Structural invariants of 2-D systems ⋆

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Abstract: In this paper, some important structural properties of two-dimensional (2-D) systems which remain invariant under certain transformation groups are identified and investigated. As is well known, structural invariants that follow from the definition of controlled and conditioned invariant subspaces play pivotal roles in many theoretical studies of systems theory and in the solution of several control/estimation problems. These concepts are developed within a 2-D context in this paper for the first time.

1. INTRODUCTION

The fundamental notion upon which classic geometric control theory hinges is that of controlled invariance. For one-dimensional (1-D) systems governed by the standard linear time-invariant (LTI) state difference equation

$$x_{k+1} = A\,x_k + B\,u_k, \tag{1}$$

a controlled invariant subspace can be defined as the locus of the state trajectories generated by (1), in the sense that:

(a) if the initial state x_0 of (1) lies on a controlled invariant subspace $\mathscr{V}$, a control u_k exists that maintains the entire state trajectory on the same subspace $\mathscr{V}$;

(b) conversely, the subspace of smallest dimension containing any state trajectory generated by (1) is controlled invariant.

Moreover, in the usual 1-D context controlled invariance enjoys a fundamental feedback property:

(c) the control inputs that maintains the state trajectory of (1) on a controlled invariant subspace can always be expressed in terms of a static state feedback input $u_k = F\,x_k$.

Controlled invariant subspaces – also known as (A,B)-invariant subspaces – were introduced for the first time in the 1-D case in the pioneering paper Basile and Marro (1969) as the subspaces that satisfy

$$A\,\mathscr{V} \subseteq \mathscr{V} + \operatorname{im} B. \tag{2}$$

Conditioned invariance for 1-D systems was also introduced in Basile and Marro (1969) as the dual of controlled invariance. In the last forty years, controlled and conditioned invariant subspaces have played a pivotal role in the solution of a very vast number of control and estimation problems, including disturbance decoupling, unknown-input observation problems, model matching problems, fault detection and non-interaction problems, and optimal control/filtering problems, see e.g. Wonham (1985); Basile and Marro (1992); Trentelman et al. (2001) and the references therein. Several efforts have been devoted to extend the notion of controlled invariance to 2-D systems. The first paper containing a definition of controlled invariance for 2-D Fornasini-Marchesini first-order models

$$x_{i+1,j+1} = A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B_1\,u_{i+1,j} + B_2\,u_{i,j+1}, \tag{3}$$

introduced in Fornasini and Marchesini (1978) is Conte et al. (1988). According to this definition, a controlled invariant subspace for (3) is any subspace $\mathscr{V}$ which satisfies

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \mathscr{V} \subseteq (\mathscr{V} \oplus \mathscr{V}) + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{4}$$

This definition extends the 1-D counterpart (2) in a natural way. While it is true that given boundary conditions on a subspace $\mathscr{V}$ satisfying (4) a control input can always be designed to maintain the entire solution of (3) on $\mathscr{V}$, and that such control can be expressed as a static feedback $u_{i,j} = F\,x_{i,j}$, property **(b)** now does not hold, i.e., a control might exist maintaining the solution of (3) on a certain subspace $\mathscr{W}$ without $\mathscr{W}$ necessarily being controlled invariant for (3). Hence, this definition of controlled invariance, while useful to describe control laws in terms of pure static feedback as shown in Ntogramatzidis et al. (2008), does not characterise the set of trajectories generated by (3) univocally.

A second definition of controlled invariance was given in Karamancioğlu and Lewis (1992) for the singular model

$$E\,x_{i+1,j+1} = A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B\,u_{i,j}. \tag{5}$$

The reason why this model was considered is that the purpose was to ultimately characterise the solutions of a singular 2-D system in Roesser form. For the aims of this paper, we can consider E to be the identity matrix. According to the definition given in Karamancioğlu and Lewis (1992), a controlled invariant subspace for (5) with $E = I_n$ is a subspace $\mathscr{V}$ satisfying

$$A_1\,\mathscr{V} + A_2\,\mathscr{V} \subseteq \mathscr{V} + \operatorname{im} B. \tag{6}$$

Both properties **(a)** and **(b)** extend to this definition. But the drawback here is the lack of a characterisation of controlled invariance in terms of static feedback control laws, since for (5) an input $u_{i,j} = F\,x_{i,j}$ is not well defined. To combine the advantages of these two definitions without incurring in their drawbacks, in this paper we propose a definition of controlled invariance for the Fornasini-Marchesini original model

$$x_{i+1,j+1} = A_0\,x_{i,j} + A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B\,u_{i,j}, \tag{7}$$

Fornasini and Marchesini (1976), which can realise any strictly causal bivariate rational function but where, differently from

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(3), the input appears only once. Moreover, this model is closed under static feedback controls $u_{i,j} = F\,x_{i,j}$; furthermore, unlike (3), its dual is also well-defined, so that conditioned invariance can be defined in a natural way. In this paper we show that the definition (6) can be trivially extended to models in the form (7), thus retaining the fundamental properties **(a)** and **(b)** of the 1-D case. Even though model (7) is closed under static feedback, this definition of controlled invariance does not imply that the feedback property **(c)** automatically holds for these subspaces. However, we will show that it is possible to introduce a well-characterised restriction of the set of controlled invariant subspaces of (7) which is locus of solutions of (7) generated by the input $u_{i,j} = F\,x_{i,j}$. In other words, we show that, similarly to what happens for example for systems over rings, Hautus (1982), there is no exact counterpart of the definition (2) for two-dimensional systems that retains all three properties **(a), (b), (c)**, but we have to distinguish between the subspaces for which **(a)** and **(b)** are satisfied (thus leading to the notion of controlled invariance) and then restrict this set of subspaces to identify those where the solutions of (7) generated by a static feedback input lie (and this will lead to the notion of controlled invariance of feedback type). This definition, with respect to the one in Conte et al. (1988), characterises univocally and in finite terms the subspace of trajectories of a 2-D system that are generated by static feedback controls. This enables for example a solution of the classic disturbance decoupling problem to be derived in terms of constructive conditions that eliminate the conservativeness of the solutions proposed so far in the literature, see Conte et al. (1988) and Ntogramatzidis et al. (2008).

2. PROBLEM FORMULATION

Consider the Fornasini-Marchesini two-dimensional model

$$x_{i+1,j+1} = A_0\,x_{i,j} + A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B\,u_{i,j}, \qquad (8)$$

$$y_{i,j} = C\,x_{i,j}, \qquad (9)$$

introduced in Fornasini and Marchesini (1976), where, for all integers $i,j \in \mathbb{Z}$, the vector $x_{i,j} \in \mathbb{R}^n$ is the *local state* of the system, $u_{i,j} \in \mathbb{R}^m$ is the input, $y_{i,j} \in \mathbb{R}^p$ is the output. Here, A_0, A_1, $A_2 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. For the sake of brevity, we identify the system (8-9) with the quintuple $\Sigma \stackrel{\text{def}}{=} (A_0,A_1,A_2;B;C)$. To define appropriate boundary conditions for this model, we introduce, for each $k \in \mathbb{Z}$, the separation set

$$\mathfrak{C}_k \stackrel{\text{def}}{=} \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \mid i+j = k\}.$$

A suitable set of boundary conditions for (8-9) is given by the assignment of the local state over two consecutive separation sets, i.e., $x_{i,j} = \bar{x}_{i,j} \in \mathbb{R}^n$ for all $(i,j) \in \mathfrak{C}_{-1} \cup \mathfrak{C}_0$. The fact that for a valid definition of a boundary condition the local state must be assigned over two adjacent separation sets is due to the intrinsically second order recursion in (8). Defining

$$\mathfrak{Q}_i \stackrel{\text{def}}{=} (\{i\} \times \{j \in \mathbb{Z} \,|\, j \ge i\}) \cup (\{j \in \mathbb{Z} \,|\, j \ge i\} \times \{i\}).$$

another valid set of boundary conditions for (8-9) is given by the assignment $x_{i,j} = \bar{x}_{i,j}$ for all $(i,j) \in \mathfrak{Q}_0$.

To treat these boundary conditions in a unified framework, we introduce the symbol $\mathfrak{B}$ to identify any boundary condition that is valid for (8), including $\mathfrak{C}_{-1} \cup \mathfrak{C}_0$ or $\mathfrak{Q}_0$. We use the symbol $\mathfrak{B}_1$ to denote the first forward boundary, i.e., $\mathfrak{C}_1$ or $\mathfrak{Q}_1$, respectively. Similarly, $\mathfrak{B}_i$ denotes the i-th forward boundary. We also denote by $\mathfrak{B}^+$ the set of indexes defined by iterating the recursion (8) starting from indexes over $\mathfrak{B}$. Hence

- if $\mathfrak{B} = \mathfrak{C}_{-1} \cup \mathfrak{C}_0$, then $\mathfrak{B}^+ = \bigcup_{i=-1}^{\infty} \mathfrak{C}_i$;
- if $\mathfrak{B} = \mathfrak{Q}_0$, then $\mathfrak{B}^+ = \bigcup_{i=0}^{\infty} \mathfrak{Q}_i$.

Given a subspace $\mathscr{W}$ of $\mathbb{R}^n$, we say that (8) has a $\mathscr{W}$-valued boundary condition if $x_{i,j} \in \mathscr{W}$ for all $(i,j) \in \mathfrak{B}$, and that (8) has a $\mathscr{W}$-valued solution if $x_{i,j} \in \mathscr{W}$ for all $(i,j) \in \mathfrak{B}^+$.

3. CONTROLLED AND CONDITIONED INVARIANT SUBSPACES

A subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is controlled invariant for Σ if it is simultaneously 1-D controlled invariant for the pairs (A_0,B), (A_1,B), (A_2,B). Hence, $\mathscr{V}$ is controlled invariant for Σ if and only if

$$A_i\,\mathscr{V} \subseteq \mathscr{V} + \operatorname{im} B, \quad i \in \{0,1,2\}. \qquad (10)$$

The set of controlled invariants is always non-empty, as it contains at least $\{0\}$ and $\mathbb{R}^n$. The following theorem is a simple extension of Theorem 2.2 in Karamancioğlu and Lewis (1992).

Theorem 3.1. Let $\mathscr{V}$ be a subspace of $\mathbb{R}^n$. Equation (8) has a $\mathscr{V}$-valued solution for any $\mathscr{V}$-valued boundary condition if and only if $\mathscr{V}$ is controlled invariant for (8).

Proof: **(Necessity).** Suppose (10) does not hold. Then, there exist ξ', ξ'', ξ''' in $\mathscr{V}$ such that no $\omega \in \mathbb{R}^m$ exists for which

$$A_0\,\xi' + A_1\,\xi'' + A_2\,\xi''' + B\,\omega \in \mathscr{V} \qquad (11)$$

holds. By choosing $x_{0,0} = \xi'$, $x_{1,0} = \xi''$ and $x_{0,1} = \xi'''$, (11) says that a control $u_{0,0}$ cannot be found such that $x_{1,1} \in \mathscr{V}$. Hence, (8) does not have a $\mathscr{V}$-valued solution with this $\mathscr{V}$-valued boudary condition.
(Sufficiency). Obvious. ∎

Remark 3.1. The definition of controlled invariance proposed in Conte et al. (1988), while convenient in the definition of friends, does not guarantee that a subspace $\mathscr{V}$ for which the system has a $\mathscr{V}$-valued solution is controlled invariant. In other words, for the first-order Fornasini-Marchesini model (3) and the definitions currently available for controlled invariance, the *only if* part of Theorem 3.1 does not necessarily hold.

The dual concept of controlled invariance is called conditioned invariance. A subspace $\mathscr{S}$ of $\mathbb{R}^n$ is conditioned invariant for Σ if it is simultaneously conditioned invariant with respect to the three triples (A_0,C), (A_1,C) and (A_2,C), i.e., if and only if

$$A_i\,(\mathscr{S} \cap \ker C) \subseteq \mathscr{S}, \qquad i \in \{0,1,2\}.$$

The duality between 2-D controlled and conditioned invariant subspaces can be stated in precise terms as follows. Let $\Sigma^\top$ identify the dual system of (8-9), which is associated with the quintuple $(A_0^\top, A_1^\top, A_2^\top; C^\top; B^\top)$.

Lemma 3.1. The orthogonal complement of a controlled invariant for Σ is conditioned invariant for $\Sigma^\top$, and *vice-versa.*

The proof follows closely that of Proposition 4.1.3 in Basile and Marro (1992) and is therefore omitted.

4. OUTPUT-NULLING AND INPUT-CONTAINING SUBSPACES

In many control problems it is of interest to derive control laws that maintain certain outputs of a system at zero. The most famous example is the disturbance decoupling problem, Basile and Marro (1992). This requirement leads to the notion of output-nulling subspace. An output-nulling subspace for Σ is such that (8-9) have a $\mathscr{V}$-valued solution and an identically zero output for any $\mathscr{V}$-valued boundary condition. A solution of (8-9) yields zero output if and only if for all $(i,j) \in \mathfrak{B}^+$ the local state $x_{i,j}$ lies in $\ker C$. Hence, an output-nulling subspace is simply a controlled invariant subspace contained in $\ker C$. The set of output-nulling subspaces of (8-9) is denoted by $\mathscr{V}(\Sigma)$. This set is easily seen to be closed under subspace addition. Therefore, it admits a maximum $\mathscr{V}^\star$ given by the sum of all elements of $\mathscr{V}(\Sigma)$, i.e., $\mathscr{V}^\star \stackrel{\text{def}}{=} \max \mathscr{V}(\Sigma) = \sum_{\mathscr{V}\in\mathscr{V}(\Sigma)} \mathscr{V}$.

The following lemma extends the famous algorithm for the computation of $\mathscr{V}^\star$ introduced in Basile and Marro (1969).

Lemma 4.1. Subspace $\mathscr{V}^\star$ is the last term of the monotonically non-increasing sequence of subspaces $\{\mathscr{V}_i\}_i$ given by

$$\begin{cases} \mathscr{V}_0 = \ker C \\ \mathscr{V}_i = \displaystyle\bigcap_{j=0}^{2} A_j^{-1}(\mathscr{V}_{i-1} + \operatorname{im} B) \cap \ker C \qquad i \in \{1,2,\ldots,k\}, \end{cases}$$

where the integer $k \le n-1$ is determined by the condition $\mathscr{V}_{k+1} = \mathscr{V}_k$, i.e., $\mathscr{V}^\star = \mathscr{V}_k$.

Proof: First, we show by induction that the sequence $\{\mathscr{V}_i\}_i$ is monotonically non-increasing. Trivially $\mathscr{V}_0 \supseteq \mathscr{V}_1$. Let $\mathscr{V}_{h-1} \supseteq \mathscr{V}_h$. We show that $\mathscr{V}_h \supseteq \mathscr{V}_{h+1}$. From $\mathscr{V}_{h-1} \supseteq \mathscr{V}_h$ we get

$$\begin{aligned} \mathscr{V}_h &= \bigcap_{j=0}^{2} A_j^{-1}(\mathscr{V}_{j-1} + \operatorname{im} B) \cap \ker C \\ &\supseteq \bigcap_{j=0}^{2} A_j^{-1}(\mathscr{V}_j + \operatorname{im} B) \cap \ker C = \mathscr{V}_{h+1}. \end{aligned}$$

Now, we show that $\mathscr{V}^\star$ is output-nulling. First, we show that it is controlled invariant. For $\mathscr{V}^\star$ there holds

$$\mathscr{V}^\star = \bigcap_{j=0}^{2} A_j^{-1}(\mathscr{V}^\star + \operatorname{im} B) \cap \ker C. \tag{12}$$

Let $h \in \{0,1,2\}$. By applying A_h to both sides of (12) we obtain

$$\begin{aligned} A_h \mathscr{V}^\star &= A_h \Big(\bigcap_{j=0}^{2} A_j^{-1}(\mathscr{V}^\star + \operatorname{im} B) \cap \ker C \Big) \\ &\subseteq \bigcap_{j=0}^{2} A_h A_j^{-1}(\mathscr{V}^\star + \operatorname{im} B) \cap (A_h \ker C) \\ &\subseteq A_h A_h^{-1}(\mathscr{V}^\star + \operatorname{im} B) \subseteq (\mathscr{V}^\star + \operatorname{im} B) \cap \operatorname{im} A_h \subseteq \mathscr{V}^\star + \operatorname{im} B. \end{aligned}$$

Therefore, $\mathscr{V}^\star$ is 1-D controlled invariant for (A_i, B), with $i \in \{0,1,2\}$. Hence, it is also a 2-D controlled invariant subspace. Given the way $\{\mathscr{V}_i\}_i$ has been constructed, $\mathscr{V}_i \subseteq \ker C$, so that $\mathscr{V}^\star$ is also output-nulling. We show that $\mathscr{V}^\star$ is the largest output-nulling for Σ. Let $\tilde{\mathscr{V}}$ be another output-nulling subspace, so that $\tilde{\mathscr{V}}$ is controlled invariant for (A_i, B) for $i \in \{0,1,2\}$ and $\tilde{\mathscr{V}}$ is contained in $\ker C$. Hence, $\tilde{\mathscr{V}} \subseteq A_i^{-1}(\tilde{\mathscr{V}} + \operatorname{im} B)$. Since this is true for each $i \in \{0,1,2\}$, we find

$$\tilde{\mathscr{V}} \subseteq \bigcap_{j=0}^{2} A_j^{-1}(\tilde{\mathscr{V}} + \operatorname{im} B) \cap \ker C. \tag{13}$$

Now we show that every term of $\{\mathscr{V}_i\}_i$ contains $\tilde{\mathscr{V}}$, so that, in particular, $\mathscr{V}^\star \supseteq \tilde{\mathscr{V}}$. Clearly, $\mathscr{V}_0 \supseteq \tilde{\mathscr{V}}$. Suppose $\mathscr{V}_i \supseteq \tilde{\mathscr{V}}$. Thus

$$\mathscr{V}_{i+1} = \bigcap_{j=0}^{2} A_j^{-1}(\mathscr{V}_i + \operatorname{im} B) \cap \ker C \supseteq \bigcap_{j=0}^{2} A_j^{-1}(\tilde{\mathscr{V}} + \operatorname{im} B) \cap \ker C$$

which in turn contains $\tilde{\mathscr{V}}$ in view of (13). Hence, $\mathscr{V}^\star \supseteq \tilde{\mathscr{V}}$. ■

The duals of 2-D output-nulling subspaces are the 2-D *input-containing* subspaces. An input-containing subspace $\mathscr{S}$ is a conditioned invariant subspace of Σ that contains $\operatorname{im} B$.

The set of input-containing subspaces of Σ is denoted by $\mathscr{S}(\Sigma)$. This set is closed under intersection. Therefore, it admits a minimum given by $\mathscr{S}^\star = \min \mathscr{S}(\Sigma) = \bigcap_{\mathscr{S}\in\mathscr{S}(\Sigma)} \mathscr{S}$. By dualising the algorithm for $\mathscr{V}^\star$, we have the following.

Lemma 4.2. Subspace $\mathscr{S}^\star$ is the last term of the monotonically non-decreasing sequence of subspaces $\{\mathscr{S}_i\}_i$ given by

$$\begin{cases} \mathscr{S}_0 = \operatorname{im} B \\ \mathscr{S}_i = \displaystyle\sum_{j=0}^{2} A_j(\mathscr{S}_{i-1} \cap \ker C) + \operatorname{im} B \qquad i \in \{1,2,\ldots,k\}, \end{cases}$$

where the integer $k \le n-1$ is determined by the condition $\mathscr{S}_{k+1} = \mathscr{S}_k$, i.e., $\mathscr{S}_k = \mathscr{S}^\star$.

Using Lemma 3.1 it is straightforward to prove that the orthogonal complement of an output-nulling subspace for Σ is input-containing for $\Sigma^\top$ and *vice-versa*.

5. CONTROLLED INVARIANT AND OUTPUT-NULLING SUBSPACES OF FEEDBACK TYPE

The Fornasini-Marchesini model (8-9), differently from the model used in Karamancioğlu and Lewis (1992), is closed under the feedback $u_{i,j} = F x_{i,j}$, which gives rise to

$$x_{i+1,j+1} = (A_0 + BF)\, x_{i,j} + A_1\, x_{i+1,j} + A_2\, x_{i,j+1}. \tag{14}$$

It is easy to see that the notion of controlled invariance alone is not sufficient to guarantee the existence of a feedback matrix F that maintains the local state $x_{i,j}$ on a controlled invariant subspace for $\mathscr{V}$-valued boundary conditions. We define the controlled invariant subspaces of feedback type as the subspaces of solutions of (8) that can be generated by the input $u_{i,j} = F x_{i,j}$. This means that **i)** given a controlled invariant subspace of feedback type $\mathscr{W}$, it is always possible to find F such that $u_{i,j} = F x_{i,j}$ generates a trajectory entirely contained in $\mathscr{W}$; **ii)** given $u_{i,j} = F x_{i,j}$, the subspace of minimum dimension containing the corresponding local state trajectory is controlled invariant of feedback type. In this case, matrix F is called a *friend* of the controlled invariant subspace of feedback type. The characterisation **ii)** is missing in all other definitions given so far for 2-D controlled invariant subspaces.

Theorem 5.1. The subspace $\mathscr{W}$ of $\mathbb{R}^n$ is a controlled invariant subspace of feedback type for Σ if and only if

- $\mathscr{W}$ is controlled invariant for (A_0, B);
- $\mathscr{W}$ is both A_1 and A_2-invariant.

Proof: **(Necessity).** By virtue of (14), given a friend F of a controlled invariant subspace of feedback type, the inclusion

$$[\, A_0 + BF \;\; A_1 \;\; A_2 \,]\, (\mathscr{W} \oplus \mathscr{W} \oplus \mathscr{W}) \subseteq \mathscr{W} \tag{15}$$

must be satisfied, else it is possible to find $x_{i,j}, x_{i+1,j}, x_{i,j+1} \in \mathscr{W}$ such that $x_{i+1,j+1}$ does not lie on $\mathscr{W}$; (15) can be written in matrix notation using a basis W of $\mathscr{W}$ (i.e., $\operatorname{im} W = \mathscr{W}$ and $\ker W = \{0\}$), by saying that X_0, X_1, X_2 exist such that

$$[\,A_0 W \;\; A_1 W \;\; A_2 W\,] = W\,[\,X_0 \;\; X_1 \;\; X_2\,] - [\,B F W \;\; O \;\; O\,].$$

This means that there exist X_0, X_1, X_2 and $\Omega = -F W$ such that W satisfies $A_0 W = W X_0 + B\,\Omega$, $A_1 W = W X_1$ and $A_2 W = W X_2$. The first equality says that $\mathscr{W}$ is controlled invariant for (A_0, B), and the last two imply that $\mathscr{W}$ is both A_1 and A_2-invariant.
(Sufficiency). Let $\mathscr{W}$ be a controlled invariant subspace of feedback type, and let W be a basis of $\mathscr{W}$. Since $\mathscr{W}$ is controlled invariant for (A_0, B), two matrices X_0 and Ω exist such that $A_0 W = W X_0 + B\,\Omega$. Since $\mathscr{W}$ is both A_1 and A_2-invariant, two matrices X_1 and X_2 exist such that $A_i W = W X_i$, with $i = \{1,2\}$. Since W is full column-rank, $\Omega = -F W$ can be solved in F, so that (15) holds for $\mathscr{W}$, and F is a friend of $\mathscr{W}$. ■

The notion of controlled invariant subspaces of feedback type can be extended to output-nulling subspaces. A subspace $\mathscr{W}$ of $\mathbb{R}^n$ is output-nulling of feedback type if a feedback F exists such that, for any $\mathscr{W}$-valued initial condition, the solution of Σ generated by a static feedback control is $\mathscr{W}$-valued and the output is identically zero. The set of output-nulling subspaces of feedback type, denoted by $\mathscr{W}(\Sigma)$, is closed under addition. Hence, the maximum output-nulling subspace $\mathscr{W}^\star$ of feedback type can still be defined as the sum of all the elements of $\mathscr{W}(\Sigma)$. An algorithm for the computation of $\mathscr{W}^\star$ is given as follows.

Lemma 5.1. Subspace $\mathscr{W}^\star$ is the last term of the monotonically non-increasing sequence of subspaces $\{\mathscr{W}_i\}_i$ given by

$$\begin{cases} \mathscr{W}_0 = \ker C \\ \mathscr{W}_i = \ker C \cap A_0^{-1}(\mathscr{W}_{i-1} + \operatorname{im} B) \cap A_1^{-1}\,\mathscr{W}_{i-1} \cap A_2^{-1}\,\mathscr{W}_{i-1} \\ \hfill i \in \{1,2,\ldots,k\}, \end{cases}$$

where the integer $k \le n-1$ is determined by the condition $\mathscr{W}_{k+1} = \mathscr{W}_k$, i.e., $\mathscr{W}^\star = \mathscr{W}_k$.

The proof can be carried out along the same lines of that of Lemma 4.1 with the obvious modifications.

6. CONDITIONED INVARIANT SUBSPACES OF OUTPUT-INJECTION TYPE

We consider the dual of controlled invariance of feedback type, here referred to as conditioned invariance of output-injection type, and we relate it to the existence of certain types of observers that can estimate some linear combinations of the components of the local state vector in presence of unknown inputs; this approach closely follows the 1-D one developed in Willems (1981), and discussed in Chapter 5 of Trentelman et al. (2001). This approach was considered in the 2-D framework of Fornasini-Marchesini first-order models in Ntogramatzidis et al. (2008). Given the subspace $\mathscr{Z}$ or $\mathbb{R}^n$ and a matrix Q such that we can write $\mathscr{Z} = \ker Q$, we define a $\mathscr{Z}$-observer for (8-9) with $B = 0$ as a system ruled by

$$\omega_{i+1,j+1} = K_0\,\omega_{i,j} + K_1\,\omega_{i+1,j} + K_2\,\omega_{i,j+1} + L y_{i,j} \qquad (16)$$

such that if $\omega_{i,j} = Q x_{i,j}$ for all $(i,j) \in \mathfrak{B}$, then $\omega_{i,j} = Q x_{i,j}$ for all $(i,j) \in \mathfrak{B}^+$. In other words, a $\mathscr{Z}$-observer recovers the components of the local state that are external to $\mathscr{Z}$, i.e., it is such that if $\omega_{i,j} = x_{i,j}/\mathscr{Z}$ on the boundary, then $\omega_{i,j} = x_{i,j}/\mathscr{Z}$ everywhere. Thus, if the initial conditions of the system and of the observer are equal modulo $\mathscr{Z}$, the state of the observer is always equal to the local state of the system modulo $\mathscr{Z}$.

A subspace $\mathscr{Z}$ is conditioned invariant of output-injection type for Σ if there exists a $\mathscr{Z}$-observer for Σ with $B = 0$. The following theorem provides a geometric characterisation of conditioned invariance of output-injection type.

Theorem 6.1. The subspace $\mathscr{Z}$ of $\mathbb{R}^n$ is a conditioned invariant subspace of output-injection type for Σ if and only if

- $\mathscr{Z}$ is conditioned invariant for (A_0, C);
- $\mathscr{Z}$ is both A_1 and A_2-invariant.

Proof: **(Sufficiency).** Let $\mathscr{Z}$ be conditioned invariant for (A_0, C), and invariant with respect to A_1 and A_2. We can write $\mathscr{Z} = \ker Q$, where Q satisfies the linear equations

$$Q A_0 = \Gamma_0\,Q + \Lambda C, \qquad (17)$$

$$Q A_1 = \Gamma_1\,Q, \qquad (18)$$

$$Q A_2 = \Gamma_2\,Q. \qquad (19)$$

Consider (16) with $K_i = \Gamma_i$ for $i \in \{0,1,2\}$, and $L = \Lambda$. Define the observation error as $e_{i,j} \stackrel{\text{def}}{=} Q x_{i,j} - \omega_{i,j}$. Since it is assumed that $\omega_{i,j} = Q x_{i,j}$ over the boundary $\mathfrak{B}$, we get $e_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}$. Thus,

$$\begin{aligned} e_{i+1,j+1} &= Q x_{i+1,j+1} - \omega_{i+1,j+1} \\ &= Q A_0\,x_{i,j} + Q A_1\,x_{i+1,j} + Q A_2\,x_{i,j+1} \\ &\quad -\Gamma_0\,\omega_{i,j} - \Gamma_1\,\omega_{i+1,j} - \Gamma_2\,\omega_{i,j+1} - \Lambda C x_{i,j} \\ &= \Gamma_0\,e_{i,j} + \Gamma_1\,e_{i+1,j} + \Gamma_2\,e_{i,j+1}. \end{aligned}$$

Since these dynamics are autonomous, the estimation error is zero everywhere if it is zero over $\mathfrak{B}$.

(Necessity). There exists a $\mathscr{Z}$-observer for Σ with $B = 0$. Therefore, given $\omega_{i,j} = Q x_{i,j}$ over the boundary $\mathfrak{B}$, we have $\omega_{i,j} = Q x_{i,j}$ over $\mathfrak{B}^+$. Let the boundary condition of (8) be such that $x_{0,0} \in \mathscr{Z} \cap \ker C$, $x_{1,0} \in \mathscr{Z}$ and $x_{0,1} \in \mathscr{Z}$. The boundary condition of the $\mathscr{Z}$-observer is such that $\omega_{0,0} = \omega_{1,0} = \omega_{0,1} = 0$. This is compatible with the fact that $\omega_{i,j} = Q x_{i,j}$ for $(i,j) \in \{(0,0),(1,0),(0,1)\}$, since for such pairs of indexes we have $x_{i,j} \in \mathscr{Z}$, and hence $Q x_{i,j} = 0$. Therefore, from (16) we get

$$\omega_{1,1} = K_0\,\omega_{0,0} + K_1\,\omega_{1,0} + K_2\,\omega_{0,1} + L C x_{0,0},$$

which is equal to zero since $x_{0,0} \in \ker C$. On the other hand,

$$Q x_{1,1} = Q A_0\,x_{0,0} + Q A_1\,x_{1,0} + Q A_2\,x_{0,1} = \omega_{1,1},$$

which is zero as shown above. For the arbitrariness of $x_{0,0}$, $x_{1,0}$ and $x_{0,1}$ we get $Q A_0(\mathscr{Z} \cap \ker C) + Q A_1\,\mathscr{Z} + Q A_2\,\mathscr{Z} = \{0\}$, which implies that $A_0(\mathscr{Z} \cap \ker C) + A_1\,\mathscr{Z} + A_2\,\mathscr{Z} \subseteq \mathscr{Z}$. Hence, $\mathscr{Z}$ is a conditioned invariant subspace of output-injection type. ■

The definition of conditioned invariance of output-injection type is new, and with respect to the definition of conditioned invariance given so far for 2-D systems it univocally characterises $\mathscr{Z}$-observers. Indeed, conditioned invariant subspaces as defined in Conte et al. (1988) guarantee the existence of a $\mathscr{Z}$-observer, but the vice-versa is not necessarily true, and the condition in Theorem 6.1 for that definition is only sufficient.

We can define an input-containing subspace of output-injection type as a subspace $\mathscr{Z}$ for which a $\mathscr{Z}$-observer exists for Σ with B not necessarily zero. Hence, a subspace $\mathscr{Z}$ is an input-containing subspace of output-injection type if and only if it is conditioned invariant for (A_0, C), it contains the image of B, and it is both A_1 and A_2-invariant. It is easily seen that $\mathscr{L}$ is conditioned invariant (resp. input-containing) of output-injection type if and only if its orthogonal complement $\mathscr{L}^\perp$ is controlled invariant (resp. output-nulling) of feedback type for $\Sigma^\top$. Therefore, the following lemma holds.

Lemma 6.1. Subspace $\mathscr{Z}^\star$ is the last term of the monotonically non-decreasing sequence of subspaces $\{\mathscr{Z}_i\}_i$ given by

$$\begin{cases} \mathscr{Z}_0 = \operatorname{im} B \\ \mathscr{Z}_i = \operatorname{im} B + A_0(\mathscr{Z}_{i-1} \cap \ker C) + A_1\, \mathscr{Z}_{i-1} + A_2\, \mathscr{Z}_{i-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad i \in \{1,2,\ldots,k\}, \end{cases}$$

where the integer $k \le n-1$ is determined by the condition $\mathscr{Z}_{k+1} = \mathscr{Z}_k$, i.e., $\mathscr{Z}^\star = \mathscr{Z}_k$.

6.1 Friends and Stabilisation

In this section we show how to characterise the set of friends of a controlled invariant subspace of feedback type. First, recall that the subspace $\mathscr{W}$ is a controlled invariant subspace of feedback type if there exist X_0, X_1, X_2 and Ω such that $A_0 W = W X_0 + B\Omega$, $A_1 W = W X_1$, $A_2 W = W X_2$. These identities can be written together as

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} W = \left[\begin{array}{cc|cc} W & B & O & O \\ \hline O & O & W & O \\ O & O & O & W \end{array}\right] \left[\begin{array}{c} X_0 \\ \Omega \\ \hline X_1 \\ X_2 \end{array}\right]. \qquad (20)$$

To find the complete set of friends of $\mathscr{W}$, we write the set of solutions of (20) as [1]

$$\left[\begin{array}{c} X_0 \\ \Omega \\ \hline X_1 \\ X_2 \end{array}\right] = \left[\begin{array}{cc|cc} W & B & O & O \\ \hline O & O & W & O \\ O & O & O & W \end{array}\right]^\dagger \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} W + \left[\begin{array}{c} H_0 \\ H_1 \\ \hline H_2 \\ H_3 \end{array}\right] K_1 \qquad (21)$$

where $\left[\, H_0^\top \; H_1^\top \; H_2^\top \; H_3^\top \,\right]^\top$ is a basis matrix of the subspace $\ker[\,W \;\; B\,] \oplus \ker W \oplus \ker W$, and K_1 is an arbitrary matrix of suitable size. Finally, given any of such solutions, we can construct F as a solution of the linear equation $\Omega = -F\,W$, which can be written as $F = -\Omega\,(W^\top W)^{-1}\,W^\top + K_2 Z$, where Z is a basis of $\ker W$ and K_2 is another arbitrary matrix of suitable size. Thus, as for the 1-D case, there are two degrees of freedom in the computation of a friend F, given by K_1 and K_2. This freedom can be exploited to stabilise the trajectories of the model that are internal and/or external to $\mathscr{W}$. Consider the change of basis $T = [\,W \;\; W_c\,]$, where W_c is such that T is non-singular. Then

$$T^{-1}(A_0 + BF)\,T = \begin{bmatrix} L_1(K_1,K_2) & L_2(K_1,K_2) \\ O & L_3(K_1,K_2) \end{bmatrix},$$

$$T^{-1}A_1\,T = \begin{bmatrix} M_1 & M_2 \\ O & M_3 \end{bmatrix},\; T^{-1}A_2\,T = \begin{bmatrix} N_1 & N_2 \\ O & N_3 \end{bmatrix}.$$

Lemma 6.2. Matrix $L_1(K_1,K_2)$ does not depend on K_2, and matrix $L_3(K_1,K_2)$ does not depend on K_1.

The proof of Lemma 6.2 can be carried out along the same lines of that of Lemma 3.3 in Ntogramatzidis et al. (2008).

From Lemma 6.2, defining $\begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} \stackrel{\text{def}}{=} T^{-1}x_{i,j}$, the closed-loop update equation can be written in the new basis as

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} L_1(K_1) & L_2(K_1,K_2) \\ O & L_3(K_2) \end{bmatrix} \begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix}$$
$$\qquad + \begin{bmatrix} M_1 & M_2 \\ O & M_3 \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix} + \begin{bmatrix} N_1 & N_2 \\ O & N_3 \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix}.$$

[1] Here the symbol † is used to denote the Moore-Penrose pseudo-inverse.

The coordinates x' represent the components of the local state on the controlled invariant subspace of feedback type $\mathscr{W}$, while the coordinates x'' represent the components of the local state that are external to $\mathscr{W}$. The zeros in the matrices above confirm that the local state of the closed loop system lies on $\mathscr{W}$ for any $\mathscr{W}$-valued boundary condition. In fact, a $\mathscr{W}$-valued boundary condition can be written in these coordinates as $x''_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}$. Hence, the components $x''_{i,j}$ are identically zero on $\mathfrak{B}^+$ since the update equation $x''_{i+1,j+1} = L_3(K_2)\,x''_{i,j} + M_3\,x''_{i+1,j} + N_3\,x''_{i,j+1}$ with the boundary condition $x''_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}$ generates $x''_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}^+$.

A controlled invariant subspace of feedback type $\mathscr{W}$ is said to be internally stabilisable if for any $\mathscr{W}$-valued boundary condition a friend F can be found so that the local state $x_{i,j}$ lies on $\mathscr{W}$ for all $(i,j) \in \mathfrak{B}^+$ and converges to zero as the bi-index (i,j) evolves away from the boundary $\mathfrak{B}$. Similarly, $\mathscr{W}$ is said to be externally stabilisable if for any boundary condition a friend of $\mathscr{W}$ exists such that the corresponding local state converges to $\mathscr{W}$ as (i,j) evolves away from the boundary $\mathfrak{B}$. Using the standard change of coordinates described above, we see that

- $\mathscr{W}$ is internally stabilisable if and only if $x'_{i,j}$ converges to zero as (i,j) evolves away from $\mathfrak{B}$, i.e., if and only if the triple $(L_1(K_1), M_1, N_1)$ is asymptotically stable;
- $\mathscr{W}$ is externally stabilisable if and only if $x''_{i,j}$ converges to zero as (i,j) evolves away from $\mathfrak{B}$, i.e., if and only if the triple $(L_3(K_2), M_3, N_3)$ is asymptotically stable.

Therefore, the first degree of freedom K_1 affects the internal stabilisability of a controlled invariant subspace of feedback type, but not the external stabilisability, while the second degree of freedom K_2 affects the external stabilisability of a controlled invariant subspace of feedback type, but not the internal stabilisability. Let us now focus on the internal stabilisation of a controlled invariant subspace of feedback type. If we construct the friend F with (21) and then solve $\Omega = -F\,W$, we find that the matrices X_0, X_1 and X_2 are the matrices of the Fornasini-Marchesini subsystem that represent the internal dynamics of (8) restricted to $\mathscr{W}$. In fact, let us consider an r-dimensional controlled invariant subspace of feedback type $\mathscr{W}$, and a $\mathscr{W}$-valued boundary condition. Given $x_{i,j}, x_{i+1,j}, x_{i,j+1} \in \mathscr{W}$, there exist $z_{i,j}, z_{i+1,j}, z_{i,j+1} \in \mathbb{R}^r$ such that $z_{i,j} = W\,x_{i,j}$, $z_{i+1,j} = W\,x_{i+1,j}$ and $z_{i+1,j} = W\,x_{i+1,j}$. Therefore, we can write the closed-loop system as

$$\begin{aligned} x_{i+1,j+1} &= (A_0 + BF)\,W\,z_{i,j} + A_1\,W\,x_{i+1,j} + A_2\,W\,x_{i,j+1} \\ &= W\,(X_0\,z_{i,j} + X_1\,z_{i+1,j} + X_2\,z_{i,j+1}), \end{aligned}$$

which implies that $x_{i+1,j+1}$ lies on $\mathscr{W}$, and therefore by defining $z_{i+1,j+1} = X_0\,z_{i,j} + X_1\,z_{i+1,j} + X_2\,z_{i,j+1}$, given a solution $x_{i,j}$ on $\mathscr{W}$ we can construct the vector $z_{i,j}$ which represents the projection of the local state $x_{i,j}$ on $\mathscr{W}$. Hence, $\mathscr{W}$ is internally stabilisable if and only if $A_0\,W = W\,X_0 + B\,\Omega$, $A_1\,W = W\,X_1$ and $A_2\,W = W\,X_2$ can be solved in X_0, X_1, X_2, Ω in such a way that the triple (X_0, X_1, X_2) is asymptotically stable in the usual 2-D sense, i.e., if and only if the determinant of $I - X_0\,z_1\,z_2 - X_1\,z_2 - X_2\,z_1$ differs from zero for all (z_1,z_2) in the unit bi-disc $\{(\zeta_1,\zeta_2) \in \mathbb{C}\times\mathbb{C} \mid |\zeta_1| \le 1 \text{ and } |\zeta_2| \le 1\}$, see Fornasini and Marchesini (1978), Proposition 3. We derive an LMI condition for this to happen in terms of the degree of freedom K_1 by using the condition in Kar and Sigh (2003). This condition ensures that (X_0,X_1,X_2) is asymptotically stable if three symmetric positive semidefinite matrices P_0, P_1 and P_2 exist such that

$$\begin{bmatrix} P_0 & O & O \\ O & P_1 & O \\ O & O & P_2 \end{bmatrix} - \begin{bmatrix} X_0^\top \\ X_1^\top \\ X_2^\top \end{bmatrix} (P_0+P_1+P_2)\,[\,X_0\ \ X_1\ \ X_2\,] > 0. \quad (22)$$

From the properties of the Schur complements, this condition can be rewritten as

$$\left[\begin{array}{c|c} \begin{matrix} P_0 & O & O \\ O & P_1 & O \\ O & O & P_2 \end{matrix} & \begin{bmatrix} X_0^\top \\ X_1^\top \\ X_2^\top \end{bmatrix} P \\ \hline P\,[\,X_0\ X_1\ X_2\,] & P \end{array}\right] > 0, \quad (23)$$

where $P = P_0+P_1+P_2$. Moreover, by defining

$$\begin{bmatrix} Q_0 \\ Q_0' \\ \hline Q_1 \\ Q_2 \end{bmatrix} \overset{\text{def}}{=} \left[\begin{array}{cc|cc} W & B & O & O \\ \hline O & O & W & O \\ O & O & O & W \end{array}\right]^{\dagger} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} W \quad (24)$$

we can see that from (21) we can write $X_i = Q_i + H_i K_1$ for $i \in \{0,1,2\}$, and (23) becomes

$$\left[\begin{array}{ccc|c} P_0 & \star & \star & \star \\ O & P_1 & \star & \star \\ O & O & P-P_0-P_1 & \star \\ \hline P(Q_0+H_0K_1) & P(Q_1+H_1K_1) & P(Q_2+H_2K_1) & P \end{array}\right] > 0,$$

where the symbol $\star$ is used to abbreviate off-diagonal blocks in symmetric matrices. Pre-multiplying and post-multiplying the former by the block-diagonal matrix $\text{diag}(P^{-1},P^{-1},P^{-1},P^{-1})$ and defining $\Phi = P^{-1}$, $\Psi = P^{-1}P_0P^{-1}$ and $\Theta = P^{-1}P_1P^{-1}$, along with $\Xi = K_1\Theta$ we get the LMI

$$\left[\begin{array}{ccc|c} \Psi & \star & \star & \star \\ O & \Theta & \star & \star \\ O & O & \Phi-\Psi-\Theta & \star \\ \hline Q_0\Phi+H_0\Xi & Q_1\Phi+H_1\Xi & Q_2\Phi+H_2\Xi & \Phi \end{array}\right] > 0. \quad (25)$$

Hence, we have just proved the following result, which provides a computationally sound LMI-based method to calculate the gain K_1 that stabilises a controlled invariant subspace of feedback type internally.

Theorem 6.2. The controlled invariant subspace of feedback type $\mathscr{W}$ is internally stabilisable if there exist matrices $\Phi = \Phi^\top > 0$, $\Psi = \Psi^\top > 0$, $\Theta = \Theta^\top > 0$ and Ξ of suitable sizes such that (25) holds. Given a quadruple (Φ,Ψ,Θ,Ξ) in the convex set defined by (25), a matrix K_1 such that the triple (X_0,X_1,X_2) in (21) is asymptotically stable is given by $K_1 = \Xi\,\Phi^{-1}$.

A parallel (dual) theory for the external stabilisation of conditioned invariant subspaces of output-injection type can easily be established. This is instrumental in the definitions of $\mathscr{L}$-observers for which, when there is a mismatch between the boundary conditions of Σ and (16), the estimation error converges to zero as the bi-index (i,j) evolves away from the boundary, see Ntogramatzidis et al. (2010).

CONCLUDING REMARKS

In this paper, fundamental structural invariants of 2-D systems have been introduced and discussed. The most remarkable difference with respect to the 1-D case is the need for a distinction between controlled invariance and controlled invariance of feedback type.

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