Interaction of an elastic wave with a circular crack in a fluid-saturated porous medium

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We consider interaction of a normally incident time-harmonic longitudinal plane wave with a circular crack imbedded in a porous medium governed by Biot's equations of dynamic poroelasticity. The problem is formulated in cylindrical coordinates as a system of dual integral equations for the Hankel transform of the wave field, which is then reduced to a single Fredholm integral equation of the second kind. The solution of this equation yields elastic wave dispersion and attenuation in a medium containing a random distribution of aligned cracks. These dissipation effects are caused by wave induced fluid flow between pores and cracks. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2165178]

Many porous materials, in addition to the network of pores, contain larger inhomogeneities such as inclusions, cavities, fractures, or cracks.

We present a theoretical study of the problem of the interaction of a plane longitudinal elastic wave in a poroelastic medium[1] with an open oblate spheroidal crack of radius a and small thickness $2b \ll a$ placed perpendicular to the direction of wave propagation. We restrict the analysis to so-called mesoscopic cracks[2–4] whose radius is small compared to the wavelength $2\pi/k_1$ of the normal compressional wave, but large compared to the individual pore size. Crack thickness $2b$ will be assumed smaller than the fluid diffusion length (wavelength of Biot's slow wave). In elastic media (say, in a dry porous material) such cracks would have very little effect on wave propagation. However, in porous media there may be a significant effect due to fluid diffusion in and out of the crack (as the fluid diffusion length is much smaller than the wavelength).

The corresponding problem for elasticity was solved by Robertson[5] and we apply the same approach to the equations of poroelasticity. We consider an incident plane normal (fast) compressional wave propagating in a porous medium with porosity ϕ and permeability κ along the z axis of the cylindrical co-ordinate system with the axial solid displacement $u_z^{\text{in}}=u_0\exp(ik_1z)$, where k_1 is the wave number [time dependency $\exp(-i\omega t)$ is assumed]. We aim to obtain the secondary (scattered) field $\mathbf{u}(\mathbf{r})$ resulting from interaction of the incident wave with the crack occupying the circle $r \leqslant a$ in the plane $z=0$. The total field is therefore $\mathbf{u}^T(\mathbf{r})=u_z^{\text{in}}\mathbf{e}_z+\mathbf{u}(\mathbf{r})$, where $\mathbf{e}_z$ is a unit vector in the z direction. We assume that the crack is in hydraulic communication with the surrounding pore space (the case of an impermeable crack face has been considered previously).[6] Both the scattered and total fields must each satisfy the following equations of poroelasticity in the semi-infinite poroelastic medium $z \geqslant 0$:

$$\nabla \cdot \boldsymbol{\sigma} = -\omega^2(\rho\mathbf{u}+\rho_f\mathbf{w}), \quad \nabla p = \omega^2(\rho_f\mathbf{u}+q\mathbf{w}), \tag{1}$$

where $\mathbf{w}=\phi(\mathbf{U}-\mathbf{u})$ is the so-called relative fluid displacement, $\mathbf{U}$ is the average fluid displacement, ρ_f and ρ are the densities of the fluid and of the overall porous medium, and $q(\omega)$ is a frequency dependent coefficient responsible for viscous and inertial coupling between the solid and fluid motion, while $\boldsymbol{\sigma}$ and p are the total stress tensor and fluid pressure which are related to the displacement vectors via the constitutive relations

$$\boldsymbol{\sigma} = [(H-2\mu)\nabla\cdot\mathbf{u}+\alpha M\nabla\cdot\mathbf{w}]\mathbf{I}+\mu[\nabla\mathbf{u}+(\nabla\mathbf{u})^T], \tag{2}$$

$$p = -\alpha M\nabla\cdot\mathbf{u}-M\nabla\cdot\mathbf{w}. \tag{3}$$

In Eqs. (2) and (3) μ is the shear modulus of the solid skeleton, $\alpha=1-K/K_g$ is the Biot-Willis coefficient, and M and H are poroelastic material constants related to the bulk moduli of the fluid K_f, solid K_g, and dry skeleton K by the Gassmann[7] equations

$$M=[(\alpha-\phi)/K_g+\phi/K_f]^{-1}, \quad H=K+\tfrac{4}{3}\mu+\alpha^2M.$$

The distribution of displacements and stresses in the neighborhood of the crack is the same as that produced in a semi-infinite porous medium $z \geqslant 0$ when its free surface $z=0$ is subject to the following boundary conditions:

$$\sigma_{rz}=0, \quad 0 \leqslant r < \infty \tag{4}$$

$$u_z=0, \quad a < r < \infty \tag{5}$$

$$w_z=0, \quad a < r < \infty \tag{6}$$

$$u_z+w_z=0, \quad 0 \leqslant r < a \tag{7}$$

$$\sigma_{zz}+p=-ik_1(H-\alpha M)u_0, \quad 0 \leqslant r < a. \tag{8}$$

Conditions (4)–(6) are a consequence of the symmetry of the problem and are completely analogous to the corresponding conditions in the elastic case. Condition (7) is a consequence

[a)]Electronic mail: robert.galvin@geophy.curtin.edu.au of the fact that due to the small crack thickness the fluid in the crack (but not in the pores!) can be assumed incompressible, and states that the volume-fraction average of normal displacement $(1-\phi)u_z+\phi U_z=u_z+w_z$ through the face of the crack is zero. Condition (8) is a consequence of the continuity of the *total* stress and fluid pressure at the boundary $z=0^+$ between the porous medium and the fluid in the crack,[8] that is, $\sigma_{zz}^T=-p^c$, $p^T=p^c$, where p^c is the fluid pressure in the crack. Therefore $\sigma_{zz}^T=-p^T$ or $\sigma_{zz}+p=-(\sigma_{zz}^{\mathrm{in}}+p^{\mathrm{in}})$. Using constitutive Eqs. (2) and (3) one obtains condition (8). Finally, we note that boundary conditions (5)–(7) can be combined to give

$$u_z+w_z=0, \quad 0\leqslant r<\infty. \tag{9}$$

The general solution of the equations of motion in cylindrical coordinates can be obtained by representing the four axial and radial components u_z, u_r, w_z, and w_r of the solid and relative fluid displacements in the form of an inverse Hankel transform with respect to the radial coordinate r, e.g.,

$$u_i(z,r)=\int_0^\infty \tilde{u}_i(z,y)yJ_0(yr)dy, \quad i=z,r. \tag{10}$$

Substitution of these representations into the equations of motion (1) yields a system of ordinary second-order differential equations with constant coefficients for the functions $\tilde{u}_i(z,y),\tilde{w}_i(z,y)$ in the independent variable z with radial wave number y as a parameter. These equations can be readily integrated, giving general solutions of the form

$$\tilde{u}_i(z,y)=A_{i1}(y)e^{-q_1 z}+A_{i2}(y)e^{-q_2 z}+A_{is}(y)e^{-q_s z}, \tag{11}$$

where q_1, q_2, and q_s are vertical wave numbers of the fast compressional, slow compressional, and shear waves, respectively. Then boundary conditions (4), (5), (8), and (9) yield a system of integral equations for the unknown wave amplitudes $A_{ij}(y)$. Eliminating A_{is} and A_{i2}[9] yields the following system of dual integral equations for the normal (fast) compressional wave amplitude $B(y)=2\mu(1-g)k_s^2y[2y^2-k_s^2(1-\alpha M/H)]^{-1}A_{z1}(y)$:

$$\int_0^\infty y[1+T(y)]B(y)J_0(yr)dy=-p_0, \quad 0\leqslant r<a, \tag{12}$$

$$\int_0^\infty B(y)J_0(yr)dy=0, \quad a<r<\infty, \tag{13}$$

where $T(y)$ is a known transfer function between stress and displacement arising from the mixed nature of the boundary conditions, J_0 is the zero order Bessel function of the first kind, $p_0=ik_1(H-\alpha M)u_0$ is a constant representing the magnitude of the incident wave, $g=\mu/(K+4\mu/3)$.

As shown by Noble,[10] dual equations (12) and (13) are equivalent to a single Fredholm equation of the second kind

$$B(x)+\frac{1}{\pi}\int_0^\infty R(x,y)T(y)B(y)dy=-p_0S(x), \tag{14}$$

where $R(x,y)=\sin[a(x-y)]/(x-y)-\sin[a(x+y)]/(x+y)$, and $S(x)=(2/\pi)(\sin ax-ax\cos ax)/x^2$. For mesoscopic cracks, $T(y)$ has the form

$$T(y)=M\frac{(2\alpha gy^2-k_2^2)^2-2yq_2\alpha g[k_2^2(\alpha g-2)+2\alpha y^2 g]}{2Hg(g-1)yq_2k_2^2}, \tag{15}$$

where $q_2=\sqrt{y^2-k_2^2}$ and k_2 is the wave number of Biot's slow wave. For frequencies much smaller than Biot's characteristic frequency[1] $\omega_B=\phi\eta/\kappa\rho_f$, Biot's slow wave is a diffusion wave and its wave number (inverse of the fluid diffusion length) is proportional to the square root of frequency

$$k_2=\left[\frac{i\omega\eta H}{\kappa M(K+4\mu/3)}\right]^{1/2}, \tag{16}$$

where η denotes the viscosity of the pore fluid.

Attenuation and dispersion of an elastic wave propagating in a medium with a random distribution of aligned cracks can be estimated using a Foldy-type approximation of multiple scattering[11] which expresses the effective wave number k^* for the medium with cracks in terms of the number of scatterers per unit volume n_0 and the far-field forward scattering amplitude $f(0)$ for a single scatterer,

$$k^*\approx k_1\left[1+\frac{2\pi n_0 f(0)}{k_1^2}\right], \tag{17}$$

where

$$f(0)=-\frac{ik_1}{u_0}\frac{(H-\alpha M)}{2\mu H(1-g)}\lim_{y\to 0}\frac{B(y)}{y}. \tag{18}$$

Figures 1(a) and 1(b) show this solution in terms of effective velocity $c(\omega)=\omega/\mathrm{Re}\,k^*$ (normalized) and dimensionless attenuation $Q^{-1}=2\mathrm{Im}\,k^*/\mathrm{Re}\,k^*$ as functions of dimensionless frequency .

Also shown in Figs. 1(a) and 1(b) are asymptotic solutions in the low- and high-frequency limits. For low frequencies such that $|k_2a|\ll 1$ Eq. (14) can be solved analytically by using an asymptotic expression for the transfer function $T(y)$ in the limit $k_2\ll y$, and expanding $R(x,y)$ in powers of x. This yields a separable integral equation to give

$$f_{\mathrm{low}}(0)=\frac{(5+D_{\mathrm{low}})(H-\alpha M)^2k_1^2a^3}{15\pi\mu H(1-g)}, \tag{19}$$

where

$$D_{\mathrm{low}}=\frac{M(2-4\alpha g+3\alpha^2g^2)(k_2a)^2}{2Hg(1-g)}. \tag{20}$$

By substituting Eq. (19) into Eq. (17) and taking the real component we can obtain an expression for effective velocity in the low frequency limit

$$c_0=c_1\left[1-\frac{2\varepsilon(H-\alpha M)^2}{3\mu H(1-g)}\right]. \tag{21}$$

In Eq. (21) $c_1=\omega/k_1=(H/\rho)^{1/2}$ is the velocity of the fast compressional wave in the porous host (crack-free porous medium) and $\varepsilon=n_0a^3=(3/4\pi)(a/b)\phi_c$ is the crack density parameter[12,13] where $\phi_c=(4/3)\pi a^2bn_0$ is the additional porosity present due to the cracks. For dry open cracks $K_f=M=0$, $H=K+4\mu/3$ and Eq. (21) simplifies to

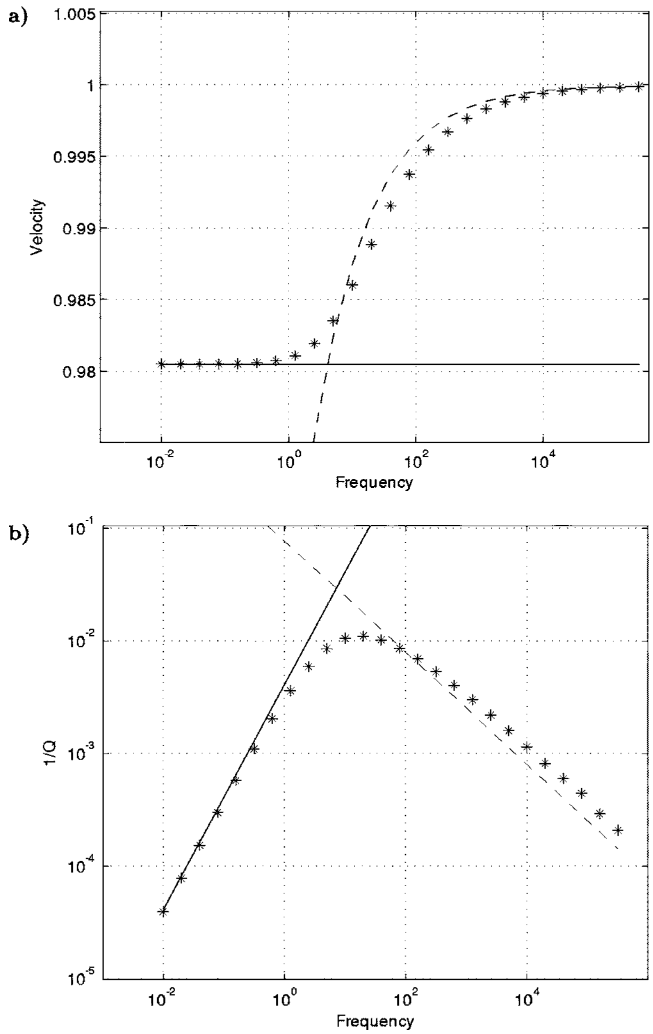

FIG. 1. Dimensionless velocity (a) and attenuation (b) as functions of dimensionless frequency: numerical solution (asterisks), low-frequency asymptotic (solid line), and high-frequency asymptotic (dashed line).

$$c_0 = c_1\left[1 - \frac{2\varepsilon}{3g(1-g)}\right], \tag{22}$$

which coincides with the well-known expression for the velocity of compressional waves propagating perpendicular to a system of dry open cracks in an elastic medium in the limit of low crack density.[12,13] Furthermore, Eq. (21) coincides with the expression for the compressional wave velocity obtained from Gassmann's exact static result for the undrained elastic moduli of an anisotropic fluid-saturated porous medium[7] with low crack density.[14] This Gassmann consistency is an important feature of the model presented here. Low-frequency attenuation Q^{-1} is defined by the imaginary part of the function $f_{\text{low}}(0)$,

$$Q_{\text{low}}^{-1} = \frac{2M(H-\alpha M)^2(2-4\alpha g+3\alpha^2 g^2)|k_2 a|^2\varepsilon}{15\mu H^2 g(1-g)^2}, \tag{23}$$

and is proportional to the first power of frequency.

In the limit of high frequencies such that the crack radius is large compared to the fluid diffusion length, but smaller than the incident wavelength, $|k_1 a| \ll 1 \ll |k_2 a|$ (while the crack thickness is still smaller then the diffusion length, $|k_2 b| \ll 1$ and the frequency is still smaller than Biot's characteristic frequency ω_B), we have

$$f_{\text{high}}(0) = \frac{i(k_1 a)^2 (H-\alpha M)^2 g}{3\mu M k_2}. \tag{24}$$

Substitution of expression (24) into Eq. (17) shows the velocity in that limit tends to the velocity in the porous crack-free medium. This result is logical as at sufficiently high frequencies the fluid has no time to move between pores and cracks, and therefore the cracks behave as if they were isolated.[15–17] Note, however, that this result is a consequence of assuming that there is an incompressible fluid occupying the cracks and a small aspect ratio b/a; the more precise validity condition is $K_f/\mu \gg b/a$, see for example, Ref. 18. In particular, the dry case is excluded, except in the static limit (22). Attenuation at high frequencies reads

$$Q_{\text{high}}^{-1} = \frac{2\sqrt{2}\pi\varepsilon(H-\alpha M)^2 g}{3\mu M|k_2 a|}, \tag{25}$$

and thus scales with $\omega^{-1/2}$. We note that both low- and high-frequency asymptotes of attenuation are consistent (but not identical) with the corresponding results for mesoscopic spherical inclusions[2,3,19–21] and for a poroelastic medium with smoothly varying material properties.[4]

The solution for intermediate frequencies was obtained numerically by the method of quadratures. This solution exhibits a typical relaxation peak around a normalized frequency ω' of about 10, or at circular frequency $f = \omega/2\pi \simeq 2\kappa M(K+4\mu/3)/H\eta a^2$, the frequency where the fluid diffusion length $1/|k_2|$ is of the order of the crack diameter a. Note that the diffusion length is proportional to $\omega^{-1/2}$ and is usually much smaller than the wavelength of the normal compressional or shear wave. This shows that the presence of cracks in a fluid-saturated porous medium can cause significant attenuation and dispersion at very low frequencies, well before the onset of elastic (Rayleigh) scattering.

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