Stability analysis and design of impulsive control systems*

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Abstract

This paper investigates the problem of impulsive feedback control of nonlinear systems. A set of impulsive stabilization criteria are established by employing the method of Lyapunov functions. These results are then applied to the control of the Lorenz chaotic system. Compared to the conventional control method in the existing literature, the impulsive controller design in this paper is very simple. Numerical experiments are carried out for the control of the Lorenz system. It is shown that small and frequent impulses need to be used in order to stabilize the Lorenz system.

Key words : Impulsive feedback control, stability, Lyapunov function, Lorenz system, impulsive stabilization.

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1. Introduction

Recently, impulsive control has attracted the interest of many researchers, see [1, 2, 7, 8, 10-14, 15, 18, 19] and relevant references therein. Such control arises naturally in a wide variety of applications, such as orbital transfer of satellite [13, 15], ecosystems management [11], dynamic portfolio management. As a representative example in dynamic portfolio management, we note that the dynamical behavior of the total stock value of a particular investor can be described by an impulsive control system. More specifically, we see that when a certain amount of stock is purchased or sold, the total stock value changes instantaneously to a new value. This situation is exactly like an impulsive control being applied to the dynamical system. The timings of the purchasing and selling of stock as well as the amount of the stock involved in each transaction can all be considered as decision variables to be determined by the investor. This is clearly an impulsive conrol problem.

In this paper, we consider a general impulsive control problem. Since stability is a fundamental issue in control system analysis and design, we shall use the method of Lyapunov functions to establish a set of impulsive stabilization criteria for the impulsive control system concerned. These results are then applied to a chaotic system, the well-known Lorenz system. It is shown that by using impulsive feedback control, all the solutions of the Lorenz system will converge to an equilibrium point. The rest of the paper is organized as follows. In Section 2, we formulate the problem of impulsive control and introduce some notions and definitions. We then establish, in Section 3, several stability criteria for impulsive differential systems. These criteria are then used in Section 4 for the designs of impulse feedback control. Several impulsive stabilization criteria are derived. As an application, impulsive control of the Lorenz system is discussed in Section 5. Finally, numerical experiments are carried out for the impulsive control of Lorenz system.

2. Problem Formulation

Consider the nonlinear system

$$\begin{cases} x' = f(t,x) \\ y = \varphi(x) \end{cases} \tag{1}$$

where $x \in R^n$ is the state variable, $y \in R^m$ is the output variable, $f(t,x)$ and $\varphi(x)$ are continuous functions in their respective domains of definition. An impulsive control law of (2.1) is given by a sequence $\{\tau_k, u_k(y(\tau_k))\}$, where

$$0 < \tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} < \cdots, \quad \tau_k \to \infty \text{ as } k \to \infty$$

and $u_k(y)$ is a continuous function which maps R^m to R^n for all $k = 1, 2, \cdots$. It works as follows. Let $x(t) = x(t, t_0, x_0)$ be a solution of system (2.1) starting at (t_0, x_0). The point $P_t(t, x(t))$ begins its motion from the initial point $P_{t_0} = (t_0, x_0)$ and moves along the curve $[(t, x); t \geq t_0, x = x(t)]$ until the time $\tau_1 > t_0$ at which the point $P_{\tau_1}(\tau_1, x(\tau_1))$ is transferred immediately to $P_{\tau_1^+} = (\tau_1, x_1^+)$, where $x_1^+ = x(\tau_1) + u_1(y(\tau_1)$. Then the point P_t continues to move further along the curve with $x(t) = x(t, \tau_1, x_1^+)$ until it triggers a second transfer at $\tau_2 > \tau_1$. Once again, the point $P_{\tau_2} = (\tau_2, x(\tau_2))$ is mapped into the point $P_{\tau_2^+} = (\tau_2, x_2^+)$, where $x_2^+ = x(\tau_2) + u_2(y(\tau_2))$. As before, the point P_t continues to move forward with $x(t) = x(t, \tau_2, x_2^+)$ as the solution of (2.1) starting at (τ_2, x_2^+). Clearly, this process continues as long as the solution of (2.1) exists and it results in a piecewise continuous trajectory $x(t)$ which satisfies the following relations.

$$\begin{cases} x' = f(t, x), & t \neq \tau_k \\ y = \varphi(x), & z \neq \tau_k \\ \triangle x = u_k(y), & t = \tau_k \\ x(t_0) = x_0, & k = 1, 2, \cdots, \end{cases} \tag{2}$$

where $\triangle x(\tau_k) = x(\tau_k^+) - x(\tau_k^-), x(\tau_k^+) = \lim_{t \to \tau_k^+} x(t)$ and $x(\tau_k^-) = \lim_{t \to \tau_k^-} x(t)$. We call (2.2) an impulsive differential system. For the basic concepts and theorems of such systems, we refer the reader to [3]. Without loss of generality, we assume $f(t, 0) \equiv 0$ and $\varphi(0) = 0$ so that system (2.1) admits a trivial solution. Our impulsive control problem may now be stated formally as follows.

Problem (P)

Subject to the dynamical system (2.1), find an impulsive control law $\{\tau_k, u_k(y_{(\tau_k)})\}$ such that the impulsive differential system (2.2) is

(i) stable, i.e., $\forall\ \varepsilon > 0$, $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t)\| < \varepsilon$, $t > t_0$, where $x(t) = x(t, t_0, x_0)$ satisfies system (2.2); or

(ii) uniformly stable, i.e. δ in (i) is independent of t_0; or

(iii) asymptotically stable, i.e. system (2.2) is stable and $\forall t_0 \in R_+$, there exists a $\sigma(t_0) > 0$ such that $\|x_0\| < \sigma$ implies $\lim_{t\to\infty} x(t) = 0$; or

(iv) uniformly asymptotically stable, i.e., system (2.2) is uniformly stable and there exists a $\sigma > 0$, for any $\eta > 0$ and $t_0 \in R_+$, we can find a $T = T(\eta) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t)\| < \eta$, $t \geq t_0 + T$.

Definition 2.1 System (2.1) is said to be impulsively

(C1) stablizable if (i) holds;

(C2) uniformly stabilizable if (ii) holds;

(C3) asymptotically stabilizable if (iii) holds;

(C4) uniformly asymptotically stabilizable if (iv) holds.

We denote by K the class of continuous functions φ: $R_+ \to R_+$ such that $\varphi(s)$ is strictly increasing and $\varphi(0) = 0$; $\sum$ the class of functions $V : R_+ \times R^n \to R_+$ such that $V(t,x)$ is continuous everywhere except possibly at a sequence of points $\{\tau_k\}$ at which $V(t,x)$ is left continuous and the right limit $V(\tau_k^+, x)$ exists for all $x \in R^n$. For $\rho > 0$, let $s(\rho) = \{x \in R^n;$ $\|x\| < \rho\}$. We need the following definition.

Definition 2.2 A function $V(t,x)$ is said to be

(i) decrescent if there exists a function $a \in K$ such that

$$V(t,x) \leq a(\|x\|), \quad (t,x) \in R_+ \times s(\rho); \tag{3}$$

(ii) positive definite if there exists a function $b \in K$ such that

$$b(\|x\|) \leq V(t,x), \quad (t,x) \in R_+ \times s(\rho), \tag{4}$$

$$\text{and} \qquad V(t,0) \equiv 0, \quad t \in R_+.$$

We define the upper right-hand generalized derivative of a function $V(t,x)$ along solutions of (2.1) by

$$D^+V(t,x) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta} \left[V(t+\delta, x + \delta f(t,x) - V(t,x)\right] \tag{5}$$

If $V(t,x)$ is continuously differentiable, then (2.5) reduces to

$$D^+V(t,x) = \frac{\partial}{\partial t}V(t,x) + \frac{\partial}{\partial x}V(t,x) \cdot f(t,x) \tag{6}$$

Let $x(t)$ be a solution of (2.1) defined on some interval I and define $m(t) = V(t, x(t))$, $t \in I$. Then the upper right-hand Dini derivative of $m(t)$ is defined by

$$D^+m(t) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta}\left[m(t+\delta) - m(t)\right] \tag{7}$$

If $V(t,x)$ is locally Lipschitzian in x, then it can be shown [4] that

$$D^+m(t) = D^+V(t, x(t))$$

It can also be shown [4] that a continuous function $m(t)$ is nondecreasing if and only if $D^+m(t) \geq 0$.

3. Stability Criteria

In this section, we shall establish some stability criteria for system (2.2). Let $I_k(x) = u_k(\varphi(x))$. Then, system (2.2) can be rewritten as:

$$\begin{cases} x' = f(t,x), & t \neq \tau_k \\ \Delta x = I_k(x), & t = \tau_k \\ x(t_0) = x_0, & k = 1, 2, \cdots \end{cases} \tag{8}$$

Let us first establish a theorem on stability.

Theorem 3.1 Assume that

(i) $V \in \sum$, $V(t,x)$ is locally Lipschitzian in x, positive definite;

(ii) there exist $\mu_k \in R$ and $c_k \in K$ such that

$$D^+V(t,x) \leq \frac{\mu_k}{\Delta \tau_k} c_k(V(t,x)), \quad (t,x) \in (\tau_{k-1}, \tau_k) \times s(\rho);$$

(iii) there exist $v_k \in R$ and $d_k \in K$ such that

$$V(\tau_k^+,\ x + I_k(x)) \leq V(\tau_k, x) + v_k d_k(V(\tau_k, x)), \quad x \in s(\rho);$$

(iv) $\mu_k + v_k \leq 0$, for $s \in (0, \rho)$, $c_k(s) \leq d_k(s)$ if $v_k < 0$ and $d_k(s) \leq c_k(s)$ if $u_k < 0$;

(v) there exists a $\rho_1 \in (0, \rho)$ such that $x \in s(\rho_1)$ implies $x + I_k(x) \in s(\rho)$.

Then system (3.1) is stable.

Proof. Since $V(t, x)$ is positive definite, there exists a function $b \in K$ such that

$$V(t, x) \geq b(\|x\|), \quad (t, x) \in R_+ \times s(\rho). \tag{9}$$

Let $\varepsilon \in (0, \rho_1)$ be given. Choose $\alpha_1 = b(\varepsilon)e^{-1}$. By conditions (iii) and (iv) we get, for $(\tau_k, x) \in R_+ \times s(\rho)$,

$$-V(\tau_k, x) \leq v_k d_k(V(\tau_k, x)) \leq -\mu_k d_k(V(\tau_k, x)) \tag{10}$$

which implies

$$\int_{\alpha_1}^{b(\varepsilon)} \frac{ds}{d_k(s)} \geq \int_{\alpha_1}^{b(\varepsilon)} \frac{ds}{s} = \mu_k, \quad k = 1, 2, \cdots \tag{11}$$

Similarly, for $\alpha_2 = \alpha_1 e^{-1}$, we get

$$\int_{\alpha_2}^{\alpha_1} \frac{ds}{d_k(s)} \geq \mu_k, \quad k = 1, 2, \cdots \tag{12}$$

Since $\alpha_1 \in K$, there exists a $\alpha_3 = \alpha_3(\varepsilon) > 0$ such that

$$\alpha_3 + |v_1| \quad \alpha_{11}(\alpha_3) < \alpha_1. \tag{13}$$

Choose $\alpha = \min\{\alpha_2, \alpha_3\}$. Then, there exists a $\delta = \delta(\varepsilon, t_0) > 0$, $\delta \in (0, \varepsilon)$ such that $\|x_0\| < \delta$ implies

$$V(t_0, x_0) < \alpha \tag{14}$$

Let $x(t) = x(t, t_0, x_0)$ be a solution of (3.1) with $\|x_0\| < \delta$. We are going to show that $\|x(t)\| < \varepsilon$ for all $t \geq t_0$. Suppose for the sake of contradiction that there exists a $t_1 > t_0$ such that

$$\left\|x(t_1^+)\right\| \geq \varepsilon \quad \text{and} \quad \|x(t)\| < \varepsilon, \quad t \in [t_0, t_1) \tag{15}$$

By the choice of ε, we see that $x(t_1) \in s(\rho_1)$, and hence, by condition (v), $x(t_1^+) \in s(\rho)$. Using (3.2), we get

$$V(t_1^+, x(t_1^+)) \geq b(\left\|x(t_1^+)\right\|) \geq b(\varepsilon) \tag{16}$$

There are two cases to consider.

Case 1 $\quad t_1 \in (t_0, \tau_1)$. $V(t_1^+, x(t_1^+)) = V(t_1, x(t_1))$ if $\mu_1 \leq 0$, then by condition (ii), $V(t, x(t))$ is nonincreasing in $[t_0, t_1]$ and hence

$$V(t_1, x(t_1)) \leq V(t_0, x_0) < \alpha$$

which contradicts (3.9). If $\mu_1 > 0$, then by condition (iv), $v_1 < 0$. Thus, by conditions (ii) and (iv), we get

$$\int_{\alpha}^{V(t_1,x(t_1))} \frac{ds}{d_1(s)} \leq \int_{V(t_0,x_0)}^{V(t_1,x(t_1))} \frac{ds}{c_1(s)} \leq u_1 \frac{t_1 - t_0}{\Delta\tau_1} \leq u_1$$

which implies, in view of (3.5), that

$$V(t_1, x(t_1)) < \alpha_1 < b(\varepsilon).$$

This again contradicts (3.9).

Case 2 $\quad t_1 \in [\tau_m, \tau_{m+1}), \quad m \geq 1$.

If $u_1 \leq 0$, then condition (ii) implies that $V(t, x(t))$ is nonincreasing in $[t_0, \tau_1]$, and hence

$$V(\tau_1, x(\tau_1)) \leq V(t_0, x_0) < \alpha \tag{17}$$

This, together with condition (iii) and (3.6), implies that

$$\begin{aligned} V(\tau_1^+, x(\tau_1^+)) &\leq V(\tau_1, x(\tau_1)) + v_1 d_1(V(\tau_1, x(\tau_1)) \\ &\leq \alpha + |v_1|\, d_1(\alpha) < \alpha_1 \end{aligned} \tag{18}$$

If $\mu_1 > 0$, then $v_1 < 0$, and hence

$$V(\tau_1^+, x(\tau_1^+)) \leq V(\tau_1, x(\tau_1)) \tag{19}$$

By conditions (ii) and (iv), we get

$$\int_{\alpha}^{V(\tau_1,x(\tau_1))} \frac{ds}{d_1(s)} \leq \int_{V(t_0,x_0)}^{V(\tau_1,x(\tau_1))} \frac{ds}{c_1(s)} \leq \mu_1 \frac{\tau_1 - t_0}{\Delta\tau_1} = \mu_1$$

which implies, in view of (3.5), that

$$V(\tau_1, x(\tau_1)) < \alpha_1.$$

Thus, by (3.12),

$$V(\tau_1^+, x(\tau_1^+)) < \alpha_1. \tag{20}$$

If $m > 1$, then, for $1 < k \le m$, we have

$$\int_{V(\tau_{k-1}^+, x(\tau_{k-1}^+))}^{V(\tau_k^+, x(\tau_k^+))} \frac{ds}{c_k(s)} = \int_{V(\tau_{k-1}^+, x(\tau_{k-1}^+))}^{V(\tau_k, x(\tau_k))} \frac{ds}{c_k(s)} + \int_{V(\tau_k, x(\tau_k))}^{V(\tau_k^+, x(\tau_k^+))} \frac{ds}{c_k(s)}.$$

By condition (ii), we have

$$\int_{V(\tau_{k-1}^+, x(\tau_{k-1}^+))}^{V(\tau_k, x(\tau_k))} \frac{ds}{c_k(s)} \le \int_{\Delta\tau_k}^{\tau_k} ds = \mu_k. \tag{21}$$

Since $c_k \in K$, we get

$$\begin{aligned} \int_{V(\tau_k, x(\tau_k))}^{V(\tau_k^+, x(\tau_k^+))} \frac{ds}{c_k(s)} &\le \int_{V(\tau_k, x(\tau_k))}^{V(\tau_k^+, x(\tau_k^+))} \frac{ds}{c_k(V(\tau_k, x(\tau_k))} \\ &\le \frac{v_k d_k(V(\tau_k, x(\tau_k)))}{c_k(V(\tau_k, x(\tau_k)))} \le v_k. \end{aligned} \tag{22}$$

Thus, it follows from (3.14) and (3.15) and condition (iv) that

$$\int_{V(\tau_{k-1}^+, x(\tau_{k-1}^+))}^{V(\tau_k^+, x(\tau_k^+))} \frac{ds}{c_k(s)} \le \mu_k + v_k \le 0$$

This implies that

$$V(\tau_k^+, x(\tau_k^+)) \le V(\tau_{k-1}^+, x(\tau_{k-1}^+))$$

and consequently

$$V(\tau_m^+, x(\tau_m^+)) \le V(\tau_1^+, x(\tau_1^+)) < \alpha_1 \tag{23}$$

Now by the same argument as used in Case 1 will lead to a contradiction. Thus, $\|x(t)\| < \varepsilon$ for all $t \ge t_0$ and hence system (3.1) is stable, completing the proof.

By strengthening the conditions on $V(t,x)$ and $d_k(s)$, we get uniform stability which is the next result.

Theorem 3.2 In addition to all conditions of Theorem 3.1, suppose further that

(vi) $V(t,x)$ is decrescent;

(vii) for any $\eta > 0$ there exists a $\sigma > 0$ such that

$$s + |v_k|\, d_k(s) < \eta, \quad \forall s \in (0, \sigma), \quad \forall k = 1, 2, \cdots .$$

Then system (3.1) is uniformly stable.

Proof. Since $V(t,x)$ is decrescent, there exists a function $a \in K$ such that

$$V(t,x) \le a(\|x\|), \quad (t,x) \in R_+ \times s(\rho). \tag{24}$$

Let $\varepsilon \in (0, \rho_1)$ be given. Choose α_1, $\alpha_2 > 0$ as in the proof of Theorem 3.1. By condition (vii), there exists a $\alpha_3 > 0$ such that

$$\alpha_3 + |v_k|\, d_k(\alpha_3) < \alpha_1, \quad k = 1, 2, \cdots \tag{25}$$

Choose $\alpha = \min\{\alpha_2, \alpha_3\}$. Then, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$a(\delta) < \min\{\alpha, b(\varepsilon)\} \tag{26}$$

Let $x(t) = x(t, t_0, x_0)$ be a solution of (3.1) with $\|x_0\| < \delta$. The rest of the proof remains the same as that of Theorem 3.1. Since δ is chosen to be independent of t_0, it follows that system (3.1) is uniformly stable.

By strengthening condition (iv) of Theorem 3.1, we get a result on asymptotic stability.

Theorem 3.3 Assume that all conditions of Theorem 3.1 hold. Suppose further that for any $\beta > 0$ the series

$$\sum_{k=1}^{\infty} (\mu_k + v_k) e_k(\beta) = -\infty$$

where $e_k(s) = \max\{c_k(s), d_k(s)\}$. Then system (3.1) is asymptotically stable.

Proof. Stability follows from Theorem 3.1. Thus, there exists a $\delta_0 = \delta_0(t_0, \rho_1) > 0$ such that $\|x_0\| < \delta_0$ implies

$$\|x(t)\| < \rho_1, \quad t \ge t_0, \tag{27}$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (3.1).

It remains to show that $\lim_{t\to\infty} x(t) = 0$. As in the proof of Theorem 3.1, we get

$$\int_{V(\tau_{k-1}^+,x(\tau_{k-1}^+))}^{V(\tau_k^+,x(\tau_k^+))} \frac{ds}{c_k(s)} \leq \mu_k + v_k \leq 0, \quad k = 1, 2, \cdots, \tag{28}$$

which implies that $V(\tau_k^+, x(\tau_k^+))$ is nonincreasing and

$$\int_{V(\tau_{k-1}^+,x(\tau_{k-1}^+))}^{V(\tau_k^+,x(\tau_k^+))} \frac{ds}{c_k(s)} \geq \left[V(\tau_k^+, x(\tau_k^+)) - V(\tau_{k-1}^+, x(\tau_{k-1}^+))\right] \frac{1}{c_k(V(\tau_k^+, x(\tau_k^+)))}.$$

This, together with (3.21), yields

$$V(\tau_k^+, x(\tau_k^+)) \leq V(\tau_{k-1}^+, x(\tau_{k-1}^+)) + (\mu_k + v_k)c_k(V(\tau_k^+, x(\tau_k^+))). \tag{29}$$

From conditions (ii) and (iv), we get

$$\frac{D^+V(t,x)}{d_k(V(t,x))} \leq \frac{\mu_k c_k(V(t,x))}{\Delta\tau_k d_k(V(t,x))} \leq \frac{\mu_k}{\Delta\tau_k}.$$

Thus,

$$\begin{aligned} \int_{V(\tau_{k-1}^+,x(\tau_{k-1}^+))}^{V(\tau_k^+,x(\tau_k^+))} \frac{ds}{d_k(s)} &= \int_{V(\tau_{k-1}^+,x(\tau_{k-1}^+))}^{V(\tau_k,x(\tau_k))} \frac{ds}{d_k(s)} + \int_{V(\tau_k,x(\tau_k))}^{V(\tau_k^+,x(\tau_k^+))} \frac{ds}{d_k(s)} \\ \leq \mu_k + \frac{v_k d_k(V(\tau_k, x(\tau_k)))}{d_k(V(\tau_k, x(\tau_k)))} &= \mu_k + v_k \leq 0 \end{aligned}$$

which implies

$$V(\tau_k^+, x(\tau_k^+)) \leq V(\tau_{k-1}^+, x(\tau_{k-1}^+)) + (\mu_k + v_k)d_k(V(\tau_k^+, x(\tau_k))). \tag{30}$$

Combining (3.22) and (3.23) gives

$$V(\tau_k^+, x(\tau_k^+)) \leq V(\tau_{k-1}^+, x(\tau_{k-1}^+)) + (\mu_k + v_k)e_k(V(\tau_k^+, x(\tau_k^+))). \tag{31}$$

Let $\lim_{k\to\infty} V(\tau_k^+, x(\tau_k^+)) = c$. If $c > 0$, then there exists a $\beta > 0$ such that $V(\tau_k^+, x(\tau_k^+)) \geq \beta$ for k sufficiently large. This implies that, for some $m > 0$,

$$\lim_{k\to\infty} V(\tau_k^+, x(\tau_k^+)) \leq V(\tau_m^+, x(\tau_m^+)) - \sum_{j=1}^{\infty}(u_{m+j} + v_{m+j})e_k(\beta) = -\infty,$$

which is a contradiction. Thus, $c = 0$.

Let $\eta > 0$ and $\sigma = \eta e^{-1}$. Then, by (3.3), we get

$$\int_{\sigma}^{\eta} \frac{ds}{d_k(s)} \geq \int_{\sigma}^{\eta} \frac{ds}{s} = \mu_k, \quad k = 1, 2, \cdots \tag{32}$$

Since $\lim_{k\to\infty} V(\tau_k^+, x(\tau_k^+)) = 0$, there exists a $M > 0$ such that

$$V(\tau_k^+, x(\tau_k^+)) < \sigma, \quad \forall k \geq M. \tag{33}$$

We are going to show that

$$V(t, x(t)) \leq \eta, \quad \forall t \geq \tau_M. \tag{34}$$

For $t \geq \tau_M$, we assume $t \in (\tau_k, \tau_{b+1}]$ for some $k \geq M$. If $\mu_{k+1} \leq 0$, then $V(t, x(t))$ is nonincreasing in $(\tau_k,\ \tau_{k+1}]$, and hence, $V(t, x(t)) \leq V(\tau_k^+, x(\tau_k^+)) < \eta$, i.e. (3.27) is true. If $\mu_{k+1} > 0$, then $v_{k+1} < 0$. Thus,

$$\int_{\sigma}^{V(\tau_{k+1}, x(\tau_{k+1}))} \frac{ds}{d_k(s)} \leq \int_{V(\tau_k^+, x(\tau_k^+))}^{V(\tau_{k+1}, x(\tau_{k+1}))} \frac{ds}{c_{k+1}(s)} \leq u_{k+1}$$

which implies, in view of (3.25), $V(\tau_{k+1}, x(\tau_{k+1})) \leq \eta$. But $V(t, x(t))$ is non-increasing in $(\tau_k, \tau_{k+1}]$. Hence,

$$V(t, x(t)) \leq V(\tau_{k+1}, x(\tau_{k+1})) \leq \eta.$$

Thus, we have shown that (3.27) is true. This implies that $\lim_{t\to\infty} V(t, x(t)) = 0$. Since $V(t, x(t))$ is positive definite, it follows that $\lim_{t\to\infty} x(t) = 0$. Thus, system (3.1) is asymptotically stable and the proof is complete.

Finally, we prove a result on uniform asymptotic stability.

Theorem 3.4 Assume that all conditions of Theorem 3.2 hold. Suppose further that the sequence $\{\Delta\tau_k\}$ is bounded and for any β, $c > 0$ there exists positive integer N such that

$$\sum_{k=q+1}^{q+N} (\mu_k + v_k)\ell_k(\beta) < -C, \quad \forall q \geq 0 \tag{35}$$

where

$$\ell_k(s) = \max\{c_k(s), d_k(s)\}.$$

Then, system (3.1) is uniformly asymptotically stable.

Proof. Uniform stability follows from Theorem 3.2. Thus, there exist a $\delta_0 = \delta_0(\rho^*)$ and $\ell^* = \min\{\rho_1, b^{-1}(\rho_1)\}$, such that

$$\|x_0\| < \delta_0 \text{ implies } \|x(t)\| < \rho^*, \quad t \geq t_0,$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (3.1).

For any $\varepsilon \in (0, \rho^*)$, define $\delta = \delta(\varepsilon)$ as in the definition of uniform stability. Clearly, we can choose δ_0 and δ so that

$$0 < \delta \leq \delta_0 \leq \rho^* \leq b^{-1}(\rho_1) \text{ and } b(\delta) < \rho_1.$$

By (3.28), we can choose a positive integer N such that

$$\sum_{k=q+1}^{q+N} (\mu_k + v_k)\ell_k(b(s)) < -a(\rho_1). \tag{36}$$

Let $\Delta\tau_k \leq \tau, \forall k = 1, 2, \cdots$. Choose $T = (N+1)\tau$. Let $x(t) = x(t, t_0, x_0)$ be any solution of (3.1) with $\|x_0\| < \delta_0$. Then, it is sufficient to show that there exists a $t^* \in [t_0, t_0 + T]$ such that

$$\|x(t^*)\| < \delta \tag{37}$$

Suppose for the sake of contradition that $\|x(t)\| \geq \delta$ for all $t \in [t_0, t_0 + T]$. Then, we have, by (3.2),

$$V(t, x(t) \geq b(\delta), \quad t \in [t_0, t_0 + T]. \tag{38}$$

Using the same arguments as those used in the proof of Theorem 3.3, we get

$$V(\tau_k^+, x(\tau_k^+)) \leq V(\tau_{k-1}^+, x(\tau_{k-1}^+)) + (\mu_k + v_k)\ell_k(V(\tau_k^+, x(\tau_k^+))), \tag{39}$$

$k = q + 1,\, q + 2, \cdots$.

Since by the definition of τ, we have

$$t_{q+N} = t_{q-1} + \sum_{k=q}^{q+N} \Delta\tau_k \leq t_0 + (N + 1)\tau = t_0 + T,$$

we get

$$V(\tau_{q+N}^+, x(\tau_{q+N}^+)) \leq a(\rho^*) + \sum_{k=q+1}^{q+N} (\mu_k + v_k)\ell_k(b(\delta)) < 0,$$

which is a contradiction. Thus, (3.30) must be true. Hence, by uniform stability,

$$\|x(t)\| < \varepsilon, \quad t \geq t_0 + T,$$

i.e. system (3.1) is uniformly asymptotically stable. The proof is complete.

Remark 3.1 Theorems 3.1-3.4 will be used for the design of impulsive feedback control in the next section. It should be noted that in a general control problem the constant μ_k is positive in most cases, which implies that the uncontrolled system is unstable. Of course, situations like $\mu_k \leq 0$ are also possible. But in such cases, impulsive controls are used mainly for the purpose of faster convergence.

Remark 3.2 In addition to their applications in impulsive control, Theorem 3.1-3.4 are of independent interests to the stability theory of impulsive differential systems. We wish to emphasize that in our results the Lyapunov function is not only allowed to increase or decrease along trajectories between impulses, but also allowed to switch back and forth between them at each impulses. This represent a significant improvement over the results given in [5, 10], where the Lyapunov function is required to decrease (increase) during the continuous portion of the trajectory and then experience a jump increase (decrease) at the impulses. For results in a similar sprit, also see [9].

4. Impulsive control design

With the stability criteria in Section 3 at our disposal, we are ready to design impulsive control laws for system (2.1). Suppose that there is a Lyapunov function $V(t,x)$, $V \in C^2[R_+ \times R^n, R_+]$, such that $V(t,x)$ is positive definite and its derivative along solutions of (2.1) satisfies

$$V'(t,x) \leq \lambda(t)G(t,x), \quad (t,x) \in R_+ \times s(\rho), \tag{40}$$

where $\lambda : R_+ \to R, G : R_+ \times R^n \to R_+$ are continuous. Moreover,

$$\frac{\partial^2 V(t,x)}{\partial x^2} \geq 0, \quad (t,x) \in R_+ \times s(\rho), \tag{41}$$

where $\dfrac{\partial^2 V(t,x)}{\partial x^2}$ is an $n \times n$ matrix and a matrix $A \geq 0$ means that the matrix A is positive semi-definite.

Theorem 4.1 System (2.1) is impulsively stabilizable if there exists an impulsive control law $\{\tau_k, u_k(y)\}$ such that

(i) there exist $v_k \in R$ and $d_k \in K$ such that

$$\frac{\partial V}{\partial x}(\tau_k, x) \cdot u_k(\varphi(x)) \leq v_k d_k(V(\tau_k, x)), \quad x \in s(\rho);$$

(ii) there exist $\mu_k \in R$ and $c_k \in K$ such that

$$\begin{aligned} \Delta\tau_k \lambda(t) &\leq \mu_k, \quad t \in (\tau_{k-1}, \tau_k], \text{ and} \\ G(t,x) &\leq c_k(V(t,x)), \quad t \in (\tau_{k-1}, \tau_k] \times s(\rho); \end{aligned}$$

(iii) $\mu_k + v_k \leq 0$, $c_k(s) \leq d_k(s)$ if $v_k < 0$ and $d_k(s) \leq c_k(s)$ if $\mu_k < 0$ for $s \in (0, \rho)$;

(iv) there exists a $\rho_1 \in (0, \rho)$ such that $x \in s(\rho_1)$ implies $x + u_k(\varphi(x)) \in s(\rho)$ for all $k = 1, 2, \cdots$.

Proof. Since $V \in C^2[R_+ \times R^n, R_+]$, we have

$$\begin{aligned} V(\tau_k, x + u_k(\varphi(x)) &= V(\tau_k, x) + \frac{\partial V}{\partial x}(\tau_k, x) \cdot u_k(\varphi(x)) \\ &+ u_k(\varphi(x))^T \frac{\partial^2 V}{\partial x^2}(\tau_k, \zeta) u_k(\varphi(x)), \quad x \in s(\rho), \end{aligned}$$

where ζ is a point on the line segment jointing 0 and x. This implies, by (4.2), that

$$V(\tau_k, x + u_k(\varphi(x))) \leq V(\tau_k, x) + \frac{\partial V}{\partial x}(\tau_k, x) u_k(\varphi(x)), \quad x \in s(\rho).$$

Thus, by condition (i), we obtain

$$V(\tau_k^+, x + u_k(\varphi(x))) \leq V(\tau_k, x) + v_k d_k(V(\tau_k, x)), \quad x \in s(\rho)$$

which is condition (iii) of Theorem 3.1.

By condition (ii) and (4.1), we get

$$V'(t,x) \leq \frac{\mu_k}{\Delta\tau_k} c_k(V(t,x)), \quad (t,x) \in (\tau_{k-1}, \tau_k) \times s(\rho),$$

which is condition (ii) of Theorem 3.1. It is easy to check that all other conditions of Theorem 3.1 are satisfied. Thus, system (3.2) is stable. Hence, system (2.1) is impulsively stabilizable.

The following results are easy consequences of Theorems 3.2-3.4 in view of Theorem 4.1. Thus, we merely state them and the proofs are omitted.

Theorem 4.2 System (2.2) is impulsively uniformly stabilizable if there exists an impulsive control law $\{\tau_k, u_k(y)\}$ such that all conditions of Theorem 4.1 are satisfied and furthermore conditions (vi) and (vii) of Theorem 3.2 hold.

Theorem 4.3 System (2.1) is impulsively asymptotically stabilizable if there exists an impulsive control law $\{\tau_k, u_k(y)\}$ such that all conditions of Theorem 4.1 hold and furthermore for any $\beta > 0$ the series

$$\sum_{k=1}^{\infty}(\mu_k + v_k)\ell_k(\beta) = -\infty, \tag{42}$$

where $\ell_k(s) = \max\{c_k(s), d_k(s)\}$.

Theorem 4.4 System (2.1) is impulsively uniformly asymptotically stabilizable if there exists an impulsive control law $\{\tau_k, u_k(y)\}$ such that all conditions of Theorems 3.1-3.2 hold and furthermore for any β, $c > 0$, there exists a positive integer N such that (3.28) holds.

Remark 4.1 If $c_k(s) = d_k(s) = s$, then condition (iii) of Theorem 4.1 is reduced to

$$\mu_k + v_k \leq 0 \quad \text{and} \quad v_k \geq -1 \tag{43}$$

This is the case for the impulsive control of the Lorenz system. The details will be discussed in the next section.

5. Control of Lorenz system

Chaotic systems occur in many practical systems such as heated fluid, chemical reaction and other dynamic processes. Such systems have attracted the attention of many researchers from a variety of disciplines and are becoming an interesting and important research areas. Chaotic systems are deterministic systems, but their behavious are much too complex, and hence are unpredictable. A well-known example is the Lorenz system. It is obtained from a fluid layer heated from below and cooled from above in such a way that a temperature difference is established across it. The convection motion is described by the Navier-Stokes equation. Taking Fourier expansion of these equations along two spatial direction, and truncating to retain only three modes will lead to the Lorenz system. The Lorenz system is given by

$$\begin{cases} x_1' = \sigma(x_2 - x_1), \\ x_2' = \rho x_1 - x_2 - x_1 x_3, \\ x_3' = -\beta x_3 + x_1 x_2, \end{cases} \tag{44}$$

where σ, ρ, β are real positive parameters denoting the Prandt number, the Rayleigh number, and a geometric factor, respectively. The state variables x_1, x_2 and x_3 represent measures of fluid velocity and the spatial temperature distribution in the fluid layer under gravity. The Lorenz system has the following properties; if ρ is in [0, 1], the origin of the system is a stable equilibrium point. If ρ is between 1 and some large number $\rho^*(\sigma, \beta)$, there is one unstable equilibrium point at the origin and two stable equilibrium points at

$$\left[\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1\right], \left[-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho - 1\right].$$

If ρ is larger than the number $\rho^*(\sigma, \beta)$, there are no stable equilibrium points and the trajectory of the system will be in chaotic behaviour. For example, when the parameters in (5.1) are chosen as $\sigma = 10$, $\rho = 28$ and $\beta = \frac{8}{3}$, the Lorenz system is chaotic. See Figure 1 and Figure 2.

The Lorenz system is a paradigm of chaotic systems because it captures many of the characteristics of chaotic dynamics [6, 17]. Many researchers have investigated the control of the Lorenz systems in recent years. For example, a bang-bang controller is designed in [17] to regulate the Lorenz system to one of the unstable equilibrium points; in [16] the Lorenz system is controlled to become periodic oscillations; a suboptimal feedback controller is designed in [20] for the Lorenz system using a multilayer feedforward neural network. In this section, we shall design a nonlinear impulsive feedback controller for the Lorenz system. As it can be seen later, the controller design is simple and direct and the proposed controller can stabilize the Lorenz system so that all trajectories will converge to the zero equilibrium point.

For simplicity, we assume that all the state variables are measurable, i.e. $\varphi(x) = x$. Choose $V(t, x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$. Then $V(t, x)$ is positive definite, decrescent and $\frac{\partial^2 V}{\partial x^2}(t, x) = I$, where I is an 3×3 identity matrix.

Along solutions of (5.1), we have

$$\begin{aligned} V'(t,x) &= -\sigma x_1^2 - x_2^2 + (\sigma+\rho)x_1x_2 - \beta x_3^2 \\ &\leq \lambda^* V(t,x), \end{aligned} \tag{45}$$

where $\lambda^* = \max\{-(\sigma+1) + \sqrt{(\sigma-1)^2 + (\sigma+\rho)^2},\ -2\beta\}$.

Choose an impulsive control law $\{\tau_k, u_k(x)\}$ such that

$$u_k(x) = -\frac{1}{2}\Delta\tau_k\lambda(x_1, x_2, x_3)^T. \tag{46}$$

where $\lambda \leq \lambda^*$ is a constant.

Then we have the following result.

Theorem 5.1 The Lorenz system (5.1) is impulsively uniformly asymptotically stabilizable if $\Delta\tau_k\lambda \leq 1$ and $\lambda < \lambda^*$.

Proof. By the definition of $V(t,x)$, we have

$$\frac{\partial V}{\partial x} = (x_1, x_2, x_3)^T.$$

Thus, we get

$$\frac{\partial V}{\partial x}(\tau_k, x) \cdot u_k(x) = -\Delta\tau_k\lambda V(\tau_k, x).$$

This implies that $v_k = -\Delta\tau_k\lambda$ and $d_k(s) = s$. By (5.2), we may choose $\mu_k = \Delta\tau_k\lambda^*$ and $c_k(s) = s$. Now in view of condition (iii) of Theorem 4.1 and (4.3), we get

$$\Delta\tau_k\lambda^* - \Delta\tau_k\lambda < 0 \quad \text{and} \quad s - \Delta\tau_k\lambda s \geq 0$$

which implies $\lambda < \lambda^*$ and $\Delta\tau_k\lambda \leq 1$. Then the conclusion follows from Theorem 4.2 and Theorem 4.4.

Remark 4.1 For the Lorenz system, the constant μ_k, and consequently the constant λ, are artificial. Hence the impulsive stabilizability of the Lorenz system depends essentially on the choice of τ_k's.

Finally, we shall do some numerical experiment and verify our impulsive control algorithms. For this purpose, we choose $\sigma = 10, \rho = 28$ and $\beta = \frac{8}{3}$. Then, $\lambda^* = 28.051248$. $\Delta\tau_k < \frac{1}{\lambda^*} \approx 0.035649$. For evenly spaced impulses, the numerical results are shown in Figures 3 - 10. In Figures 3, 5, 7 and 9, the solid line shows $x_1(t)$, the dash-dotted line shows $x_2(t)$, the dotted line shows $x_3(t)$. For an initial value (5, 10, -20), one can see that the trajectory asymptotically approaches the origin with a settling time of, respectively, about 1.2 in Figure 3, about 0.7 in Figure 5 and about 0.2 in Figure 7. Small impulses are used in Figures 3 - 4, medium impulses in Figures 5 - 6 and large impulses in Figures 7 - 8. It can be seen from Figures 3, 5 and 7 that the settling time gets shorter as the impulses get larger. However if the impulses are too large, then the Lorenz system becomes unstable. See Figures 9 - 10.

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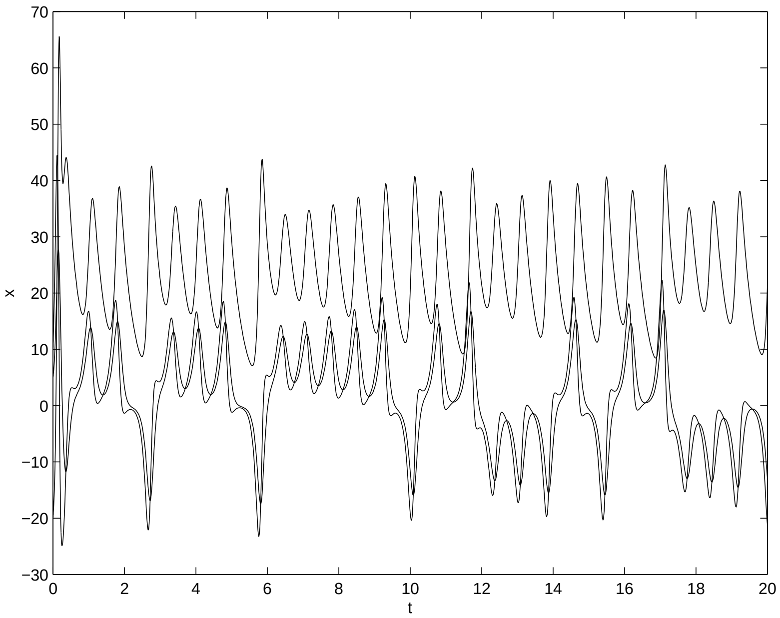

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Figure 1: Chaos behavor for Lorenz system

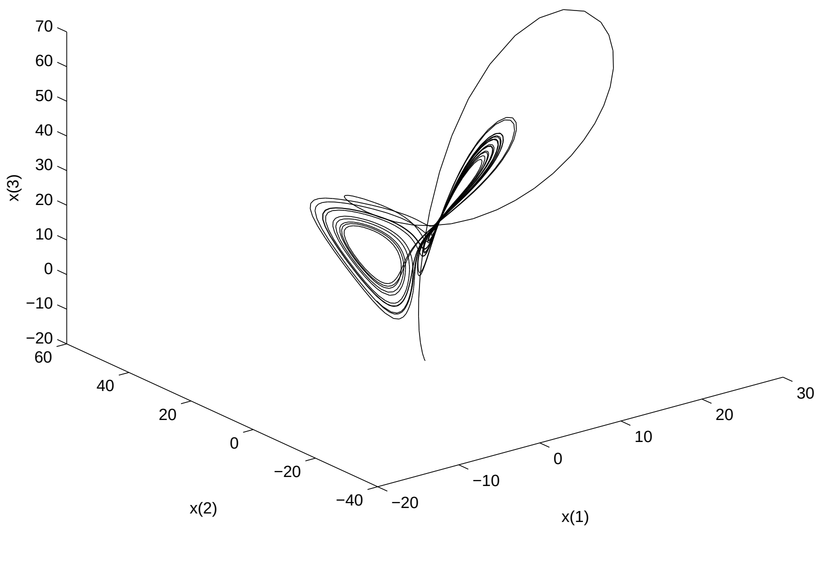

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Figure 2: Phase portait for Lorenz system

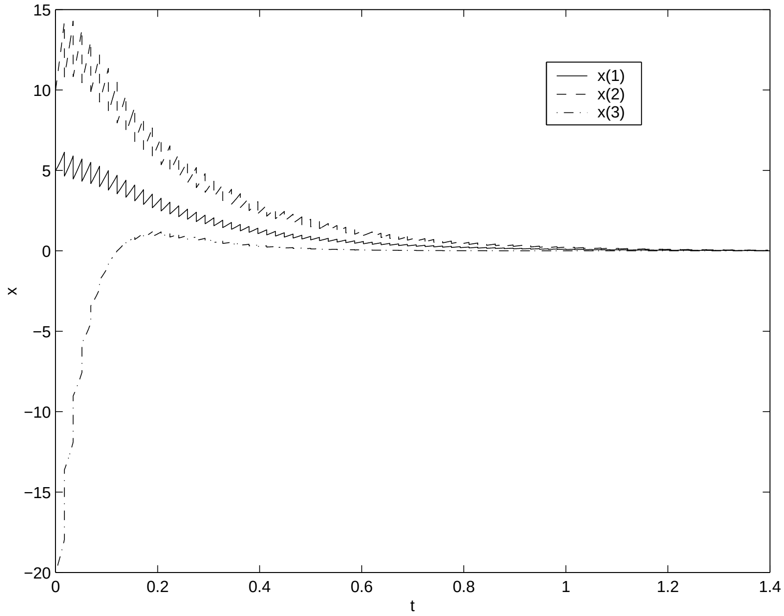

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Figure 3: Controled Lorenz system with small impulses

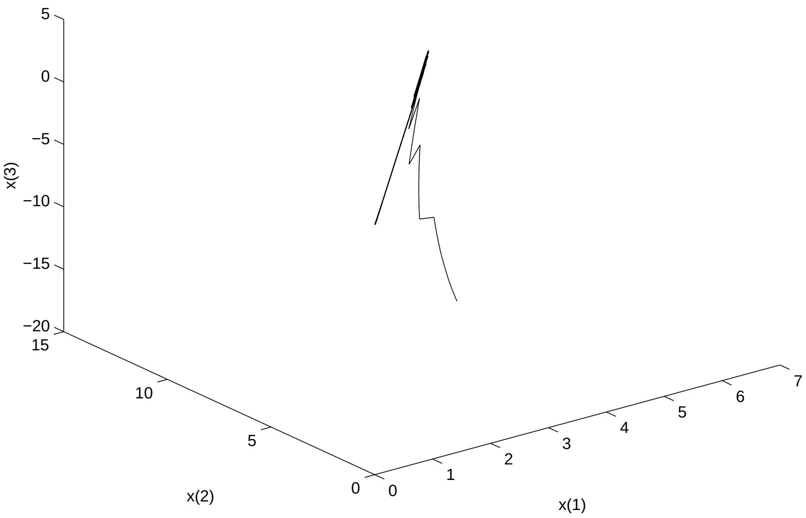

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Figure 4: Phase portrait of controled Lorenz system with small impulses

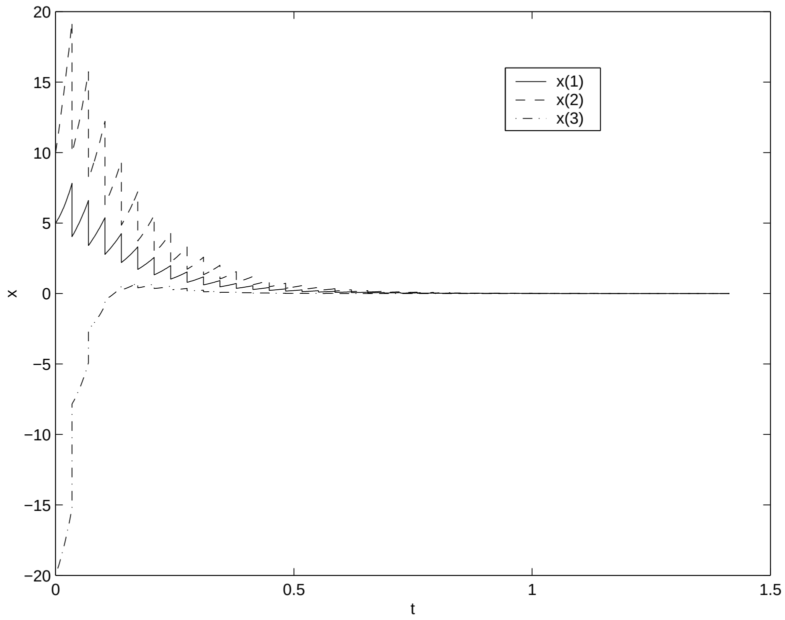

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Figure 5: Controled Lorenz system with medium impulses

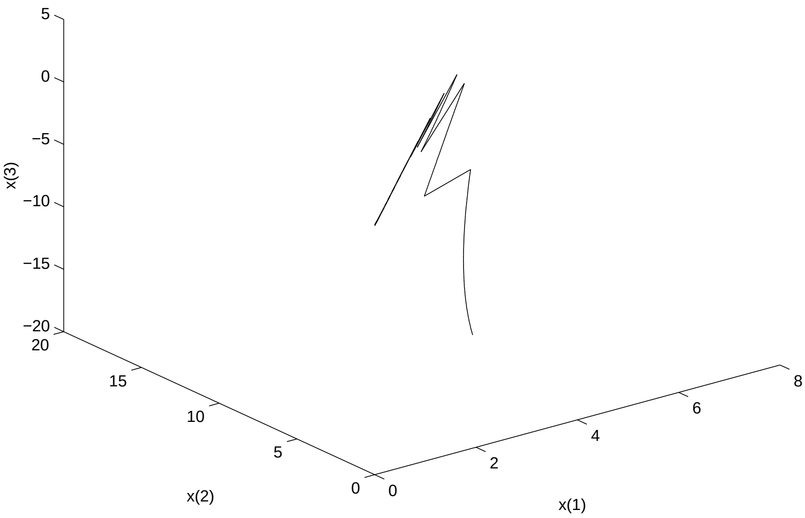

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Figure 6: Phase portrait of controled Lorenz system with medium impulses

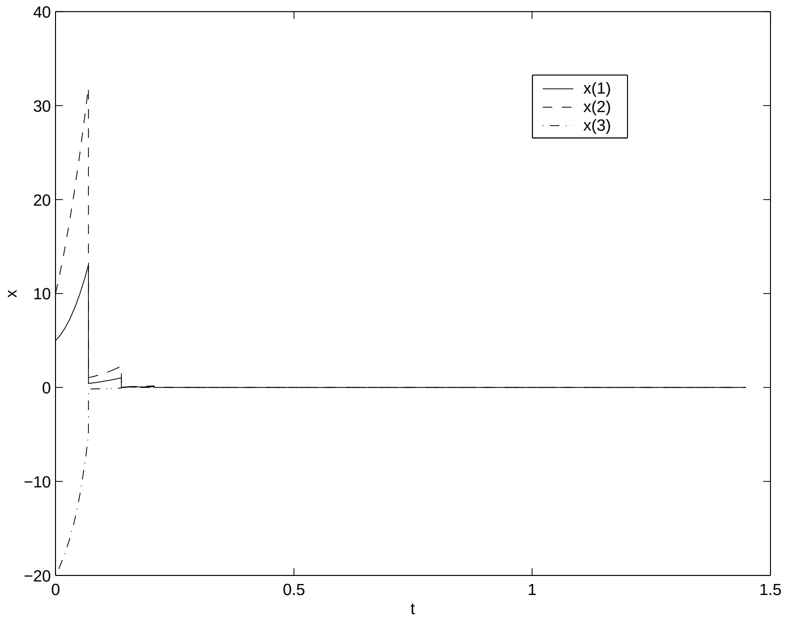

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Figure 7: Controled Lorenz system with large impulses

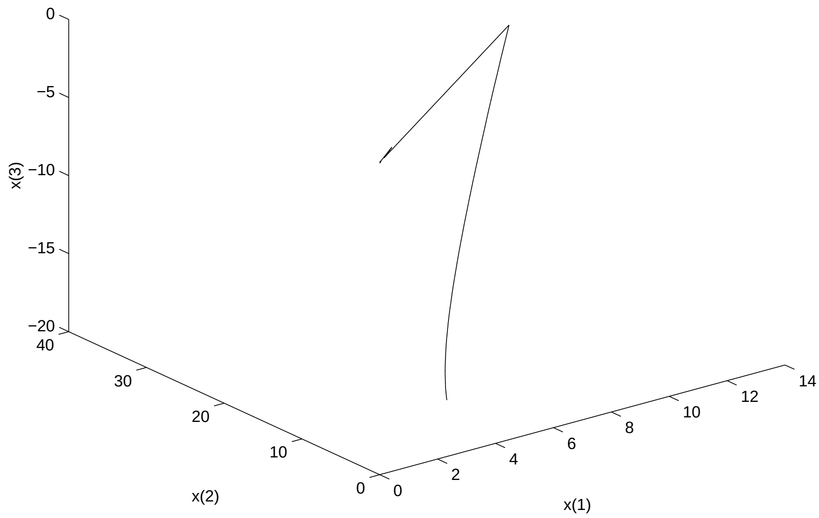

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Figure 8: Phase portrait of controled Lorenz system with large impulses

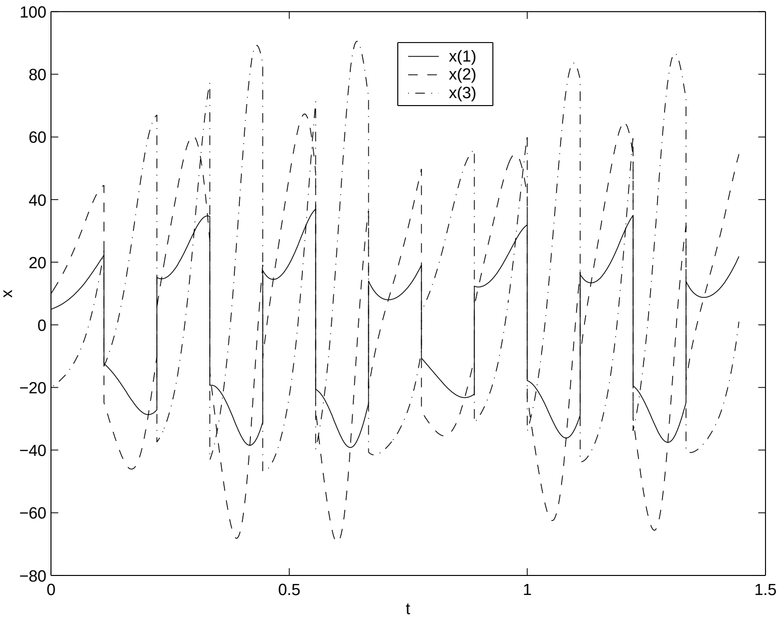

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Figure 9: Unstable impulsive Lorenz system

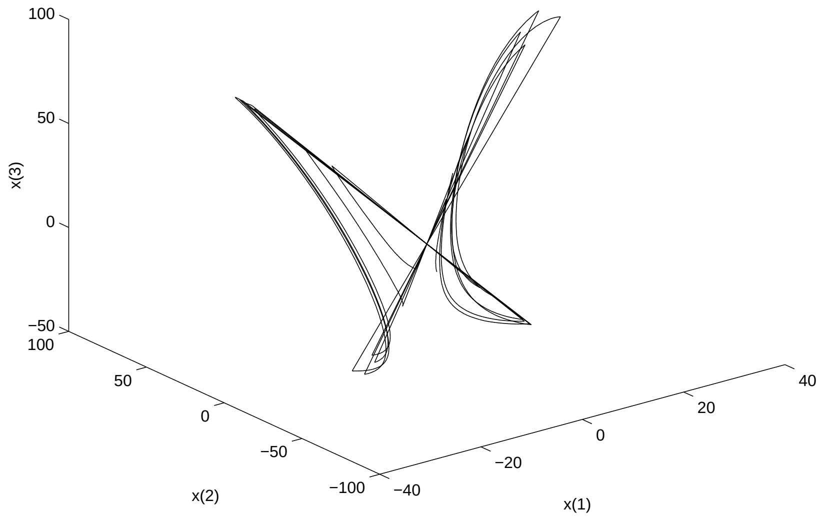

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Figure 10: Phase portrait of unstable impulsive Lorenz system