The Effect of Distribution for a Moving Force

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ABSTRACT

Many types of slender or thin walled structures experience forces which traverse across them. For example: vehicles passing over a bridge, overhead crane operations and liquid "slug" movement in spanning pipelines. This moving force can initiate a large dynamic stress within the structure and is often important for assessing structural fatigue. For many of these force/structure scenarios, modelling of the force as a concentrated point force would be an adequate simplifying approximation. In some cases, however, it may not be appropriate to simplify the distributed force into a single point force. For instance, slug lengths in pipelines can be significant in relation to span lengths. There is currently no guidance in the literature regarding the distribution effect of the force on the response of a structure under a moving force. This paper investigates the dynamic response of an elastic, simply supported beam under the influence of a moving distributed force, with varying distribution to span length ratios. In addition, it examines the speed of the traversing force, which is also highly influential on the dynamic response of the beam. This investigation is undertaken using an explicit transient dynamic finite element formulation of a simply supported beam. Guidelines are provided to discriminate between those scenarios when it is appropriate to simplify a distributed moving force as a concentrated force, and those when it must be modelled as the original distributed force.

INTRODUCTION

The majority of previous literature on the dynamic response of beam like structures subjected to a moving load, treat the load as a concentrated force (Yang,Yau & Wu, 2004)as shown schematically in Figure 1. The case of a distributed load, as shown schematically in Figure 2, has yet to be addressed in the current literature.

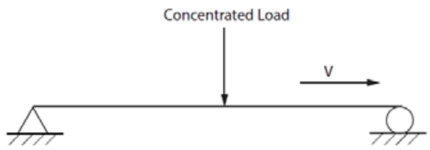

Figure 1: Simply Supported Beam Subjected to a Concentrated Load

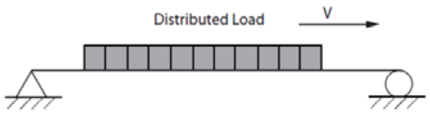

Figure 2: Simply Supported Beam Subjected to a Distributed Moving Load

(Wu, Whittaker & Cartmell, 2002) presented a finite element technique to investigate the dynamic response of structures under a concentrated moving load, where the inertia of the concentrated moving load is ignored.

(Rieker & Whittaker, 1999) studied the dynamic response of a simply supported beam under the passage of a distributed moving mass, where the inertia of the distributed load is considered. (Rieker & Whittaker, 1999) simplified the distributed moving mass to a sequence of equal concentrated loads moving at constant speed. The study concluded that simplifying the moving mass to a concentrated mass, instead of modelling the actual length of the moving mass, is a conservative approximation of the true response.

Recently (Reda, Forbes & Sultan, 2011) investigated the dynamic response of a simply supported beam further, considering the bending moment along the length of the beam span at various speed parameters and damping ratios under a concentrated moving force. This work was undertaking using combinations of analytical and finite elements approaches. Results presented by (Reda, Forbes & Sultan, 2011) indicated that the maximum bending moment does not necessarily occur at the mid-point of the simply supported beam. The location of the maximum bending moment depends on both the speed of the moving force and the damping within the structure. On the contrary, the maximum deflection always occurs at approximately the mid-point of the simply supported beam irrespective of the moving load speed or damping present.

The aim of this paper is to investigate the dynamic response of an elastically simply supported beam, subjected to a distributed moving load. The response is also compared with that due to a concentrated moving load to determine when it is appropriate to model a distributed load as a concentrated point force and whether this simplification provides a more conservative response form.

A finite element formulation is presented for Euler-Bernoulli beam elements subjected to a distributed moving load. The formulation uses a modified shape function to account for the presence of the distributed load as well as to calculate the equivalent nodal forces and moments.

An explicit integration scheme is used in the transient finite element formulation. The explicit integration formulation is an exact formulation in contrast to the implicit form. These two competing methods are broadly similar, with the main difference being the exact nature and conditional stability of the explicit form as opposed to the implicit formulation.

The finite element code within this paper was implemented in Matlab, as in spite of the sophistication of the commercial finite element codes such as ANSYS and ABAQUS, these commercial packages can require enormous effort to simulate either a moving force or moving mass problem.

The results presented are prepared by performing multiple runs of a finite element model, while varying the length of the distributed load and the speed of the traversing force. A total of six distributed moving length combinations and one concentrated load are run at speed ratios varying from 0.2 to 1 (of the critical speed). The results also showed that the assumption of modelling or simplifying the moving load as a concentrated force rather than its original distributed length is robust and provides a conservative approximation of the dynamic response.

A DISTRIBUTED LOAD MOVING ALONG A SIMPLY SUPPORTED BEAM

This section presents the formulations which are used to investigate the dynamic response of an elastically simply supported beam subjected to a moving distributed load. In the view of the nature of the distributed moving load, it is important to know the location of the front and the rear of the distributed moving load. This allows for the modelling of the following three possible scenarios:

Case-1: The distributed moving load enters the simply supported beam as illustrated in Figure 3 .

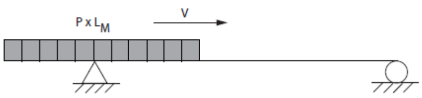

Figure 3: Arrival of the Distributed Moving Load to the Simply Supported Beam

Case-2a: the distributed load travels along the simply supported beam, as illustrated in Figure 4 (distributed load is shorter than span length).

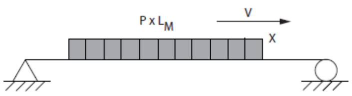

Figure 4: The Distributed Moving Load Travels along the Simply Supported Beam (Distributed load is shorter than span length)

Case-2b: The distributed load length is longer than the simply supported beam length, as illustrated in Figure 5.

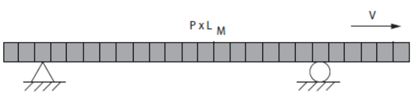

Figure 5: The Distributed Moving Load Travels along the Simply Supported Beam (Distributed load is longer than span length)

Case-3: The distributed load exits the simply supported beam as illustrated in Figure 6 .

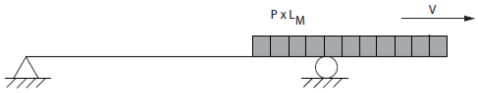

Figure 6: The Distributed Moving Load Exits the Simply Supported Beam

EQUATIONS OF MOTION

Before embarking on any structural vibration analysis, it is crucial to formulate the equation of motion. The equation of motion for a simply supported beam subject to a passage of continuous moving load can be written in the following form:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f_o(t)\} \quad (1)$$

Where:

$[M]$: Mass matrix of the simply supported beam element.

$[C]$: Damping matrix of the simply supported beam element

$[K]$: Stiffness matrix of the simply supported beam element

$\{f_o(t)\}$: External force vector.

$\{\ddot{x}\}$: Acceleration vector of the simply supported beam.

$\{\dot{x}\}$: Velocity vector of the simply supported beam.

$\{x\}$: Displacement vector of the simply supported beam.

SELECTING THE TIME STEP

When applying the explicit central difference method in any transient dynamic analysis, it is important to select the appropriate time step size on the basis of the shortest period which a wave within the FE mesh can travel. The shortest period also corresponds to the highest natural frequency in the system. The time step size should be small in order to guarantee that the time history of the force excitation is adequately captured. The time step size is selected using the following equation (Cook et al., 2002):

$$\Delta t \leq \frac{2}{\omega_{max}} \quad (2)$$

DEFINITION OF THE SHAPE FUNCTION

The nodal forces for any given load within a finite element model are given by (Cook et al., 2002):

$$f_o = P\int_a^b N.\, d\varepsilon \quad (3)$$

Where

N: Element shape function

P = Load vector

a-b :Limits of the load vector

ε: Non-dimensional element length

The nodal forces can be calculated by integrating the shape functions to account for the effect of distributed load.

Beam element cubic shape functions (Cook et al., 2002) for each element are used and defined as:

$$N_1(\varepsilon) = 1 - 3\varepsilon^2 + 2\varepsilon^3 \quad (4)$$

$$N_2(\varepsilon) = (\varepsilon - 2\varepsilon^2 + \varepsilon^3)L_E \quad (5)$$

$$N_3(\varepsilon) = 3\varepsilon^2 - 2\varepsilon^3 \quad (6)$$

$$N_4(\varepsilon) = (-\varepsilon^2 + \varepsilon^3)L_E \quad (7)$$

Where:

$$\varepsilon = \frac{x}{L_E} \quad (8)$$

x : Distance along the element to the point load application.

L_E : Element length.

The modified shape functions are calculated as follows (Cook et al., 2002):

$$N_{x_{modified}} = L_E \int N_x \, d\varepsilon \quad (9)$$

The vector of nodal forces equivalent to distributed load is obtained as:

$$f_o = P\int_a^b N_{x_{modified}} . d\varepsilon = P(N.\varepsilon(b) - N.\varepsilon(a)) \quad (10)$$

Integrating the shape function in regards to ε will result in the following:

$$N_{1\_mod}(\varepsilon) = (2\varepsilon - 2\varepsilon^3 + \varepsilon^4)\frac{L_E}{2} \quad (11)$$

$$N_{2\_mod}(\varepsilon) = (6\varepsilon^2 - 8\varepsilon^3 + 3\varepsilon^4)\frac{L_E^2}{12} \quad (12)$$

$$N_{3\_mod}(\varepsilon) = (2\varepsilon^3 - \varepsilon^4)\frac{L_E}{2} \quad (13)$$

$$N_{4\_mod}(\varepsilon) = (-4\varepsilon^3 + 3\varepsilon^4)\frac{L_E^2}{12} \quad (14)$$

DETERMINATION OF THE EQUIVALENT NODAL FORCES TO DISTRIBUTED LOAD

Knowing the location of the front and the rear edges of the moving load is quite important to facilitate the modelling of the distributed moving load, as stated earlier. At any given time, the position of the rear and front moving continuous load in relation to the left end of the beam are calculated by:

$$x_{p1}(t) = V * i\Delta t \quad (15)$$

$$x_{p2}(t) = V * i\Delta t - \text{distributed load length} \quad (16)$$

Where:

V: Continuous load speed.

i: Time step.

Δt: Time step size.

x_{p1}: Acting positions of the rear of the moving distributed load.

x_{p2}: Acting positions of the front of the moving distributed load.

The approach, adopted in this work to determine the location of the front and rear edges of the moving load, is similar to the way adopted by(Wu, Whittaker & Cartmell, 2000) to determine the location of the concentrated load in reference to the first node of the beam.

Letting S1 and S2 denote the element number that the continuous moving load is applied to at any given time for the rear and front of the distributed moving load, are given by (Wu, Whittaker & Cartmell, 2000):

$$S1 = \left(\text{Integer part of } \frac{x_{p1}(t)}{l}\right) + 1 \quad (17)$$

$$S2 = \left(\text{Integer part of } \frac{x_{p2}(t)}{l}\right) + 1 \quad (18)$$

It should be noted that the modified shape functions, equations 9-12, are expressed in terms of the local x coordinate. Therefore, it is important to modify ε in terms of the global $x_{p1}(t)$ and $x_{p2}(t)$. The modified ε_1 and ε_2 for the rear and the front respectively of the distributed moving load are given by (Wu, Whittaker & Cartmell, 2000):

$$\varepsilon_1 = \frac{x_{p1}(t) - (S1-1)L_E}{L_E} \quad (19)$$

$$\varepsilon_2 = \frac{x_{p2}(t) - (S2-1)L_E}{L_E} \quad (20)$$

$$\varepsilon_3 = 1 \quad (21)$$

Where ε_3 is an integer, which is used in the simulation when the distributed moving load is located between two nodes. i.e, the distributed moving load covers the entire element length.

ASSUMPTIONS

The following assumptions are made throughout this study:

The beam is of a constant cross section and constant unit mass per length.

The beam cross section is pipe.

The material of the beam is homogenous and isotropic.

The deflections are small compared to the cross-sectional dimensions.

No initial curvature exits.

The effect of transverse shear deformation is negligible.

The mass of the moving load is smaller than that of the mass of the beam.

No damping is used.

INPUT DATA

The input data used for the comparative study is shown in Table 1.

Table 1: Input data used for analysis

Parameter	Unit	Value
Span Length	m	50
Pipe Outside Diameter	mm	323.9
Wall Thickness	mm	12.7
Material Young's Modulus	GPa	205
Steel Density	kg/m^3	7850
Pipe Unit Weight	kg/m	97.47
Total Damping Ratio	--	0
Speed Parameter	--	0.2-1
Distributed Moving Load Length (ratio of span)	%	0/10/30/50/70 /90/200
Total distributed Moving Load Weight	N	2000

In order to illustrate the principles presented to determine the external forces vectors and bending moment vectors, consider a 50 m long simply supported beam with the properties presented in **Table 1** with 30 elements (31 nodes). The 30 elements are equally spaced along the simply supported beam.

In the first case, the simply supported beam is subjected to a concentrated load of -2000 N. In the second case, the simply supported beam is subjected to a distributed load of 15m length and with unit weight of (2000 N/15 m). In both cases, the loads travel with the same constant speed. It is obvious that the time required for the distributed load to travel from one end to the other end is greater than that of the concentrated load.

Figure 7 and Figure **8**highlight the force-time for three consecutive nodes for the concentrated and distributed loads respectively. The Y-axis represents the value of the load while the X-axis represents the total number of time steps.

Figure 7 illustrates the force-time graph for three consecutive nodes of the simply supported beam subjected to a concentrated load. It is clear that forces on each node are zero for all time steps except when the concentrated force traverses the respective elements either side of the node of interest.

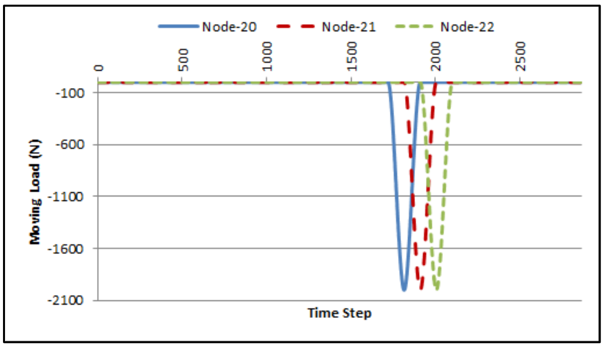

Figure 7: Force-Time Step Graph for Nodes-20/21/22 for the Concentrated Load

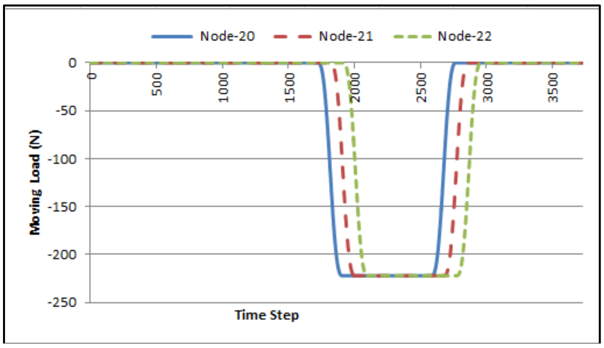

Figure 8**: Force-Time Step Graph for Nodes-20/21/22 for the Distributed Load**

Figure 8 illustrates the force-time graph for the same three consecutive nodes of the simply supported beam subjected to a distributed load. It is also clear that forces on each node are zero for all time steps except when the distributed force travels across the elements either side of the node of interest. It can be seen that the load is constant as the distributed load passes over the node of interest. It can be seen from comparing Figure 7 with Figure 8 that the model captures correctly the nature of the distributed moving load.

Figure 9 and Figure 10 highlight the bending moment time for the same consecutive three nodes for the concentrated load and distributed load respectively. The Y-axis represents the value of the bending moment while the X-axis once again represents the total number of time step.

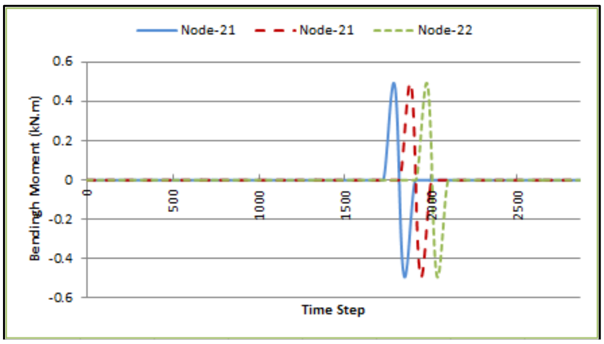

Figure 9: Bending Moment-Time Step Graph for Nodes-20/21/22 for the Concentrated Load

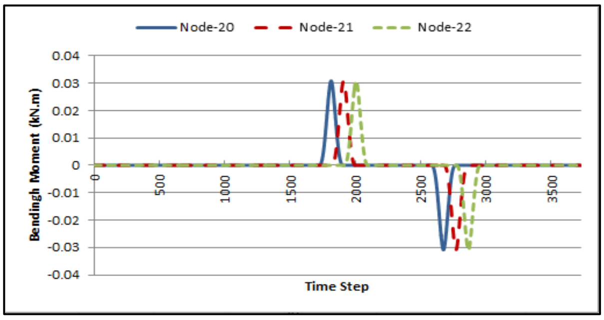

Figure 10: Bending Moment-Time Step Graph for Nodes-20/21/22 for the Distributed Load

DETERMINATION OF THE RESPONSE HISTORY USING DIRECT EXPLICIT METHOD

The central difference explicit method is used to determine the transient response history of displacement and bending moment. The central difference method expresses the velocity and the acceleration at time t_j in terms of displacement at times t_{j-1} , t_j and t_{j+1} The velocity and the acceleration are obtained by approximating the response curve by a quadratic polynomial within the time interval (t_{j-1} , t_{j+1}).

The following equations are used to determine the velocity and acceleration (Petyt, 2010):

$$\dot{d_t} = \frac{d_{i+1}-d_{i-1}}{2(\Delta t)} \quad (22)$$

$$\ddot{d_i} = \frac{\dot{d}_{i+1}-\dot{d}_{i-1}}{2(\Delta t)} \quad (23)$$

The response at the time t_{j+1} is then obtained by substituting equations 22 and 23 into equation 1 (equation of motion) evaluated at time t_j , which gives (Petyt, 2010)::

$$\frac{[M]}{(\Delta t)^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) + \frac{[C]}{2\Delta t}\left(u_{j+1} - u_{j-1}\right) + [K]u_j = \{f_o(j)\} \quad (24)$$

Solving for u_{j+1} gives the following (Petyt, 2010):

$$\left(\frac{[M]}{(\Delta t)^2} + \frac{[C]}{2\Delta t}\right) u_{j+1} = \{f_o(j)\} + \left(\frac{2[M]}{(\Delta t)^2} + [K]\right) u_j - \left(\frac{[M]}{(\Delta t)^2} - \frac{[C]}{2\Delta t}\right) u_{j-1} \quad (25)$$

It may be seen that the displacements u_{j+1} and can be determined provided that the displacements u_{j-1} and u_j are known. The time history of the displacements and bending moments are calculated by assuming j=1,2 ,3…….

RESULTS

Figure 11 and Figure 12 highlight the maximum deflection and the maximum bending moment, respectively, versus the speed parameter. The maximum deflection and the maximum bending moment presented in these two figures are the maximum values which occur along the simply supported beam as the moving load travels across it. (Reda, Forbes & Sultan 2011) showed that the maximum bending moment does not necessarily occur at the mid-point of the simply supported beam.

Figure 11 and Figure 12 show that the dynamic response of the concentrated load is very similar to the dynamic response of 10% (load/span length) distributed load. These figures show that the maximum deflection and the maximum bending moment decreases with the increase of the distributed moving load length. It is believed that the forced and free vibrations induce sinusoidal waves which will cancel out each other. In view of the deflection of the simply supported beam under the passage of the distributed moving load is caused by:

Forced Vibration: due to the distributed moving load acting on elements along the beam.

Free Vibration: due to elements that the moving distributed load have passed.

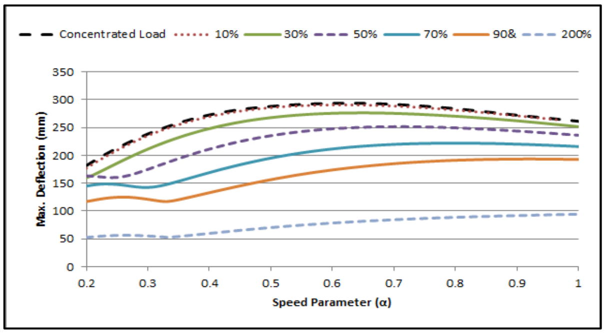

Figure 11: Maximum Deflection Vs. Speed Parameter

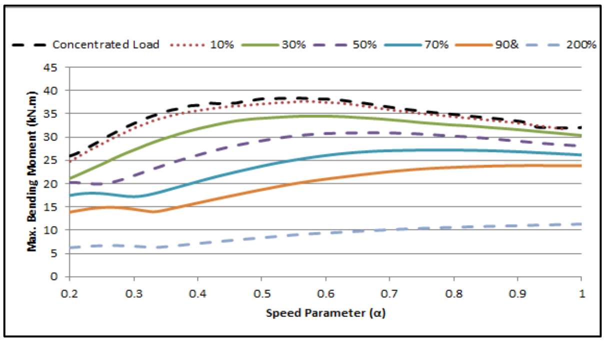

Figure 12: Maximum Bending Moment Vs. Speed Parameter

Figure 13 and Figure 14 illustrate the normalised deflection Max u(x, t)/u0 at various distributed load lengths at speed ratios of 1.0 and 0.6 respectively.

Figure 15 and Figure 16 illustrate the normalised bending moment Max M(x, t)/M0 at various distributed load lengths at speed ratios of 1.0 and 0.6 respectively.

The normalised deflection/bending moment describes the maximum dynamic deflection/ bending moment in relation to the static deflection/ bending moment that would be produced by a steady load, as the moving load travels across the beam.

The static deflection and static bending moment assume that the total load is located at the centre of the simply supported beam. In other words, the static deflection and static bending are calculated by mimicking the total distributed load at the centre of the beam. Figure 15 and Figure 16 show that the maximum bending moment does not necessarily occur at the mid-point of the beam. It is evident that the location of the maximum bending moment is much more sensitive to the speed parameter.

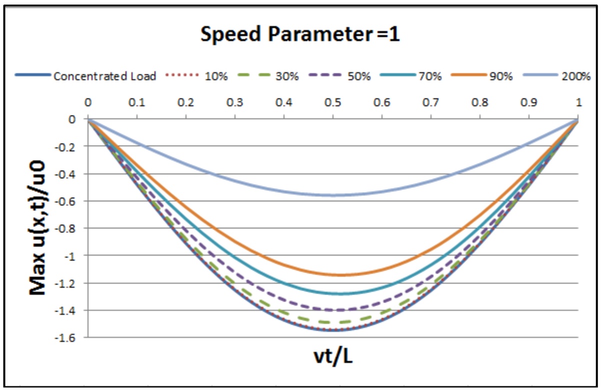

Figure 13: Normalised Deflection Max u(x,t)/u0 for Various Distributed Load Lengths at Speed Parameter of 1.0

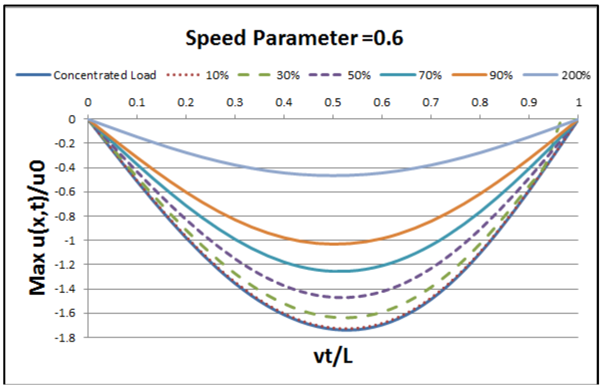

Figure 14: Normalised Deflection Max u(x,t)/u0 for Various Distributed Load Lengths at Speed Parameter of 0.6

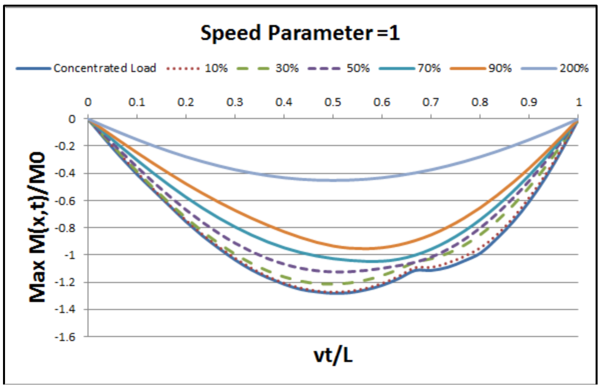

Figure 15: Normalised Bending Moment Max M(x,t)/M0 for Various Distributed Load Lengths at Speed Parameter of 1.0

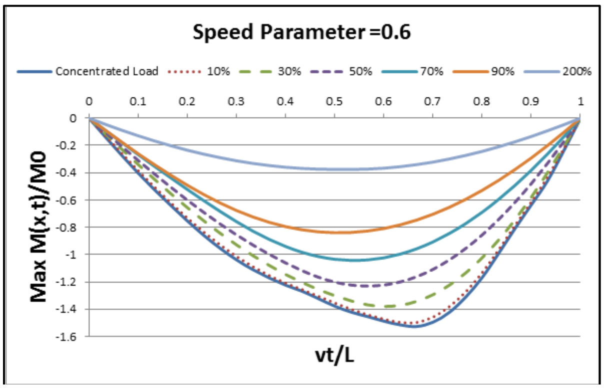

Figure 16: Normalised Bending Moment Max M(x,t)/M0 for Various Distributed Load Lengths at Speed Parameter of 0.6

SUMMARY & CONCLUSION

The dynamic response of an elastically supported beam subjected to a distributed moving load using a finite element model is presented in this paper. The technique applied to derive the shape functions, equivalent nodal forces and equivalent bending moments are only valid for a beam element, but in principle it is a general procedure and can be adopted for other element types.

A finite element code was written in order to determine the equivalent nodal forces and equivalent bending moments as a result of the distributed moving load travelling along the simply supported beam. The finite element code is used to investigate the dynamic response of the simply supported beam under either, a moving concentrated load or moving distributed load. This is in an effort to highlight the conservative approximation associated with modelling any moving load as a concentrated load rather than the actual length of the moving load.

The following conculsions are made by comparing the dynamic response of the beam subjected to either a concentrated load or distributed moving load:

The dynamic response of the beam subjected to a concentrated load represents the upper bound of the maximum deflections and the maximum bending moments at any given speed parameter. The dynamic response of the simply supported beam subjected to a concentrated moving load is shown to be very similar to the dynamic response of the simply supported beam subjected to a 10% distributed moving load. In other words, a beam subjected to small length ratio distributed loads behaves in the same manner to that of a concentrated load.

It is overly conservative to simplify any moving load into a concentrated load. However, this simplification is acceptable in the cases where the length of the moving load is unknown or difficult to predict. For instance, slug flow across unsupported pipeline spans.

The maximum bending moment does not necessarily occur at the mid-point of the simply supported beam. The location of the maximum bending moment might be an important design aspect for some areas. The nature of the distributed moving load leads to a suppression in the response compared to that of a concentrated moving force. In a view of the deflection of the simply supported beam under the passage of the distributed moving load is believed to be caused by:

Forced Vibration: due to the distributed moving load acting on elements along the beam.

Free Vibration: due to elements that the moving load have passed.

The forced and free vibrations induce sinusoidal waves which will cancel out each other.

RECOMMENDATIONS

The following recommendations are made for further work:

Investigate the behaviour of the beam subjected to a distributed moving load when the speed ratio is greater than 1.

Investigate the behaviour of a beam subjected to a distributed moving load under different damping ratios.

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APPENDIX A: DERIVATION OF NODAL FORCES

The simulation of the moving distributed load requires the application of the forces and moments to all the nodes of the beam. In an effort to investigate the dynamic response of the simply supported beam subjected to a passage of distributed moving load, and based on the nature of the distributed moving load, the distributed moving load will be discretised into the following three different cases to calculate the external forces and the bending moment vectors:

Case-1: Arrival of moving load on a simply supported beam

When the front of the continuous moving load covers part of element-1 of the beam, as shown in Figure 17, the external forces and the bending moments vectors are given by:

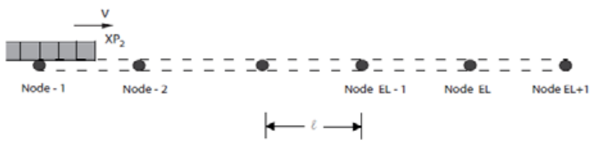

Figure 17- Arrival of the Distributed Load to the Simply Supported Beam (Covers Node-1).

Node-1:

$$F(1,i) = P\left(N_{1_{mod}}(\varepsilon_2) + N_{3_{mod}}(\varepsilon_3)\right)$$

$$M(1,i) = P\left(N_{2_{mod}}(\varepsilon_2) + N_{4_{mod}}(\varepsilon_3)\right)$$

Node-2:

$$F(2,i) = P\left(N_{3_{mod}}(\varepsilon_2) - N_{3_{mod}}(0)\right)$$

$$M(2,i) = P\left(N_{4_{mod}}(\varepsilon_2) - N_{4_{mod}}(0)\right)$$

Other nodes of the beam

$$F(j, i) = \text{zero}$$

$$M(j, i) = \text{zero}$$

When the front of the continuous moving load covers element-1 and part of element-2 of the beam, as shown in Figure 18, the external forces and the bending moment vectors are given by:

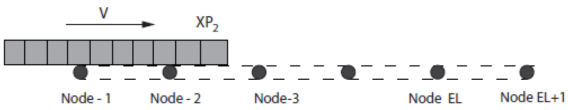

Figure 18- Arrival of the Distributed Load to the Simply Supported Beam (Covers of the 1st Element & Part of the 2nd Element).

Node-1:

$$F(1, i) = P\left(N_{1_{mod}}(\varepsilon_3) - N_{1_{mod}}(0)\right)$$

$$M(1, i) = P\left(N_{2_{mod}}(\varepsilon_3) - N_{2_{mod}}(0)\right)$$

Node-2:

$$F(2, i) = P\left(N_{1_{mod}}(\varepsilon_2) + N_{3_{mod}}(\varepsilon_3)\right)$$

$$M(2, i) = P\left(N_{2_{mod}}(\varepsilon_2) + N_{4_{mod}}(\varepsilon_3)\right)$$

Node-3:

$$F(3, i) = P\left(N_{3_{mod}}(\varepsilon_2) - N_{3_{mod}}(0)\right)$$

$$M(3, i) = P\left(N_{4_{mod}}(\varepsilon_2) - N_{4_{mod}}(0)\right)$$

Other nodes of the beam

$$F(j, i) = \text{zero}$$

$$M(j, i) = \text{zero}$$

Case-2 Departure of moving load from a simply supported beam

When the rear of the continuous moving load covers part of the last element of the beam, as shown in Figure 19 ,whilst the front of the distributed moving load left the structure, the external forces and the bending moments vectors are given by:

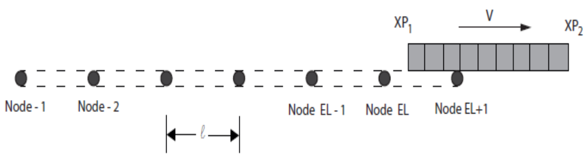

Figure 19: Departure of Moving Load from the Simply Supported Beam (Distributed Load Covers Part of the Last Element)

Node EL+1:

$$F(EL + 1, i) = P\left(N_{3_{mod}}(\varepsilon_3) - N_{3_{mod}}(\varepsilon_1)\right)$$

$$M(EL + 1, i) = P\left(N_{4_{mod}}(\varepsilon_3) - N_{4_{mod}}(\varepsilon_1)\right)$$

Node EL:

$$F(EL, i) = P\left(N_{1_{mod}}(\varepsilon_3) - N_{1_{mod}}(\varepsilon_1)\right)$$

$$M(EL, i) = P\left(N_2(\varepsilon_3) - N_{2_{mod}}(\varepsilon_1)\right)$$

Other nodes of the beam

$$F(j, i) = \text{zero}$$

$$M(j, i) = \text{zero}$$

When the rear of the continuous moving load covers the element between nodes EL and EL+1, covers the element between nodes EL and EL-1 as well as part of the element between nodes EL-1 and EL-2, as shown in Figure 20, whilst the front of the distributed moving load left the structure, the external forces and the bending moments vectors are given by:

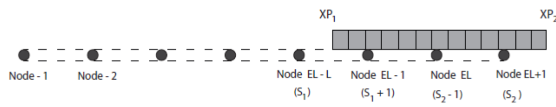

Figure 20: Departure of Moving Load from the Simply Supported Beam

Node EL+1 (S2):

$$F(EL + 1, i) = P\left(N_{3_{mod}}(\varepsilon_3) - N_{3_{mod}}(0)\right)$$

$$M(EL + 1, i) = P\left(N_{4_{mod}}(\varepsilon_3) - N_{4_{mod}}(0)\right)$$

Node EL (S2-1):

$$F(EL, i) = P\left(N_{1_{mod}}(\varepsilon_3) - N_{1_{mod}}(0)\right) + P\left(N_{3_{mod}}(\varepsilon_3) - N_{3_{mod}}(0)\right)$$

$$M(EL, i) = P\left(N_{2_{mod}}(\varepsilon_3) - N_{2_{mod}}(0)\right) + P\left(N_{4_{mod}}(\varepsilon_3) - N_{4_{mod}}(0)\right)$$

Node EL-1(S1+1):

$$F(EL - 1, i) = P\left(N_{3_{mod}}(\varepsilon_3) - N_{3_{mod}}(\varepsilon_1)\right) + P\left(N_{1_{mod}}(\varepsilon_3) - N_{1_{mod}}(0)\right)$$

$$M(EL - 1, i) = P\left(N_{4_{mod}}(\varepsilon_3) - N_{4_{mod}}(\varepsilon_1)\right) + P\left(N_{2_{mod}}(\varepsilon_3) - N_{2_{mod}}(0)\right)$$

Node EL-2 (S1):

$$F(EL-2,i) = P\left(N_{1_{mod}}(\varepsilon_3) - N_{1_{mod}}(\varepsilon_1)\right)$$

$$M(EL-2,i) = P\left(N_{2_{mod}}(\varepsilon_3) - N_{2_{mod}}(\varepsilon_1)\right)$$

Other nodes of the beam

$$F(j,i) = zero$$

$$M(j,i) = zero$$

Case-3 Steady Vibration

When the distributed moving load is located between three elements as shown in Figure 21, the external forces and bending moments vectors are given by:

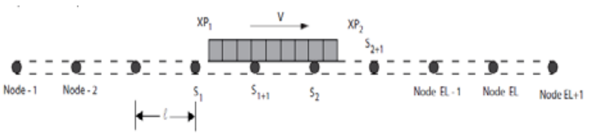

Figure 21: Distributed Load Travels along the Simply supported beam

Node-S1

$$F(S1,i) = P\left(N_{1_{mod}}(\varepsilon_3) - N_{1_{mod}}(\varepsilon_1)\right)$$

$$M(S1,i) = P\left(N_{2_{mod}}(\varepsilon_3) - N_{2_{mod}}(\varepsilon_1)\right)$$

Node-S1+1

$$F(S1+1,i) = P\left(N_{3_{mod}}(\varepsilon_3) - N_{3_{mod}}(\varepsilon_1)\right) + P\left(N_{1_{mod}}(\varepsilon_3) - N_{1_{mod}}(0)\right)$$

$$M(S1+1,i) = P\left(N_{4_{mod}}(\varepsilon_3) - N_{4_{mod}}(\varepsilon_1)\right) + P\left(N_{2_{mod}}(\varepsilon_3) - N_{2_{mod}}(0)\right)$$

Node-S2

$$F(S2,i) = P\left(N_{3_{mod}}(\varepsilon_3) - N_{3_{mod}}(0)\right) + P\left(N_{1_{mod}}(\varepsilon_2) - N_{1_{mod}}(0)\right)$$

$$M(S2,i) = P\left(N_{4_{mod}}(\varepsilon_3) - N_{4_{mod}}(0)\right) + P\left(N_{2_{mod}}(\varepsilon_2) - N_{2_{mod}}(0)\right)$$

Node-S2+1

$$F(S2+1,i) = P\left(N_{3_{mod}}(\varepsilon_2) - N_{3_{mod}}(0)\right)$$

$$M(S2+1,i) = P\left(N_{4_{mod}}(\varepsilon_2) - N_{4_{mod}}(0)\right)$$

Other nodes of the beam

$$F(j,i) = zero$$

$$M(j,i) = zero$$

The external forces and bending moments vectors at each time are determined for all the nodes of the subject beam based on the three cases presented above.