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Research Article

Painleve-Kuratowski Convergences for the Solution Sets of Set-Valued Weak Vector Variational Inequalities

Z. M. Fang,[1] S. J. Li,[1] and K. L. Teo[2]

[1] *College of Mathematics and Science, Chongqing University, Chongqing, 400044, China*
[2] *Department of Mathematics and Statistics, Curtin University of Technology, P.O. Box U1987, Perth, WA 6845, Australia*

Correspondence should be addressed to S. J. Li, lisj@cqu.edu.cn

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Painleve-Kuratowski convergence of the solution sets is investigated for the perturbed set-valued weak vector variational inequalities with a sequence of mappings converging continuously. The closedness and Painleve-Kuratowski upper convergence of the solution sets are obtained. We also obtain Painleve-Kuratowski upper convergence when the sequence of mappings converges graphically. By virtue of a sequence of gap functions and a key assumption, Painleve-Kuratowski lower convergence of the solution sets is established. Some examples are given for the illustration of our results.

1. Introduction

Since the concept of vector variational inequality (VVI) was introduced by Giannessi [1] in 1980, many important results on various kinds of vector variational inequality problems have been established, such as existence of solutions, relations with vector optimization, stability of solution set maps, gap function, and duality theories (see, e.g., [2–8] and the references cited therein).

The stability analysis of the solution set maps for the parametric (VVI) problem is of considerable interest amongst researchers in the area. Some results on the semicontinuity of the solution set maps for the parametric (VVI) problem with the parameter perturbed in the space of parameters are now available in the literature. In [4], Khanh and Luu proved the upper semicontinuity of the solution set map for two classes of parametric vector quasivariational inequalities. In [7], Li et al. established the upper semicontinuity property of the solution set map for a perturbed generalized vector quasivariational inequality problem and also obtained the lower semicontinuity property of the solution set map for a perturbed classical scalar variational inequality. In [9], Cheng and Zhu investigated the upper and lower semicontinuities of the solution set map for a parameterized weak vector variational inequality in a finite dimensional Euclidean space by using a scalarization method. In [6], Li and Chen obtained the closedness and upper semicontinuity of the solution set map for a parametric weak vector variational inequality under weaker conditions than those assumed in [9]. Then, under a key assumption, they proved a lower semicontinuity result of the solution set map in a finite dimensional space by using a parametric gap function.

However, there are few investigations on the convergence of the sequence of mappings. In particular, almost no stability results are available for the perturbed (VVI) problem with the sequence of mappings converging continuously or graphically. It appears that the only relevant paper is [10], where Lignola and Morgan considered generalized variational inequality in a reflexive Banach space with a sequence of operators converging continuously and graphically and obtained the convergence of the solution sets under an assumption of pseudomonotonicity. Since the perturbed (VVI) problem with a sequence of mappings converging is different from the parametric (VVI) problem with the parameter perturbed in a space of parameters, these results do not apply to the parametric (VVI) problem with the parameter perturbed in a space of parameters. Thus, it is important to study Painleve-Kuratowski upper and lower convergences of the sequence of solution sets.

In passing, it is worth noting that some stability results are available for the vector optimization and vector equilibrium problems with a sequence of sets converging in the sense of Painleve-Kuratowski (see [11–13]). It is well known that the vector equilibrium problem is a generalization of (VVI) problem. However, if the results obtained for the vector equilibrium problem are to be applied to the (VVI) problem, the required assumptions are on the (VVI) problem as a whole. There is no information about the conditions that are required on the functions defining the (VVI) problem. Clearly, this is unsatisfactory. Our study of the stability properties for the perturbed (VVI) problem with a sequence of converging mappings is under appropriate assumptions on the function defining the (VVI) problem rather than on the (VVI) problem as a whole.

In this paper, we should establish Painleve-Kuratowski upper and lower convergences of the solution sets of the perturbed set-valued weak variational inequity (SWVVI) with a sequence of converging mappings in a Banach space. We first discuss Painleve-Kuratowski upper convergence and closedness of the solution sets. To obtain Painleve-Kuratowski lower convergence of the solution sets, we introduce a sequence of gap functions based on the nonlinear scalarization function introduced by Chen et al. in [14] and a key assumption (H_g) imposed on the sequence of gap functions. Then, we obtain Painleve-Kuratowski lower convergence of the solution sets for $(\text{SWVVI})_n$.

The rest of the paper is organized an follows. In Section 2, we introduce problems (SWVVI) and $(\text{SWVVI})_n$, and recall some definitions and important properties of these problems. In Section 3, we investigate Painleve-Kuratowski upper convergence and the closedness of the solution sets. In Section 4, we introduce respective gap functions for problems (SWVVI) and $(\text{SWVVI})_n$ and then establish Painleve-Kuratowski lower convergence of the solution sets under a key assumption.

2. Preliminaries

Let X and Y be two Banach spaces and let $L(X, Y)$ be the set of all linear continuous mappings from X to Y. The value of a linear mapping $t \in L(X, Y)$ at $x \in X$ is denoted by $\langle t, x\rangle$. Let $C \subset Y$

be a closed and convex cone with nonempty interior, that is, $\operatorname{int} C \neq \emptyset$. We define the ordering relations as follows.

For any $y_1, y_2 \in Y$,

$$y_1 \leq_{\operatorname{int} C} y_2 \Longleftrightarrow y_2 - y_1 \in \operatorname{int} C,$$
$$y_1 \nleq_{\operatorname{int} C} y_2 \Longleftrightarrow y_2 - y_1 \notin \operatorname{int} C. \tag{2.1}$$

Consider the set-valued weak vector variational inequality (SWVVI) problem for finding $x \in K$ and $t \in T(x)$ such that

$$\langle t, y - x \rangle \in Y \setminus -\operatorname{int} C \quad \forall y \in K, \tag{2.2}$$

where $K \subset X$ is a nonempty subset and $T : K \to 2^{L(X,Y)}$ is a set-valued mapping.

For a sequence of set-valued mappings $T_n : K_n \to 2^{L(X,Y)}$, we define a sequence of set-valued weak vector variational inequality $(\text{SWVVI})_n$ problems for finding $x_n \in K_n$ and $t_n \in T_n(x_n)$ such that

$$\langle t_n, y - x_n \rangle \in Y \setminus -\operatorname{int} C \quad \forall y \in K_n, \tag{2.3}$$

where $K_n \subset X$ is a sequence of nonempty subsets.

We denote the solution sets of problems (SWVVI) and $(\text{SWVVI})_n$ by $I(T)$ and $I(T_n)$, respectively, that is,

$$I(T) = \{x \in K \mid \exists t \in T(x), \text{ s.t. } \langle t, y - x \rangle \in Y \setminus -\operatorname{int} C \, \forall y \in K\},$$
$$I(T_n) = \{x_n \in K_n \mid \exists t_n \in T_n(x_n), \text{ s.t. } \langle t_n, y - x_n \rangle \in Y \setminus -\operatorname{int} C \, \forall y \in K_n\}. \tag{2.4}$$

Throughout this paper, we assume that $I(T) \neq \emptyset$ and $I(T_n) \neq \emptyset$. The stability analysis is to investigate the behaviors of the solution sets $I(T)$ and $I(T_n)$.

Now we recall some basic definitions and properties of problems (SWVVI) and $(\text{SWVVI})_n$. For each $\varepsilon > 0$ and a subset $A \subset X$, let the open ε-neighborhood of A be defined as $U(A, \varepsilon) = \{x \in X \mid \exists a \in A, \text{ s.t. } \|a - x\| < \varepsilon\}$. The notation $B(\lambda, \delta)$ denotes the open ball with center λ and radius $\delta > 0$.

In the following, we introduce some concepts of the convergence of set sequences and mapping sequences which will be used in the sequel. Define

$$\begin{aligned} \mathcal{N}_\infty &:= \{N \subset \mathcal{N} \mid \mathcal{N} \setminus N \text{ finite}\} \\ &= \{\text{subsequences of } \mathcal{N} \text{ containing all } n \in \mathcal{N} \text{ beyond some } \overline{n}\}, \\ \mathcal{N}_\infty^\sharp &:= \{N \subset \mathcal{N} \mid N \text{ infinite}\} \\ &= \{\text{all subsequences of } \mathcal{N}\}, \end{aligned} \tag{2.5}$$

where $\mathcal{N}$ denotes the set of all positive integer numbers and $\overline{n}$ is an integer in $\mathcal{N}$.

Definition 2.1 (see [11, 15]). Let X be a normed space. A sequence of sets $\{D_n \subset X : n \in N\}$ is said to converge in the sense of Painleve-Kuratowski (P.K.) to D (i.e., $D_n \overset{\text{P.K.}}{\longrightarrow} D$) if

$$\limsup_{n\to\infty} D_n \subset D \subset \liminf_{n\to\infty} D_n \tag{2.6}$$

with

$$\begin{aligned} \liminf_{n\to\infty} D_n &:= \left\{x \mid \exists N \in \mathcal{N}_\infty, \exists x_n \in D_n \ (n \in N) \text{ with } x_n \longrightarrow x\right\}, \\ \limsup_{n\to\infty} D_n &:= \left\{x \mid \exists N \in \mathcal{N}^\sharp_\infty, \exists x_n \in D_n \ (n \in N) \text{ with } x_n \longrightarrow x\right\}. \end{aligned} \tag{2.7}$$

It is said that the sequence $\{D_n\}$ upper converges in the sense of Painleve-Kuratowski to D if $\limsup_{n\to\infty} D_n \subset D$. Similarly, the sequence $\{D_n\}$ is said to lower converge in the sense of Painleve-Kuratowski to D if $D \subset \liminf_{n\to\infty} D_n$.

Definition 2.2 (see [15]). A set-valued mapping $S : X \rightarrow 2^Y$ is outer semicontinuous (osc) at $\overline{x}$ if $\limsup_{x\to\overline{x}} S(x) \subset S(\overline{x})$ with $\limsup_{x\to\overline{x}} S(x) := \bigcup_{x_n\to\overline{x}} \limsup_{n\to\infty} S(x_n)$.

On the other hand, it is inner semicontinuous (isc) at $\overline{x}$ if $\liminf_{x\to\overline{x}} S(x) \supset S(\overline{x})$ with $\liminf_{x\to\overline{x}} S(x) := \bigcap_{x_n\to\overline{x}} \liminf_{n\to\infty} S(x_n)$.

The set-valued mapping is said to be continuous at $\overline{x}$, written as $S(x) \rightarrow S(\overline{x})$ as $x \rightarrow \overline{x}$ if it is both outer semicontinuous and inner semicontinuous.

Definition 2.3 (see [15]). Let $S_n : X \rightarrow 2^Y$ be a sequence of set-valued mappings and $S : X \rightarrow 2^Y$ be a set-valued mapping. It is said that the sequence $\{S_n\}$ converges continuously to S at x if

$$\limsup_{n\to\infty} S_n(x_n) \subset S(x) \subset \liminf_{n\to\infty} S_n(x_n) \quad \forall \text{ sequence } x_n \longrightarrow x. \tag{2.8}$$

If $\{S_n\}$ converges continuously to S at every $x \in X$, then it is said that $\{S_n\}$ converges continuously to S on X.

Let $S : X \rightarrow 2^Y$ be a set-valued map, the graph of S is defined as

$$\operatorname{gph} S = \left\{(x, u) \mid u \in S(x)\right\}. \tag{2.9}$$

Applying set convergence theory to the graphs of the mappings, we obtain the graphical convergence of the sequence of mappings.

Definition 2.4 (see [15]). For a sequence of mappings $S_n : X \rightarrow 2^Y$, the graphical outer limit, denoted by $g - \limsup_n S_n$, is the mapping which has as its graph the set $\limsup_n (\operatorname{gph} S_n)$:

$$\begin{aligned} \operatorname{gph}\left(g - \limsup_n S_n\right) &= \limsup_n \left(\operatorname{gph} S_n\right), \\ \left(g - \limsup_n S_n\right)(x) &= \left\{u \mid \exists N \in \mathcal{N}^\sharp_\infty,\ x_n \overset{N}{\longrightarrow} x,\ u_n \overset{N}{\longrightarrow} u,\ u_n \in S_n(x_n)\right\}. \end{aligned} \tag{2.10}$$

The graphical inner limit, denote by $g - \liminf_n S_n$, is the mapping having as its graph the set $\liminf_n(\operatorname{gph} S_n)$:

$$\operatorname{gph}\left(g - \liminf_n S_n\right) = \liminf_n (\operatorname{gph} S_n),$$
$$\left(g - \liminf_n S_n\right)(x) = \left\{u \mid \exists N \in \mathcal{N}_\infty,\ x_n \xrightarrow{N} x,\ u_n \xrightarrow{N} u,\ u_n \in S_n(x_n)\right\}. \tag{2.11}$$

If the outer and inner limits of the mappings S_n agree, it is said that their graphical limit, $g - \lim_n S_n$, exists. In this case, the notation $S_n \xrightarrow{g} S$ is used, and the sequence $\{S_n\}$ of mappings is said to converge graphically to S. Clearly, $S_n \xrightarrow{g} S \Leftrightarrow \operatorname{gph} S_n \xrightarrow{\text{P.K.}} \operatorname{gph} S$.

Proposition 2.5 (see [15]). *For any sequence of mappings $S_n : X \to 2^Y$, it holds that*

$$\left(g - \liminf_n S_n\right)(x) = \bigcup_{\{x_n \to x\}} \liminf_{n\to\infty} S_n(x_n),$$
$$\left(g - \limsup_n S_n\right)(x) = \bigcup_{\{x_n \to x\}} \limsup_{n\to\infty} S_n(x_n), \tag{2.12}$$

where the unions are taken over all sequences $x_n \to x$. Thus, the sequence $\{S_n\}$ converges graphically to S if and only if, at each point $\overline{x} \in X$, it holds that

$$\bigcup_{\{x_n \to \overline{x}\}} \limsup_{n\to\infty} S_n(x_n) \subset S(\overline{x}) \subset \bigcup_{\{x_n \to \overline{x}\}} \liminf_{n\to\infty} S_n(x_n). \tag{2.13}$$

From Proposition 2.5 and Definition 2.3, the following proposition follows readily.

Proposition 2.6. *Let $S_n : X \to 2^Y$ be a sequence of set-valued mappings and $S : X \to 2^Y$ be a set-valued mapping. Then, the sequence $\{S_n\}$ outer converges graphically to S if and only if $\{S_n\}$ outer converges continuously to S, that is,*

$$g - \limsup_n S_n \subset S \Longleftrightarrow \limsup_n S_n(x_n) \subset S(x) \quad \text{for any } x \in X,\ \forall \text{ sequences } x_n \longrightarrow x. \tag{2.14}$$

Definition 2.7 (see [10]). Given a sequence of mappings S_n, $\{S_n\}$ is said to be uniformly bounded if for any sequence x_n contained in a bounded set, there exists a positive number k such that for any sequence u_n with $u_n \in S_n(x_n)$ for all $n \in N$, it holds that

$$\|u_n\| \le k \quad \forall n \in N. \tag{2.15}$$

Proposition 2.8 (see [16]). *For any fixed $e \in \operatorname{int} C$, $y \in Y$, $r \in R$, and the nonlinear scalarization function $\xi_e : Y \to R$ defined by $\xi_e(y) = \min\{t \in R : y \in te - C\}$:*

(i) *ξ_e is a continuous and convex function on Y;*

(ii) *$\xi_e(y) < r \Leftrightarrow y \in re - \operatorname{int} C$;*

(iii) *$\xi_e(y) \ge r \Leftrightarrow y \notin re - \operatorname{int} C$.*

3. Painleve-Kuratowski upper convergence of the solution sets

In this section, our focus is on the Painleve-Kuratowski upper convergence and the closedness of the solution sets.

Theorem 3.1. *Suppose that*

(i) T_n *outer converges continuously to* T*, that is,*

$$\limsup_{n\to\infty} T_n(x_n) \subset T(x) \quad \textit{for any sequence } \{x_n\} \textit{ with } x_n \longrightarrow x; \tag{3.1}$$

(ii) $K_n \overset{\text{P.K.}}{\longrightarrow} K$;

(iii) T_n *are uniformly bounded.*

Then, $\limsup_{n\to\infty} I(T_n) \subset I(T)$*, that is to say for any subsequence* $\{x_{n_k}\}$ *of solutions to* $(SWVVI)_n$*, if* $x_{n_k} \to x$*, then* x *is a solution to* $(SWVVI)$*.*

Proof. The proof is listed on contradiction arguments. On a contrary, suppose that $\exists x \in \limsup_{n\to\infty} I(T_n)$ but $x \notin I(T)$.

From $x \in \limsup_{n\to\infty} I(T_n)$, we have $x = \lim_{k\to\infty} x_{n_k}$, where $x_{n_k} \in I(T_{n_k})$ and $\{n_k\}$ is a subsequence of N. Then, there exists $t_{n_k} \in T_{n_k}(x_{n_k})$ such that

$$\langle t_{n_k}, z - x_{n_k}\rangle \in Y \setminus -\operatorname{int} C \quad \forall z \in K_{n_k}. \tag{3.2}$$

Since $K \subset \liminf_{n\to\infty} K_n$, it is clear that for any $z' \in K$, there exists a sequence $\{z_{n_k}\}$ with $\{z_{n_k}\} \subset K_{n_k}$ and $z_{n_k} \to z'$, as $k \to \infty$. Thus,

$$\langle t_{n_k}, z_{n_k} - x_{n_k}\rangle \in Y \setminus -\operatorname{int} C. \tag{3.3}$$

Since $\limsup_{n\to\infty} K_n \subset K$ and $x_{n_k} \in K_{n_k}$, we have $x \in K$. Now, we note that $x \notin I(T)$. Thus, for all $t \in T(x)$, there exists $z_t \in K$ such that

$$\langle t, z_t - x\rangle \in -\operatorname{int} C. \tag{3.4}$$

From the uniform boundedness of T_n, we may assume, without loss of generality, that $t_{n_k} \to t_0$ (though a subsequence of $\{t_{n_k}\}$ if necessary). By (i), we get $t_0 \in T(x)$. Thus,

$$\langle t_{n_k}, z_{n_k} - x_{n_k}\rangle \longrightarrow \langle t_0, z' - x\rangle, \quad \text{as } k \longrightarrow +\infty. \tag{3.5}$$

It follows from (3.3) and the closedness of $Y \setminus -\operatorname{int} C$ that

$$\langle t_0, z' - x\rangle \in Y \setminus -\operatorname{int} C \quad \forall z' \in K, \tag{3.6}$$

which is a contradiction to (3.4). This completed the proof. □

Remark 3.2. Let $X = E$ and $Y = E^*$, where E is a reflexive Banach space and E^* is its dual. If we take $C = R^+$, $(\text{SWVVI})_n$ reduce to the generalized variational inequality problems with perturbed operators $(\text{GVI})_n$ considered in [10, Section 3]. The convergence for the solution sets of $(\text{GVI})_n$ was studied under the the pseudomonotonicity assumption in [10]. Furthermore, if T and T_n are vector-valued mappings, then $(\text{SWVVI})_n$ reduce to $(\text{VI})_n$ considered in [10, Section 2]. We also notice that the Painleve-Kuratowski upper convergence of the solution sets of $(\text{SWVVI})_n$ is obtained under weaker assumptions than these assumed in [10, Proposition 2.1] for obtaining convergence of the solution sets.

From Proposition 2.5 and Theorem 3.1, we obtain readily the following corollary.

Corollary 3.3. *Suppose that*

(i) T_n *outer converges graphically to* T*, written as* $g - \limsup_n T_n \subset T$*, that is,*

$$\limsup_{n\to\infty} (gph\, T_n) \subset gph\, T; \tag{3.7}$$

(ii) $K_n \xrightarrow{\text{P.K.}} K$;

(iii) $\{T_n\}$ *is uniformly bounded.*

Then, $\limsup_{n\to\infty} I(T_n) \subset I(T)$.

Remark 3.4. Let $X = Y = R^m$. Then, problems $(\text{SWVVI})_n$ reduce to the generalized variational inequalities with perturbed operators considered in [10, Proposition 3.1] and the convergence was obtained under the assumption that the operators converge graphically.

Theorem 3.5. *Suppose that*

(i) T *is osc on* K*, that is, for all* $x \in K$, $\limsup_{n\to\infty} T(x_n) \subset T(x)$ *for any sequence* $x_n \to x$;

(ii) K *and* $T(K)$ *are compact sets.*

Then, $I(T)$ *is a compact set.*

Proof. First, we prove that $I(T)$ is a closed set. Take any sequence $x_n \in I(T)$ with $x_n \to x$. Then, there exists $t_n \in T(x_n)$ such that

$$\langle t_n, z - x_n \rangle \in Y \setminus -\text{int}\, C \quad \forall z \in K. \tag{3.8}$$

It follows from the compactness of K that $x \in K$. Suppose that $x \notin I(T)$, we have

$$\forall t \in T(x), \quad \exists z_0 \in K, \text{ s.t. } \langle t, z_0 - x \rangle \in -\text{int}\, C. \tag{3.9}$$

Since $T(K)$ is a compact set, without loss of generality, we assume that there exists a t_0 such that $t_n \to t_0$. Thus, we have $\langle t_n, z - x_n \rangle \to \langle t_0, z - x \rangle$. By (i), we get a $t_0 \in T(x)$. It follows from (3.8) and the closedness of $Y \setminus -\text{int}\, C$ that

$$\langle t_0, z - x \rangle \in Y \setminus -\text{int}\, C \quad \forall z \in K, \tag{3.10}$$

which contradicts with (3.9). Hence, $x_0 \in I(T)$ and $I(T)$ is a closed set. Next, it follows from $I(T) \subset K$ and the compactness of K that $I(T)$ is a compact set. The proof is completed. □

Similarly, we have the following result.

Theorem 3.6. *For any n, suppose that*

(i) *T_n is osc on K_n, that is, $\forall x \in K_n$*

$$\limsup_{m\to\infty} T_n(x_m) \subset T_n(x) \quad \text{for any sequence } x_m \longrightarrow x; \tag{3.11}$$

(ii) *K_n and $T_n(K_n)$ are compact sets.*

Then, $I(T_n)$ is a compact set.

4. Painleve-Kuratowski lower convergence of the solution sets

In this section, we focus on the lower convergence of the solution sets. Assume that K and K_n are compact sets and that for each $x \in X$, $T(x)$ and $T_n(x)$ are compact sets. Let $g : K \to R$ and $g_n : K_n \to R$ be functions defined by

$$\begin{aligned} g(x) &= \max_{t\in T(x)} \min_{y\in K} \xi_e(\langle t, y - x\rangle), \quad x \in K, \\ g_n(x_n) &= \max_{t_n\in T_n(x_n)} \min_{y\in K_n} \xi_e(\langle t_n, y - x_n\rangle), \quad x_n \in K_n. \end{aligned} \tag{4.1}$$

Since K, K_n, $T(x)$, and $T_n(x)$ are compact sets and $\xi_e(\cdot)$ is continuous, $g(x)$ and $g_n(x_n)$ are well defined.

Proposition 4.1. (i) *$g(x) \le 0$ for all $x \in K$;*
(ii) *$g_n(x_n) \le 0$ for all $x_n \in K_n$;*
(iii) *$g(x_0) = 0$ if and only if $x_0 \in I(T)$;*
(iv) *$g_n(x_n) = 0$ if and only if $x_n \in I(T_n)$.*

Proof. Define

$$\overline{g}(x,t) = \min_{y\in K} \xi_e(\langle t, y - x\rangle), \quad x \in K,\ t \in T(x). \tag{4.2}$$

We first prove that $\overline{g}(x,t) \le 0$. On a contrary, we suppose that this is false. Then, there exist $\overline{x} \in K$ and $\overline{t} \in T(\overline{x})$ such that $\overline{g}(\overline{x},\overline{t}) > 0$. Thus,

$$0 < \overline{g}(\overline{x},\overline{t}) \le \xi_e(\langle \overline{t}, y - \overline{x}\rangle) \quad \forall y \in K, \tag{4.3}$$

which is impossible when $y = \overline{x}$. Therefore,

$$g(x) = \max_{t\in T(x)} \overline{g}(x,t) \le 0 \quad \forall x \in K. \tag{4.4}$$

By the same taken, we can show that

$$g_n(x_n) = \max_{t_n \in T_n(x_n)} \min_{y \in K_n} \xi_e(\langle t_n, y - x_n \rangle) \leq 0 \quad \forall x_n \in K_n. \tag{4.5}$$

On the other hand, if $g(x_0) = 0$, then there exists a $t_0 \in T(x_0)$ such that $\overline{g}(x_0, t_0) = 0$, that is,

$$\min_{y \in K} \xi_e(\langle t_0, y - x_0 \rangle) = 0, \quad x_0 \in K. \tag{4.6}$$

From Proposition 2.8, (4.6) is valid if and only if for any $y \in K$,

$$\xi_e(\langle t_0, y - x_0 \rangle) \geq 0. \tag{4.7}$$

Clearly, (4.7) holds if and only if for any $y \in K$, $\langle t_0, y - x_0 \rangle \in Y \setminus -\operatorname{int} C$, that is, $x_0 \in I(T)$. This proves that (iii) holds.

Similarly, we can show that (iv) holds.

The functions g_n are called the gap functions for $(\text{SWVVI})_n$ if properties (ii) and (iv) of Proposition 4.1 are satisfied.

In view of hypothesis (H_g) of [6, 17, 18], we introduce the following key assumption:

(H_g): given the sequence $\{T_n\}$ for any $\epsilon > 0$, there exist an $\alpha > 0$ and an $\overline{n}$ such that $g_n(x_n) \leq -\alpha$ for all $n > \overline{n}$ and for all $x_n \in K_n \setminus U(I(T_n), \epsilon)$.

Geometrically, the hypothesis (H_g) means that given a sequence of mappings $\{T_n\}$, we can find for any small positive number $\epsilon > 0$, a small positive number $\alpha > 0$ and a large-enough positive number $\overline{n} > 0$ such that for all $n > \overline{n}$, if a feasible point x_n is away from the solution sets $I(T_n)$ by distance of at least ϵ, then the values of all gap functions for $(\text{SWVVI})_n$ is less than or equal to at least some "$-\alpha$."

To illustrate assumption (H_g), we give the following example.

Example 4.2. Let

$$\begin{gathered} X = R, \qquad Y = R^2, \\ T_n(x) = \begin{pmatrix} 1 \\ \left[1, 1 + \dfrac{1}{n} + x^2\right] \end{pmatrix}, \\ T(x) = \begin{pmatrix} 1 \\ [1, 1 + x^2] \end{pmatrix}, \\ K = K_n = [0, 1], \qquad C = R_+^2. \end{gathered} \tag{4.8}$$

Consider problems $(\text{SWVVI})_n$. From direct computation, we obtain $I(T_n) = \{0\}$. To check assumption (H_g), we take $e = (1,1)^T \in \operatorname{int} R_+^2$. Then,

$$\begin{aligned} g_n(x_n) &= \max_{t_n \in T_n(x_n)} \min_{y \in K_n} \xi_e(\langle t_n, y - x_n \rangle) \\ &= \max_{t_n \in T_n(x_n)} \min_{y \in K_n} \max_{1 \le i \le 2} [\langle t_n, y - x_n \rangle]_i \\ &= \max_{z_n \in [1, 1 + (1/n) + x_n^2]} \min_{y \in K_n} \max \{y - x_n, z_n(y - x_n)\} \\ &= -x_n. \end{aligned} \tag{4.9}$$

For any given $0 < \epsilon$, we take $\alpha = \epsilon > 0$ and $N = 1$. Then, for all $n > N$ and for all $x_n \in K_n \setminus \cup(I(T_n), \epsilon)$, we have $g_n(x_n) = -x_n \le -\alpha$. Hence, assumption (H_g) is valid. □

Lemma 4.3. *Suppose that*

(i) *T_n inner converges continuously to T, that is,*

$$T(x) \subset \liminf_{n \to \infty} T_n(x_n) \quad \textit{for any sequence } \{x_n\} \textit{ with } x_n \longrightarrow x; \tag{4.10}$$

(ii) *$K_n \xrightarrow{\text{P.K.}} K$;*

(iii) *$\bigcup_{n=1}^{\infty} K_n$ is a compact set.*

Then, for any $\delta \ge 0$, $x_0 \in K$ and sequence $\{x_n\}$ with $x_n \in K_n$ and $x_n \to x_0$, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ and $N > 0$ such that $g_{n_l}(x_{n_l}) \ge g(x_0) - \delta$ for all $l \ge N$.

Proof. Let $\tilde{g} : K \times L(X, Y) \to R$ be a function defined by

$$\tilde{g}(x, t) = \min_{y \in K} \xi_e(\langle t, y - x \rangle), \quad x \in K,\ t \in T(x). \tag{4.11}$$

From the continuity of $\xi_e(\langle t, y - x \rangle)$ with respect to (x, t, y), the compactness of K and [19, Chapter 3, Section 1, Proposition 23], we have that $\tilde{g}(x, t)$ is continuous with respect to (x, t). Thus, from the compactness of $T(x_0)$, there exists a $t_0 \in T(x_0)$ such that

$$g(x_0) = \max_{t \in T(x_0)} \min_{y \in K} \xi_e(\langle t, y - x_0 \rangle) = \max_{t \in T(x_0)} \tilde{g}(x_0, t) = \min_{y \in K} \xi_e(\langle t_0, y - x_0 \rangle). \tag{4.12}$$

From assumption (i), there exists a sequence $\{t_n\}$ satisfying $t_n \in T_n(x_n)$ such that

$$t_n \longrightarrow t_0. \tag{4.13}$$

It follows from the compactness of K_n that there exists $\{y_n\}$ with $y_n \in K_n$ such that

$$\min_{y \in K_n} \xi_e(\langle t_n, y - x_n \rangle) = \xi_e(\langle t_n, y_n - x_n \rangle). \tag{4.14}$$

Since $\bigcup_{n=1}^{\infty} K_n$ is compact, we assume, without loss of generality, that $y_n \longrightarrow y_0$. Thus, it follows from (ii) that $y_0 \in K$. Consequently,

$$\lim_{n\to\infty} \xi_e(\langle t_n, y_n - x_n\rangle) = \xi_e(\langle t_0, y_0 - x_0\rangle) \geq \min_{y\in K} \xi_e(\langle t_0, y - x_0\rangle) = g(x_0). \tag{4.15}$$

So, for any $\delta > 0$, there exists an $N > 0$ such that $\xi_e(\langle t_n, y_n - x_n\rangle) \geq g(x_0) - \delta$ for all $n \geq N$. By (4.14), we have

$$\begin{aligned} g_n(x_n) &= \max_{t_n\in T_n(x_n)} \min_{y\in K_n} \xi_e(\langle t_n, y - x_n\rangle) \\ &\geq \min_{y\in K_n} \xi_e(\langle t_n, y - x_n\rangle) \\ &= \xi_e(\langle t_n, y_n - x_n\rangle) \\ &\geq g(x_0) - \delta \quad \forall n \geq N. \end{aligned} \tag{4.16}$$

Hence, the result holds. □

Set $T_0 = T$ and $K_0 = K$. We have the following lemma.

Lemma 4.4. *Suppose that for $n = 0, 1, 2, \ldots$, T_n is osc on K_n, that is, for $n = 0, 1, 2, \ldots$,*

$$\limsup_{m\to\infty} T_n(x_m) \subset T_n(x) \quad \text{for any sequence } \{x_m\} \text{ with } x_m \longrightarrow x. \tag{4.17}$$

Then, $I(T) \subset \liminf_{n\to\infty} I(T_n)$ if and only if for all $\epsilon > 0$, $\exists N > 0$ such that $I(T) \subset U(I(T_n), \epsilon)$ for all $n > N$.

Proof. We assume $I(T) \subset \liminf_{n\to\infty} I(T_n)$, but there exists an $\epsilon > 0$ such that for all $N > 0$, there exists an $N_n \geq N$ satisfying $I(T) \not\subset U(I(T_{N_n}), \epsilon)$. Then, there exists a sequence $\{x_n\}$ with $x_n \in I(T)$, but $x_n \notin U(I(T_{N_n}), \epsilon)$. From Theorem 3.5, we note that $I(T)$ is a compact set. Without loss of generality, we assume $x_n \longrightarrow x$ and $x \in I(T)$. Thus, for any sequence $\{y_n\}$ satisfying $y_n \longrightarrow y$ with $y_n \in I(T_n)$, we have $\|y_{N_n} - x_n\| \geq \epsilon > 0$. Letting $n \longrightarrow \infty$, we get $\|y - x\| \geq \epsilon > 0$. Therefore, there does not exist any sequence $y_n \in I(T_n)$ satisfying $y_n \longrightarrow x$. This is a contradiction to $I(T) \subset \liminf_{n\to\infty} I(T_n)$.

Conversely, suppose that for any $\epsilon > 0$, $\exists N > 0$ such that $I(T) \subset U(I(T_n), \epsilon)$ for all $n \geq N$. From Theorem 3.6, we note that $I(T_n)$ is compact for all n. Thus, for any $x \in I(T)$, there exists $x_n \in I(T_n)$ such that $\|x_n - x\| = d(x, I(T_n)) \leq \epsilon$ for all $n \geq N$. So, we have $x_n \longrightarrow x$ and $I(T) \subset \liminf_{n\to\infty} I(T_n)$. Therefore, the result of the lemma follows readily. □

Now, we are in a position to state and prove our main result in the following theorem.

Theorem 4.5. *Suppose that assumption (H_g) holds and that the following conditions are satisfied:*

(i) *T_n is osc on K_n for $n = 0, 1, 2, \ldots$, that is, for $n = 0, 1, 2, \ldots$,*

$$\limsup_{m\to\infty} T_n(x_m) \subset T_n(x) \quad \text{for any sequence } \{x_m\} \text{ with } x_m \longrightarrow x; \tag{4.18}$$

(ii) T_n *inner converge continuously to* T*, that is,*

$$T(x) \subset \liminf_{n \to \infty} T_n(x_n) \quad \textit{for any sequence } \{x_n\} \textit{ with } x_n \longrightarrow x; \tag{4.19}$$

(iii) $\bigcup_{n=1}^{\infty} K_n$ *is a compact set;*

(iv) $K_n \xrightarrow{\text{P.K.}} K$.

Then, $I(T) \subset \liminf_{n\to\infty} I(T_n)$.

Proof. We prove the result via contradiction. On a contrary, we assume, by Lemma 4.4, that there exists an $\epsilon > 0$ such that for any $\overline{N} > 0$, we have $N_n \geq \overline{N}$ satisfying

$$I(T) \not\subset U(I(T_{N_n}), \epsilon), \tag{4.20}$$

that is, there exists a sequence $\{x_{N_n}\}$ satisfying

$$x_{N_n} \in I(T) \setminus U(I(T_{N_n}), \epsilon). \tag{4.21}$$

From the compactness of $I(T)$, we can assume, without loss of generality, that $x_{N_n} \to x \in I(T)$. Then, there exists an $\overline{N}_1 > 0$ such that $\|x_{N_n} - x\| \leq \epsilon/4$ forall $n > \overline{N}_1$. It is clear that $B(x, \epsilon/N_n) \cap K \neq \varnothing$ for any positive integer n. Since $K \subset \liminf_{n\to\infty} K_n$, there exist a sequence $\{y_{N_n}\} \subset K_{N_n}$ satisfying $y_{N_n} \to x$. Then, there exists an $\overline{N}_2 > 0$ such that $y_{N_n} \in B(x, \epsilon/N_n) \cap K_{N_n}$ for all $n > \overline{N}_2$.

Now, we note that $y_{N_n} \notin U(I(T_{N_n}), \epsilon/4)$. Otherwise, there would exist a sequence $\{z_{N_n}\}$ with $z_{N_n} \in I(T_{N_n})$ such that $\|y_{N_n} - z_{N_n}\| < \epsilon/4$. Thus, for $\overline{N}_0 = \max\{\overline{N}_1, \overline{N}_2\}$, we have

$$\|x_{N_n} - z_{N_n}\| \leq \|x_{N_n} - x\| + \|x - y_{N_n}\| + \|y_{N_n} - z_{N_n}\| \leq \frac{\epsilon}{4} + \frac{\epsilon}{N_n} + \frac{\epsilon}{4} < \epsilon \quad \forall n > \overline{N}_0. \tag{4.22}$$

This implies that $x_{N_n} \in U(I(T_{N_n}), \epsilon)$, which contradicts with (4.21). Thus,

$$y_{N_n} \in K_{N_n} \setminus U\left(I(T_{N_n}), \frac{\epsilon}{4}\right). \tag{4.23}$$

By hypothesis (H_g), there exist, for any $\epsilon > 0$, an $\alpha > 0$ and an $\overline{N}$ such that for all $n > \overline{N}$ and for all $x \in K_n \setminus U(I(T_n), \epsilon)$, $g_n(x) \leq -\alpha$. In particular, it follows from (4.23) that

$$g_{N_n}(y_{N_n}) \leq -\alpha \quad \text{for } n \text{ large enough.} \tag{4.24}$$

By virtue of Lemma 4.3, there exists, for any $\delta > 0$, a subsequence $\{y_{N_{n_k}}\}$ of $\{y_{N_n}\}$ and $\widetilde{N} > 0$ such that

$$g_{N_{n_k}}(y_{N_{n_k}}) \geq g(x) - \delta \quad \forall k > \widetilde{N}. \tag{4.25}$$

We can take δ such that $-\alpha + \delta < 0$. Thus,

$$g(x) \leq g_{N_{n_k}}\left(y_{N_{n_k}}\right) + \delta \leq -\alpha + \delta < 0, \tag{4.26}$$

that is,

$$\max_{t \in T(x)} \min_{y \in K} \xi_e\left(\langle t, y - x \rangle\right) < 0. \tag{4.27}$$

So, for any $t \in T(x)$, $\min_{y \in K} \xi_e(\langle t, y - x \rangle) < 0$. Thus, there exists a $y \in K$ such that

$$\xi_e\left(\langle t, y - x \rangle\right) < 0. \tag{4.28}$$

Consequently, by Proposition 2.8, we have $\langle t, y - x \rangle \in -\text{int}\, C$, which shows that $x \notin I(T)$. This contradicts with $x \in I(T)$. Therefore, our result follows readily. □

Now, we explain the applicability of Theorem 4.5 through an example.

Example 4.6. Consider Example 4.2. It follows from a direct computation that $I(T_n) = I(T) = \{0\}$. It is easy to testify that assumption (H_g) holds and so are conditions (i)–(v) of Theorem 4.5. Obviously, the solution sets of problem $(\text{SWVVI})_n$ lower converge in the sense of Painleve-Kuratowski.

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