Well-Posedness for Set Optimization Problems[1]

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Abstract: In this paper, three kinds of well-posedness for set optimization are first introduced. By virtue of a generalized Gerstewitz's function, the equivalent relations between the three kinds of well-posedness and the well-posedness of three kinds of scalar optimization problems are established, respectively. Then, sufficient and necessary conditions of well-posedness for set optimization problems are obtained by using a generalized forcing function, respectively. Finally, various criteria and characterizations of well-posedness are given for set optimization problems.

Keywords: Well-posedness; Set optimization; Gerstewitz's function.

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1 Introduction

Well-posedness of optimization problems was first studied by Tykhonov [1] in 1966. Since then, the notions of well-posedness have been extended to different kinds of optimization problems (see [2, 3, 4, 5, 6, 7]). In the book edited by Lucchetti and Revalski [8], Loridan gave a survey on some theoretical results of well-posedness, approximate solutions and variational principles in vector optimization. Based on the ε-minimal solutions, Bednarczuk [9] investigated several well-posedness for vector optimization problems. Huang [2] introduced two kinds of extended well-posedness for set-valued optimization problems and investigated a series of their characterizations and criteria. Also, some authors discussed the well-posedness of variational inequality and equilibrium problems, see [7, 10, 11].

It is worth noting that there is a kind of optimization problems called set optimization problems, which was firstly introduced by Kuroiwa (see [12]). Comparing with the usual set-valued optimization problems, set optimization problems consider relationship among image sets, but not look for efficient points of the set of all image sets. Thus, set optimization problems often play more natural roles. Until now there have been many papers to study them (see [12, 13, 14, 15, 16, 17, 18]). Kuroiwa [13] showed six relations among sets, and obtained duality theorems of set optimization. Kuroiwa [15] introduced efficiencies for a family of sets and investigated existence results of such efficient sets. Using the concept of cone extension and the Mordukhovich coderivative, Ha [16] studied some variants of the Ekeland's variational principle for a set-valued mapping under various continuity assumptions. Alonso and Rodríguez-Marín [18] discussed the optimality conditions for set optimization. But to the best of our knowledge, there is still no paper concerning well-posedness for set optimization problems.

In this paper, we shall first introduce three kinds of well-posedness for a set optimization problem, i.e., k_0-well-posedness at a minimizer, generalized k_0-well-posedness and extended k_0-well-posedness. Then, using a generalized version of so-called nonlinear scalarization functional (see [17]), we establish equivalent relations between the three kinds of well-posedness for the set optimization problem and well-posedness of the three kinds of scalar optimization problems, respectively. Finally, base on these scalar results, we extend some basic results of well-posedness of scalar optimization problems to set optimization problems and derive some criteria and characterizations for the three types of well-posedness of the set optimization problem.

The rest of the paper is organized as follows. In Section 2, we present the concepts of three kinds of well-posedness for a set optimization problem and give examples to illustrate them. In Section 3, we prove the equivalent relations between three kinds of well-posedness of the set optimization problem and well-posedness of the three kinds of scalar optimization problems, respectively, and extend many basic results of scalar optimization problems to the set optimization problem. In Section 4, we give some characterizations and criteria to the three kinds of well-posedness for the set optimization problem.

2 Preliminaries and Well-Posedness of (X, I)

Let (X, d) be a metric space, and Y be a real topological linear space ordered by a convex closed and pointed cone $C \subset Y$ with its topological interior $\text{int}C \neq \emptyset$. Let $k_0 \in \text{int}C$ and $e = -k_0$. We say that a nonempty set $A \subset Y$ is C-proper if $A + C \neq Y$ and denote by $\mathcal{P}_{0C}(Y)$ the family of all C-proper subsets of Y. $A \subset Y$ is said to be C-closed if $A + C$ is a closed set, C-bounded if for each neighborhood U of zero in Y there is some positive number t such that $A \subset tU + C$ and C-compact if any cover of A of the form $\{U_\alpha + C : U_\alpha \text{ are open}\}$ admits a finite subcover. $A \subset Y$ is said to be bounded if for each neighborhood U of zero in Y there is some positive number t such that $A \subset tU$. Suppose that $A, B \subset Y$. By $A \leq_C B$ and $A \leq_{intC} B$ we denote $B \subset A + C$ and $B \subset A + \text{int}C$, respectively. Similarly, by $A \nleq_C B$ and $A \nleq_{intC} B$ we denote $B \not\subset A + C$ and $B \not\subset A + \text{int}C$, respectively.

Assume that $I : X \to 2^Y$ is a set-valued mapping with nonempty values at each point in X. I is said to be bounded (closed, compact, convex, C-closed, C-bounded, C-compact)-valued if for each $x \in X$, $I(x)$ is a bounded (closed, compact, convex, C-closed, C-bounded, C-compact) set. I is said to be C-lower semi-continuous iff, for any $A \subset Y$, the set $\{x \in X : I(x) \leq_C A\}$ is closed. I is said to be upper semi-continuous (*u.s.c.* for short) at $x \in X$ if for any open set $U \supset I(x)$, there exists a neighborhood V of x such that $\bigcup_{x \in V} I(x) := I(V) \subset U$. I is said to be *u.s.c.* on X if I is *u.s.c.* at every point of X. I is said to be lower semi-continuous (*l.s.c.* for short) at $x^0 \in X$ if for any $y^0 \in I(x^0)$ and any neighborhood U of y^0, there exists a neighborhood V of x^0 such that $\forall x \in V$, $I(x) \cap U \neq \emptyset$. I is said to be *l.s.c.* on X if I is *l.s.c.* at every point of X. I is said to be continuous, if I is both *l.s.c.* and *u.s.c.* on X.

Consider the following set optimization problem:

$$(X, I) : \text{minimize} \;\; I(x), \;\, x \in X.$$

A point $y \in X$ is said to be a minimizer of (X, I) if and only if $\forall x \in X$, $I(x) \leq_C I(y)$ implies $I(y) \leq_C I(x)$, and the set of all minimizers is denoted by $\operatorname{argmin}(X, I)$. A point $y \in X$ is said to be a weak minimizer of (X, I) if and only if $I(x) \not\leq_{intC} I(y)$, for all $x \in X$, and the set of all weak minimizers is denoted by $\operatorname{argwmin}(X, I)$. Clearly, a minimizer of (X, I) must be a weak minimizer of (X, I), but the reverse may not hold. In this paper, we always assume that $\operatorname{argmin}(X, I) \neq \emptyset$ and $\operatorname{argwmin}(X, I) \neq \emptyset$.

Now we introduce three kinds of well-posedness for set optimization problem (X, I).

Definition 2.1 *(X, I) is said to be k_0-well-posed at $v \in$ argmin(X, I), if for each sequence $\{x_n\}$, which satisfies that $\exists \varepsilon_n > 0$, $\varepsilon_n \to 0$ such that*

$$I(x_n) \leq_C I(v) + \varepsilon_n k_0, \tag{1}$$

it holds that $x_n \to v$.

The sequence $\{x_n\}$ as in (1) is called a k_0-minimizing sequence to $v \in$ argmin(X, I).

Well-posedness defined in Definition 2.1 is a notion to study the behavior of the function at one point in $\operatorname{argmin}(X, I)$. The following two definitions investigate the behavior of the variables when the corresponding objective function values are approached in different means to the sets of minimizers.

Definition 2.2 *(X, I) is said to be generalized k_0-well-posed, if for each sequence $\{x_n\}$, which satisfies that $\exists \varepsilon_n > 0$, $\varepsilon_n \to 0$ and $\exists z_n \in$ argmin(X, I) such that*

$$I(x_n) \leq_C I(z_n) + \varepsilon_n k_0, \tag{2}$$

there exist a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $x^* \in$ argmin(X, I) such that $x_{n_k} \to x^*$.

The sequence $\{x_n\}$ as in (2) is called a generalized k_0-minimizing sequence.

Definition 2.3 *(X, I) is said to be extended k_0-well-posed, if for every sequence $\{x_n\}$, which satisfies that $\exists \varepsilon_n > 0$, $\varepsilon_n \to 0$, $\forall x \in X$,*

$$I(x_n) \not\subset I(x) + \varepsilon_n k_0 + intC, \tag{3}$$

there exist a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $x^* \in$ argwmin(X, I) such that $x_{n_k} \to x^*$.

The sequence $\{x_n\}$ as in (3) is called an extended k_0-minimizing sequence.

If problem (X, I) is neither generalized k_0-well-posed nor extended k_0-well-posed, then it is called k_0-ill-posed.

Let us illustrate these definitions by the following examples.

Example 2.1 *Let $X = R$, $C = R^2_+$ and $k_0 = (1, 1)$.*

(i) *Let the mapping $F_1 : [0, 4\pi] \to R^2$ be defined by*

$$F_1(x) = \begin{cases} (x, sinx) + \{(y_1, y_2) \mid y_1^2 + y_2^2 = \frac{1}{9}\}, \ if \ x \in [0, 2\pi), \\ (x, sinx) + \{(y_1, y_2) \mid y_1^2 + y_2^2 = \frac{1}{4}\}, \ if \ x \in [2\pi, 4\pi]. \end{cases}$$

Then, set optimization problem (X, F_1) is k_0-well-posed at $x = \frac{3\pi}{2}$ and $x = \frac{7\pi}{2}$.

(ii) *Let the mapping $F_2 : [0, +\infty) \to R^2$ be defined by*

$$F_2(x) = \{\lambda(0, 1) + (1 - \lambda)(x, 0) : 0 \le \lambda \le 1\}.$$

Then set optimization problem (X, F_2) is generalized k_0-well-posed.

(iii) *Let the mapping $F_3 : (0, +\infty) \to R^2$ be defined by*

$$F_3(x) = (x - n, n) + [0, 1] \times [0, 1], \ x \in (n, n + 1], \ n = 0, 1, \ldots$$

Then, set optimization problem (X, F_3) is extended k_0-well-posed.

(iv) *Let the mapping $F_4 : (-\infty, 0] \to R^2$ be defined by*

$$F_4(x) = \begin{cases} \{(0, u), 0 \le u \le 1\}, \ if \ x = 0, \\ \{(2^x, u), 0 \le u \le \frac{1}{1-2^x}\}, \ if \ x \in (-\infty, 0). \end{cases}$$

Then, set optimization problem (X, F_4) is k_0-ill-posed.

Remark 2.1 **(1)** *It is not difficult to see that k_0-minimizing sequence to $v \in argmin(X, I)$ and generalized k_0-minimizing sequence are always exist. From Proposition 3.1 of [16], when $I(X)$ is C-bounded, extended k_0-minimizing sequence exists.*

(2) *Suppose that (X, I) is k_0-well-posed at $v \in argmin(X, I)$, then there exists no point $u \in X$, $u \ne v$ satisfying both $I(v) \le_C I(u)$ and $I(u) \le_C I(v)$.*

(3) *Problem (X, I) is generalized k_0-well-posed iff $argmin(X, I)$ is compact and $d(x_n, argmin(X, I)) \to 0$ for every generalized k_0-minimizing sequence $\{x_n\}$. (X, I) is extended k_0-well-posed iff $argwmin(X, I)$ is compact and $d(x_n, argwmin(X, I)) \to 0$ for every extended k_0-minimizing sequence $\{x_n\}$.*

(4) *Assume that argmin*(X, I) *is compact and* I *is continuous. If problem* (X, I) *is* k_0*-well-posed at* y *for each* $y \in$ *argmin*(X, I)*, then* (X, I) *is generalized* k_0*-well-posed.*

Next we recall the definitions of well-posedness and and generalized well-posedness for a scalar optimization problem in [19]. Let $f : X \to R$ be a real-valued function. Consider the following scalar optimization problem:

$$(X, f) : \quad \min_{x \in X} f(x).$$

- (X, f) is called Tykhonov well-posed iff f has an unique minimizer on X towards which every sequence $u_n \in X$ such that $f(u_n) \to \inf f(X)$ converges.
- (X, f) is called generalized well-posed in the scalar sense iff the set of minimizers of (X, f) is not empty, and every sequence $\{u_n\} \subset X$ such that $f(u_n) \to \inf f(X)$ has some subsequence $\{u_{n_k}\}$ converging to a minimizer of (X, f).

Remark 2.2 *If* $Y = R$*,* $C = R_+$*,* $k_0 = 1$ *and* I *is single-valued, then* k_0*-well-posed at* $v \in$ *argmin*(X, I) *reduces to the Tykhonov well-posedness in [19]. The generalized* k_0*-well-posedness and extended* k_0*-well-posedness for* (X, I) *reduce to the generalized well-posedness in the scalar sense (see [19]).*

3 Scalarization and Well-posedness of (X, I)

In this section, we recall the Gerstewizt's function studied in [20] and discuss the equivalent relations of three kinds of well-posedness between set optimization problems and scalar optimization problems, respectively.

Definition 3.1 *([20]) Let* $a \in Y$*. It is said that* $\phi_{e,a} : Y \to R$ *defined by*

$$\phi_{e,a}(y) = \min\{t \in R : y \in te + a + C\}, \text{ for } y \in Y,$$

is the Gerstewizt's function.

The Gerstewizt's function is continuous and strictly decreasing on Y. This function is also called nonlinear scalarization functional. It plays important roles in many areas.

Based on Definition 3.1, Hern*á*ndez and Rodríguez-Marín (see [17]) introduced a generalized Gerstewitz's function as follows.

Definition 3.2 *([17]) Let the function* $G_e(\cdot,\cdot)$*:* $\mathcal{P}_{0C}(Y)^2 \to R \cup \{\infty\}$ *defined by setting*

$$G_e(A,B) = \sup_{b\in B}\{\phi_{e,A}(b)\}, \ \textit{for } (A,B) \in \mathcal{P}_{0C}(Y)^2,$$

where the function $\phi_{e,A} : Y \to R \cup \{-\infty\}$ *is defined by*

$$\phi_{e,A}(y) = \inf\{t \in R : y \in te + A + C\}, \ \textit{for } y \in Y.$$

Note that when $A = \{a\}$ and $B = \{y\}$, the function $G_e(A,B)$ reduces to the function $\phi_{e,a}(y)$.

From Proposition 3.2, Theorem 3.6 and Theorem 3.9 of [17], we immediately obtain the following important properties of $G_e(\cdot,\cdot)$.

Lemma 3.1 *([17]) Let* A *be a* C*-bounded set and* $B \in \mathcal{P}_{0C}(Y)$*. Then* B *is* C*-bounded if and only if* $G_e(A,B) < \infty$*.*

Lemma 3.2 *([17])* A *is* C*-closed and* B *is* C*-bounded. Then the following equality holds:*

$$G_e(A,B) = \min\{r \mid B \subset re + A + C\}.$$

Lemma 3.3 *([17]) Assume that* $r \in R$*,* A *is* C*-closed and* B *is* C*-bounded. Then,*

(i) $G_e(A,B) \le r \Leftrightarrow B \subset re + A + C$*;*

(ii) *If* B_1 *and* B_2 *are* C*-compact sets and* $B_2 \le_{intC} B_1$*, then* $G_e(A,B_1) < G_e(A,B_2)$*;*

(iii) $G_e(A,A) = 0$*.*

Lemma 3.4 *Assume* $A, A_1, A_2, B \in \mathcal{P}_{0C}(Y)$*,* $r \in R$*,* A, A_1, A_2 *are* C*-closed and* B *is* C*-bounded. Then,*

(i) $G_e(A + \varepsilon k_0, B) = G_e(A,B) + \varepsilon$*, for all* $\varepsilon \ge 0$*;*

(ii) $G_e(A, B + \varepsilon k_0) = G_e(A,B) - \varepsilon$*, for all* $\varepsilon \ge 0$*;*

(iii) *If* $A_1 \le_C A_2$*, then* $G_e(A_1,B) \le G_e(A_2,B)$*;*

(iv) $G_e(A,B) < r \Leftrightarrow B \subset re + A + intC$*.*

Proof. (i) From the properties of the function $G_e(\cdot,\cdot)$,

$$\begin{aligned} G_e(A+\varepsilon k_0, B) &= \min\{r \mid B \subset re + A + \varepsilon k_0 + C\} \\ &= \min\{r \mid B \subset (r-\varepsilon)e + A + C\} \\ &= \min\{r-\varepsilon \mid B \subset (r-\varepsilon)e + A + C\} + \varepsilon \\ &= G_e(A,B) + \varepsilon. \end{aligned}$$

(ii) From the properties of the function $G_e(\cdot,\cdot)$,

$$\begin{aligned} G_e(A, B+\varepsilon k_0) &= \min\{l \mid B + \varepsilon k_0 \subset le + A + C\} \\ &= \min\{l \mid B \subset (l+\varepsilon)e + A + C\} \\ &= \min\{l+\varepsilon \mid B \subset (l+\varepsilon)e + A + C\} - \varepsilon \\ &= G_e(A,B) - \varepsilon. \end{aligned}$$

(iii) Since

$$G_e(A_2, B) = \min\{r \mid B \subset re + A_2 + C\},$$

$$B \subset G_e(A_2, B)e + A_2 + C.$$

Let $A_1 \leq_C A_2$. Then we have

$$B \subset G_e(A_2, B)e + A_1 + C \text{ and } G_e(A_1, B) \leq G_e(A_2, B).$$

(iv) Suppose $G_e(A,B) < r$. Then there exists an $\lambda < r$ such that $G_e(A,B) \leq \lambda$. It follows from Lemma 3.3 (i) that

$$\begin{aligned} B &\subset \lambda e + A + C \\ &= (\lambda - r + r)e + A + C \\ &= re + A - (r-\lambda)e + C. \end{aligned}$$

By $-(r-\lambda)e \in \text{int}C$, we have $B \subset re + A + \text{int}C$.

Conversely, if $B \subset re + A + \text{int}C$, then for every $b \in B$, there exist $a \in A$ and $c \in \text{int}C$ such that $b = re + a + c$. Since Y is a linear topological space, one can find a real number $t > 0$ such that $c + te \in C$. Set $\lambda := r - t$, we get

$$b = re + a + c = (t+\lambda)e + a + c = \lambda e + a + c + te.$$

Hence $y \in \lambda e + a + C$. This implies $B \subset \lambda e + A + C$. Thus, from Lemma 3.3 (i), we have $G_e(A,B) \leq \lambda < r$. The proof is complete. $\square$

Theorem 3.1 *Suppose that $v \in$ argmin(X, I) and I is C-bounded-valued and C-closed-valued. Then,*

(i) *the problem (X, I) is k_0-well-posed at v if and only if the problem $(X, G_e(I(\cdot), I(v)))$ is Tykhonov well-posed;*

(ii) *the problem (X, I) is generalized k_0-well-posed if and only if argmin(X, I) is compact and the scalar problem $(X, \inf_{v \in argmin(X,I)} G_e(I(\cdot), I(v)))$ is generalized well-posed in the scalar sense;*

(iii) *if I is C-compact-valued, the problem (X, I) is extended k_0-well-posed if and only if argwmin(X, I) is compact and the scalar problem $(X, -\inf_{x \in X} G_e(I(x), I(\cdot)))$ is generalized well-posed in the scalar sense.*

Proof. First, we prove that for $v \in \text{argmin}(X, I)$ and every $x \in X$, $G_e(I(x), I(v)) \geq 0$. On a contrary, suppose that there exist $x \in X$ and $r < 0$, such that $G_e(I(x), I(v)) < r$. From Lemma 3.4 (iv), we have

$$I(v) \subset re + I(x) + \text{int}C \subset I(x) + \text{int}C.$$

Since v is a minimizer of (X, I), $I(x) \subset I(v) + C$. Therefore, $I(x) \subset I(x) + \text{int}C$, which is a contradiction.

Next, we prove that for $x \in X$, $G_e(I(x), I(v)) = 0$ if and only if $I(x) \leq_C I(v)$ and $I(v) \leq_C I(x)$. In fact, combining with $G_e(I(x), I(v)) = 0$ and $v \in \text{argmin}(X, I)$, we have that

$$I(v) \subset I(x) + C \text{ and } I(x) \subset I(v) + C,$$

i.e.,

$$I(x) \leq_C I(v) \text{ and } I(v) \leq_C I(x).$$

Conversely, assume $I(x) \subset I(v) + C$ and $I(v) \subset I(x) + C$. From $I(v) \subset I(x) + C$, we have $G_e(I(x), I(v)) \leq 0$. If $G_e(I(x), I(v)) < 0$, then there exists $r < 0$ such that $G_e(I(x), I(v)) < r$. From Lemma 3.4 (iv), we have $I(v) \subset re + I(x) + \text{int}C$. Thus,

$$I(x) \subset I(v) + C \subset re + I(x) + \text{int}C \subset I(x) + \text{int}C,$$

which is a contradiction.

Finally, we prove that (i), (ii) and (iii) hold, respectively.

(i) Assume that $\{x_n\}$ is a sequence satisfying

$$G_e(I(x_n), I(v)) \to \min_{x\in X} G_e(I(x), I(v)) = 0.$$

It follows from Lemma 3.3 (ii) that we may assume $G_e(I(x_n), I(v)) = \varepsilon_n \geq 0$ and $\varepsilon_n \to 0$, which implies $I(v) \subset \varepsilon_n e + I(x_n) + C$. Then, $\{x_n\}$ is a k_0-minimizing sequence to v. Since (X, I) is k_0-well-posed at v, we have $x_n \to v$.

Now we show that v is the unique minimizer of the scalar problem $(X, G_e(I(\cdot), I(v))$. In fact, if there exists $u \neq v$ such that $G_e(I(u), I(v)) = 0$, we take $x_n = u$ for all n. Then, it follows from Lemma 3.3 (i) that $\{x_n\}$ is a k_0-minimizing sequence to v. So $x_n \to v$, which is a contradiction. Thus, $(X, G_e(I(\cdot), I(v))$ is Tykhonov well-posed.

Conversely, we assume that $\{x_n\}$ is a sequence such that there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ satisfying $I(x_n) \leq_C I(v) + \varepsilon_n k_0$. Then

$$0 = \min_{x\in X} G_e(I(x), I(v)) \leq G_e(I(x_n), I(v)) \leq G_e(I(v) + \varepsilon_n k_0, I(v)) = \varepsilon_n.$$

Since the scalar problem $(X, G_e(I(\cdot), I(v)))$ is well-posed, $\{x_n\}$ converges to v, i.e., the problem (X, I) is k_0-well-posed at v.

(ii) Assume that $\{x_n\}$ is a sequence satisfying

$$\inf_{v\in argmin(X,I)} G_e(I(x_n), I(v)) \to \inf_{x\in X} \inf_{v\in argmin(X,I)} G_e(I(x), I(v)) = 0.$$

Then there exist $\varepsilon'_n > 0$, $\varepsilon'_n \to 0$ and $v_{x_n} \in \operatorname{argmin}(X, I)$ such that

$$0 \leq G_e(I(x_n), I(v_{x_n})) = \min\{r \mid I(v_{x_n}) \subset re + I(x_n) + C\} < \varepsilon'_n.$$

It follows from Lemma 3.3 (ii) that there exists $\varepsilon_n \in [0, \varepsilon'_n)$ satisfying $G_e(I(x_n), I(v_{x_n})) = \varepsilon_n$, which implies $I(v_{x_n}) \subset \varepsilon_n e + I(x_n) + C$. Since the problem (X, I) is generalized k_0-well-posed, $\{x_n\}$ has a subsequence converging to some point $u \in \operatorname{argmin}(X, I)$. Thus, the scalar optimization problem $(X, \inf_{v\in argmin(X,I)} G_e(I(\cdot), I(v)))$ is generalized well-posed in the scalar sense.

Conversely, assume that $\{x_n\}$ is a sequence, which satisfies that $\exists \varepsilon_n > 0$, $\varepsilon_n \to 0$ and $v_n \in \operatorname{argmin}(X, I)$ such that

$$I(x_n) \leq_C I(v_n) + \varepsilon_n k_0. \tag{4}$$

Then, from Lemma 3.4 (iii), we have

$$0 = \inf_{x\in X} \inf_{v\in argmin(X,I)} G_e(I(x), I(v))$$

$$
\begin{aligned}
&\leq \inf_{v\in argmin(X,I)} G_e(I(x_n), I(v)) \\
&\leq G_e(I(x_n), I(v_n)) \\
&\leq G_e(I(v_n)+\varepsilon_n k_0, I(v_n)) \\
&= \varepsilon_n.
\end{aligned}
$$

Since $\inf_{v\in argmin(X,I)} G_e(I(\cdot), I(v))$ is generalized well-posed in the scalar sense, $\{x_n\}$ has some subsequence converging to some point $u \in \text{argmin}(X, I)$.

(iii) Firstly, we assume that $\{x_n\}$ is a sequence, which satisfies $I(x_n) \not\subset I(x) + \varepsilon_n k_0 + \text{int}C$, for all $x \in X$. Then, from Lemma 3.4 (iv), we have

$$G_e(I(x), I(x_n)) \geq -\varepsilon_n, \forall x \in X,$$

i.e.

$$-\inf_{x\in X} G_e(I(x), I(x_n)) \leq \varepsilon_n.$$

If there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $-\inf_{x\in X} G_e(I(x), I(x_{n_k})) < 0$, then

$$\forall x \in X, \quad G_e(I(x), I(x_{n_k})) > 0,$$

which means that

$$I(x_{n_k}) \not\subset I(x) + \text{int}C, \quad \forall x \in X.$$

Thus, $x_{n_k}, k = 1, 2, \cdots$, are the weak minimizers of (X, I). From the compactness of argwmin(X, I), there exists a subsequence of $\{x_{n_k}\}$ converging to a point in argwmin(X, I).

If there exists no subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$-\inf_{x\in X} G_e(I(x), I(x_{n_k})) < 0,$$

then, there is a subsequence $\{x'_{n_k}\} \subset \{x_n\}$ such that

$$0 \leq -\inf_{x\in X} G_e(I(x), I(x'_{n_k})) \leq \varepsilon_{n_k}.$$

Since for all $v \in \text{argwmin}(X, I)$, $\inf_{x\in X} G_e(I(x), I(v)) = 0$ and $(X, -\inf_{x\in X} G_e(I(x), I(\cdot)))$ is well-posed, there exists a subsequence of $\{x'_{n_k}\}$ converging to a point of argwmin(X, I). Therefore, the problem (X, I) is extended k_0-well-posed.

Conversely, suppose that $\{x_n\}$ is a sequence, which satisfies that $\exists \varepsilon_n \geq 0$ with $\varepsilon_n \to 0$ such that

$$-\inf_{x\in X} G_e(I(x), I(x_n)) \leq \inf_{v\in X}[-\inf_{x\in X} G_e(I(x), I(v))] + 2\varepsilon_n.$$

So for all $v \in X$, we have

$$\inf_{x\in X} G_e(I(x), I(x_n)) \geq \inf_{x\in X} G_e(I(x), I(v)) - 2\varepsilon_n.$$

Then, for any $v \in X$ and n, there exists a point $y_{n,v} \in X$ such that

$$G_e(I(x), I(x_n)) \geq G_e(I(y_{n,v}), I(v)) - \varepsilon_n, \forall x \in X.$$

Especially, taking $x = y_{n,v}$, we have

$$G_e(I(y_{n,v}), I(x_n)) \geq G_e(I(y_{n,v}), I(v)) - \varepsilon_n.$$

It follows from Lemma 3.3 (ii) and Lemma 3.4 (ii) that

$$I(x_n) \not\subset I(v) + \varepsilon_n k_0 + \mathrm{int}C.$$

By the arbitrariness of v and n, we get

$$I(x_n) \not\subset I(v) + \varepsilon_n k_0 + \mathrm{int}C, \forall v \in X \text{ and } n.$$

Since (X, I) is extended k_0-well-posed, there exists a subsequence of $\{x_n\}$ converging to a point $x_0 \in \mathrm{argwmin}(X, I)$.

Now we show $x_0 \in \mathrm{argmin}(X, -\inf_{x\in X} G_e(I(x), I(\cdot)))$. In fact, from $x_0 \in \mathrm{argwmin}(X, I)$, we have that for every $x \in X$, $G_e(I(x), I(x_0)) \geq 0$. Arbitrarily choosing $y \in X$, we obtain

$$\inf_{x\in X} G_e(I(x), I(x_0)) \geq G_e(I(y), I(y))$$

by Lemma 3.3 (iii). Then, for every $y \in X$,

$$-\inf_{x\in X} G_e(I(x), I(y)) \geq -\inf_{x\in X} G_e(I(x), I(x_0)),$$

i.e., $x_0 \in \mathrm{argmin}(X, -\inf_{x\in X} G_e(I(x), I(\cdot)))$. Thus, the scalar problem $(X, -\inf_{x\in X} G_e(I(x), I(\cdot)))$ is generalized well-posed in the scalar sense. This completes the proof. □

Corollary 3.1 *Let X be a compact space, I be a C-bounded and C-lower semi-continuous mapping defined on X. Suppose that for every $v \in$ argmin(X, I), there exists no point $u \in X$, $u \neq v$ satisfying both $I(v) \leq_C I(u)$ and $I(u) \leq_C I(v)$. Then, (X, I) is k_0-well-posed at v.*

Proof. Naturally, for any real number α,

$$\{x \in X : G_e(I(x), I(v)) \leq \alpha\} = \{x \in X : I(v) \subset \alpha e + I(x) + C\}.$$

From the closedness of the set $\{x \in X : A \subset I(x) + C, \forall A \subset Y\}$, we have that the set

$$\{x \in X : G_e(I(x), I(v)) \leq \alpha\}$$

is also closed for any α. So, $G_e(I(\cdot), I(v))$ is lower semi-continuous on X. From the compactness of X and Example 6 of [19, p.3], the scalar problem $(X, G_e(I(\cdot), I(v)))$ is Tykhonov well-posed. Thus, by Theorem 3.1 (i), we get that (X, I) is k_0-well-posed at v. $\square$

Corollary 3.2 *Let X be a locally compact metric space. Assume that $v \in$ argmin(X, I), I is C-bounded-valued and C-closed-valued and $\{x \in X : \forall \varepsilon > 0, \exists v \in$ argmin$(X, I), I(x) \leq_C I(v) + (t + \varepsilon)k_0\}$ is connected. Then, the following three assertions are equivalent:*

(i) *$\exists t > 0$, $\{x \in X : \forall \varepsilon > 0, \exists v \in$ argmin$(X, I), I(x) \leq_C I(v) + (t + \varepsilon)k_0\}$ is compact;*

(ii) *(X, I) is generalized k_0-well-posed;*

(iii) *argmin(X, I) is nonempty and compact.*

Proof. We know the set $\{x \in X : \inf_{v \in argmin(X,I)} G_e(I(x), I(v)) \leq t\}$ is equivalent to the set $\{x \in X : \forall \varepsilon > 0, \exists v \in \text{argmin}(X, I),\ G_e(I(x), I(v)) \leq t + \varepsilon\}$, which is also equivalent to the set $\{x \in X : \forall \varepsilon > 0, \exists v \in \text{argmin}(X, I),\ I(x) \leq_C I(v) + (t + \varepsilon)k_0\}$. So, assumption (i) is equivalent to the conclusion that

$$\{x \in X : \inf_{v \in argmin(X,I)} G_e(I(x), I(v)) \leq t\}$$

is compact for some $t > \inf_{x \in X} \inf_{v \in argmin(X,I)} G_e(I(x), I(v)) = 0$. From Proposition 37 of [19, p.25] and Theorem 3.1 (ii), we conclude that (i), (ii) and (iii) are equivalent. $\square$

Remark 3.1 *When I is scalar-valued, $C = R_+$ and $k_0 = 1$, Corollary 3.1 reduces to Example 6 of [19, p.3] and Corollary 3.2 reduces to Proposition 37 of [19, p.25].*

As [19], we introduce a function $c : R_+ \to R$, which is called a generalized forcing function if and only if $c(t) \geq 0$, $c(0) = 0$ and

$$t_n \geq 0,\ c(t_n) \to 0 \ \Rightarrow\ \exists \{t_{n_k}\} \subset \{t_n\} \text{ such that } t_{n_k} \to 0 \ \ (k \to \infty). \tag{5}$$

Theorem 3.2 *Assume that X is a metric space and I is C-bounded-valued and C-closed-valued. Then,*

(i) *if (X, I) is generalized k_0-well-posed, for any fixed $x \in X$, there exists some generalized forcing function c satisfying, for all $x' \in argmin(X, I)$,*

$$I(x') \not\subset I(x) + c(d(x, argmin(X, I))e + intC. \tag{6}$$

Conversely, if $argmin(X, I) \neq \emptyset$ and $argmin(X, I)$ is compact, (6) holds for some c satisfying (5), then (X, I) is generalized k_0-well-posed;

(ii) *(X, I) is k_0-well-posed at $v \in argmin(X, I)$ if and only if there exists a generalized forcing function c satisfying $I(v) \not\subset I(x) + c(d(x, v))e + intC$.*

Proof. (i) Let (X, I) be generalized k_0-well-posed. Define

$$c(t) = \inf\{G_e(I(x), I(x')) : d(x, \mathrm{argmin}(X, I)) = t,\ x' \in \mathrm{argmin}(X, I)\}.$$

It is easy to see that $c(t) \geq 0$. We conclude that $c(0) = 0$ since $\mathrm{argmin}(X, I)$ is compact.

Now let $t_n \geq 0$ with $c(t_n) \to 0$. Then $\exists\, x_n \in X$, $x'_n \in \mathrm{argmin}(X, I)$, $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$d(x_n, \mathrm{argmin}(X, I)) = t_n$$

and

$$G_e(I(x_n),\ I(x'_n)) \leq \varepsilon_n.$$

Thus, we have

$$I(x'_n) \subset \varepsilon_n e + I(x_n) + C.$$

From the definition of generalized k_0-well-posedness, we have that there exists a subsequence $\{x_{n_k}\}$ such that $d(x_{n_k}, \mathrm{argmin}(X, I)) \to 0$, namely, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \to 0$.

In addition, by the definition of $c(t)$, we have $\forall\, x' \in \mathrm{argmin}(X, I)$, such that

$$G_e(I(x), I(x')) \geq c(d(x, \mathrm{argmin}(X, I)),$$

which implies

$$I(x') \not\subset I(x) + c(d(x, \mathrm{argmin}(X, I))e + \mathrm{int}C.$$

Conversely, if for $x_n \in X$, $\varepsilon_n > 0$ with $\varepsilon_n \to 0$, $\exists\, x'_n \in \mathrm{argmin}(X, I)$, such that

$$I(x'_n) \subset I(x_n) + \varepsilon_n e + C,$$

then we have

$$\varepsilon_n \geq G_e(I(x_n), I(x'_n)) \geq c(d(x_n, \text{argmin}(X, I)).$$

So,

$$d(x_n, \text{argmin}(X, I)) \to 0.$$

From the compactness of argmin(X, I), we conclude that (X, I) is generalized k_0-well-posed.

(ii) The proof of (ii) is similar to that of (i). So we omit it. $\square$

4 Criteria and Characterizations of Well-Posedness

Now we consider some characterizations and criteria of well-posedness for set optimization problems. For every bounded set $A \subset X$, we recall the Kuratowski measure of noncompactness of A (see [2]):

$$\alpha(A) = \inf\{k > 0 : A \ has \ a \ finite \ cover \ of \ sets \ with \ diameter \ < k\}.$$

The generalized k_0-well-posedness (extended k_0-well-posedness) can be characterized by the behavior of $\varepsilon -$ argmin(X, I) ($\varepsilon -$ argmin$'(X, I)$) as $\varepsilon \to 0$, which is defined as

$$\varepsilon - \text{argmin}(X, I) = \{x \in X \mid I(z) + \varepsilon k_0 \subset I(x) + C, \exists z \in \text{argmin}(X, I)\}$$

$$(\varepsilon - \text{argmin}'(X, I) = \{x \in X \mid I(z) + \varepsilon k_0 \not\leq_{intC} I(x), \forall z \in X\}).$$

It is clear that $\varepsilon -$argmin$(X, I) \neq \emptyset$, and if $I(X)$ is C-bounded, then $\varepsilon -$argmin$'(X, I) \neq \emptyset$.

Proposition 4.1 *Suppose that $I : X \to 2^Y$ is compact-valued and $\varepsilon -$ argmin(X, I) is closed. Then the problem (X, I) is generalized k_0-well-posed (extended k_0-well-posed) and argmin(X, I) (argmin$'(X, I)$) is compact if and only if*

$$\alpha[\varepsilon - \textit{argmin}(X, I)] \to 0 \ (\alpha[\varepsilon - \textit{argmin}'(X, I)] \to 0)$$

as $\varepsilon \to 0$.

Proof. Put

$$L(\varepsilon) = \varepsilon - \text{argmin}(X, I).$$

Similar to Theorem 3.2 in [2], we only need to verify argmin$(X, I) = \cap_{\varepsilon > 0} L(\varepsilon)$.

Let $x_1 \in \bigcap_{\varepsilon>0} L(\varepsilon)$ and take any $\varepsilon > 0$. Then there exists $z_0 \in \text{argmin}(X, I)$ such that

$$I(z_0) + \varepsilon k_0 \subset I(x_1) + C.$$

Let $\varepsilon \to 0$. Since I is compact-valued, we have

$$I(z_0) \subset I(x_1) + C.$$

By virtue of $z_0 \in \text{argmin}(X, I)$, we deduce that the point x_1 must be in $\text{argmin}(X, I)$. Thus, $\bigcap_{\varepsilon>0} L(\varepsilon) \subset \text{argmin}(X, I)$. On the other hand, it is clear that

$$\text{argmin}(X, I) \subset \bigcap_{\varepsilon>0} L(\varepsilon).$$

So, we have proved that $\text{argmin}(X, I) = \bigcap_{\varepsilon>0} L(\varepsilon)$.

Similarly, let $L'(\varepsilon) = \varepsilon - \text{argmin}'(X, I)$. We only need to verify

$$\text{argwmin}(X, I) = \cap_{\varepsilon>0} L'(\varepsilon).$$

The inclusion relation $\text{argwmin}(X, I) \subset \cap_{\varepsilon>0} L'(\varepsilon)$ holds obviously. Now we show that $\cap_{\varepsilon>0} L'(\varepsilon) \subset \text{argwmin}(X, I)$. In fact, if $x \in \cap_{\varepsilon>0} L'(\varepsilon)$ but $x \notin \text{argwmin}(X, I)$, then there exist $z_0 \in X$, $\varepsilon_n \to 0^+$ and $y_n \in I(x)$ such that

$$I(x) \subset I(z_0) + \text{int}C, \tag{7}$$

and

$$y_n \notin I(z_0) + \varepsilon_n k_0 + \text{int}C.$$

Since I is compact-valued, without loss of generality, we may assume that $y_n \to y_0 \in I(x)$. So, $y_0 \notin I(z_0) + \text{int}C$, which contradicts (7). The proof is complete. □

Let

$$L(v, \alpha) := \{x \in X \mid I(x) \leq_C I(v) + \alpha k_0\}.$$

Proposition 4.2 *Assume $I : X \to 2^Y$ is compact-valued. Then, the set optimization problem (X, I) is k_0-well-posed at $v \in$ argmin(X, I) if and only if $\inf_{\alpha}$ diam $L(v, \alpha) = 0$.*

Proof. The proof is similar to that of Theorem 11 [19, p.5]. So it is omitted. □

Let X^* be the topological dual space of X and $C^* = \{b \in X^* \mid b(c) \geq 0, \forall c \in C\}$. We define the function $I_\varepsilon = I + \varepsilon d(\cdot, v) k_0$, where $v \in$ *argmin*(X, I) and $\varepsilon > 0$. Obviously, $v \in \text{argmin}(X, I_\varepsilon)$.

Theorem 4.1 *Assume that $I : X \to 2^Y$ is convex-valued, compact-valued and bounded valued. Then the set optimization problem (X, I_ε) is k_0-well-posed at $v \in argmin(X, I)$.*

Proof. From Proposition 4.2, we only need to prove that there exists a constant d, which is independent of α, such that

$$L_\varepsilon(v, \alpha) := \{x \in X \mid I_\varepsilon(x) \leq_C I(v) + \alpha k_0\} \subset B(v, \alpha d),$$

where $B(v, \alpha d)$ is a ball at v with radius αd. Let $x \in L(v, \alpha)$ with $x \neq v$.

If $I(x) \not\leq_C I(v)$, then there exists $y \in I(v)$ such that $y \cap (I(x) + C) = \emptyset$. Especially, we have $y \cap I(x) = \emptyset$. Since I is convex valued, there exists an $b \in C^*$ such that for all $y_1 \in I(x)$,

$$b(y) < b(y_1).$$

Then,

$$\min b(I(v)) < \min b(I(x)).$$

Let $\bar{b} \in C^*$ satisfying $\bar{b}(k_0) > 0$. Since I is compact-valued, we may choose an $\beta > 0$ such that

$$\min(b + \beta\bar{b})(I(v)) < \min(b + \beta\bar{b})(I(x)).$$

It follows from $x \in L_\varepsilon(v, \alpha)$ that

$$\begin{aligned}
\min(b + \beta\bar{b})(I(x) + \varepsilon d(x, v)k_0) &\leq \min(b + \beta\bar{b})(I(v) + \varepsilon d(v, v)k_0) + \alpha(b + \beta\bar{b})(k_0) \\
&< \min(b + \beta\bar{b})(I(x)) + \alpha(b + \beta\bar{b})(k_0).
\end{aligned}$$

Hence, $\varepsilon d(x, v)(b + \beta\bar{b})(k_0) < \alpha(b + \beta\bar{b})(k_0)$, i.e.,

$$d(x, v) \leq \alpha/\varepsilon.$$

If $I(x) \leq_C I(v)$, then $I(v) \leq_C I(x)$. Since $x \in L(v, \alpha)$, we conclude that

$$I(v) + \varepsilon d(x, v)k_0 \leq_C I(x) + \varepsilon d(x, v)k_0 = I_\varepsilon(x) \leq_C I(v) + \alpha k_0,$$

i.e.,

$$I(v) + \alpha k_0 \subset I(v) + \varepsilon d(x, v)k_0 + C.$$

If $\alpha < \varepsilon d(x, v)$, then,

$$I(v) \subset I(v) + (\varepsilon d(x, v) - \alpha)k_0 + C \subset I(v) + intC,$$

which is a contradiction. So, we have that $\varepsilon d(x,v) \leq \alpha$, i.e., $d(x,v) \leq \alpha/\varepsilon$. Let $d = 1/\varepsilon$. We conclude that $d(x,v) \leq \alpha d$, for each $x \in L(v,\alpha)$. It is said that $L(v,\alpha) \subset B(v,\alpha d)$. $\square$

For any $k_0 \in \mathrm{int}C$, let

$$M(\varepsilon) = \varepsilon - \mathrm{argmin}(X, I),$$

$$M'(\varepsilon, v) = \{x \in X \mid I(v) + \varepsilon k_0 \subset I(x) + C\},$$

$$M''(\varepsilon) = \varepsilon - \mathrm{argmin}'(X, I).$$

Proposition 4.3 **(i)** *$M(\cdot)$ is u.s.c. at 0 and argmin(X, I) is compact iff the problem (X, I) is generalized k_0-well-posed.*

(ii) *$M'(\cdot, v)$ is u.s.c. at 0 and $M'(0, v) = \{v\}$ iff the problem (X, I) is k_0-well-posed at v.*

(iii) *$M''(\cdot)$ is u.s.c. at 0 and argwmin$(X, I) \neq \emptyset$ is compact iff the problem (X, I) is extended k_0-well-posed.*

Proof. We only prove that (i) holds, since the proofs of (ii) and (iii) are similar to that of (i).

First of all, it is easy to prove that

$$M(0) = \mathrm{argmin}(X, I).$$

Suppose that (X, I) is generalized k_0-well-posed. It follows from Remark 2.1 that argmin(X, I) is compact. So, we only need to prove that $M(\cdot)$ is *u.s.c.* at 0. Suppose $M(\cdot)$ is not *u.s.c.*. Then there is a neighborhood $N(M(0))$ of $M(0)$, such that for any neighborhood U of 0, there exists ε' satisfying

$$\{x \in X \mid I(z) + \varepsilon' k_0 \subset I(x) + C, \exists z \in \mathrm{argmin}(X, I)\} \not\subset N(M(0)).$$

Thus, we can choose $\varepsilon_n \to 0$ satisfying

$$\exists x_n \in X, \;\; z_n \in \mathrm{argmin}(X, I), \tag{8}$$

such that

$$I(z_n) + \varepsilon_n k_0 \subset I(x_n) + C \text{ and } x_n \notin N(M(0)). \tag{9}$$

We deduce from (8) and (9) that $\{x_n\}$ is a generalized k_0-minimizing sequence. Therefore, there exist a subsequence $\{x_{n_k}\}$ and $z \in \mathrm{argmin}(X, I)$ such that $x_{n_k} \to z$. From (9), $x_{n_k} \notin N(M(0))$, which is a contradiction.

Conversely, suppose that $\exists\, \varepsilon_n \geq 0$ with $\varepsilon_n \to 0$, $x_n \in X,\ z_n \in \operatorname{argmin}(X, I)$ such that

$$I(z_n) + \varepsilon_n k_0 \subset I(x_n) + C.$$

Since $M(\cdot)$ is *u.s.c.* at 0, for any $N(M(0))$, when n is sufficiently large, we get that $x_n \in N(M(0))$. Therefore, for every neighborhood W of 0, there exists $n_0 \in N$ such that $x_n \in \operatorname{argmin}(X, I) + W, \forall n > n_0$. By the compactness of $\operatorname{argmin}(X, I)$, we obtain that the problem (X, I) is generalized k_0-well-posed. □

References

[1] A. N. Tykhonov, On the stability of the functional optimization problem, USSR Compt. Math. Math. Phys., 6 (1966) 28-33.

[2] X. X. Huang, Extended and strongly extended well-posedness of set-valued optimization problems, Math. Methods Oper. Res., 53 (2001) 101-116.

[3] X. X. Huang, Pointwise well-posedness of perturbed vector optimization problems in a vector-valued variational principle, J. Optim. Theory Appl., 108(3) (2001) 671-686.

[4] E. Miglierina, E. Molho, M. Rocca, Well-posedness and scalarization in vector optimization, J. Optim. Theory Appl., 126(2) (2005) 391-409.

[5] D. Dentcheva, S. Helbig, On variational principles, level sets, well-posedness and ε-solutions in vector optimization, J. Optim. Theory Appl., 89 (1996) 329-349.

[6] R. Hu, Y. P. Fang, Set-valued increasing-along-rays maps and well-posed set-valued star-shaped optimization, J. Math. Anal. Appl., 331 (2007) 1371-1383.

[7] M. B. Lignola, Well-posedness and L-well-posedness for quasivariational inequalities, J. Optim. Theory Appl., 128(1) (2006) 119-138.

[8] R. Lucchetti, I. Revalski, (Eds.) Recent development in well-posed variational problems, Kluwer Academic Publishers, Dordrecht, 1995.

[9] E. M. Bednarczuk, An approach to well-posedness in vector optimization problems: consequences to stability, Control and Cybernetics, 23(1994) 107-122.

[10] M. Rocca, Well-posed vector optimization problems and vector variational inequalities. J. Inf. Optim. Sci., 27(2) (2006) 259–270.

[11] X. X. Huang, X. Q. Yang, Levitin-Polyak well-posedness in generalized variational inequality problems with functional constraints, J. Ind. Manag. Optim., 3(4) (2007) 671-684.

[12] D. Kuroiwa, Some duality theorems of set-valued optimization with natural criteria, in: Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis, World Scientific, River Edge, NJ, (1999) 221-228.

[13] D. Kuroiwa, On set-valued optimization, Nonlinear Anal., 47 (2001) 1395-1400.

[14] D. Kuroiwa, Existence theorems of set optimization with set-valued maps, J. Inf. Optim. Sci., 24 (2003) 73-84.

[15] D. Kuroiwa, Existence of efficient points of set optimization with weighted criteria, J. Nonlinear Convex Anal., 4 (2003) 117-123.

[16] T. X. D. Ha, Some variants of Ekeland variational priciple for a set-valued map, J. Optim. Theory Appl., 124(1) (2005) 187-206.

[17] E. Hernández, L. Rodríguez-Marín, Nonconvex scalarization in set optimization with set-valued maps, J. Math. Anal. Appl., 325(1) (2007) 1-18.

[18] M. Alonso, L. Rodríguez-Marín, Set-relations and optimality conditions in set-valued maps, Nonlinear Anal., 63 (2005) 1167-1179.

[19] A. L. Dontchev, T. Zolezzi, Well-posed optimization Problems. In: Lecture Notes in Mathematics, 1543, Springer-Verlag, Berlin, 1993.

[20] A. Göpfort, H. Riahi, C. Tammer, C. Zălinescu, Variational methods in partially ordered spaces, Springer, New York, 2003.