

## INTEGER APERTURE LEAST-SQUARES ESTIMATION

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### ABSTRACT

GNSS carrier phase ambiguity resolution is the key to fast and high-precision satellite positioning and navigation. It applies to a great variety of current and future models of GPS, modernized GPS and Galileo. In [Teunissen, 2003] we described the general principle of integer aperture (IA) ambiguity estimation. In the present contribution we introduce one particular IA estimator, the integer aperture least-squares (IALS) estimator. The motivation for studying this estimator stems from the known optimality of the integer least-squares estimator. It is shown how the IALS estimator extends the integer least-squares estimator and how its performance can be measured by means of its fail-rate and success-rate.

### 1 INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of double difference (DD) carrier phase data. Its practical importance becomes clear when one realizes the great variety of current and future GNSS models to which it applies. An overview of GNSS models, together with their applications in surveying, navigation, geodesy and geophysics, can be found in textbooks such as [Hofmann-Wellenhof et al., 2001], [Leick, 1995], [Misra and Enge, 2001], [Parkinson and Spilker, 1996], [Strang and Borre, 1997] and [Teunissen and Kleusberg, 1998].

In [Teunissen, 2003] we introduced the class of integer aperture (IA) estimators for carrier phase ambiguity resolution. This class allows one to design ambiguity estimators such that the ambiguity resolution process will have a user-defined *fixed fail-rate*. In this contribution we will introduce the integer aperture least-squares (IALS) estimator as an extension of the well-known integer least-squares estimator. We start with a brief review of integer estimation and of integer least-squares estimation in particular. Then we describe the general principle of integer aperture estimation and introduce the integer aperture least-squares estimator. It is shown how the framework of integer aperture estimation incorporates the important problem of ambiguity discernibility. By setting the size and shape of the integer aperture pull-in region, the user has control over the fail-rate of the

integer aperture estimator and thus also over the amount of discernibility. In case of the IALS estimator the aperture pull-in region is chosen as a down-sized version of the integer least-squares pull-in region. It is shown how the aperture of the pull-in region governs the fail-rate and the success-rate of the IALS estimator and how lower bounds and upper bounds of these probabilities can be computed.

## 2 INTEGER LEAST-SQUARES ESTIMATION

### 2.1 THE GNSS MODEL

As our point of departure we take the following system of linear observation equations

$$E\{y\} = Aa + Bb, \quad a \in Z^n, \quad b \in R^p \quad (1)$$

with  $E\{\cdot\}$  the mathematical expectation operator,  $y$  the  $m$ -vector of observables,  $a$  the  $n$ -vector of unknown integer parameters and  $b$  the  $p$ -vector of unknown real-valued parameters. The  $m \times (n + p)$  design matrix  $(A, B)$  is assumed to be of full rank.

All the linear(ized) GNSS models can in principle be cast in the above frame of observation equations. The data vector  $y$  will then usually consist of the 'observed minus computed' single- or dual- frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector  $a$  are then the DD carrier phase ambiguities, expressed in units of cycles rather than range, while the entries of the vector  $b$  will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

The procedure which is usually followed for solving the GNSS model can be divided into three steps. In the *first* step one simply discards the integer constraints  $a \in Z^n$  and performs a standard least-squares (LS) adjustment. As a result one obtains the LS-estimators of  $a$  and  $b$  as

$$\begin{cases} \hat{a} = (\bar{A}^T Q_y^{-1} \bar{A})^{-1} \bar{A}^T Q_y^{-1} y \\ \hat{b} = (\bar{B}^T Q_y^{-1} \bar{B})^{-1} \bar{B}^T Q_y^{-1} y \end{cases} \quad (2)$$

with  $Q_y$  the vc-matrix of the observables,  $\bar{A} = P_B^\perp A$ ,  $\bar{B} = P_A^\perp B$ , and the two orthogonal projectors  $P_B^\perp = I_m - B(B^T Q_y^{-1} B)^{-1} B^T Q_y^{-1}$  and  $P_A^\perp = I_m - A(A^T Q_y^{-1} A)^{-1} A^T Q_y^{-1}$ . This solution is usually referred to as the 'float' solution.

In the *second* step the 'float' estimator  $\hat{a}$  is used to compute the corresponding integer estimator  $\check{a} \in Z^n$ . This implies that a mapping  $S$  from the  $n$ -dimensional space of reals to the  $n$ -dimensional space of integers is introduced such that

$$\check{a} = S(\hat{a}), \quad S: R^n \mapsto Z^n \quad (3)$$

This integer estimator is then used in the final and *third* step to adjust the 'float' estimator  $\hat{b}$ . As a result one obtains the so-called 'fixed' estimator of  $b$  as

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \quad (4)$$

in which  $Q_{\hat{a}}$  denotes the vc-matrix of  $\hat{a}$  and  $Q_{\hat{b}\hat{a}}$  denotes the covariance matrix of  $\hat{b}$  and  $\hat{a}$ . This 'fixed' estimator can alternatively be expressed as  $\check{b} = (B^T Q_y^{-1} B)^{-1} B^T Q_y^{-1} (y - A\check{a})$ .

Note that only two of the three steps are needed in case one only would be interested in obtaining an integer solution for  $a$ . In the case of GNSS, however, one is particularly interested in the solution of the third step as it contains the solution for the baseline coordinates. All three steps are therefore needed in case of GNSS.

## 2.2 INTEGER AMBIGUITY ESTIMATION

The above three-step procedure is still ambiguous in the sense that it leaves room for choosing the integer map  $S$ . It will be clear that the map  $S$  will not be one-to-one due to the discrete nature of  $Z^n$ . Instead it will be a many-to-one map. This implies that different real-valued vectors will be mapped to one and the same integer vector. One can therefore assign a subset  $S_z \subset R^n$  to each integer vector  $z \in Z^n$ :

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \quad (5)$$

The subset  $S_z$  contains all real-valued vectors that will be mapped by  $S$  to the same integer vector  $z \in Z^n$ . This subset is referred to as the *pull-in region* of  $z$ . It is the region in which all vectors are pulled to the same integer vector  $z$ .

Since the pull-in regions define the integer estimator completely, one can define classes of integer estimators by imposing various conditions on the pull-in regions. One such class was introduced by Teunissen (1999a) as follows.

### Definition 1 (*Integer estimators*)

The mapping  $\tilde{a} = S(\hat{a})$  is said to be an integer estimator if its pull-in regions satisfy

- (i)  $\bigcup_{z \in Z^n} S_z = R^n$
- (ii)  $Int(S_{z_1}) \cap Int(S_{z_2}) = \emptyset, \quad \forall z_1, z_2 \in Z^n, z_1 \neq z_2$
- (iii)  $S_z = z + S_0, \quad \forall z \in Z^n$

This definition is motivated as follows. Each one of the above three conditions describe a property of which it seems reasonable that it is possessed by an arbitrary integer estimator. The first condition states that the pull-in regions should not leave any gaps and the second that they should not overlap. The absence of gaps is needed in order to be able to map any float solution  $\hat{a} \in R^n$  to  $Z^n$ , while the absence of overlaps is needed to guarantee that the float solution is mapped to just one integer vector. Note that we allow the pull-in regions to have common boundaries. This is permitted if we assume to have zero probability that  $\hat{a}$  lies on one of the boundaries. This will be the case when the probability density function (PDF) of  $\hat{a}$  is continuous.

The third and last condition of the definition follows from the requirement that  $S(x+z) = S(x) + z, \forall x \in R^n, z \in Z^n$ . Also this condition is a reasonable one to ask for. It states that when the float solution  $\hat{a}$  is perturbed by  $z \in Z^n$ , the corresponding integer solution is perturbed by the same amount. This property allows one to apply the *integer remove-restore* technique:  $S(\hat{a} - z) + z = S(\hat{a})$ . It therefore allows one to work with the fractional parts of the entries of  $\hat{a}$ , instead of with its complete entries.

Using the pull-in regions, one can give an explicit expression for the corresponding integer estimator  $\tilde{a}$ . It reads

$$\tilde{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \quad \text{with} \quad s_z(\hat{a}) = \begin{cases} 1 & \text{if } \hat{a} \in S_z \\ 0 & \text{if } \hat{a} \notin S_z \end{cases} \quad (6)$$

Note that the  $s_z(\hat{a})$  can be interpreted as weights, since  $\sum_{z \in Z^n} s_z(\hat{a}) = 1$ . The integer estimator  $\check{a}$  is therefore equal to a weighted sum of integer vectors with binary weights.

### 2.3 INTEGER LEAST-SQUARES AMBIGUITY ESTIMATION

Different choices for  $S$  will lead to different integer estimators  $\check{a}$  and thus also to different baseline estimators  $\check{b}$ . One can therefore now think of constructing integer maps which possess certain desirable properties. Examples are integer rounding, integer bootstrapping and integer least-squares. In this contribution we will make use of the integer least-squares (ILS) estimator. It is defined as

$$\check{a}_{LS} = \arg \min_{z \in Z^n} \|\hat{a} - z\|_{Q_{\hat{a}}}^2 \quad (7)$$

In contrast to integer rounding and integer bootstrapping, an integer search is needed to compute  $\check{a}_{LS}$ . The ILS-estimator was introduced in [Teunissen, 1993], see also [Teunissen, 1995]. The ILS procedure is mechanized in the LAMBDA method, which is currently one of the most applied methods for GNSS carrier phase ambiguity resolution. Practical results obtained with it can be found, for example, in [Boon and Ambrosius, 1997], [Boon et al., 1997], [Cox and Brading, 1999], [de Jonge and Tiberius, 1996b], [de Jonge et al., 1996], [Han, 1995], [Jonkman, 1998], [Peng et al., 1999], [Tiberius and de Jonge, 1995], [Tiberius et al., 1997].

To determine the ILS pull-in regions we need to know the set of float solutions  $\hat{a} \in R^n$  that are mapped to the same integer vector  $z \in Z^n$ . This set is described by all  $x \in R^n$  that satisfy  $z = \arg \min_{u \in Z^n} \|x - u\|_{Q_{\hat{a}}}^2$ . The ILS pull-in-region that belongs to the integer vector  $z$  follows therefore as

$$S_{LS,z} = \{x \in R^n \mid \|x - z\|_{Q_{\hat{a}}}^2 \leq \|x - u\|_{Q_{\hat{a}}}^2, \forall u \in Z^n\} \quad (8)$$

It consists of all those points which are closer to  $z$  than to any other integer point in  $R^n$ . The metric used for measuring these distances is determined by the vc-matrix  $Q_{\hat{a}}$ . An alternative representation of the ILS pull-in regions is

$$S_{LS,z} = \bigcap_{c_i \in Z^n} \{x \in R^n \mid |c_i^T Q_{\hat{a}}^{-1}(x - z)| \leq \frac{1}{2} \|c_i\|_{Q_{\hat{a}}}^2\}, \forall z \in Z^n \quad (9)$$

This shows that the ILS pull-in regions are constructed from intersecting half-spaces. One can show that at most  $2^n - 1$  pairs of such half spaces are needed for constructing the pull-in region. The ILS pull-in regions are convex, symmetric sets of volume 1, which satisfy the conditions of Definition 1. They are hexagons in the two-dimensional case. Two-dimensional examples of the pull-in regions of integer least-squares are given in Figure 1.

### 2.4 PROBABILITY OF CORRECT INTEGER ESTIMATION: THE AMBIGUITY SUCCESS-RATE

For the evaluation of the fixed ambiguities one needs the distribution of the integer estimator  $\check{a}$ . This distribution is of the discrete type and it will be denoted as  $P(\check{a} = z)$ . It is a probability mass function, having zero masses at nongrid points and nonzero masses

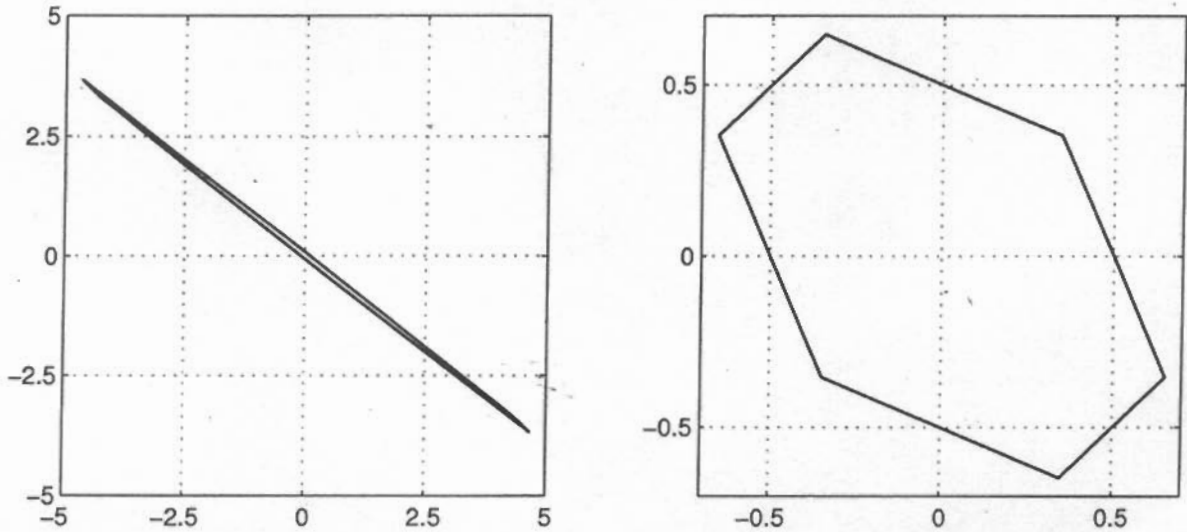


Figure 1: Two-dimensional pull-in regions of integer least-squares. Left: high correlation; Right: low correlation.

at some or all grid points. This distribution is obtained from integrating the probability density function (PDF) of  $\hat{a}$  over the pull-in regions,

$$P(\check{a} = z) = \int_{S_z} f_{\hat{a}}(x) dx, \quad z \in Z^n \quad (10)$$

This distribution is of course dependent on the pull-in regions  $S_z$  and thus on the chosen integer estimator. Since various integer estimators exist which are admissible, some may be better than others. Having the problem of GNSS ambiguity resolution in mind, one is particularly interested in the estimator which maximizes the probability of correct integer estimation. This probability equals  $P(\check{a} = a)$ , but it will differ for different ambiguity estimators. The answer to the question which estimator maximizes the probability of correct integer estimation is given by the following theorem.

**Theorem 1** (*Optimal integer estimation*)

Let  $f_{\hat{a}}(x | a)$  be the PDF of the float solution  $\hat{a}$  and let

$$\check{a}_{ML} = \arg \max_{a \in Z^n} f_{\hat{a}}(\hat{a} | a) \quad (11)$$

be an integer estimator. Then

$$P(\check{a}_{ML} = a) \geq P(\check{a} = a) \quad (12)$$

for any arbitrary integer estimator  $\check{a}$ .

*Proof:* see [Teunissen, 1999]

Note that we have denoted the PDF of  $\hat{a}$  for the occasion as  $f_{\hat{a}}(x | a)$  instead of as  $f_{\hat{a}}(x)$ . This has been done to explicitly show the dependence of the pdf on the true but unknown ambiguity vector  $a \in Z^n$ . In this contribution we will make a limited use of this notation. We will use the notation  $f_{\hat{a}}(x | a)$  only when it is really needed to show the dependence on  $a$  explicitly.

The above theorem holds true for an arbitrary pdf of the float ambiguities  $\hat{a}$ . In most GNSS applications however, one assumes the data to be normally distributed. The estimator  $\hat{a}$  will then be normally distributed too, with mean  $a \in Z^n$  and vc-matrix  $Q_{\hat{a}}$ ,  $\hat{a} \sim N(a, Q_{\hat{a}})$ . In this case the optimal estimator becomes identical to the integer least squares estimator

$$\check{a}_{LS} = \arg \min_{a \in Z^n} \|x - a\|_{Q_{\hat{a}}}^2 \quad (13)$$

The above theorem therefore gives a probabilistic justification for using the ILS estimator when the pdf is Gaussian. For GNSS ambiguity resolution it shows, that one is better off using the ILS estimator than any other admissible integer estimator. Due to the rather complicated geometry of the integer least-squares pull-in region, no exact and easy-to-compute expression exists for the ILS success rate. However, there do exist sharp bounds for this success rate.

**Corollary 1** (*Bounds for the ILS success rate*)

Let the float solution be distributed as  $\hat{a} \sim N(a, Q_{\hat{a}})$ . Then

$$\prod_{i=1}^n \left( 2\Phi \left( \frac{1}{2\sigma_{i|I}} \right) - 1 \right) \leq P(\check{a}_{LS} = a) \leq P \left( \chi^2(n, 0) \leq \frac{c_n}{\text{ADOP}^2} \right) \quad (14)$$

with  $\sigma_{i|I}$  the conditional standard deviation of the  $i$ th ambiguity conditioned on the previous  $I = 1, \dots, i-1$  ambiguities,  $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}v^2\} dv$ ,  $c_n = (\frac{n}{2}\Gamma(\frac{n}{2}))^{2/n}/\pi$ ,  $\text{ADOP} = \sqrt{\det Q_{\hat{a}}^{-1}}$ , and where  $\Gamma$  denotes the gamma function and  $\chi^2(n, 0)$  the central Chi-square distribution with  $n$  degrees of freedom.

The lower bound was introduced in [Teunissen, 1997] and proven in [Teunissen, 1999]. The upper bound was introduced in [Hassibi and Boyd, 1998] and proven in [Teunissen, 2000]. Although other lower bounds exist, the above lower bound is presently the sharpest one available, see e.g. [Thomsen, 2000] and [Verhagen, 2003]. In [Teunissen, 1998] it has been shown that the lower bound is the exact success rate of integer bootstrapping. The above bounds will be used later in this contribution for obtaining bounds for one of the integer aperture estimators.

### 3 INTEGER APERTURE LEAST-SQUARES ESTIMATION

#### 3.1 INTEGER APERTURE ESTIMATION

In this section we will extend the theory of integer (I) estimation to integer aperture (IA) estimation. The outcome of an I estimator is always integer. It may happen, however, that one is not willing to accept the integer outcome. This will be the case when one is doubtful about the correctness of the integer outcome. In that case one would rather prefer the non-integer float solution instead. The decision whether or not to make use of the integer outcome can be made in different ways. One approach is to base the decision on the probability of correct integer estimation, the success-rate. The decision is then made in favor of the float solution if this probability falls below a certain user-defined threshold. This approach can be referred to as being model-driven, since the probability of correct integer estimation depends on the strength of the underlying mathematical model but not on the actual outcome of the estimator. With this approach, the decision whether or to make use of the I estimator can thus be made before the actual measurements are

collected and processed. Next to the model-driven approach, one can also make use of a more data-driven approach. In many GNSS ambiguity resolution procedures we already have such data-driven approaches in place. They are referred to as the 'discernibility tests'. They come to reject the integer outcome when it appears difficult, using the float solution, to discern between the 'best' and the 'second best' integer solution. In the case of a rejection the decision is made in favor of the float solution. As with the model-driven approach, the rationale of the 'discernibility tests' is that one wants to avoid the situation of having to work with an incorrect integer solution. Since the aim of both approaches is essentially the same, one may wonder whether or not it is possible to formulate an overall framework in which both approaches find their natural place. This indeed turns out to be possible. The required framework is given by the class of integer aperture (IA) estimators as introduced in Teunissen (2003). The IA estimators are defined by dropping one of the three conditions of Definition 1, namely the condition that the pull-in regions should cover  $R^n$  completely. The pull-in regions of the IA estimators are therefore allowed to have gaps, thus making it possible that their outcomes could be equal to the float solution as well.

In order to introduce the new class of ambiguity estimators from first principles, let  $\Omega \subset R^n$  be the region of  $R^n$  for which  $\hat{a}$  is mapped to an integer if  $\hat{a} \in \Omega$ . It seems reasonable to ask of the region  $\Omega$  that it has the property that if  $\hat{a} \in \Omega$  then also  $\hat{a} + z \in \Omega$ , for all  $z \in Z^n$ . If this property would not hold, then float solutions could be mapped to integers whereas their fractional parts would not. We thus require  $\Omega$  to be translational invariant with respect to an arbitrary integer vector:  $\Omega + z = \Omega$ , for all  $z \in Z^n$ . Knowing  $\Omega$  is however not sufficient for defining our estimator.  $\Omega$  only determines whether or not the float solution is mapped to an integer, but it does not tell us yet to which integer the float solution is mapped. We therefore define

$$\Omega_z = \Omega \cap S_z, \forall z \in Z^n \quad (15)$$

where  $S_z$  is a pull-in region satisfying the conditions of Definition 1. Then

- (i)  $\bigcup_z \Omega_z = \bigcup_z (\Omega \cap S_z) = \Omega \cap (\bigcup_z S_z) = \Omega \cap R^n = \Omega$
- (ii)  $\Omega_{z_1} \cap \Omega_{z_2} = (\Omega \cap S_{z_1}) \cap (\Omega \cap S_{z_2}) = \Omega \cap (S_{z_1} \cap S_{z_2}) = \emptyset, \forall z_1, z_2 \in Z^n, z_1 \neq z_2$
- (iii)  $\Omega_0 + z = (\Omega \cap S_0) + z = (\Omega + z) \cap (S_0 + z) = \Omega \cap S_z = \Omega_z, \forall z \in Z^n$

This shows that the subsets  $\Omega_z \subset S_z$  satisfy the same conditions as those of Definition 1, be it that  $R^n$  has now been replaced by  $\Omega \subset R^n$ . Hence, the mapping of the IA-estimator can now be defined as follows. The IA-estimator maps the float solution  $\hat{a}$  to the integer vector  $z$  when  $\hat{a} \in \Omega_z$  and it maps the float solution to itself when  $\hat{a} \notin \Omega$ . The class of IA-estimators can therefore be defined as follows.

**Definition 2 (Integer aperture estimators)**

Integer aperture estimators are defined as

$$\hat{a}_{IA} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) \omega_z(\hat{a}) \quad (16)$$

with  $\omega_z(x)$  the indicator function of  $\Omega_z = \Omega \cap S_z$  and  $\Omega \subset R^n$  translational invariant.

Note that an IA-estimator is indeed also an IE-estimator, just like an I-estimator is. There is also resemblance between an IA-estimator and an I-estimator. Since the indicator

functions  $s_z(x)$  of the pull-in regions  $S_z$  sum up to unity,  $\sum_{z \in Z^n} s_z(x) = 1$ , the I-estimator (6) may be written as

$$\tilde{a} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) s_z(x) \quad (17)$$

Comparing this expression with that of (16) shows that the difference between the two estimators lies in their binary weights,  $s_z(x)$  versus  $\omega_z(x)$ . Since the  $s_z(x)$  sum up to unity for all  $x \in R^n$ , the outcome of an I-estimator will always be integer. This is not true for an IA-estimator, since the binary weights  $\omega_z(x)$  do not sum up to unity for all  $x \in R^n$ . The IA-estimator is therefore a hybrid estimator having as outcome either the real-valued float solution  $\hat{a}$  or an integer solution. The IA-estimator returns the float solution if  $\hat{a} \notin \Omega$  and it will be equal to  $z$  when  $\hat{a} \in \Omega_z$ . Note, since  $\Omega$  is the collection of all  $\Omega_z = \Omega_0 + z$ , that the IA-estimator is completely determined once  $\Omega_0$  is known. Thus  $\Omega_0 \subset S_0$  plays the same role for the IA-estimators as  $S_0$  does for the I-estimators. By changing the size and shape of  $\Omega_0$  one changes the outcome of the IA-estimator. The subset  $\Omega_0$  can therefore be seen as an adjustable pull-in region with two limiting cases. The limiting case in which  $\Omega_0$  is empty and the limiting case when  $\Omega_0$  equals  $S_0$ . In the first case the IA-estimator becomes identical to the float solution  $\hat{a}$ , and in the second case the IA-estimator becomes identical to an I-estimator. The subset  $\Omega_0$  therefore determines the *aperture* of the pull-in region.

In order to evaluate the performance of an IA-estimator as to whether it produces the correct integer outcome  $a \in Z^n$ , it is helpful to classify its possible outcomes. An IA-estimator can produce one of the following three outcomes

$$\hat{a}_{IA} = \begin{cases} a \in Z^n & \text{(correct integer)} \\ z \in Z^n \setminus \{a\} & \text{(incorrect integer)} \\ \hat{a} \in R^n \setminus Z^n & \text{(no integer)} \end{cases} \quad (18)$$

A correct integer outcome may be considered a *success*, an incorrect integer outcome a *failure*, and an outcome where no correction at all is given to the float solution as indeterminate or *undecided*. The probability of success, the *success-rate*, equals the integral of  $f_{\hat{a}_{IA}}(x)$  over  $\Omega_a$ , whereas the probability of failure, the *fail-rate*, equals the probability of  $f_{\hat{a}_{IA}}(x)$  over  $\Omega \setminus \Omega_a$ . The respective probabilities are therefore given as

$$\begin{cases} P_S = P(\hat{a}_{IA} = a) & = \int_{\Omega_a} f_{\hat{a}_{IA}}(x) dx & = \int_{\Omega_a} f_{\hat{a}}(x) dx & \text{(success)} \\ P_F = \sum_{z \neq a} P(\hat{a}_{IA} = z) & = \sum_{z \neq a} \int_{\Omega_z} f_{\hat{a}_{IA}}(x) dx & = \sum_{z \neq a} \int_{\Omega_z} f_{\hat{a}}(x) dx & \text{(failure)} \\ P_U = P(\hat{a}_{IA} = \hat{a}) & = 1 - \int_{\Omega} f_{\hat{a}_{IA}}(x) dx & = 1 - P_S - P_F & \text{(undecided)} \end{cases} \quad (19)$$

Note that these three probabilities are completely governed by  $f_{\hat{a}}(x)$ , the PDF of the float solution, and by  $\Omega_0$ , the aperture pull-in region which uniquely defines the IA-estimator. Depending on the type of IA-estimator one is considering, the above integrals for computing the success-rate and the fail-rate may be difficult to evaluate exactly. Whether or not an exact evaluation is possible depends to a large extent on the complexity of the geometry of the aperture pull-in region  $\Omega_0$ .

### 3.2 THE INTEGER APERTURE LEAST SQUARES-ESTIMATOR

In this section we introduce the integer aperture least-squares (IALS) estimator. It is obtained from down-scaling the pull-in region of the integer least-squares estimator. We therefore have the following definition.



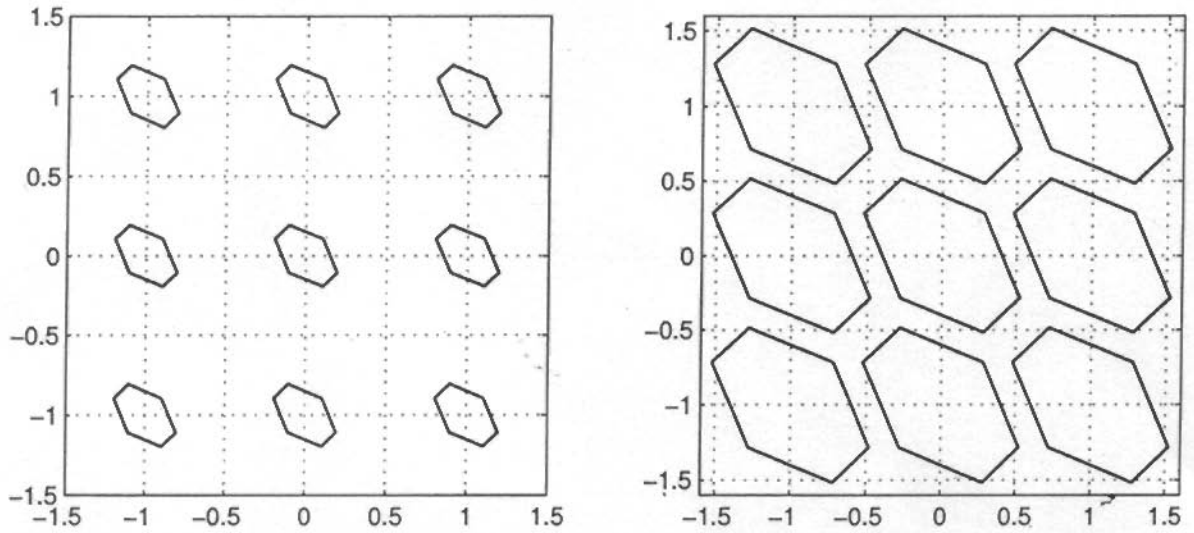


Figure 2: Two-dimensional integer aperture pull-in regions of the IALS estimator for different values of  $\lambda$ .

**Definition 3** (*IA least-squares estimation*)

The pull-in regions of the integer aperture least-squares (IALS) estimator are defined as

$$\Omega_{LS,z} = \lambda S_{LS,z} \quad \forall z \in Z^n, \quad 0 \leq \lambda \leq 1 \quad (20)$$

with

$$\begin{cases} \lambda S_{LS,z} &= \{x \in R^n \mid \frac{1}{\lambda}(x-z) \in S_{LS,0}\} \\ S_{LS,0} &= \{x \in R^n \mid \|x\|_{Q_{\hat{a}}}^2 \leq \|x-u\|_{Q_{\hat{a}}}^2, \forall u \in Z^n\} \end{cases}$$

It is easily verified that these pull-in regions indeed satisfy the conditions of integer aperture pull-in regions. Note that  $\lambda$  acts as the aperture parameter (see Figure 2). By changing  $\lambda$  one changes the size of  $\Omega_{LS,z}$  but not its shape. The shape of  $\Omega_{LS,z}$  will remain identical to that of the least-squares pull-in region  $S_{LS,z}$ . The computational steps involved are now as follows. Using the float solution  $\hat{a}$  and its vc-matrix  $Q_{\hat{a}}$ , one first computes the integer least-squares solution  $\check{a}_{LS}$ . Then the aperture parameter  $\lambda$  is used to up-scale the least-squares residual  $\check{\epsilon}_{LS} = \hat{a} - \check{a}_{LS}$  to  $\frac{1}{\lambda}\check{\epsilon}_{LS}$ . This up-scaled version of the residual vector is then used to verify whether

$$u = \arg \min_{z \in Z^n} \left\| \frac{1}{\lambda}\check{\epsilon}_{LS} - z \right\|_{Q_{\hat{a}}}^2 \quad (21)$$

equals the zero vector or not. If it equals the zero vector then  $\hat{a}_{ALS} = \check{a}_{LS}$ , otherwise  $\hat{a}_{ALS} = \hat{a}_{LS}$ .

The motivation for introducing this estimator stems from the known optimality of the integer least-squares estimator, cf. Theorem 1, the ease with which it can be computed using the LAMBDA method, and the fact that the shape of its aperture region is not affected by changes in  $\lambda$ . This latter property makes it possible to use some of the probabilistic results which are already available for the integer least-squares estimator. In the Gaussian case namely the aperture parameter  $\lambda$  acts as a scale factor on the vc-matrix. This follows from the property  $\int_{\lambda S_{LS,0}} f_{\hat{a}}(x+a)dx = \lambda^n \int_{S_{LS,0}} f_{\hat{a}}(\lambda x+a)dx$ . This

implies that if  $P_{Q_{\hat{a}}}(\tilde{a}_{LS} = a)$  denotes the success-rate of the ILS-estimator having  $Q_{\hat{a}}$  as vc-matrix, the success-rate of the IALS-estimator is given as  $P_S = P_{\frac{1}{\lambda^2}Q_{\hat{a}}}(\tilde{a}_{LS} = a)$ . Hence, when one has routines available for computing the ILS success-rate one may use them as well for computing the success-rate of the IALS-estimator, the only difference being that one has to replace  $Q_{\hat{a}}$  by the up-scaled version  $\frac{1}{\lambda^2}Q_{\hat{a}}$ . The same holds true for the known lower bounds and upper bounds of the ILS success-rate.

As to the fail-rate of the IALS-estimator one may use ellipsoidal regions to bound  $\lambda S_{LS,0}$ . In this way one can obtain bounds on the fail-rate which are based on noncentral Chi-square distributions. The following theorem gives such bounds for both the success-rate and the fail-rate of the IALS-estimator.

**Theorem 2** (*Probabilistic lower bounds and upper bounds for the IALS-estimator*)

Let the float solution be distributed as  $\hat{a} \sim N(a, Q_{\hat{a}})$  and let the lower bounds and the upper bounds of the fail-rate and success-rate of the IALS-estimator be denoted as  $L_F \leq P_F \leq U_F$  and  $L_S \leq P_S \leq U_S$ . Then

$$\begin{cases} L_F = \sum_{z \in Z^n \setminus \{0\}} P(\chi^2(n, \lambda z) \leq l_F) \\ U_F = \sum_{z \in Z^n \setminus \{0\}} P(\chi^2(n, \lambda z) \leq u_F) \\ L_S = \prod_{i=1}^n \left( 2\Phi\left(\frac{\lambda}{2\sigma_{i|I}}\right) - 1 \right) \\ U_S = P\left(\chi^2(n, 0) \leq \frac{\lambda^2 c_n}{\text{ADOP}^2}\right) \end{cases} \quad (22)$$

with  $\lambda z = z^T Q_{\hat{a}}^{-1} z$ ,  $l_F = \frac{1}{4} \lambda^2 \min_{z \in Z^n \setminus \{0\}} \|z\|_{Q_{\hat{a}}}^2$ ,  $u_F = \lambda^2 \max_{x \in S_0} \|x\|_{Q_{\hat{a}}}^2$ ,  $c_n = \left(\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right)^{2/n} / \pi$  and  $\text{ADOP} = \sqrt{\det Q_{\hat{a}}}$ .

*Proof:* We first prove the lower bound for the fail-rate. Define the subset

$$E_{\epsilon_l, 0} = \{x \in S_{LS,0} \mid \|x\|_{Q_{\hat{a}}}^2 \leq \epsilon_l^2\}$$

and let  $\epsilon_l$  be chosen such that

$$E_{\epsilon_l, 0} \subset \lambda S_{LS,0}$$

Then

$$\begin{aligned} P_F &= \sum_{z \in Z^n \setminus \{0\}} P(\hat{a} - a \in \lambda S_{LS,z}) \\ &= \sum_{z \in Z^n \setminus \{0\}} P(\hat{a} - a - z \in \lambda S_{LS,0}) \\ &\geq \sum_{z \in Z^n \setminus \{0\}} P(\hat{a} - a - z \in E_{\epsilon_l, 0}) \\ &= \sum_{z \in Z^n \setminus \{0\}} P(\|\hat{a} - a - z\|_{Q_{\hat{a}}}^2 \leq \epsilon_l^2) \\ &= \sum_{z \in Z^n \setminus \{0\}} P(\chi^2(n, \lambda z) \leq \epsilon_l^2) \end{aligned}$$

In order to determine the value for  $\epsilon_l$ , we make use of the two equivalent representations of  $\lambda S_0$ ,

$$\|x\|_{Q_{\hat{a}}}^2 \leq \|x - \lambda z\|_{Q_{\hat{a}}}^2 \Leftrightarrow |z^T Q_{\hat{a}}^{-1} x| \leq \frac{1}{2} \lambda \|z\|_{Q_{\hat{a}}}, \forall z \in Z^n$$

and of

$$E_{\epsilon_l, 0} \subset E'_{\epsilon_l, 0} \quad \text{with} \quad E'_{\epsilon_l, 0} : |z^T Q_{\hat{a}}^{-1} x| \leq \epsilon_l \|z\|_{Q_{\hat{a}}}, \forall z \in Z^n$$

This shows that  $E_{\epsilon_l, 0} \subset \lambda S_0$  if  $\epsilon_l$  is chosen as  $\epsilon_l \leq \frac{1}{2} \lambda \min_{z \in Z^n \setminus \{0\}} \|z\|_{Q_{\hat{a}}}$ . This concludes the proof of the lower bound of  $P_F$ . The proof of the upper bound goes along similar lines. It makes use of the fact that  $\lambda S_{LS,0} \subset E_{\epsilon_u, 0}$  if  $\epsilon_u$  is chosen as  $\epsilon_u \geq \max_{x \in \lambda S_0} \|x\|_{Q_{\hat{a}}} = \lambda \max_{x \in S_0} \|x\|_{Q_{\hat{a}}}$ .

The given bounds on the success-rate  $P_S$  are of a different type. They follow directly from using the aperture parameter  $\lambda$  as scale factor in the known bounds of the ILS success-rate, see Corollary 1. *End of proof.*

#### 4 REFERENCES

- Boon, F., B. Ambrosius (1997): Results of real-time applications of the LAMBDA method in GPS based aircraft landings. *Proceedings KIS97*, pp. 339-345.
- Boon, F., P.J. de Jonge, C.C.J.M. Tiberius (1997): Precise aircraft positioning by fast ambiguity resolution using improved troposphere modelling. *Proceedings ION GPS-97*, Vol. 2, pp. 1877-1884.
- Cox, D.B. and J.D.W. Brading (1999): Integration of LAMBDA ambiguity resolution with Kalman filter for relative navigation of spacecraft. *Proc ION NTM 99*, pp. 739-745.
- de Jonge P.J., C.C.J.M. Tiberius (1996a): The LAMBDA method for integer ambiguity estimation: implementation aspects. Publications of the Delft Computing Centre, *LGR-Series* No. 12.
- de Jonge, P.J., C.C.J.M. Tiberius (1996b): Integer estimation with the LAMBDA method. *Proceedings IAG Symposium No. 115, 'GPS trends in terrestrial, airborne and spaceborne applications'*, G. Beutler et al. (eds), Springer Verlag, pp. 280-284.
- de Jonge, P.J., C.C.J.M. Tiberius, P.J.G. Teunissen (1996): Computational aspects of the LAMBDA method for GPS ambiguity resolution. *Proceedings ION GPS-96*, pp. 935-944.
- Han, S. (1995): Ambiguity resolution techniques using integer least-squares estimation for rapid static or kinematic positioning. Symposium *Satellite Navigation Technology: 1995 and beyond*, Brisbane, Australia, 10 p.
- Hassibi, A., S. Boyd (1998): Integer parameter estimation in linear models with applications to GPS. *IEEE Transactions on Signal Processing*, vol. 46, no. 11, pp. 2938-2952.
- Hofmann-Wellenhof, B., H. Lichtenegger, J. Collins (2001): *Global Positioning System: Theory and Practice*. 5th edition. Springer Verlag.
- Jonkman, N.F. (1998): Integer GPS ambiguity estimation without the receiver-satellite geometry. Publications of the Delft Geodetic Computing Centre, *LGR-Series*, No. 18.
- Leick, A. (1995): *GPS Satellite Surveying*. 2nd edition, John Wiley, New York.
- Misra, P., P. Enge (2001): *Global Positioning System: Signals, Measurements, and Performance*. Ganga-Jamuna Press, 2001.
- Parkinson, B., J.J. Spilker (eds) (1996): *GPS: Theory and Applications*, Vols 1 and 2, AIAA, Washington DC.
- Peng, H.M., F.R. Chang, L.S. Wang (1999): Attitude determination using GPS carrier phase and compass data. *Proc ION NTM 99*, pp. 727-732.

- Strang, G., K. Borre (1997): *Linear Algebra, Geodesy, and GPS*, Wellesley-Cambridge Press.
- Teunissen, P.J.G., A. Kleusberg (eds) (1998): *GPS for Geodesy*, 2nd enlarged edition, Springer Verlag.
- Teunissen, P.J.G. (1993): Least-squares estimation of the integer GPS ambiguities. Invited Lecture, Section IV Theory and Methodology, IAG General Meeting, Beijing, China, August 1993. Also in: *LGR Series*, No. 6, Delft Geodetic Computing Centre.
- Teunissen, P.J.G. (1995): The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation. *Journal of Geodesy*, 70: 65-82.
- Teunissen, P.J.G. (1997): Some remarks on GPS ambiguity resolution. *Artificial Satellites*, Vol. 32, No.3, pp. 119-130.
- Teunissen, P.J.G. (1998): The success probability of integer GPS rounding and bootstrapping. *Journal of Geodesy*, 72: 606-612.
- Teunissen, P.J.G. (1999): An optimality property of the integer least-squares estimator. *Journal of Geodesy*, 73: 587-593.
- Teunissen, P.J.G. (2000): ADOP based upperbounds for the bootstrapped and the least-squares ambiguity success rates. *Artificial Satellites*, Vol. 35, No. 4, pp. 171-179.
- Teunissen, P.J.G. (2003): Integer Aperture GNSS Ambiguity Resolution. *Artificial Satellites*, Vol. 38, No.3, pp. 79-88.
- Thomsen, H.F. (2000): Evaluation of upper and lower bounds on the success probability. *Proc. ION GPS 2000*, Salt Lake City UT, pp. 183-188.
- Tiberius, C.C.J.M. and P. de Jonge (1995): Fast positioning with the LAMBDA method. *Proc DSNS 95*, Bergen, Norway, paper No. 30.
- Verhagen, S. (2003): On the approximation of the integer least-squares success rate: which lower or upper bound to use. *Journal of Global Positioning Systems*, Vol. 2, No. 2, pp. 117-124.
- Verhagen, S. and P.J.G. Teunissen (2004): A new GNSS ambiguity resolution method and its comparison with existing approaches. *Submitted to Journal of Guidance, Control, and Dynamics*.

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