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THE TOTAL RUN LENGTH OF A WORD

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ABSTRACT. A *run* in a word is a periodic factor whose length is at least twice its period and which cannot be extended to the left or right (by a letter) to a factor with greater period. In recent years a great deal of work has been done on estimating the maximum number of runs that can occur in a word of length n . A number of associated problems have also been investigated. In this paper we consider a new variation on the theme. We say that the *total run length* (TRL) of a word is the sum of the lengths of the runs in the word and that $\tau(n)$ is the maximum TRL over all words of length n . We show that $n^2/8 < \tau(n) < 47n^2/72 + 2n$ for all n . We also give a formula for the average total run length of words of length n over an alphabet of size α , and some other results.

1. INTRODUCTION

We use notation for combinatorics on words. A word of n elements is $x = x[1..n]$, with $x[i]$ being the i th element and $x[i..j]$ the *factor* of elements from position i to position j . If $i = 1$ then the factor is a *prefix* and if $j = n$ it is a *suffix*. The letters in x come from some *alphabet* A . The *length* of x , written $|x|$, is the number of letters x contains and the number of occurrences of a letter a in x is denoted by $|x|_a$. Two or more adjacent identical factors form a so-called *power*. A word which is not a power is said to be *primitive*. A word x or factor x is *periodic* with period p if $x[i] = x[i + p]$ for all i such that $x[i]$ and $x[i + p]$ are in x . A periodic word with least period p and length n is said to have *exponent* n/p . For example, the word *ababa* has exponent $5/2$ and can be written as $(ab)^{5/2}$. If $x = x[1..n]$ then the *reverse* of x , written $R(x)$, is $x[n]x[n-1] \cdots x[1]$. A word that equals its own reverse is called a *palindrome*.

In this paper we are concerned with runs. A *run* (or *maximal periodicity*) in a word x is a factor $x[i..j]$ having minimum period p , length at least $2p$ and such that neither $x[i-1..j]$ nor $x[i..j+1]$ is a factor with period p . Runs are important because of their applications in data compression and computational biology (see, for example, [9]). In recent years a number of papers have appeared concerning the function $\rho(n)$ which is the maximum number of runs that can occur in a word of length n . In 2000 Kolpakov and Kucherov [9] showed that $\rho(n) = O(n)$ but their method did not give any information about the size of the implied constant. They conjectured that $\rho(n) < n$ for all n which has become known as the Runs Conjecture. In [18] Rytter showed that $\rho(n) < 5n$. This bound was improved progressively in [17] and [3] and most recently by Crochemore, Ilie and Tinta [4] to $1.029n$. Their method is difficult and heavily computational. Giraud [8] has produced weaker results using a much simpler technique. He also showed that $\lim_{n \rightarrow \infty} \rho(n)/n$ exists. In the other direction Franek *et al.* [7] showed that this limit is greater than 0.927 , a result that was improved by Kusano *et al.* [10] and Simpson [20] to 0.944 . We therefore have

$$0.944 < \lim_{n \rightarrow \infty} \rho(n)/n < 1.029.$$

These investigations have prompted authors to investigate a number of associated problems. Baturó and coauthors looked at runs in Sturmian words [2]. Puglisi and Simpson [16] gave formulas for the expected number of runs in a word of length n .

This depends on the alphabet size, with binary alphabets giving the highest values. Kusano and Shinohara [12] obtained similar results for necklaces (words with their ends joined). Crochemore [5] and others have investigated runs whose length is at least three times the period. Rather than counting the number of runs one can consider the sum of the exponents of the runs. The word $ababaabaa$ has runs $(ab)^{5/2}$, $(aba)^{7/3}$ and two copies of a^2 , so it contains 4 runs with sum of exponents $53/6$. Let $\epsilon(n)$ be the maximum sum of the exponents of runs in a word of length n . It is known [5] that for large n

$$2.035n < \epsilon(n) < 4.1n.$$

In this paper we introduce a new variation on this theme. The *total run length* of a word is the sum of the lengths of the runs in the word. The word given above contains runs aa (twice), $ababa$ and $abaabaa$ of lengths 2, 5, and 7 so its total run length is 16. We write $TRL(w)$ for the total run length of a word w and $\tau(n)$ for $\max\{TRL(w) : |w| = n\}$. In the next section we give some minor results about *total run length* (TRL for short) and obtain a lower bound on $\tau(n)$. An upper bound is given in Section 3 and formula for the expected TRL in Section 4. In the final section we discuss our results and suggest some areas for further research.

2. A LOWER BOUND ON $\tau(n)$

Table 1 (below) gives values of $\tau(n)$ for small n (under the assumption that these values are attained by binary words) and examples of words that attain these values.

TABLE 1. Values of $\tau(n)$ assuming these values are attained by binary words.

n	$\tau(n)$	$\tau(n)/n^2$	Example
1	0	0	a
2	2	0.5	aa
3	3	0.333	aaa
4	4	0.250	$aaaa$
5	6	0.240	$aabab$
6	10	0.278	$aabaab$
7	12	0.245	$aabaabb$
8	16	0.250	$aabbaabb$
9	19	0.235	$abaaabaab$
10	29	0.290	$aababaabab$
11	32	0.264	$abaababaaba$
12	37	0.257	$abaababaabab$
13	42	0.249	$ababbababbaba$
14	47	0.240	$aaabaabaaabaab$
15	53	0.236	$abaabababaababa$
16	60	0.234	$aabaababaabaabab$
17	70	0.242	$ababaabababaababa$
18	73	0.225	$aababaabababaababa$
19	80	0.222	$abaababaabaababaaba$
20	85	0.212	$abaababaabaababaabab$
21	92	0.209	$ababaababababaabababa$
22	99	0.205	$aababaababaaababaababa$

We do not know whether binary words are best, though this seems likely. The same uncertainty exists for $\rho(n)$ and is discussed in [1]. Note that if a binary word is optimal

with respect to TRL, then so is its reverse and its *complement* (formed by interchanging the letters a and b). In most cases, the binary words attaining the values in Table 1 are unique up to reversal and complementation. Also note that, in many cases, binary words having maximum TRL are palindromes.

If we use an alphabet of size greater than 2 we can construct words of any length containing no runs (see Section 3.1.2 of [14]) and so having TRL equal to 0. For binary words the minimum values of $TRL(w)$ for w of length up to 5 are 0,0,0,2,2. For larger values of n we have the following.

Theorem 1. *The minimum value of $TRL(w)$ for binary words of length n , $n \geq 6$, is $n - 4$, and is attained by the word $aba^{n-4}ba$.*

Proof. Clearly the given word has TRL equal to $n - 4$. We must show no binary word of length n has a lower TRL. Suppose the word w does. Then it has $TRL(w) \leq n - 5$ and therefore there are at least 5 letters in w which do not belong to any run. Consider the middle of these 5, and without loss of generality suppose it's a . Its neighbours cannot equal a as then it would belong to a run. Therefore it is the central a of some factor $ab^{k_1}ab^{k_2}a$. If $k_1 \leq k_2$ then we have a preficial run $ab^{k_1}ab^{k_1}$ so the central a does belong to a run, contradicting the hypothesis. If $k_1 > k_2$ an analogous argument applies. \square

Theorem 2. *For $n > 1$ we have $\tau(n) > n^2/8$.*

Proof. From Table 1 we see this holds up to $n = 5$. For even n greater than 5 let $u(k) = ((ab)^ka)^2$. We find

$$n = |u(k)| = 4k + 2$$

and

$$\begin{aligned} TRL(u(k)) &= 2k^2 + 8k + 4 \\ &= (n^2 + 4n + 12)/8 \end{aligned}$$

so $\tau(n) \geq (n^2 + 4n + 12)/8$ and the theorem holds for even n . For odd n note that $\tau(n) > \tau(n - 1) \geq (n^2 + 2n + 9)/8$ so the bound also holds in this case too. \square

3. AN UPPER BOUND FOR $\tau(n)$

We first assemble some lemmas.

Lemma 3. [6] (The Periodicity Lemma) *If x is a word having two periods p and q and $|x| \geq p + q - \gcd(p, q)$ then x also has period $\gcd(p, q)$.*

Lemma 4. (Lemma 8.1.1 of [14]) *Let \mathbf{a} be a word having two periods p and q with $q < p$. Then the suffix and prefix of length $|\mathbf{a}| - q$ both have period $p - q$.*

Lemma 5. (Lemma 8.1.2 of [14]) *Let \mathbf{a} , \mathbf{b} and \mathbf{c} be words such that \mathbf{ab} and \mathbf{bc} have period p and $|\mathbf{b}| \geq p$. Then the word \mathbf{abc} has period p .*

Lemma 6. *If w is a word for which $w[1..2p]$ has period p and $w[k + 1..k + 2p + 2]$ has period $p + 1$, where $0 \leq k \leq p$ then w has the form:*

$$(1) \quad w = Xx^{p-k}Xx^{p-k+1}Xx$$

where x is a letter and $|X| = k$.

Proof. Note that $w[k+1..2p]$ has periods p and $p+1$. By Lemma 4 its prefix $w[k+1..p]$ (which is empty if $k=p$) has period 1. Say this is x^{p-k} . Then by the p periodicity $w[p+k+1..2p] = x^{p-k}$. By the $p+1$ periodicity $w[k+2p+2] = w[p+k+1] = x$ and $w[2p+1] = w[p] = x$. Let $w[1..k] = X$ so that $|X| = k$. Then by the p periodicity $w[p+1..p+k] = X$ and then, by the $p+1$ periodicity, $w[2p+2..2p+k+1] = X$. Assembling all this gives (1). \square

Remark. It is clear that if $w[1..2p+2]$ has period $p+1$ and $w[k+3..2p+2]$ has period p then w is the reverse of the right hand side of (1).

Theorem 7. *It is not possible for a letter to simultaneously belong to two distinct runs with period p and two distinct runs of period $p+1$.*

Proof. The proof is by contradiction. If there exists a counterexample to the theorem then it has a prefix and a suffix each of length $2p$ and each of which has a length $2p$ prefix with period p and a length $2p+2$ suffix with period $p+1$, or vice versa, and is such that the four periodic factors have at least one letter in common.

Let

$$\alpha = xXx\underline{x^sXx^sX}$$

and

$$\beta = yYy\underline{y^tYy^tY}$$

where

$$(2) \quad |X| + s = |Y| + t = p.$$

By Lemma 6 each of α and β has a prefix of length $2p+2$ with period $p+1$ and a suffix of length $2p$ and period p . The intersection of these two squares is underlined. We write $R(\alpha)$ and $R(\beta)$ for the reverses of α and β . We consider four cases.

Case 1. A word w has prefix α and suffix $R(\beta)$ with the underlined factors having non-empty intersection.

Case 2. A word w has prefix $R(\alpha)$ and suffix β with the underlined factors having non-empty intersection.

Case 3. A word w has prefix α and suffix β with the underlined factors having non-empty intersection.

Case 4. A word w has prefix $R(\alpha)$ and suffix $R(\beta)$ with the underlined factors having non-empty intersection.

If the statement of the Theorem is incorrect then a word belonging to one of these cases must exist with the stated periods being minimal and the four squares belonging to four different runs. We will show that in each case this cannot occur.

Case 1. We have

$$\alpha = xXx\underline{x^sXx^sX} \quad \text{and} \quad R(\beta) = Yy^t\underline{Yy^tYy^tY}.$$

Let d be the length of the intersection of these two words. The intersection must have length less than p else, by Lemma 5, the two period p squares would belong to the same run of period p . We must also have $d > |X| + |Y|$ else the underlined factors would not intersect. Recall that $|x^sX| = |y^tY| = p$ so the intersection is a suffix x^iX of x^sX and a prefix Yy^j of Yy^t . Thus

$$|x^i| + |X| > |X| + |Y|$$

so $i > |Y|$ which implies that $x = y$ and that of X and Y is a power of x . It follows that w is a power of x and the four squares belong to a single run.

Case 2. We have

$$R(\alpha) = Xx^sXx^sXx \quad \text{and} \quad \beta = yY\underline{yy^tYy^tY}.$$

Let d be the length of the intersection of these two words. Now $R(\alpha)$ and β have, respectively, a suffix with period $p + 1$ and a prefix with period $p + 1$. As in Case 1 we must have $d < p + 1$. In order that the underlined factors intersect we must also have $d > 4 + |X| + |Y|$. The intersection is thus

$$x^iXx = yYy^j$$

where $i + 1 + |X| > |X| + |Y| + 4$ implying $i > |Y| + 3$. As in Case 1 this implies that $x = y$ and that X and Y are powers of x as is the whole word w .

Case 3. We have

$$\alpha = xXx\underline{x^sXx^sX} \quad \text{and} \quad \beta = yY\underline{yy^tYy^tY}.$$

This case is more complicated than the others. Let us add another y to the right hand end of β . Set $\beta' = \beta y$ and $Y' = Yy$. Then

$$\beta' = yY'\underline{yy^{t-1}Y'y^{t-1}Y'}.$$

This has the same form as β but its underlined factor is one letter shorter and begins one position further to the right. Suppose we have a word with prefix α and suffix β in which the underlined factors intersect. By iterating the construction just described we can arrange that the two underlined factors have an intersection of length one. This will be the final x in the underlined factor of α and the initial y in the underlined factor of β . Thus $x = y$. We suppose, without loss of generality, that this is the case with our word.

X and Y may have prefixes or suffixes which are powers of x . We set

$$\begin{aligned} X &= x^a U x^b \\ Y &= x^c V x^d \end{aligned}$$

for non-negative integers a, b, c and d , where U and V neither begin nor end with x . Equations (2) now become

$$(3) \quad |U| + a + b + s = |V| + c + d + t = p.$$

and we have

$$\begin{aligned} \alpha &= x^{a+1} U x^{b+1} \underline{x^{a+s} U x^{b+s}} x^a U x^b \\ \beta &= x^{c+1} V x^{d+1} \underline{x^{c+t} V x^{d+t}} x^c V x^d. \end{aligned}$$

By our assumption the last x in the underlined section of α coincides with the first x in the underlined section of β . This means that $x^{c+1} V x^{d+1}$ is a suffix of $x^{a+1} U x^{b+1} \underline{x^{a+s} U x^{b+s-1}}$ so that $U = V$ and $b + s - 1 = d + 1$. It also means that $x^a U x^b$ is a prefix of $\underline{x^{c+t-1} V x^{d+t}} x^c V x^d$ so that $a = c + t - 1$. Together this gives $a + b + s - 1 = c + d + t$, which contradicts (3).

Case 4. This is just the reverse of Case 3 and need not be separately considered. The proof is complete. □

Theorem 8. For all n we have $\tau(n) < 47n^2/72 + 2n$.

Proof. Periods of runs in a word of length n must be less than or equal to $n/2$.

Consider runs with periods in $\{2q - 1, 2q\}$ for $1 \leq q \leq \lfloor n/6 \rfloor$. By Theorem 7 no letter can belong to more than three such runs so the contribution to the TRL is at most $3n$ for each such pair, and the contribution from all such pairs is at most $3n \lfloor n/6 \rfloor$.

Now consider runs with periods in $\{2q - 1, 2q\}$ for $\lfloor n/6 \rfloor + 1 \leq q \leq \lceil n/4 \rceil$. The upper bound here ensures that the maximum value of $2q$ is at least equal to $\lfloor n/2 \rfloor$. For some values of n we will be counting more runs than we need. The number of pairs $\{2q - 1, 2q\}$ is $\lceil n/4 \rceil - \lfloor n/6 \rfloor$.

We first show that there can be at most one run in a word of length n for each period in the set under consideration. Let p be such a period. Then $p \geq 2(\lfloor n/6 \rfloor + 1) > n/3$. If we had two runs with period p their intersection would have length at least $4p - n$ which is greater than p . This is impossible by Lemma 5. So we have at most one run for each period p . Suppose there is a run of length x with period $2q - 1$ and a run of length y with period $2q$. These have intersection of length at least $x + y - n$. By Lemma 3, this must be less than

$$2q + 2q - 1 - \gcd(2q - 1, 2q) = 4q - 2$$

else the runs will collapse into a single run with period 1. So $x + y \leq n + 4q - 3$. The contribution from all such pairs to the TRL is at most

$$\sum_{q=\lfloor n/6 \rfloor+1}^{\lceil n/4 \rceil} n + 4q - 3 = (\lceil n/4 \rceil - \lfloor n/6 \rfloor)(n - 3 + 2(\lceil n/4 \rceil + \lfloor n/6 \rfloor + 1)).$$

Adding this to the bound for the shorter periods we see that the TRL is less than

$$(\lceil n/4 \rceil - \lfloor n/6 \rfloor)(n - 3 + 2(\lceil n/4 \rceil + \lfloor n/6 \rfloor + 1)) + 3n \lfloor n/6 \rfloor.$$

We can show that this is less than the bound in the theorem by considering values of n in each residue class modulo 12. □

4. THE EXPECTED VALUE OF TRL

Theorem 9. The expected TRL for a word of length n on an alphabet of size α is

$$(4) \quad \frac{(\alpha - 1)^2}{\alpha^2} \sum_{p=1}^{\lfloor (n-2)/2 \rfloor} P(p) \sum_{i=1}^{n-2p-1} \sum_{k=2p}^{n-i-1} k\alpha^{-k} + 2\frac{\alpha - 1}{\alpha} \sum_{p=1}^{\lfloor (n-1)/2 \rfloor} P(p) \sum_{k=2p}^{n-1} k\alpha^{-k} + \frac{n}{\alpha^n} \sum_{p=1}^{\lfloor n/2 \rfloor} P(p),$$

where $P(p) = \sum_{d|p} \alpha^d \mu(p/d)$ is the number of length p primitive words on an alphabet of size α (see [13, Eq. 1.3.7]) and μ is the Möbius function.

Proof. We count the sum of the TRLs of all words of length n on an alphabet of size α . We first sum the TRLs of those runs which are neither prefixes nor suffixes.

Consider runs of the form $x[i + 1..i + k]$, where $1 \leq i$ and $i + k < n$, which have period p . For such runs $x[1..i - 1]$ can be any word, so there are α^{i-1} possibilities for this factor. The letter $x[i]$ must be chosen so that the run does not extend to the left of $x[i + 1]$. There are $\alpha - 1$ such choices. The factor $x[i + 1..i + p]$ is the generator of the run and can be any primitive word of length p , for which there are $P(p)$ choices. The rest of the run is then determined by its periodicity. The letter $x[i + k + 1]$ is chosen in one of $\alpha - 1$ ways to avoid the run extending to the right. This leave the final factor $x[i + k + 2..n]$ which can be chosen in $\alpha^{n-i-k-1}$ ways. The number of words having a run of the required form is therefore

$$\alpha^{i-1}(\alpha - 1)P(p)(\alpha - 1)\alpha^{n-i-k-1} = (\alpha - 1)^2\alpha^{n-k-2}P(p).$$

The variable i can take any value from 1 to $n - 2p - 1$ and, for each such i , k can take the values $2p$ to $n - i - 1$. The length of the run is k so the sum of total run lengths of all runs which are not suffixes are prefixes, which have period p , in all words of length n is:

$$\begin{aligned} P(p) & \sum_{i=1}^{n-2p-1} \sum_{k=2p}^{n-i-1} (\alpha - 1)^2 \alpha^{n-k-2} k \\ &= (\alpha - 1)^2 \alpha^{n-2} P(p) \sum_{i=1}^{n-2p-1} \sum_{k=2p}^{n-i-1} \alpha^{-k} k. \end{aligned}$$

Now consider those runs which are prefixes of x but not suffixes (that is, their length is less than n). Say $x[1..k]$ is such a run with period p . We have $P(p)$ choices for $x[1..p]$, $x[p+1..k]$, $x[k+1]$ can be chosen in $\alpha - 1$ ways and the rest of the word in α^{n-k-1} ways. The run length k can take any value from $2p$ to $n - 1$. The sum of the total run lengths of all prefix runs with period p , in all words of length n is:

$$P(p) \sum_{k=2p}^{n-1} (\alpha - 1) \alpha^{n-k-1} k = (\alpha - 1) \alpha^{n-1} P(p) \sum_{k=2p}^{n-1} \alpha^{-k} k.$$

By symmetry this is also the total for runs which are suffixes but not prefixes. Finally the number of runs which cover the whole word is just $P(p)$ and these all have length n . The sum of the total run length of all runs with period p is therefore:

$$(\alpha - 1)^2 \alpha^{n-2} P(p) \sum_{i=2}^{n-2p-1} \sum_{k=2p}^{n-i-1} \alpha^{-k} k + 2(\alpha - 1) \alpha^{n-1} P(p) \sum_{k=2p}^{n-1} \alpha^{-k} k + P(p)n.$$

A complication arises here because the maximum period p depends on which of the four cases we are considering. It is not hard to see that if the run is neither a prefix nor a suffix then its period is at most $\lfloor (n - 2)/2 \rfloor$, if it is a prefix but not a suffix, or vice versa, then its period p is at most $\lfloor (n - 1)/2 \rfloor$ and when it is both a prefix and a suffix, p is at most $\lfloor n/2 \rfloor$. Allowing for these different bounds, summing over p and dividing through by α^n (the number of words of length n) gives the required formula. \square

Let us say that the *TRL-density* of a word x is $TRL(x)/|x|$.

Corollary 10. *As n tends to infinity the expected density of a word on alphabet size α tends to*

$$\lim_{n \rightarrow \infty} \sum_{p=1}^n P(p) \frac{2p(\alpha - 1) + 1}{\alpha^{2p+1}}$$

where $P(p)$ is as defined in Theorem 9.

Proof. We write $S_1(n)$, $S_2(n)$ and $S_3(n)$ for the three terms in (4), that is,

$$(5) \quad \begin{aligned} S_1(n) &= \frac{(\alpha - 1)^2}{\alpha^2} \sum_{p=1}^{\lfloor (n-2)/2 \rfloor} P(p) \sum_{i=1}^{n-2p-1} \sum_{k=2p}^{n-i-1} k\alpha^{-k}, \\ S_2(n) &= 2 \frac{\alpha - 1}{\alpha} \sum_{p=1}^{\lfloor (n-1)/2 \rfloor} P(p) \sum_{k=2p}^{n-1} k\alpha^{-k}, \\ S_3(n) &= \frac{n}{\alpha^n} \sum_{p=1}^{\lfloor n/2 \rfloor} P(p). \end{aligned}$$

We write $S(n)$ for $S_1(n) + S_2(n) + S_3(n)$. We will obtain $\lim_{n \rightarrow \infty} S(n+1) - S(n)$ and show that this is a finite constant depending only on α . It follows that this limit equals $\lim_{n \rightarrow \infty} S(n)/n$ which is the required expected density. It is easy to show that $\lim_{n \rightarrow \infty} S_3(n)$ equals 0. This is not surprising since $S_3(n)$ counts the contribution to $S(n)$ from words which are themselves runs. Such words are rare among the α^n words of length n . It follows that

$$(6) \quad \lim_{n \rightarrow \infty} S_3(n+1) - S_3(n) = 0.$$

Now consider the term $S_2(n)$. This equals

$$\frac{2}{\alpha - 1} \sum_{p=1}^{\lfloor (n-1)/2 \rfloor} P(p) \left\{ \frac{2p(\alpha - 1) + 1}{\alpha^{2p}} - \frac{n(\alpha - 1) + 1}{\alpha^n} \right\}.$$

As n goes to infinity the sum of the second term in the parentheses goes to 0 so we have

$$\lim_{n \rightarrow \infty} S_2(n) = \frac{2}{\alpha - 1} \sum_{p=1}^{\lfloor (n-1)/2 \rfloor} P(p) \frac{2p(\alpha - 1) + 1}{\alpha^{2p}}.$$

Noting that $1 \leq P(p) \leq \alpha^p$ we see that this limit exists and is finite. For $\alpha = 2$ it equals 10. It follows that

$$(7) \quad \lim_{n \rightarrow \infty} S_2(n+1) - S_2(n) = 0.$$

Next consider $S_1(n)$. A change in order of summation gives

$$S_1(n) = \frac{(\alpha - 1)^2}{\alpha^2} \sum_{i=1}^{n-3} \sum_{k=2}^{n-1-i} k\alpha^{-k} \sum_{p=1}^{\lfloor k/2 \rfloor} P(p).$$

Then $S_1(n+1) - S_1(n)$ equals

$$\frac{(\alpha - 1)^2}{\alpha^2} \left\{ \sum_{k=2}^2 k\alpha^{-k} \sum_{p=1}^{\lfloor k/2 \rfloor} P(p) + \sum_{i=1}^{n-3} (n-i)\alpha^{-n+i} \sum_{p=1}^{\lfloor (n-i)/2 \rfloor} P(p) \right\}.$$

The first term in the parentheses is the $i = n - 2$ term in the first sum in (5). The second term corresponds to the terms with $k = n - i$. Since $P(1) = \alpha$ for any α this becomes, after changing the index of summation to $j = n - i$,

$$\frac{(\alpha - 1)^2}{\alpha^2} \left\{ \frac{2}{\alpha} + \sum_{j=3}^{n-1} j\alpha^{-j} \sum_{p=1}^{\lfloor j/2 \rfloor} P(p) \right\}.$$

Changing the order of summation again gives

$$\frac{(\alpha - 1)^2}{\alpha^2} \left\{ \frac{2}{\alpha} + \sum_{p=1}^{\lfloor (n-1)/2 \rfloor} P(p) \sum_{j=\max(3,2p)}^{n-1} j\alpha^{-j} \right\}.$$

To simplify matters we start the second sum at $j = 2p$. This means we are including an unwanted term corresponding to $j = 2, p = 1$. This is $2P(1)/\alpha^2 = 2/\alpha$, which equals the first term in the parentheses. We thus have

$$\begin{aligned} S_1(n+1) - S_1(n) &= \frac{(\alpha - 1)^2}{\alpha^2} \sum_{p=1}^{\lfloor (n-1)/2 \rfloor} P(p) \sum_{j=2p}^{n-1} j\alpha^{-j} \\ &= \frac{1}{\alpha} \sum_{p=1}^{\lfloor (n-1)/2 \rfloor} P(p) \left\{ \frac{2p(\alpha - 1) + 1}{\alpha^{2p}} - \frac{n(\alpha - 1) + 1}{\alpha^n} \right\}. \end{aligned}$$

We now take the limit as n goes to infinity. The second term makes no contribution to this limit since it's dominated by α^{-n} . So we have

$$\lim_{n \rightarrow \infty} S_3(n+1) - S_3(n) = \lim_{n \rightarrow \infty} \sum_{p=1}^n P(p) \frac{2p(\alpha - 1) + 1}{\alpha^{2p+1}}.$$

Summing this with (6) and (7) completes the proof. □

Some values of expected TRL-density are given in Table 2 below, along with corresponding results for the number of runs and the sum of exponents of runs.

TABLE 2. The columns show the expected number of runs per units length of a word, the expected sum of exponents per units length and the expected TRL per unit length for various alphabet sizes. The values come from [16], [11] and Corollary 10 respectively.

Alphabet size	Runs	Exponents	TRL
2	0.4116	1.1310	1.9775
3	0.3049	0.7382	1.0290
5	0.1933	0.4304	0.5208
10	0.0991	0.2087	0.2296

5. DISCUSSION

Theorems 2 and 8 show that, for all n ,

$$\frac{1}{8} < \frac{\tau(n)}{n^2} < \frac{47}{72} + \frac{2}{n}.$$

Both these bounds might be improved. Lower bounds for $\rho(n)$ and $\epsilon(n)$ were obtained by constructing words which were rich in the appropriate way. The word $u(n)$ of Theorem 2 is comparatively simple. One could look for something better using the techniques of [15] or some combinatorial heuristic such as simulated annealing or genetic algorithms.

The upper bound is probably far from best when n is large, though from Table 1 we suspect that the maximum value of $\tau(n)/n^2$ occurs when $n = 2$ and it would seem

that $\lim_{n \rightarrow \infty} \tau(n)/n^2$ exists, but we have not been able to prove it. Giraud's method [8] for showing the existence of $\lim_{n \rightarrow \infty} \rho(n)/n$ does not seem applicable to our situation. His method also showed that the limit is the supremum of the function. In our case it may be the infimum of $\{\tau(n)/n^2 : n > 1\}$. Extending Table 1 might give insight into these questions.

Acknowledgements: We thank the referee for careful reading and helpful comments.

REFERENCES

- [1] H. Bannai, M. Giraud, K. Kusano, W. Matsubara, A. Shinohara, J. Simpson, *The number of runs in a ternary word*, Proceedings of the Prague Stringology Conference 2010, Prague, Czech Republic, 2010, pp. 178–181.
- [2] P. Baturou¹, M. Piatkowski¹, and W. Rytter, *The number of runs in Sturmian words*, Proceedings of CIAA 2008, Lecture Notes in Computer Science, vol. 5148, Springer-Verlag, Berlin, 2008, pp. 252–261.
- [3] M. Crochemore, L. Ilie, *Maximal repetitions in strings*, J. Comput. System Sci. 74 (2008) 796–807.
- [4] M. Crochemore, L. Ilie, *The 'runs' conjecture*, Theoret. Comput. Sci. 412 (2011) 2931–2941.
- [5] M. Crochemore, M. Kubica, J. Radoszewski, W. Rytter, T. Walen, *On the maximal sum of exponents of runs in a string*, J. Discrete Algorithms 14 (2012), 29–36.
- [6] N. J. Fine, H. S. Wilf, *Uniqueness theorem for periodic functions*, Proc. Amer. Math. Soc. 16 (1965) 109–114.
- [7] F. Franek, J. Simpson and W. F. Smyth, *The maximum number of runs in a string*, Proceedings of the 14th Australasian Workshop on Combinatorial Algorithms (AWOCA), 2003, pp. 26–35.
- [8] M. Giraud, *Not so many runs in strings*, Proceedings of the 3rd International Conference on Language and Automata Theory and Applications (LATA), 2008, pp. 245–252.
- [9] R. Kolpakov and G. Kucherov, *On maximal repetitions in words*, J. Discrete Algorithms 1 (2000) 159–186.
- [10] K. Kusano, W. Matsubara, A. Ishino, H. Bannai and A. Shinohara, *New lower bounds for the maximum number of runs in a string*, Proceedings of the Prague Stringology Conference 2008, Prague, Czech Republic, 2008, pp. 140–145.
- [11] K. Kusano, W. Matsubara, A. Ishino, A. Shinohara, *Average Value of Sum of Exponents of Runs in a String*, Internat. J. Found. Comput. Sci. 20 (2009) 1135–1146.
- [12] K. Kusano, A. Shinohara, *Average Number of Runs and Squares in Necklace*, Proceedings of the Prague Stringology Conference 2010, Prague, Czech Republic, 2010.
- [13] M. Lothaire, *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, vol. 17, Cambridge University Press, 1997.
- [14] M. Lothaire, *Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, 2002.
- [15] W. Matsubara, K. Kusano, H. Bannai, A. Shinohara, *A series of run-rich strings*, Proceedings of the 3rd International Conference on Language and Automata Theory and Applications (LATA 2009), Lecture Notes in Computer Science, vol. 5457, Springer-Verlag, Berlin, 2009, pp. 578–587.
- [16] S. J. Puglisi and J. Simpson, *The expected number of runs in a word*, Australas. J. Combin. 42 (2008) 45–54.
- [17] S. J. Puglisi, J. Simpson and W. F. Smyth, *How many runs can a string contain?*, Theoret. Comput. Sci. 401 (2008) 165–171.
- [18] W. Rytter, *The number of runs in a string: improved analysis of the linear upper bound*, in B. Durand and W. Thomas, editors, STACS 2006, Lecture Notes in Computer Science, vol. 884, Springer-Verlag, Berlin, 2006, pp. 184–195.
- [19] W. Rytter, *The number of runs in a string*, Inform. Comput. 205 (2007) 1459–1469.
- [20] J. Simpson, *Modified Padovan words and the maximum number of runs in a word*, Australas. J. Combin. 46 (2010) 129–145.

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