Robust Filtering for Nonlinear Nonhomogeneous Markov Jump Systems by Fuzzy Approximation Approach

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***Abstract*—This paper addresses the problem of robust fuzzy $L_2 - L_\infty$ filtering for a class of uncertain nonlinear discrete-time Markov jump systems (MJSs) with nonhomogeneous jump processes. The Takagi–Sugeno fuzzy model is employed to represent such nonlinear nonhomogeneous MJS with norm-bounded parameter uncertainties. In order to decrease conservation, a polytope Lyapunov function which evolves as a convex function is employed, and then, under the designed mode-dependent and variation-dependent fuzzy filter which includes the membership functions, a sufficient condition is presented to ensure that the filtering error dynamic system is stochastically stable and that it has a prescribed $L_2 - L_\infty$ performance index. Two simulated examples are given to demonstrate the effectiveness and advantages of the proposed techniques.**

***Index Terms*—Fuzzy $L_2 - L_\infty$ filtering, Markov jump system (MJS), nonhomogeneous processes, uncertain nonlinear system.**

I. Introduction

SINCE many mathematical models of physical systems are nonlinear with complex uncertainties, causing much difficulties in the control and analysis [1], researchers have been trying to seek effective methods for controlling nonlinear systems. With the advent of Takagi–Sugeno (T-S) fuzzy model [2], T-S fuzzy model based approach has been applied to the study of control problems for nonlinear systems. It has been shown that a complex nonlinear system can be described in terms of a family of IF-THEN rules. Since, the T-S model behaves like a linear system, existing results for linear systems can be applied to the analysis and control of nonlinear systems. To date, some stability and stabilization results are obtained for T-S fuzzy systems (see [3], [4]), control issues [5]–[10], filtering [11], and fault detection [12], [13] of T-S fuzzy-based systems are also studied.

In practice, many dynamical systems have random changes in structures and parameters. They are caused by component failures or repairs, sudden environmental disturbance, or change of operation points. Markov jump systems (MJSs) are most suitable for describing such systems. In MJSs, the random jump of system parameters is governed by a Markov process or Markov chain. This class of systems can represent many processes, such as those in aerospace industry, manufacturing systems, economic systems, and electrical systems [14]. Hence, researchers have been paying remarkable attention to the problems of analysis and synthesis for MJSs. Results obtained so far cover a large variety of problems such as stochastic stability and stabilization [15], control [16]–[19], and fault detection [20]. These existing results for MJSs can be roughly divided into two types: 1) results on linear MJSs and 2) issues on nonlinear ones [21]. Obviously, it has been recognized that the results on nonlinear MJSs are generally more realistic and have better applicability. However, almost all the obtained results for MJSs are under the assumption that the transition probabilities are time invariant, namely, MJSs evolve as a homogeneous Markov process or Markov chain. This assumption is not valid in some real situations. One typical example is networked systems, where packet dropouts and network delays in such systems should be modeled by Markov processes, and the networked systems should be considered as MJSs [22], [23]. This is due to the fact that delays and packet dropouts are different in different periods, so the transition rates vary through the whole working region and they are uncertain. This leads to time-varying transition probabilities. Another example is the helicopter system [24], where the airspeed variation in such system matrices are ideally modeled as homogeneous Markov chain. However, the probabilities of the transition of these multiple airspeeds are not fixed when the weather changes. There are similar phenomena in other practical problems, such as in robotic manipulators [25], teleoperators [26], and wheeled mobile manipulators systems [27].

Manuscript received August 28, 2013; revised March 25, 2014 and June 23, 2014; accepted August 3, 2014. Date of publication November 20, 2014; date of current version August 14, 2015. This work was supported in part by the Australian Research Council under Grant DP140102180 and Grant LP140100471, in part by the 111 Project under Project B12018, in part by the National Natural Science Foundation of China under Grant 61273087 and Grant 61403169, in part by the Program for Excellent Innovative Team of Jiangsu Higher Education Institutions, and in part by the Fundamental Research Funds for the Central Universities under Grant JUSRP11459. This paper was recommended by Associate Editor W. J. Wang.

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In such situations, it is reasonable to model the system as MJS with nonhomogeneous jump process (chain), that is, the transition probabilities are time varying. One feasible assumption is to use a polytope set to describe the characteristic of uncertainties caused by time-varying transition probabilities. The main reason is that although the transition probability of the Markov process is not exactly known, one can evaluate values in some working points. Thus, we can model these time-varying transition probabilities by a polytope, which is a convex set. Due to this motivation, a polytope is applied to deal with a class of T-S fuzzy model based nonlinear Markov systems with time-varying transition probabilities.

Over the past several years, many results related to filtering and estimation have been reported for stochastic systems with time invariant transition probabilities, such as Kalman filtering [28], robust filtering [29], H_∞ filtering [30], and nonlinear fuzzy filtering [31], [32]. It is well known that L_2-L_∞ filtering works very well when dealing with external unknown noises. Therefore, we shall study the fuzzy L_2-L_∞ filtering problem for nonhomogeneous nonlinear systems. The results will cover the case involving the time-invariant transition probability matrix.

In this paper, the robust fuzzy L_2-L_∞ filtering problem is studied for uncertain nonhomogeneous nonlinear MJSs in discrete-time domain, which has not been well discussed in previous works. The T-S fuzzy model is employed to represent such nonlinear system by using IF-THEN rules and the time-varying jump transition probability matrix is described as a polytope. The rest of this paper is organized as follows. Problem statement and preliminary results are presented in Section II. In Section III, stochastic stability analysis of the resulting filtering error dynamic fuzzy system is given. In Section IV, L_2-L_∞ performance for the resulting fuzzy error dynamic system is discussed. In Section V, the robust fuzzy filter is designed such that error dynamic system is stochastically stable and satisfies the prescribed L_2-L_∞ performance index and also a few examples are given to illustrate the effectiveness of our approach. Finally, some concluding remarks are made in Section VI.

The notation $\mathbb{R}^n$ stands for an n-dimensional Euclidean space; the transpose of the matrix A is denoted by A^{T}; $E\{\cdot\}$ denotes the mathematical statistical expectation; $L_2^n[0,\infty)$ stands for the space of n-dimensional square integrable functions over $[0,\infty)$; a positive-definite matrix is denoted by $P>0$; I is the unit matrix with appropriate dimension; and $*$ means the symmetric term in a symmetric matrix.

II. Problem Statement and Preliminaries

Consider a probability space (M,F,P) where M, F, and P represent, respectively, the sample space, the algebra of events, and the probability measure defined on F. We consider an uncertain discrete-time nonlinear MJS with time-varying transition probability over the space (M,F,P)

$$\begin{cases} x_{k+1}=\iota\left(r_k,x_k,w_k\right)\\ y_k=\varsigma\left(r_k,x_k,w_k\right)\\ z_k=\chi\left(r_k,x_k\right)\end{cases}$$

where $\iota(\cdot)$, $\varsigma(\cdot)$, and $\chi(\cdot)$ are nonlinear functions, $\{r_k,k\geq 0\}$ is the concerned time-discrete Markov stochastic process, which takes values in a finite state set

$$\Lambda=\{1,2,3,\ldots,N\}$$

and r_0 represents the initial mode, the transition probability matrix is defined as $\Pi(k)=\{\pi_{mn}(k)\}$, $m,\ n\in\Lambda$, $\pi_{mn}(k)=P(r_{k+1}=n|r_k=m)$ is the transition probability from mode m at time k to mode n at time $k+1$, which satisfies $\pi_{mn}(k)\geq 0$ and $\sum_{n=1}^{N}\pi_{mn}(k)=1$, $x_k\in\mathbb{R}^l$ is the state vector of the system, $y_k\in\mathbb{R}^f$ is the output vector of the system, $z_k\in\mathbb{R}^p$ is the controlled output vector of the system, and $w_k\in L_2^q[0,\infty)$ is the external disturbance vector of the system.

The concerned system is described by the following fuzzy model:

Plant rule i
IF θ_{1k} is $M_{i1},\ldots,$ and θ_{gk} is M_{ig}
THEN

$$\begin{cases} x_{k+1}=A_i\left(r_k\right)x_k+B_i\left(r_k\right)w_k+\vartheta_i\left(x_k,r_k\right)\\ y_k=C_i\left(r_k\right)x_k+D_i\left(r_k\right)w_k\\ z_k=L_i\left(r_k\right)x_k\end{cases}\tag{1}$$

where $i\in\mathbb{S}=\{1,2,3,\ldots,v\}$, M_{ij} is the fuzzy set, $j\in\{1,2,3,\ldots,g\}$, v is the number of IF-THEN rules, $\theta_{1k},\ldots,\theta_{gk}$ are the premise variables, g is used as a number of premise variables, $A_i(r_k)$, $B_i(r_k)$, $C_i(r_k)$, $D_i(r_k)$, and $L_i(r_k)$ are mode-dependent constant matrices with appropriate dimensions at the working instant k, $\vartheta_i(\cdot)$ is time-dependent and norm-bounded uncertainty.

Assumption 1: The norm-bounded uncertainty $\vartheta_i(\cdot)$ in system (1) is assumed to satisfy

$$\vartheta_i\left(x_k,r_k\right)=\triangle A_i\left(r_k\right)x_k$$

and

$$\triangle A_i\left(r_k\right)=F_i(r_k)\Upsilon_i\left(r_k\right)N_i\left(r_k\right)$$

where $F_i(r_k)$ and $N_i(r_k)$ are constant matrices with appropriate dimensions and $\Upsilon_i(r_k)$ is an unknown matrix with Lebesgue measurable elements satisfying $\|\Upsilon_i(r_k)\|\leq 1$.

For simplicity, when $r_k=r$, $r\in\Lambda$, the matrices $A_i(r_k)$, $\triangle A_i(r_k)$, $B_i(r_k)$, $C_i(r_k)$, $D_i(r_k)$, $L_i(r_k)$, $F_i(r_k)$, and $N_i(r_k)$ are denoted as $A_i(r)$, $\triangle A_i(r)$, $B_i(r)$, $C_i(r)$, $D_i(r)$, $L_i(r)$, $F_i(r)$, and $N_i(r)$. The Markov jump fuzzy system (MJFS) is inferred as follows:

$$\begin{cases} x_{k+1}=\dfrac{\sum_{i=1}^{v}\mu_i(\theta_k)[(A_i(r)+\triangle A_i(r))x_k+B_i(r)w_k]}{\sum_{i=1}^{v}\mu_i(\theta_k)}\\ y_k=\dfrac{\sum_{i=1}^{v}\mu_i(\theta_k)[C_i(r)x_k+D_i(r)w_k]}{\sum_{i=1}^{v}\mu_i(\theta_k)}\\ z_k=\dfrac{\sum_{i=1}^{v}\mu_i(\theta_k)L_i(r)x_k}{\sum_{i=1}^{v}\mu_i(\theta_k)}\end{cases}\tag{2}$$

where $\theta_k=\left[\theta_{1k}\ \theta_{2k}\cdots\theta_{gk}\right]$, $\mu_i(\theta_k)=\prod_{j=1}^{g}M_{ij}(\theta_{jk})$, and $M_{ij}(\theta_{jk})$ is the grade of membership of θ_{jk} in M_{ij}.

It is assumed that

$$h_i(\theta_k) = \frac{\mu_i(\theta_k)}{\sum_{i=1}^{v} \mu_i(\theta_k)}$$

then, we can show that

$$h_i(\theta_k) \geq 0 \quad and \quad \sum_{i=1}^{v} h_i(\theta_k) = 1.$$

Thus, system (2) can be written as

$$\begin{cases} x_{k+1} = \sum_{i=1}^{v} h_i(\theta_k)\left[(A_i(r) + \triangle A_i(r))\, x_k + B_i(r) w_k\right] \\ y_k = \sum_{i=1}^{v} h_i(\theta_k)\left[C_i(r) x_k + D_i(r) w_k\right] \\ z_k = \sum_{i=1}^{v} h_i(\theta_k) L_i(r) x_k. \end{cases} \tag{3}$$

In order to estimate the signal z_k in system (2), if θ_{1k} is M_{i1}, ..., and θ_{gk} is M_{ig}, then, a general filter is constructed as follows:

$$\begin{cases} \hat{x}_{k+1} = A_{fi}(r)\hat{x}_k + B_{fi}(r) y_k \\ \hat{z}_k = L_{fi}(r)\hat{x}_k \end{cases} \tag{4}$$

and the fuzzy filter is

$$\begin{cases} \hat{x}_{k+1} = \sum_{i=1}^{v} h_i(\theta_k)\left[A_{fi}(r)\hat{x}_k + B_{fi}(r) y_k\right] \\ \hat{z}_k = \sum_{i=1}^{v} h_i(\theta_k) L_{fi}(r)\hat{x}_k \end{cases} \tag{5}$$

where $\hat{x}_k$ is the filter state vector, y_k is the input of the filter, and $A_{fi}(r)$, $B_{fi}(r)$, and $L_{fi}(r)$ are filter gains to be determined. It is seen from system (3) that the considered filter is mode-dependent. Suppose that the augmenting system (3) includes the states of the filter, then, we obtain the following fuzzy error dynamic system:

$$\begin{cases} \bar{x}_{k+1} = \sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \left[\bar{A}_{ij}(r)\bar{x}_k + \bar{B}_{ij}(r) w_k\right] \\ \bar{z}_k = \sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \bar{L}_{ij}(r)\bar{x}_k \end{cases} \tag{6}$$

where

$$\bar{z}_k = z_k - \hat{z}_k, \bar{x}_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}$$

$$\bar{A}_{ij}(r) = \begin{bmatrix} A_i(r) + \triangle A_i(r) & 0 \\ \mathscr{A}_i(r) & A_{fj}(r) \end{bmatrix}$$

$$\mathscr{A}_i(r) = A_i(r) + \triangle A_i(r) - A_{fj}(r) - B_{fj}(r) C_j(r)$$

$$\bar{B}_{ij}(r) = \begin{bmatrix} B_i(r) \\ B_i(r) - B_{fj}(r) D_j(r) \end{bmatrix}$$

$$\bar{L}_{ij}(r) = \begin{bmatrix} L_i(r) - L_{fj}(r) & L_{fj}(r) \end{bmatrix}.$$

Noting that if $\Pi(k)$ is a constant matrix, the MJS evolves as a homogeneous jump process. Clearly, if the transition probability matrix is time-varying, then it corresponds to a nonhomogeneous MJS, and the system evolves as a nonhomogeneous Markov process. A time-variant transition probability matrix of system (3), which is considered as a polytope, is given as

$$\Pi(k) = \sum_{s=1}^{\varrho} \alpha_s(k)\Pi^s$$

where Π^s are given matrices representing the vertices of the polytope, $s = 1, \ldots, \varrho$, ϱ represents the number of the selected vertices, $0 \leq \alpha_s(k) \leq 1$ and $\sum_{s=1}^{\varrho} \alpha_s(k) = 1$.

To proceed further, we may now state the definitions and lemmas for system (6) given below are needed.

Definition 1: For any initial mode r_0, and a given initial state $\bar{x}_0$, MJFS (6) (with $w_k = 0$) is said to be robustly stochastically stable if it holds that

$$\lim_{m\to\infty} E\left\{\sum_{k=0}^{m} \bar{x}_k^{\mathrm{T}}\bar{x}_k | \bar{x}_0, r_0\right\} < \infty. \tag{7}$$

Lemma 1 [33]: Let Q, W, S, and V be real matrices with appropriate dimensions, and let S be such that $S^{\mathrm{T}}S \leq I$. Then, for a positive scalar $\alpha > 0$, it holds that

$$Q + WSV + V^{\mathrm{T}}S^{\mathrm{T}}W^{\mathrm{T}} \leq Q + \alpha^{-1}WW^{\mathrm{T}} + \alpha V^{\mathrm{T}}V.$$

Lemma 2 [34]: Let $R(r) > 0$ be given symmetric matrices, and let W_c, $c = 1, 2, \ldots, h$, be matrices with appropriate dimension. If $0 \leq \varepsilon_c \leq 1$ and $\sum_{c=1}^{h} \varepsilon_c = 1$, then

$$\left(\sum_{c=1}^{h} \varepsilon_c W_c\right)^{\mathrm{T}} R(r) \left(\sum_{c=1}^{h} \varepsilon_c W_c\right) \leq \sum_{c=1}^{h} \varepsilon_c W_c^{\mathrm{T}} R(r) W_c.$$

Definition 2: For a given constant $\gamma > 0$, system (6) is said to be robustly stochastically stable and satisfies a $L_2 - L_\infty$ performance index γ, if it is robustly stochastically stable and the following condition holds:

$$E\|\bar{z}_k\|_\infty^2 \leq \gamma^2 E\|w_k\|_2^2 \tag{8}$$

where

$$E\|\bar{z}_k\|_\infty^2 = E\left\{\sup_{k>0}\left[\bar{z}_k^{\mathrm{T}}\bar{z}_k\right]\right\}, E\|w_k\|_2^2 = E\left\{\sum_{k=0}^{\infty} w_k^{\mathrm{T}} w_k\right\}.$$

Remark 1: For many complex practical dynamic systems, the construction of their exact mathematical model may not be possible. Thus, it is necessary to introduce $\vartheta(\cdot)$ so as to compensate for the inaccuracy caused in the mathematical modeling of the dynamical system concerned.

We may now state the aim of this paper as follows. Consider MJFS (3) with time-varying jump transition probabilities. Design a mode-dependent and parameter-dependent fuzzy filter (5), such that the resulting fuzzy filtering error system (6) is stochastically stable with a prescribed $L_2 - L_\infty$ performance index.

III. $L_2 - L_\infty$ Error Performance Analysis

Let us first address the stochastic stability of the filtering error system (6) which evolves as a nonhomogeneous jump process.

Theorem 1: For a given initial condition $\bar{x}_0$, the fuzzy filtering error system (6) (with $w_k = 0$) is stochastically stable, if there exists a set of positive definite symmetric matrices $\bar{P}_s(r)$ and $\bar{P}_q(n)$ such that

$$\Xi_{sq}(r) = -4\sum_{s=1}^{\varrho}\alpha_s(k)\bar{P}_s(r) + \sum_{n=1}^{N}\sum_{s=1}^{\varrho}\sum_{q=1}^{\varrho}\alpha_s(k)\beta_q(k)\pi_{mn}^{s}\Xi(r) < 0 \quad (9)$$

where

$$\Xi(r) = \sum_{i=1}^{v}\sum_{j=1}^{v}h_ih_j\hat{A}_{ij}^{\mathrm{T}}(r)\bar{P}_q(n)\hat{A}_{ij}(r)$$

$$0 \le \alpha_s(k) \le 1, \quad \sum_{s=1}^{\varrho}\alpha_s(k) = 1$$

$$0 \le \beta_q(k) \le 1, \quad \sum_{q=1}^{\varrho}\beta_q(k) = 1$$

$$\hat{A}_{ij}(r) = \bar{A}_{ij}(r) + \bar{A}_{ji}(r), \quad 1 \le i \le j \le v.$$

Proof: The difference equations of system (6) (with $w_k = 0$) can be written as

$$\bar{x}_{k+1} = \sum_{i=1}^{v}\sum_{j=1}^{v}h_ih_j\bar{A}_{ij}(r)\bar{x}_k. \quad (10)$$

Construct a parameter-dependent and mode-dependent Lyapunov function as

$$V(\bar{x}_k, r) = \sum_{s=1}^{\varrho}\alpha_s(k)\bar{x}_k^{\mathrm{T}}\bar{P}_s(r)\bar{x}_k \quad (r \in \Lambda) \quad (11)$$

where

$$0 \le \alpha_s(k) \le 1, \quad \sum_{s=1}^{\varrho}\alpha_s(k) = 1, \quad \bar{P}_s(r) > 0.$$

We obtain

$$\Delta V(\bar{x}_k, r) = E\{V(\bar{x}_{k+1}, r)\} - V(\bar{x}_k, r)$$
$$= \tfrac{1}{4}\sum_{n=1}^{N}\sum_{s=1}^{\varrho}\sum_{s=1}^{\varrho}\alpha_s(k)\alpha_s(k+1)\pi_{rn}^{s}\bar{x}_k^{\mathrm{T}} \left[\left(\sum_{i=1}^{v}\sum_{j=1}^{v}h_ih_j\hat{A}_{ij}^{\mathrm{T}}(r)\right)\bar{P}_s(n)\left(\sum_{i=1}^{v}\sum_{j=1}^{v}h_ih_j\hat{A}_{ij}(r)\right)\right]\bar{x}_k - \sum_{s=1}^{\varrho}\alpha_s(k)\bar{x}_k^{\mathrm{T}}\bar{P}_s(r)\bar{x}_k.$$

Define

$$\sum_{s=1}^{\varrho}\alpha_s(k+1)\bar{P}_s(n) = \sum_{q=1}^{\varrho}\beta_q(k)\bar{P}_q(n).$$

Then, we have

$$\Delta V(\bar{x}_k, r) = \tfrac{1}{4}\sum_{n=1}^{N}\sum_{s=1}^{\varrho}\sum_{q=1}^{\varrho}\alpha_s(k)\beta_q(k)\pi_{rn}^{s}\bar{x}_k^{\mathrm{T}} \left[\left(\sum_{i=1}^{v}\sum_{j=1}^{v}h_ih_j\hat{A}_{ij}^{\mathrm{T}}(r)\right)\bar{P}_q(n)\left(\sum_{i=1}^{v}\sum_{j=1}^{v}h_ih_j\hat{A}_{ij}(r)\right)\right]\bar{x}_k - \sum_{s=1}^{\varrho}\alpha_s(k)\bar{x}_k^{\mathrm{T}}\bar{P}_s(r)\bar{x}_k.$$

By Lemma 2, it gives

$$\Delta V(\bar{x}_k, r) < \bar{x}_k^{\mathrm{T}}\Xi_{sq}(r)\bar{x}_k.$$

For system (10), it follows from condition (9) that:

$$\Delta V(\bar{x}_k, r) < 0 \qquad (r \in \Lambda).$$

Let

$$\eta = \min_k\{\lambda_{\min}\left(-\Xi_{sq}(r)\right)\} \quad \forall r \in \Lambda$$

where $\lambda_{\min}(-\Xi_{sq}(r))$ is the minimal eigenvalue of $-\Xi_{sq}(r)$. Then

$$\Delta V(\bar{x}_k, r) \le -\eta\bar{x}_k^{\mathrm{T}}\bar{x}_k.$$

Thus

$$E\left\{\sum_{k=0}^{T}\Delta V(\bar{x}_k, r)\right\} = E\{V(\bar{x}_{T+1}, r)\} - V(\bar{x}_0, r) \le -\eta E\left\{\sum_{k=0}^{T}\|\bar{x}_k\|^2\right\}$$

and the following inequality holds:

$$E\left\{\sum_{k=0}^{T}\|\bar{x}_k\|^2\right\} \le \tfrac{1}{\eta}\{V(\bar{x}_0, r) - E\{V(\bar{x}_{T+1}, r)\}\} \le \tfrac{1}{\eta}V(\bar{x}_0, r)$$

which, in turn, implies that

$$\lim_{T\to\infty}E\left\{\sum_{k=0}^{T}\|\bar{x}_k\|^2\right\} \le \frac{1}{\eta}V(\bar{x}_0, r).$$

Therefore, by Definition 1, system (6) (with $w_k = 0$) is robustly stochastically stable. This concludes the proof. ■

Next, we consider the $L_2 - L_\infty$ performance for the fuzzy filtering error system (6).

In order to minimize the influences of the disturbances, $L_2 - L_\infty$ performance index is analyzed for system (6) subject to all admissible disturbances. This leads to the conclusion that system (6) is robustly stochastically stable and satisfies a prescribed $L_2 - L_\infty$ index γ.

Theorem 2: Consider system (6) (with $w_k \neq 0$) and let $\gamma > 0$ be a given constant. Suppose that there exists a set of positive definite symmetric matrices $\tilde{P}_s(r)$ and $\tilde{P}_q(n)$ such that

$$\Theta_{1sq}(r) = \begin{bmatrix} -\tilde{P}_q(n) & \tilde{P}_q(n)\hat{A}_{ij}(r) & \tilde{P}_q(n)\hat{B}_{ij}(r) \\ * & -4\tilde{P}_s(r) & 0 \\ * & * & -4I \end{bmatrix} < 0 \quad (12)$$

$$\Theta_{2sq}(r) = \begin{bmatrix} -\tilde{P}_s(r) & \hat{L}_{ij}^{\mathrm{T}}(r) \\ * & -4\gamma^2 I \end{bmatrix} < 0$$
$$\forall r \in \Lambda, \quad 1 \le i \le j \le v \tag{13}$$

where

$$\tilde{P}_q(n) = \sum_{n=1}^{N}\sum_{s=1}^{\varrho}\sum_{q=1}^{\varrho} \alpha_s(k)\beta_q(k)\pi_{rn}^{s}\bar{P}_q(n)$$
$$\tilde{P}_s(r) = \sum_{s=1}^{\varrho}\alpha_s(k)\bar{P}_s(r), \quad \hat{A}_{ij}(r) = \bar{A}_{ij}(r) + \bar{A}_{ji}(r)$$
$$\hat{B}_{ij}(r) = \bar{B}_{ij}(r) + \bar{B}_{ji}(r), \quad \hat{L}_{ij}(r) = \bar{L}_{ij}(r) + \bar{L}_{ji}(r).$$

Then, system (6) is stochastically stable and satisfies a prescribed $L_2 - L_\infty$ performance index γ.

Proof: Consider the Lyapunov function (11) for system (6). We can show that

$$\begin{aligned}
\Delta V(\bar{x}_k, r) &= E\{V(\bar{x}_{k+1}, r)\} - V(\bar{x}_k, r) \\
&= \frac{1}{4}\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \left(\left(\bar{A}_{ij}(r) + \bar{A}_{ji}(r)\right)\bar{x}_k\right. \\
&\quad + \left.\left(\bar{B}_{ij}(r) + \bar{B}_{ji}(r)\right)w_k\right)^{\mathrm{T}}\tilde{P}_q(n) \\
&\quad \sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \left(\left(\bar{A}_{ij}(r) + \bar{A}_{ji}(r)\right)\bar{x}_k\right. \\
&\quad + \left.\left(\bar{B}_{ij}(r) + \bar{B}_{ji}(r)\right)w_k\right) - \bar{x}_k^{\mathrm{T}}\tilde{P}_s(r)\bar{x}_k \\
&= \bar{x}_k^{\mathrm{T}}\left[\frac{1}{4}\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j\left(\bar{A}_{ij}(r) + \bar{A}_{ji}(r)\right)^{\mathrm{T}}\tilde{P}_q(n)\right. \\
&\quad \left.\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j\left(\bar{A}_{ij}(r) + \bar{A}_{ji}(r)\right) - \tilde{P}_s(r)\right]\bar{x}_k \\
&\quad + 2\bar{x}_k^{\mathrm{T}}\left[\frac{1}{4}\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j\left(\bar{A}_{ij}(r) + \bar{A}_{ji}(r)\right)^{\mathrm{T}}\tilde{P}_q(n)\right. \\
&\quad \left.\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j\left(\bar{B}_{ij}(r) + \bar{B}_{ji}(r)\right)\right]w_k \\
&\quad + w_k^{\mathrm{T}}\left[\frac{1}{4}\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j\left(\bar{B}_{ij}(r) + \bar{B}_{ji}(r)\right)^{\mathrm{T}}\tilde{P}_q(n)\right. \\
&\quad \left.\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j\left(\bar{B}_{ij}(r) + \bar{B}_{ji}(r)\right)\right]w_k.
\end{aligned}$$

To establish the $L_2 - L_\infty$ performance for the system, the following cost function is introduced for system (6):

$$J(T) = E\{V(\bar{x}_k, r)\} - E\left\{\sum_{k=0}^{T} w_k^{\mathrm{T}} w_k\right\}. \tag{14}$$

Under zero initial condition, index $J(T)$ can be written as

$$J(T) \le E\left\{\sum_{k=0}^{T}\left[-w_k^{\mathrm{T}} w_k + \Delta V(\bar{x}_k, r)\right]\right\}. \tag{15}$$

Thus, we have

$$\begin{aligned}
J(T) &\le E\left\{\sum_{k=0}^{T}\left[-w_k^{\mathrm{T}} w_k + \Delta V(\bar{x}_k, r)\right]\right\} \\
&= E\sum_{k=0}^{T}\left\{\bar{x}_k^{\mathrm{T}}\left(\frac{1}{4}\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \hat{A}_{ij}^{\mathrm{T}}(r)\tilde{P}_q(n)\right.\right. \\
&\quad \left.\left.\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \hat{A}_{ij}(r) - \tilde{P}_s(r)\right)\bar{x}_k\right\} \\
&\quad + E\sum_{k=0}^{T}\left\{2\bar{x}_k^{\mathrm{T}}\cdot\frac{1}{4}\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \hat{A}_{ij}^{\mathrm{T}}(r)\tilde{P}_q(n)\right. \\
&\quad \left.\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \hat{B}_{ij}(r)\cdot w_k\right\} \\
&\quad + E\sum_{k=0}^{T}\left\{w_k^{\mathrm{T}}\cdot\frac{1}{4}\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \hat{B}_{ij}^{\mathrm{T}}(r)\tilde{P}_q(n)\right. \\
&\quad \left.\sum_{i=1}^{v}\sum_{j=1}^{v} h_i h_j \hat{B}_{ij}(r)\cdot w_k - w_k^{\mathrm{T}} w_k\right\}.
\end{aligned}$$

Recalling Schur complement, it shows that

$$J(T) \le \tilde{x}_k^{\mathrm{T}}\Theta_{1sq}(r)\tilde{x}_k$$

where

$$\tilde{x}_k = \begin{bmatrix} \bar{x}_k^{\mathrm{T}} & w_k^{\mathrm{T}} \end{bmatrix}.$$

Under the assumption that $w_k = 0$, $\Theta_{1sq}(r) < 0$ implies inequality (9). Following a similar argument given in the proof of Theorem 1, we can show that system (6) is stochastically stable.

Then, by condition (12), we have

$$E\left\{\bar{x}_k^{\mathrm{T}}\tilde{P}_s(r)\bar{x}_k\right\} \le E\{V(\bar{x}_k, r)\} < E\left\{\sum_{k=0}^{T} w_k^{\mathrm{T}} w_k\right\}.$$

On the other hand, by condition (13), we can show that

$$E\left\{\bar{z}_k^{\mathrm{T}}\bar{z}_k\right\} < \gamma^2 E\left\{\bar{x}_k^{\mathrm{T}}\tilde{P}_s(r)\bar{x}_k\right\} < \gamma^2 E\left\{\sum_{k=0}^{T} w_k^{\mathrm{T}} w_k\right\}$$

for $T \to \infty$. Since $\Theta_{2sq}(r) < 0$, it follows that:

$$E\|\bar{z}_k\|_\infty^2 \le \gamma^2 E\|w_k\|_2^2. \tag{16}$$

By Definition 2, system (6) is robustly stochastically stable and satisfies a prescribed $L_2 - L_\infty$ performance. This completes the proof. ■

Remark 2: Noting that sufficiently stochastically stable conditions are given in Theorem 1, following by Theorem 1, a $L_2 - L_\infty$ performance index is considered in Theorem 2, and sufficient conditions for the existence of $L_2 - L_\infty$ filter for system (6) is given in Theorem 2. It is worth mentioning that by setting

$$\sum_{s=1}^{w}\alpha_s(k)\bar{P}_s(r) = \bar{P}(r)$$

the result obtained above can be applied to general stochastic systems with homogeneous jump process.

IV. Robust Fuzzy $L_2 - L_\infty$ Filter Design

Sufficient conditions for the existence of an admissible mode-dependent fuzzy $L_2 - L_\infty$ filter in the form of (5) for system (3) is given in the following theorems.

Theorem 3: Consider system (6) with time-varying jump transition probabilities, and let $\gamma > 0$ be a given constant. Suppose that there exists a set of positive definite symmetric matrices $\bar{P}_s(r)$, $\bar{P}_q(n)$ and mode-dependent matrices $X(r)$ such that

$$\Omega_{1sq}(r) = \begin{bmatrix} \overrightarrow{\Omega}_{1sq}(r) & X(r)\hat{A}_{ij}(r) & X(r)\hat{B}_{ij}(r) \\ * & -4\bar{P}_s(r) & 0 \\ * & * & -4I \end{bmatrix} < 0 \tag{17}$$

$$\Omega_{2sq}(r) = \begin{bmatrix} -\bar{P}_s(r) & \hat{L}_{ij}(r) \\ * & -4\gamma^2 I \end{bmatrix} < 0 \tag{18}$$

where

$$\overrightarrow{\Omega}_{1sq}(r) = -X(r) - X^{\mathrm{T}}(r) + \hat{P}_q(n)$$
$$\hat{P}_q(n) = \sum_{n=1}^{N} \pi_{rn}^s \bar{P}_q(n), \quad 1 \le i \le j \le v.$$

Then, system (6) is robustly stochastically stable and satisfies a prescribed $L_2 - L_\infty$ performance index γ.

Proof: Noting that a sufficient condition for system (6) to be stochastically stable and has a prescribed $L_2 - L_\infty$ performance index is that all the vertices of the polytope satisfy the stability requirements as shown in Theorem 2. Hence, by Theorem 2, $\Theta_{1sq}(r) < 0$ implies that

$$\Omega_{3sq}(r) = \begin{bmatrix} -\check{P}_q(n) & \check{P}_q(n)\hat{A}_{ij}(r) & \check{P}_q(n)\hat{B}_{ij}(r) \\ * & -4\bar{P}_s(r) & 0 \\ * & * & -4I \end{bmatrix} < 0 \tag{19}$$

where

$$\check{P}_q(n) = \sum_{n=1}^{N} \sum_{q=1}^{\varrho} \beta_q(k) \pi_{rn}^s \bar{P}_q(n)$$

which, in turn, implies that

$$\Omega_{4sq}(r) = \begin{bmatrix} -\hat{P}_q(n) & \hat{P}_q(n)\hat{A}_{ij}(r) & \hat{P}_q(n)\hat{B}_{ij}(r) \\ * & -4\bar{P}_s(r) & 0 \\ * & * & -4I \end{bmatrix} < 0. \tag{20}$$

In order to avoid the cross coupling of matrix product terms caused by model variation in condition (20), a slack matrix is introduced. Then, after standard matrix manipulation, condition (17) is obtained. On the other hand, by condition (13), condition (18) is obtained and this completes the proof. ■

Therefore, the sufficient conditions, which ensure that system (6) is stochastically stable and satisfies a prescribed $L_2 - L_\infty$ performance index, are obtained from Theorem 3.

Next, by Theorem 3, we will design the robust fuzzy $L_2 - L_\infty$ filter for system (3), so that the resulting error dynamic system (6) is stochastically stable and achieves a prescribed $L_2 - L_\infty$ performance index.

Theorem 4: Consider system (6) with time-varying jump transition probabilities, and let $\gamma > 0$ be a given constant. Suppose that there exist matrices $P_{1s}(r) > 0$, $P_{2s}(r) > 0$, and matrices $P_{3s}(r)$, $R(r)$, $Y(r)$, $Z(r)$, $A_{Fij}(r)$, $B_{Fij}(r)$, and $L_{Fij}(r)$ such that the following condition has a feasible solution:

$$\Gamma_{1sq}(r) = \begin{bmatrix} a_1 & a_2 & a_4 & A_{Fij}(r) & a_6 & b_1 & R(r)M_j(r) + Y(r)M_j(r) \\ * & a_3 & a_5 & A_{Fij}(r) & a_7 & b_2 & Z(r)M_j(r) + Y(r)M_j(r) \\ * & * & -4P_{1s}(r) + a_8 & -4P_{2s}(r) & 0 & 0 & 0 \\ * & * & * & -4P_{3s}(r) & 0 & 0 & 0 \\ * & * & * & * & -4I & 0 & 0 \\ * & * & * & * & * & -\alpha_i(r) & 0 \\ * & * & * & * & * & * & -\alpha_j(r) \end{bmatrix} < 0 \tag{21}$$

$$\Gamma_{2sq}(r) = \begin{bmatrix} -P_{1s}(r) & -P_{2s}(r) & L_{ij}^{\mathrm{T}}(r) - L_{Fij}^{\mathrm{T}}(r) \\ * & -P_{3s}(r) & L_{Fij}^{\mathrm{T}}(r) \\ * & * & -4\gamma^2 I \end{bmatrix} < 0 \tag{22}$$

where

$$a_1 = -R(r) - R^{\mathrm{T}}(r) + P_{1q}(n)$$
$$a_2 = -Y(r) - Z^{\mathrm{T}}(r) + P_{2q}(n)$$
$$a_3 = -Y(r) - Y^{\mathrm{T}}(r) + P_{3q}(n)$$
$$a_4 = R(r)A_{ij}(r) + Y(r)A_{ij}(r) - A_{Fij}(r) - B_{Fij}(r)C_{ij}(r)$$
$$a_5 = Z(r)A_{ij}(r) + Y(r)A_{ij}(r) - A_{Fij}(r) - B_{Fij}(r)C_{ij}(r)$$
$$a_6 = R(r)B_{ij}(r) + Y(r)B_{ij}(r) - B_{Fij}(r)D_{ij}(r)$$
$$a_7 = Z(r)B_{ij}(r) + Y(r)B_{ij}(r) - B_{Fij}(r)D_{ij}(r)$$
$$a_8 = \alpha_i(r)N_i^{\mathrm{T}}(r)N_i(r) + \alpha_j(r)N_j^{\mathrm{T}}(r)N_j(r)$$
$$b_1 = R(r)F_i(r) + Y(r)F_i(r)$$
$$b_2 = Z(r)F_i(r) + Y(r)F_i(r)$$
$$A_{ij}(r) = A_i(r) + A_j(r), \quad B_{ij}(r) = B_i(r) + B_j(r)$$
$$C_{ij}(r) = C_i(r) + C_j(r), \quad D_{ij}(r) = D_i(r) + D_j(r)$$
$$L_{ij}(r) = L_i(r) + L_j(r), \quad A_{Fij}(r) = A_{Fi}(r) + A_{Fj}(r)$$
$$B_{Fij}(r) = B_{Fi}(r) + B_{Fj}(r), \quad L_{Fij}(r) = L_{Fi}(r) + L_{Fj}(r).$$

Then, a mode-dependent fuzzy filter (5) is obtained such that the resulting filtering error system (6) is stochastically stable and satisfies a prescribed $L_2 - L_\infty$ performance index γ. Moreover, the gain matrices of the filter are given by

$$A_{fij}(r) = Y^{-1}(r)A_{Fij}(r), B_{fij}(r) = Y^{-1}(r)B_{Fij}(r)$$
$$L_{fij}(r) = L_{Fij}(r).$$

Proof: Consider the filtering error system (6). Denote

$$\bar{P}_s(r) = \begin{bmatrix} P_{1s}(r) & P_{2s}(r) \\ * & P_{3s}(r) \end{bmatrix}, X(r) = \begin{bmatrix} R(r) & Y(r) \\ Z(r) & Y(r) \end{bmatrix}.$$

Then, by Theorem 3, $\Omega_{1sq}(r) < 0$ implies

$$\Gamma_{3sq}(r) = \begin{bmatrix} a_1 & a_2 & a_9 & A_{Fij}(r) & a_6 \\ * & a_3 & a_{10} & A_{Fij}(r) & a_7 \\ * & * & -4P_{1s}(r) & -4P_{2s}(r) & 0 \\ * & * & * & -4P_{3s}(r) & 0 \\ * & * & * & * & -4I \end{bmatrix} < 0 \tag{23}$$

where

$$\begin{aligned} a_9 &= R(r)\left(A_{ij}(r) + \Delta A_{ij}(r)\right) + Y(r)\left(A_{ij}(r) + \Delta A_{ij}(r)\right) \\ &\quad - A_{Fij}(r) - B_{Fij}(r)C_{ij}(r) \\ a_{10} &= Z(r)\left(A_{ij}(r) + \Delta A_{ij}(r)\right) + Y(r)\left(A_{ij}(r) + \Delta A_{ij}(r)\right) \\ &\quad - A_{Fij}(r) - B_{Fij}(r)C_{ij}(r) \\ \Delta A_{ij}(r) &= F_i(r)\Upsilon(r)N_i(r) + F_j(r)\Upsilon(r)N_j(r). \end{aligned}$$

Clearly, $\Gamma_{3sq}(r) < 0$ is equivalent to

$$\begin{aligned} &\Gamma_{4sq}(r) + T_1(r)\Upsilon(r)T_2(r) + T_2^{\mathrm{T}}(r)\Upsilon^{\mathrm{T}}(r)T_1^{\mathrm{T}}(r) \\ &\quad + T_3(r)\Upsilon(r)T_4(r) + T_4^{\mathrm{T}}(r)\Upsilon^{\mathrm{T}}(r)T_3^{\mathrm{T}}(r) < 0 \end{aligned}$$

where

$$\Gamma_{4sq}(r) = \begin{bmatrix} a_1 & a_2 & a_4 & A_{Fij}(r) & a_6 \\ * & a_3 & a_5 & A_{Fij}(r) & a_7 \\ * & * & -4P_{1s}(r) & -4P_{2s}(r) & 0 \\ * & * & * & -4P_{3s}(r) & 0 \\ * & * & * & * & -4I \end{bmatrix} < 0 \tag{24}$$

$$\begin{aligned} T_1^{\mathrm{T}}(r) &= \big[F_i^{\mathrm{T}}(r)R^{\mathrm{T}}(r) + F_i^{\mathrm{T}}(r)Y^{\mathrm{T}}(r) \\ &\qquad F_i^{\mathrm{T}}(r)Z^{\mathrm{T}}(r) + F_i^{\mathrm{T}}(r)Y^{\mathrm{T}}(r) \quad 0 \quad 0 \quad 0 \big] \\ T_2^{\mathrm{T}}(r) &= \begin{bmatrix} 0 & 0 & N_i(r) & 0 & 0 \end{bmatrix} \\ T_3^{\mathrm{T}}(r) &= \big[F_j^{\mathrm{T}}(r)R^{\mathrm{T}}(r) + F_j^{\mathrm{T}}(r)Y^{\mathrm{T}}(r) \\ &\qquad F_j^{\mathrm{T}}(r)Z^{\mathrm{T}}(r) + F_j^{\mathrm{T}}(r)Y^{\mathrm{T}}(r) \quad 0 \quad 0 \quad 0 \big] \\ T_4^{\mathrm{T}}(r) &= \begin{bmatrix} 0 & 0 & N_j(r) & 0 & 0 \end{bmatrix}. \end{aligned}$$

Denote

$$Y(r)A_{fij}(r) = A_{Fij}(r), Y(r)B_{fij}(r) = B_{Fij}(r), L_{fij}(r) = L_{Fij}(r).$$

Then, by Lemma 1 and recalling Schur complement, $\Gamma_{3sq}(r) < 0$ holds if $\Gamma_{1sq}(r) < 0$.

On the other hand, let $L_{fij}(r) = L_{Fij}(r)$. Then, $\Omega_{2sq}(r) < 0$ implies $\Gamma_{2sq}(r) < 0$.

Therefore, if conditions (21) and (22) hold, the filtering error system (6) is stochastically stable and satisfies a prescribed $L_2 - L_\infty$ performance index γ. Moreover, the parameters of the admissible filter are given by

$$\begin{aligned} &A_{fij}(r) = Y^{-1}(r)A_{Fij}(r), B_{fij}(r) = Y^{-1}(r)B_{Fij}(r) \\ &L_{fij}(r) = L_{Fij}(r). \end{aligned}$$

This completes the proof. ■

Remark 3: By Schur complement and linear matrix inequalities (LMIs) tool box, and following Theorem 3, a $L_2 - L_\infty$ filter is designed which guarantees that the filtering error system (6) is stochastically stable and satisfies a prescribed $L_2 - L_\infty$ performance index γ. Noting that in order to calculate the optimal $L_2 - L_\infty$ performance index γ for system (6), we set $\gamma^2 = \varepsilon$. Then, Theorem 4 can be cast as an optimization problem as follows:

$$\min \quad \varepsilon \tag{25}$$

subject to LMIs

$$\Gamma_{5sq}(r) = \begin{bmatrix} a_1 & a_2 & a_4 & A_{Fij}(r) & a_6 & b_1 & R(r)M_j(r) + Y(r)M_j(r) \\ * & a_3 & a_5 & A_{Fij}(r) & a_7 & b_2 & Z(r)M_j(r) + Y(r)M_j(r) \\ * & * & -4P_{1s}(r) + a_8 & -4P_{2s}(r) & 0 & 0 & 0 \\ * & * & * & -4P_{3s}(r) & 0 & 0 & 0 \\ * & * & * & * & -4I & 0 & 0 \\ * & * & * & * & * & -\alpha_i(r) & 0 \\ * & * & * & * & * & * & -\alpha_j(r) \end{bmatrix} < 0 \tag{26}$$

$$\Gamma_{6sq}(r) = \begin{bmatrix} -P_{1s}(r) & -P_{2s}(r) & L_{ij}^{\mathrm{T}}(r) - L_{Fij}^{\mathrm{T}}(r) \\ * & -P_{3s}(r) & L_{Fij}^{\mathrm{T}}(r) \\ * & * & -4\varepsilon I \end{bmatrix} < 0. \tag{27}$$

Remark 4: By solving (25)–(27), we obtain the filter corresponding to the optimal $L_2 - L_\infty$ performance index. It is worth mentioning that the time-invariant jump probability matrix is a special case of time-varying ones.

V. Simulation Results

Example 1: We first consider nonhomogeneous discrete-time MJSs, which are aggregated into two modes, where

$$A_1(1) = \begin{bmatrix} 0.45 & -0.45 \\ 0.8 & 0.3 \end{bmatrix}, \quad A_1(2) = \begin{bmatrix} 0.36 & -0.25 \\ 0.2 & 0.5 \end{bmatrix}$$

$$A_2(1) = \begin{bmatrix} 0.5 & -0.5 \\ 0.7 & 0.5 \end{bmatrix}, \quad A_2(2) = \begin{bmatrix} 0.26 & -0.35 \\ 0.25 & 0.6 \end{bmatrix}$$

$$B_1(1)=\begin{bmatrix}-0.03\\0.25\end{bmatrix},\quad B_1(2)=\begin{bmatrix}-0.05\\0.3\end{bmatrix}$$
$$B_2(1)=\begin{bmatrix}-0.01\\0.32\end{bmatrix},\quad B_2(2)=\begin{bmatrix}-0.05\\0.22\end{bmatrix}$$
$$C_1(1)=\begin{bmatrix}0.5 & -0.4\end{bmatrix},\quad C_1(2)=\begin{bmatrix}0.3 & -0.1\end{bmatrix}$$
$$C_2(1)=\begin{bmatrix}0.25 & -0.2\end{bmatrix},\quad C_2(2)\begin{bmatrix}0.15 & -0.3\end{bmatrix}$$
$$D_1(1)=-0.3,\quad D_2(1)=-0.2$$
$$D_1(2)=-0.2,\quad D_2(2)=-0.1$$
$$L_1(1)=\begin{bmatrix}0.3 & -0.2\end{bmatrix},\quad L_2(1)=\begin{bmatrix}0.2 & -0.2\end{bmatrix}$$
$$L_1(2)=\begin{bmatrix}0.1 & 0.5\end{bmatrix},\quad L_2(2)=\begin{bmatrix}0.1 & 0.5\end{bmatrix}$$
$$M_1(1)=M_2(1)=\begin{bmatrix}0.1\\0.1\end{bmatrix},\quad M_1(2)=M_2(2)=\begin{bmatrix}0.1\\0.1\end{bmatrix}$$
$$N_1(1)=N_2(1)=\begin{bmatrix}0.1 & 0.1\end{bmatrix}$$
$$N_1(2)=N_2(2)=\begin{bmatrix}0.1 & 0.1\end{bmatrix}.$$

The vertices of the time-varying transition probability matrix are given by

$$\Pi^1=\begin{bmatrix}0.2 & 0.8\\0.35 & 0.65\end{bmatrix},\quad \Pi^2=\begin{bmatrix}0.55 & 0.45\\0.48 & 0.52\end{bmatrix}$$
$$\Pi^3=\begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\end{bmatrix},\quad \Pi^4=\begin{bmatrix}0.4 & 0.6\\0.9 & 0.1\end{bmatrix}.$$

Our purpose is to design a fuzzy L_2-L_∞ filter for system (3) such that the resulting filtering error system (6) is stochastically stable with a L_2-L_∞ attenuation performance index.

The state of the system is given as $x_k=\begin{bmatrix}x_{1k}\ x_{2k}\end{bmatrix}^{\mathrm{T}}$ and the membership functions are given by

$$h_1(x_{2k})=\frac{-x_{2k}^2+3}{6},\quad h_2(x_{2k})=\frac{x_{2k}^2+3}{6}.$$

Based on Theorem 4, set $\gamma=0.9$. We obtain the following filter matrices:

$$A_{f1}(1)=\begin{bmatrix}0.0318 & 0.0037\\-0.0049 & 0.0118\end{bmatrix}$$
$$A_{f1}(2)=\begin{bmatrix}-0.2660 & -0.1833\\0.1271 & -0.1212\end{bmatrix}$$
$$A_{f2}(1)=\begin{bmatrix}0.0327 & 0.0077\\-0.0120 & 0.0196\end{bmatrix}$$
$$A_{f2}(2)=\begin{bmatrix}-0.2429 & -0.0140\\0.0669 & -0.0070\end{bmatrix}$$
$$B_{f1}(1)=\begin{bmatrix}-0.0316\\-0.0668\end{bmatrix},\quad B_{f1}(2)=\begin{bmatrix}0.0013\\0.001\end{bmatrix}$$
$$B_{f2}(1)=\begin{bmatrix}-0.1009\\-0.1097\end{bmatrix},\quad B_{f2}(2)=\begin{bmatrix}0.0020\\0.0017\end{bmatrix}$$
$$L_{f1}(1)=\begin{bmatrix}0.1126 & -0.0753\end{bmatrix}$$
$$L_{f1}(2)=\begin{bmatrix}0.0391 & 0.1786\end{bmatrix}$$
$$L_{f2}(1)=\begin{bmatrix}0.0761 & -0.0745\end{bmatrix}$$
$$L_{f2}(2)=\begin{bmatrix}0.041 & 0.1926\end{bmatrix}.$$

TABLE I
L_2-L_∞ PERFORMANCE INDEX γ

case	mode-dependent	mode-independent
γ_{min}	0.36	1.32

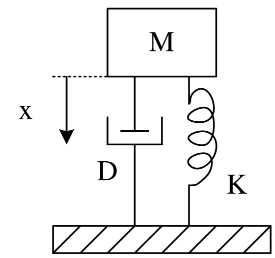

Fig. 1. Nonlinear spring-damper system.

Then, the fuzzy L_2-L_∞ filter for system (3) is obtained such that the resulting filtering error system (6) is stochastically stable and satisfies a L_2-L_∞ performance index.

Remark 5: By solving the optimization problem (25)–(27), one can obtain the optimal value of the L_2-L_∞ performance index. The mode-independent L_2-L_∞ performance index γ is also given in Table I, and it is obvious that mode-independent controller is more conservative.

Example 2: Next, we consider a nonlinear mass-spring-damper mechanical system [35] as shown in Fig. 1, where M is the mass, D and K are system parameters. The system model is represented as

$$x(k+2)=-0.1x^3(k+1)-0.02x(k)-0.67x^3(k)$$

where $x(k)\in\begin{bmatrix}-1.5\ 1.5\end{bmatrix}$, $x(k+1)\in\begin{bmatrix}-1.5\ 1.5\end{bmatrix}$.

With the T-S fuzzy model represents the nonlinear system, and considering abrupt uncertainties in the system, its jumping parameters are

$$A_1(1)=\begin{bmatrix}0 & -0.02\\1 & 0\end{bmatrix},\quad A_1(2)=\begin{bmatrix}-0.225 & -0.02\\1 & 0\end{bmatrix}$$
$$A_2(1)=\begin{bmatrix}0 & -1.5275\\1 & 0\end{bmatrix},\quad A_2(2)=\begin{bmatrix}-0.25 & -1.5275\\1 & 0\end{bmatrix}$$
$$B_1(1)=\begin{bmatrix}-0.15\\0.23\end{bmatrix},\quad B_1(2)=\begin{bmatrix}-0.05\\0.3\end{bmatrix}$$
$$B_2(1)=\begin{bmatrix}-0.01\\0.32\end{bmatrix},\quad B_2(2)=\begin{bmatrix}-0.05\\0.22\end{bmatrix}$$
$$C_1(1)=\begin{bmatrix}0.5 & -0.4\end{bmatrix},\quad C_1(2)=\begin{bmatrix}0.3 & -0.1\end{bmatrix}$$
$$C_2(1)=\begin{bmatrix}0.25 & -0.2\end{bmatrix},\quad C_2(2)\begin{bmatrix}0.15 & -0.3\end{bmatrix}$$
$$D_1(1)=-0.3,\quad D_2(1)=-0.2$$
$$D_1(2)=-0.2,\quad D_2(2)=-0.1$$
$$L_1(1)=\begin{bmatrix}0.3 & -0.2\end{bmatrix},\quad L_2(1)=\begin{bmatrix}0.2 & -0.2\end{bmatrix}$$
$$L_1(2)=\begin{bmatrix}0.1 & 0.5\end{bmatrix},\quad L_2(2)=\begin{bmatrix}0.1 & 0.5\end{bmatrix}$$
$$M_1(1)=M_2(1)=\begin{bmatrix}0.1\\0.1\end{bmatrix},\quad M_1(2)=M_2(2)=\begin{bmatrix}0.1\\0.1\end{bmatrix}$$
$$N_1(1)=N_2(1)=\begin{bmatrix}0.1 & 0.1\end{bmatrix}$$
$$N_1(2)=N_2(2)=\begin{bmatrix}0.1 & 0.1\end{bmatrix}.$$

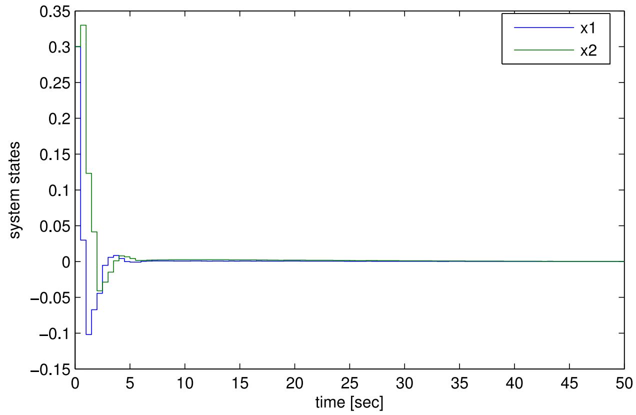

Fig. 2. Trajectory of system states.

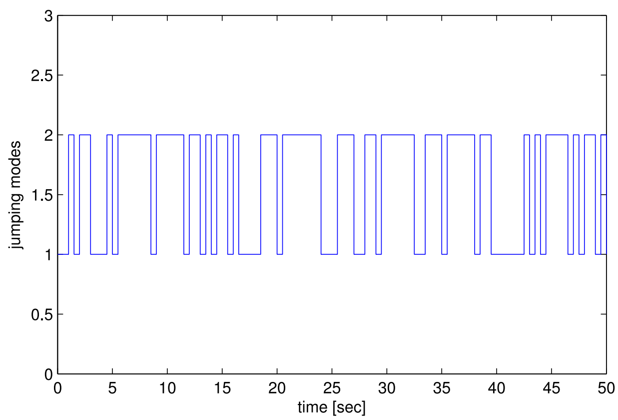

Fig. 3. System jumping modes.

The vertices of the time-varying transition probability matrix are given below

$$\Pi^1(k) = \begin{bmatrix} 0.3 & 0.7 \\ 0.35 & 0.65 \end{bmatrix}, \quad \Pi^2(k) = \begin{bmatrix} 0.55 & 0.45 \\ 0.48 & 0.52 \end{bmatrix}$$
$$\Pi^3(k) = \begin{bmatrix} 0.67 & 0.33 \\ 0.53 & 0.47 \end{bmatrix}, \quad \Pi^4(k) = \begin{bmatrix} 0.47 & 0.53 \\ 0.19 & 0.81 \end{bmatrix}.$$

Set $\gamma^2 = 0.5$, the initial condition of the system as $x_0 = \begin{bmatrix} 0.3 \ 0.3 \end{bmatrix}^{\mathrm{T}}$, the initial condition of the filter as $\begin{bmatrix} 0 \ 0 \end{bmatrix}^{\mathrm{T}}$, and the noise signal as $w_k = 0.5\mathtt{exp}(-0.1k)\mathtt{sin}(0.01\pi k)$, then, the state trajectories of system, jumping modes, filtering error response, and disturbance of the resulting filtering error system are shown in Figs. 2–5. This example shows that the designed filter is feasible and effective.

Remark 6: In [36], Gaussian distribution is used to describe uncertain transition probabilities. However, it was stated in [36] that such condition is difficult to be met/used in practice. In this paper, a new technique is proposed to improve such deficiency to make the theoretic results more practical. That is, the uncertain transition probabilities are described by a nonhomogeneous process, modeled as a polytope set. In addition, robust fuzzy $L_2 - L_\infty$ filtering is considered for a class of nonhomogeneous nonlinear MJSs, which has not been extensively studied in the past.

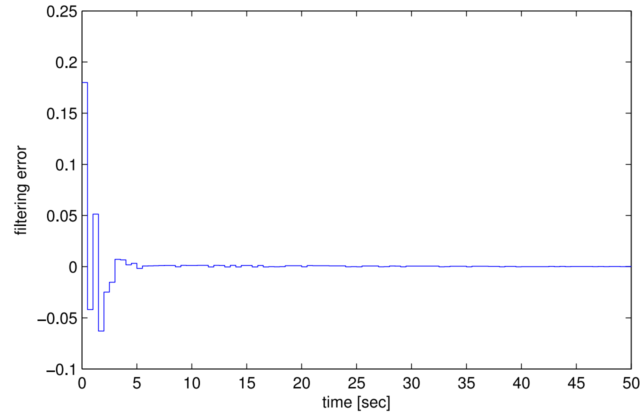

Fig. 4. Filtering error.

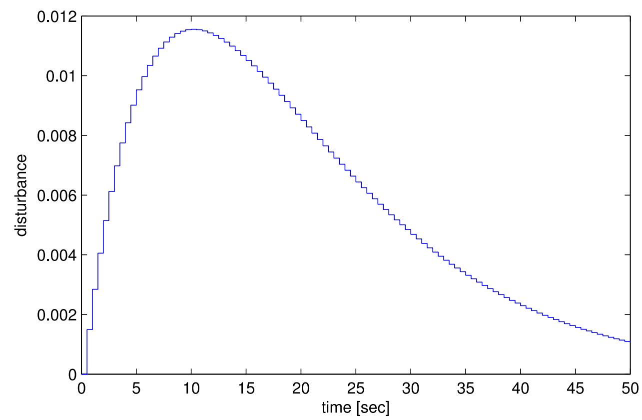

Fig. 5. Disturbance.

VI. Conclusion

In this paper, the robust fuzzy $L_2 - L_\infty$ filtering design problem for a class of uncertain discrete-time nonhomogeneous nonlinear MJSs is addressed. Its transition probabilities are expressed as a polytope, in which vertices are given *a priori*. The fuzzy filter design ensures that the resulting fuzzy filtering error dynamic system is stochastically stable and achieves a prescribed $L_2 - L_\infty$ performance index. The simulation results confirm the potential of the proposed techniques. As known in practical systems, actuator saturation is an unavoidable problem, so, in our future work, filter design for systems subject to saturation nonlinear will be considered and investigated.

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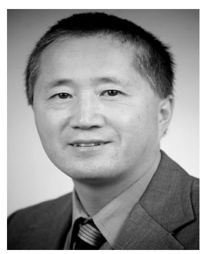

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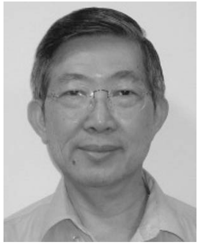

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