h_n^m\}$ is bounded and we can assume that

$$\|d_n\|/h_n^m \to \alpha \geqslant 0. \tag{10}$$

By (2) and (10), we have

$$\begin{aligned}&\frac{(x_n, y_n) - (x_0, y_0) - h_n(u_1 - x_0, v_1 - y_0) - \cdots - h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0)}{h_n^m}\\ &\quad \to (x, y - \alpha d').\end{aligned} \tag{11}$$

By (11) and the definition of the mth contingent derivative,

$$y - \alpha d' \in D^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x),$$

and so

$$\begin{aligned}&D_S^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x)\\ &\quad \subset D^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) + S\end{aligned}$$

and the conclusion follows readily. □

4. Higher-order Mond–Weir duality

Consider the following generalized vector optimization problem:

$$\begin{aligned}\text{(GVOP)}\quad &\min \quad F(x)\\ &\text{s.t.} \quad G(x) \cap (-D) \neq \emptyset,\end{aligned} \tag{12}$$

i.e., to find all $x_0 \in Q$ for which there exists a $y_0 \in F(x_0)$ such that $y_0 \in \min_{\operatorname{int} S} F(Q)$, where $Q = \{x \in X | G(x) \cap (-D) \neq \emptyset\}$. A point (x, y) is a feasible solution of Problem (GVOP) if $x \in Q$ and $y \in F(x)$.

Suppose that $(u_1, v_1), \dots, (u_{m-1}, v_{m-1}) \in \mathrm{Graph}(F+S)$ and $(u_1, w_1), \dots, (u_{m-1}, w_{m-1}) \in \mathrm{Graph}(G+D)$. We introduce a dual problem (DGVOP) of (GVOP) as follows:

$$\begin{aligned}
\max \quad & y_0 \\
\text{s.t.} \quad & lD_S^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\
& + \mu D_D^{(m)}G(x_0, z_0, u_1 - x_0, w_1 - z_0, \dots, u_{m-1} - x_0, w_{m-1} - z_0)(x) \geqslant 0, \quad x \in \Omega, && (13)\\
& \mu z_0 \geqslant 0, && (14)\\
& l \in S^+, \quad l \neq 0, && (15)\\
& \mu \in D^+, && (16)
\end{aligned}$$

where $z_0 \in G(x_0)$ and

$$\begin{aligned}
\Omega = {} & \mathrm{Dom}\, D_S^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0) \\
& \cap \mathrm{Dom}\, D_D^{(m)}G(x_0, z_0, u_1 - x_0, w_1 - z_0, \dots, u_{m-1} - x_0, w_{m-1} - z_0),
\end{aligned}$$

i.e., to find all (x_0, y_0, z_0, l, μ) which satisfy $y_0 \in \max_{\mathrm{int}\, S} H$, where

$$H = \{y_0 \in F(x_0) | (x_0, y_0, z_0, l, \mu) \text{ satisfies conditions (13)–(16)}\}.$$

A point (x_0, y_0, z_0, l, μ) satisfying (13)–(16) is called feasible for (DGVOP).

Remark 4.1. Let $Y = R^k$, $Z = R^n$, $S = R^k_+$, $D = R^n_+$, $m = 1$. Let F and G be single-valued functions $f = (f_1, f_2, \dots, f_k)$ and $g = (g_1, g_2, \dots, g_n)$ where $f_i \in C^1$ and $g_j \in C^1$. We have

$$D_S f(x_0, f(x_0))(x) = \nabla f(x_0)(x) + R^k_+$$

and

$$D_D g(x_0, g(x_0))(x) = \nabla g(x_0)(x) + R^n_+.$$

The dual problem (DGVOP) becomes

$$\begin{aligned}
\max \quad & f(x_0) \\
\text{s.t.} \quad & l\nabla f(x_0)(x) + \mu \nabla g(x_0)(x) \geqslant 0, \quad x \in \Omega, \\
& \mu g(x_0) \geqslant 0, \\
& l \in S^+, \quad l \neq 0 \\
& \mu \in D^+.
\end{aligned}$$

This is exactly one of the dual problems considered in [16]. Thus, (DGVOP) is a generality of Mond–Weir duality.

Theorem 4.1 (*Weak duality*). *Suppose that F and G are S-convex and D-convex on X, respectively. Let* $(u_1, v_1), \dots, (u_{m-1}, v_{m-1}) \in \mathrm{Graph}(F+S)$ *and* $(u_1, w_1), \dots, (u_{m-1}, w_{m-1}) \in \mathrm{Graph}(G+D)$. *Then the feasible solution* (x_0, y_0) *of* (GVOP) *and the feasible solution* $(\hat{x}, \hat{y}, \hat{z}, l, \mu)$ *of* (DGVOP) *satisfy*

$$ly_0 \geqslant l\hat{y}.$$

Proof. It follows from Proposition 3.2 that

$$y_0 - \hat{y} \in D_S^{(m)}F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \dots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})(x_0 - \hat{x}) \tag{17}$$

and

$$G(x_0) - \hat{z} \subset D_D^{(m)}G(\hat{x}, \hat{z}, u_1 - \hat{x}, w_1 - \hat{z}, \dots, u_{m-1} - \hat{x}, w_{m-1} - \hat{z})(x_0 - \hat{x}). \tag{18}$$

Since (x_0, y_0) is a feasible solution for (GVOP), $G(x_0) \cap (-D) \neq \emptyset$. Take a $z \in G(x_0) \cap (-D)$. Then, by (14), we have that

$$\mu z - \mu \hat{z} \leqslant 0. \tag{19}$$

It follows from (13) that

$$l y_0 - l \hat{y} + \mu z - \mu \hat{z} \geqslant 0.$$

Therefore, by (19), we get

$$l y_0 \geqslant l \hat{y}$$

and the proof is complete. □

Theorem 4.2 (*Strong duality*). *Suppose that the following conditions are satisfied*:

(i) *F is S-convex on X and G is D-convex on X*;
(ii) (x_0, y_0) *is a solution for* (GVOP);
(iii) $z_0 \in G(x_0)$ *and* $z_0 \notin \min_{\operatorname{int} D} G(\Omega)$;
(vi) *$G + D$ is pseudo-Lipschitzian at* (x_0, z_0);
(v) $(u_i, v_i - y_0, w_i) \in X \times (-S) \times (-D)$, *for* $i = 1, \ldots, m-1$.

Then, *there exists* $(l, \mu) \in S^+ \times D^+$ *such that* (x_0, y_0, z_0, l, μ) *is a solution of* (DGVOP).

Proof. We first prove that

$$\begin{aligned} &D^{(m)}_{S\times D}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x) \\ &\quad = D^{(m)}_S F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\ &\qquad \times D^{(m)}_D G(x_0, z_0, u_1 - x_0, w_1 - z_0, \ldots, u_{m-1} - x_0, w_{m-1} - z_0)(x). \end{aligned} \tag{20}$$

Naturally, we only need to prove

$$\begin{aligned} &D^{(m)}_S F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\ &\quad \times D^{(m)}_D G(x_0, z_0, u_1 - x_0, w_1 - z_0, \ldots, u_{m-1} - x_0, w_{m-1} - z_0)(x) \\ &\qquad \subseteq D^{(m)}_{S\times D}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x). \end{aligned}$$

Suppose that

$$\begin{aligned} (y, z) \in\ & D^{(m)}_S F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\ &\times D^{(m)}_D G(x_0, z_0, u_1 - x_0, w_1 - z_0, \ldots, u_{m-1} - x_0, w_{m-1} - z_0)(x). \end{aligned}$$

It follows from Proposition 3.1 that, for any $h_n \to 0^+$, there exists $(x_n, y_n) \to (x, y)$ such that

$$\begin{aligned} &y_0 + h_n(v_1 - y_0) + \cdots + h_n^{m-1}(v_{m-1} - y_0) + h_n^m y_n \\ &\quad \in F(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m x_n) + S. \end{aligned} \tag{21}$$

Similarly, for any $h_n \to 0^+$, there exists $(\bar{x}_n, \bar{z}_n) \to (x, z)$ such that

$$\begin{aligned} &z_0 + h_n(w_1 - z_0) + \cdots + h_n^{m-1}(w_{m-1} - z_0) + h_n^m \bar{z}_n \\ &\quad \in G(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m \bar{x}_n) + D. \end{aligned} \tag{22}$$

By the assumption (iv), there exist $M > 0$, and neighborhoods $\mathscr{W}$ of z_0 and $\mathscr{N}$ of x_0 such that

$$(G(x_1) + D) \cap \mathscr{W} \subset G(x_2) + D + M\|x_1 - x_2\|B \quad \forall x_1, x_2 \in \mathscr{N}. \tag{23}$$

Naturally, there exists $N > 0$ satisfying

$$x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m x_n \in \mathscr{N} \quad \forall n \geqslant N$$

and

$$z_0 + h_n(w_1 - z_0) + \cdots + h_n^{m-1}(w_{m-1} - z_0) + h_n^m \bar{z}_n \in \mathscr{W} \quad \forall n \geqslant N. \tag{24}$$

It follows from (22)–(24) that

$$\begin{aligned} &z_0 + h_n(w_1 - z_0) + \cdots + h_n^{m-1}(w_{m-1} - z_0) + h_n^m \bar{z}_n \\ &\quad \in G(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0 + h_n^m \bar{x}_n) + D) \cap \mathscr{W} \\ &\quad\quad \subset G(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m x_n) + D + h_n^m M\|\bar{x}_n - x_n\|B \quad \forall n \geqslant N. \end{aligned}$$

Then, there exists $z_n \to z$ such that for any $n \geqslant N$,

$$\begin{aligned} &z_0 + h_n(w_1 - z_0) + \cdots + h_n^{m-1}(w_{m-1} - z_0) + h_n^m z_n \in G(x_0 + h_n(u_1 - x_0) \\ &\quad + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m x_n) + D. \end{aligned} \tag{25}$$

It follows from (21) and (25) that

$$(y, z) \in D^{(m)}_{S\times D}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x),$$

and (20) holds.

Define

$$\begin{aligned} B = &\bigcup_{x\in\Omega} D^{(m)}_{S\times D}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x) \\ &+ (\theta, z_0). \end{aligned}$$

It follows from the convexity of $\mathrm{Graph}(F + S, G + D)$ and Proposition 2.2 that

$$T^{(m)}_{\mathrm{Graph}(F+S,G+D)}((x_0, y_0, z_0), (u_1 - x_0, v_1 - y_0, w_1 - z_0), \ldots, (u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0))$$

is a convex set. Therefore, by similar proof method for the convexity of B in Theorem 5.1 in [4], we have that B is a convex set.

We next prove that

$$B \cap [-\mathrm{int}\, S \times \mathrm{int}\, D] = \emptyset. \tag{26}$$

To arrive at a contradiction, we assume that there exists $(\hat{x}, \hat{y}, \hat{z})$ such that

$$(\hat{y}, \hat{z}) \in D^{(m)}_{S\times D}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(\hat{x}) \tag{27}$$

and

$$(\hat{y}, \hat{z} + z_0) \in -\mathrm{int}\, S \times \mathrm{int}\, D. \tag{28}$$

It follows from (27) and the definition of the mth-order contingent derivative that there exist sequences $\{h_n\}$ with $h_n \to 0^+$ and $\{(x_n, y_n, z_n)\}$ with

$$y_n \in F(x_n) + S, \quad z_n \in G(x_n) + D$$

such that

$$\frac{(x_n, y_n, z_n) - (x_0, y_0, z_0)}{h_n^m} - \frac{h_n(u_1 - x_0, v_1 - y_0, w_1 - z_0)}{h_n^m} - \cdots \frac{h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)}{h_n^m} \to (\hat{x}, \hat{y}, \hat{z}). \tag{29}$$

From (28) and (29), there exists $N > 0$ such that $h_n + \cdots + h_n^m < 1$ and

$$\frac{(y_n, z_n) - (y_0, z_0) - h_n(v_1 - y_0, w_1 - z_0) - \cdots - h_n^{m-1}(v_{m-1} - y_0, w_{m-1} - z_0)}{h_n^m} + (\theta, z_0) \in -\mathrm{int}\, S \times \mathrm{int}\, D$$

for $n \geqslant N$. Thus, we have

$$y_n - y_0 - h_n(v_1 - y_0) - \cdots - h_n^{m-1}(v_{m-1} - y_0) \in -\mathrm{int}\, S \quad \text{for } n \geqslant N$$

and

$$z_n - z_0 - h_n(w_1 - z_0) - \cdots - h_n^{m-1}(w_{m-1} - z_0) + h_n^m z_0 \in -\mathrm{int}\, D \quad \text{for } n \geqslant N.$$

Since $z_0, w_1, \ldots, w_{m-1} \in -D$ and $v_1 - y_0, \ldots, v_{m-1} - y_0 \in -S$,

$$(1 - h_n - \cdots - h_n^m)z_0 + h_n w_1 + \cdots + h_n^{m-1} w_{m-1} \in -D$$

and

$$h_n(v_1 - y_0) + \cdots + h_n^{m-1}(v_{m-1} - y_0) \in -S.$$

Thus, $z_n \in -\mathrm{int}\, D$ and $y_n - y_0 \in -\mathrm{int}\, S$. Since $z_n \in G(x_n) + D$ and $y_n \in F(x_n) + S$, there exist $\bar{z}_n \in G(x_n)$, $d_n \in D$, $\bar{y}_n \in F(x_n)$ and $s_n \in S$ such that

$$z_n = \bar{z}_n + d_n \quad \text{and} \quad y_n = \bar{y}_n + s_n \quad \text{for } n \geqslant N.$$

Naturally, $\bar{z}_n \in G(x_n) \cap -D$ and $\bar{y}_n - y_0 \in -\mathrm{int}\, S$, which contradicts that x_0 is a weak minimal solution at y_0. Thus, (26) holds. It follows from a standard separation theorem of convex sets and similar proof method of Theorem 5.1 in [4] that there exist $l \in S^+$and $\mu \in D^+$, not both zero functionals, such that

$$\mu(z_0) = \theta, \tag{30}$$
$$l(y) + \mu(z) \geqslant 0, \tag{31}$$

for all

$$(y, z) \in D^{(m)}_{S\times D}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x)$$

and $x \in \Omega$.

It follows from (20), (30) and (31) that (x_0, y_0, z_0, l, μ) satisfies (13, (4) and (16). Now we prove that the functional l satisfies (15), i.e., $l \neq 0$.

In fact, from the assumption (iii) and Proposition 3.2, there exists an $\bar{x} \in \Omega$ such that

$$D^{(m)}_D G(x_0, z_0, u_1 - x_0, w_1 - z_0, \ldots, u_{m-1} - x_0, w_{m-1} - z_0)(\bar{x}) \cap -\mathrm{int}\, D \neq \emptyset,$$

i.e., there exists $\bar{z} \in D^{(m)}_D G(x_0, z_0, u_1 - x_0, w_1 - z_0, \ldots, u_{m-1} - x_0, w_{m-1} - z_0)(\bar{x})$ and $\bar{z} \in -\mathrm{int}\, D$. Since $\mu \in D^+$, we have $\mu(\bar{z}) < 0$ if $\mu \neq 0$. Then, it follows from (31) that $l \neq 0$. So, (x_0, y_0, z_0, l, μ) is a feasible solution.

Finally, we prove that (x_0, y_0, z_0, l, μ) is a solution of (DGVOP). Suppose that (x_0, y_0, z_0, l, μ) is not a solution of (DGVOP). Then, there exists a feasible solution $(\hat{x}, \hat{y}, \hat{z}, l', \mu')$ such that

$$\hat{y} > y_0.$$

By $l' \in S^+$ and $l' \neq 0$, we have

$$l'\hat{y} > l'y_0. \tag{32}$$

Since (x_0, y_0) is a feasible solution for (GVOP), by Theorem 4.1, we have that $l'y_0 \geqslant l'\hat{y}$, which contradicts (32). Thus, the proof is complete. □

Theorem 4.3 (*Converse duality*). *Suppose that the following conditions are satisfied*:

(i) *F is S-convex on X and G is D-convex on X*;
(ii) $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1}) \in \mathrm{Graph}(F + S)$ *and* $(u_1, w_1), \ldots, (u_{m-1}, w_{m-1}) \in \mathrm{Graph}(G + D)$;
(iii) *there exist $x_0 \in X$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap (-D)$, nonzero $l \in S^+$ and $\mu \in D^+$ such that* (x_0, y_0, z_0, l, μ) *is a solution of* (DGVOP).

Then, (x_0, y_0) *is a solution of* (GVOP).

Proof. Suppose that $x \in Q$. Then, there exists $d \in G(x) \cap (-D)$. It follows from Proposition 3.2 that

$$d - z_0 \in D_D^{(m)} G(x_0, z_0, u_1 - x_0, w_1 - z_0, \ldots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0).$$

By (14), we have that $\mu z_0 \geqslant 0$. It follows from $z_0 \in G(x_0) \cap (-D)$ that $\mu z_0 \leqslant 0$. So

$$\mu z_0 = 0$$

and

$$\mu(d - z_0) = \mu(d) - \mu(z_0) = \mu(d) \leqslant 0. \tag{33}$$

Therefore, it follows from (13) and (33) that

$$l D_S^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) \geqslant 0, \quad x \in Q. \tag{34}$$

From Proposition 3.2 and (34), we have

$$l(F(Q) - y_0) \geqslant 0.$$

Since l is a nonzero positive functional, we get that $y_0 \in \min_{\mathrm{int}\, S} F(Q)$. Thus, (x_0, y_0) is a solution of (GVOP) and this completes the proof. □

Note that the following inclusion relation always holds:

$$\begin{aligned} &D^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) + S \\ &\quad \subset D_S^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x). \end{aligned} \tag{35}$$

However, converse inclusion relation may not hold. The following example explains the case.

Example 4.1. Suppose that $S = R^+$, $m = 1$ and

$$F(x) = \begin{cases} \{0\} & \text{if } x \leqslant 0, \\ \{0, -\sqrt{x}\} & \text{if } x > 0. \end{cases}$$

Then,

$$DF(0,0)(x) = \begin{cases} \{0\} & \text{if } x \neq 0, \\ \{y | y \leqslant 0\} & \text{if } x = 0 \end{cases}$$

and

$$D_S F(0,0)(x) = \begin{cases} \{y | y \geqslant 0\} & \text{if } x < 0, \\ R & \text{if } x \geqslant 0. \end{cases}$$

Obviously, when $x > 0$, we have

$$D_S F(0,0)(x) \nsubseteq DF(0,0)(x) + R^+.$$

Thus, if we use $lD^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) + \mu D^{(m)}G(x_0, z_0, u_1 - x_0, w_1 - z_0, \ldots, u_{m-1} - x_0, w_{m-1} - z_0)(x) \geqslant 0$ instead of the inequality relation in (13), we obtain the following dual problem (DGVOP1) of (GVOP):

$$\begin{aligned} \max \quad & y_0 \\ \text{s.t.} \quad & lD^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\ & + \mu D^{(m)}G(x_0, z_0, u_1 - x_0, w_1 - z_0, \ldots, u_{m-1} - x_0, w_{m-1} - z_0)(x) \geqslant 0, \quad x \in \Omega, \\ & \mu z_0 \geqslant 0, \\ & l \in S^+, \quad l \neq 0, \\ & \mu \in D^+. \end{aligned}$$

It follows from (35) that the feasible set of (DGVOP1) includes one of (DGVOP). Thus, under the assumptions of Theorem 4.2, strong duality theorem also holds for the dual problem (DGVOP1). It follows from Proposition 3.3 that if F and G satisfy the assumptions of Proposition 3.3, respectively, then the weak duality also holds for (GVOP) and (DGVOP1) under the assumptions of Theorem 4.1. Naturally, if F and G satisfy the assumptions of Proposition 3.3, respectively, then the converse duality also holds for (GVOP) and (DGVOP1) under the assumptions of Theorem 4.3.

References

[1] J.P. Aubin, I. Ekeland, Applied Nonlinear Analysis, Wiley, New York, 1984.

[2] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, 1990.

[3] H.W. Corley, Existence and Lagrangian duality for maximizations of set-valued functions, J. Optim. Theory Appl. 54 (1987) 489–501.

[4] H.W. Corley, Optimality conditions for maximizations of set-valued functions, J. Optim. Theory Appl. 58 (1988) 1–10.

[5] S. Dolecki, C. Malivert, General duality for vector optimization, Optimization 27 (1993) 97–119.

[6] S.J. Li, K.L. Teo, X.Q. Yang, Higher order optimality conditions for set-valued optimization, J. Optim. Theory Appl., to appear.

[7] D.T. Luc, J. Jahn, Axiomatic approach to duality in optimization, Numer. Funct. Anal. Optim. 13 (1992) 305–326.

[8] S.K. Mishra, S.Y. Wang, K.K. Lai, Optimality and duality in nondifferentiable and multiobjective programming under generalized d-invexity, J. Global Optim. 29 (2004) 425–438.

[9] V. Preda, K. Koller, General Mond–Weir duality for multiobjective programming with generalized (F, ρ, θ) convex set functions, Romanian J. Pure and Appl. Math. 45 (2000) 1005–1018.

[10] P.H. Sach, B.D. Craven, Invex multifunctions and duality, Numer. Funct. Anal. Optim. 12 (1991) 575–591.

[11] P.H. Sach, N.D. Yen, B.D. Craven, Generalized invexity and duality theorems with multifunctions, Numer. Funct. Anal. Optim. 15 (1994) 131–153.

[12] W. Song, Lagrangian duality for minimization of nonconvex multifunctions, J. Optim. Theory Appl. 93 (1997) 167–182.

[13] W. Song, Conjugate duality in set-valued vector optimization, J. Math. Anal. Appl. 216 (1997) 265–283.

[14] T. Tanino, Sensitivity analysis in multiobjective optimization, J. Optim. Theory Appl. 56 (1988) 479–499.

[15] T. Tanino, Conjugate duality in vector optimization, J. Math. Anal. Appl. 167 (1992) 84–97.

[16] T. Weir, B. Mond, Generalized convexity and duality in multiple objective programming, Bull. Austral. Math. Soc. 39 (1989) 287–299.