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# ON A REFINEMENT OF THE CONVERGENCE ANALYSIS FOR THE NEW EXACT PENALTY FUNCTION METHOD FOR CONTINUOUS INEQUALITY CONSTRAINED OPTIMIZATION PROBLEM

Changjun Yu[†‡], Kok Lay Teo[†], Liansheng Zhang[‡] and Yanqin Bai[‡]

[†]Department of Mathematics and Statistics
Curtin University of Technology
Kent Street, Bentley 6102, WA, Australia

[‡]Department of Mathematics
Shanghai University
99, Shangda Road, 200444, Shanghai, China

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Abstract. This note is to provide a refinement of the convergence analysis of the new exact penalty function method proposed recently.

1. **Introduction.** As in [1], we consider a class of functional inequality constrained optimization problems given below.

$$\min \; f(x) \tag{1a}$$

$$\text{subject to} \qquad \phi_j(x,\omega) \le 0, \; \forall\, \omega \in \Omega, \; j = 1, \dots, m, \tag{1b}$$

where the vector $\mathbb{R}^n$ is the parameter vector to be found, $\Omega$ is a compact interval in $\mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable in $x$, and for each $j = 1, \dots, m$, $\phi_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function in $x$ and $\omega$. Let this problem be referred to as Problem $(P)$.

Define

$$S_\epsilon = \{(x,\epsilon) \in \mathbb{R}^n \times \mathbb{R}_+ : \phi_j(x,\omega) \le \epsilon^\gamma W_j, \; \forall\, \omega \in \Omega, \; j = 1, \dots, m\} \tag{2}$$

where $\mathbb{R}_+ = \{\alpha \in \mathbb{R} \; : \; \alpha \ge 0\}$, $W_j \in (0,1)$, $j = 1, \dots, m$, are fixed constants and $\gamma$ is a positive real number. Clearly, Problem $(P)$ is equivalent to the following problem, which is denoted as Problem $(\hat{P})$.

$$\min \; f(x) \tag{3a}$$

subject to

$$(x,\epsilon) \in S_0 \tag{3b}$$

where $S_0 = S_\epsilon$ with $\epsilon = 0$. We assume that the following conditions are satisfied:

- There exists a global minimizer of Problem $(P)$, implying that $f(x)$ is bounded from below on $S_0$.

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- The number of distinct local minimum values of the objective function of Problem $(P)$ is finite.

A new exact penalty function $f_\sigma(x,\epsilon)$ defined below is introduced in [1].

$$f_\sigma(x,\epsilon) = \begin{cases} f(x) & \text{if } \epsilon = 0, \phi_j(x,\omega) \le 0 \ (\omega \in \Omega) \\ f(x) + \epsilon^{-\alpha}\Delta(x,\epsilon) + \sigma\epsilon^\beta & \text{if } \epsilon > 0 \\ +\infty & \text{otherwise} \end{cases} \tag{4}$$

where $\Delta(x,\epsilon)$, which is referred to as the constraint violation, is defined by

$$\Delta(x,\epsilon) = \sum_{j=1}^{m} \int_\Omega \Big[\max\big\{0, \phi_j(x,\omega) - \epsilon^\gamma W_j\big\}\Big]^2 d\omega \tag{5}$$

$\alpha$ and $\gamma$ are positive real numbers, $\beta > 2$, and $\sigma > 0$ is a penalty parameter. The surrogate optimization problem, which is referred to as Problem $(P_\sigma)$, is as follows.

$$\min \ f_\sigma(x,\epsilon) \tag{6a}$$

subject to

$$(x,\epsilon) \in \mathbb{R}^n \times [0,+\infty) \tag{6b}$$

2. **Convergence analysis.** For every positive integer $k$, let $(x^{(k),*}, \epsilon^{(k),*})$ be a local minimizer of Problem $(P_{\sigma_k})$. For the proof of the convergence results, the definition of constraint qualification given in Definition 2.2 of [1] should be changed to the one given below.

**Definition 1.** It is said that the constraint qualification is satisfied for the continuous inequality constraints $(1b)$ at $x = \bar{x}$ , if the following implication is valid. Suppose that

$$\int_\Omega \sum_j \varphi_j(\omega) \frac{\partial \phi_j(\bar{x},\omega)}{\partial x} d\omega = 0.$$

Then, $\varphi_j(\omega) = 0$, $\forall \omega \in \Omega$, $j = 1,\ldots,m$.

Theorem 2.3 of [1] is modified as follows.

**Theorem 2.** *Suppose that $(x^{(k),*}, \epsilon^{(k),*})$ is a local minimizer of Problem $(P_{\sigma_k})$ such that $f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*})$ is finite. If $(x^{(k),*}, \epsilon^{(k),*}) \to (x^*, \epsilon^*)$ as $k \to +\infty$, and the constraint qualification is satisfied for the continuous inequality constraints $(1b)$ at $x = x^*$, then $\epsilon^* = 0$ and $x^* \in S_0$.*

For the proof of Theorem 2, it is basically the same as that given for Theorem 2.3 of [1], except Definition 1, rather than Definition 2.2 of [1], is used.

**Remark 1.** The existence of an accumulating point of the sequence $(x^{(k),*}, \epsilon^{(k),*})$ is assured if the following condition is satisfied

$$f(x) \to \infty, \text{ as } \|x\| \to \infty.$$

Where $\|\cdot\|$ denotes the usual Euclidean norm.

**Theorem 3.** *Assume that* $\max\big\{0, \phi_j(x^{(k),*},\omega)\big\} = o((\epsilon^{(k),*})^\delta)$, $\delta > 0$, $j = 1,\ldots,m$. *Suppose that $\gamma > \alpha$, $\delta > \alpha$, $-\alpha - 1 + 2\delta > 0$, $2\gamma - \alpha - 1 > 0$. Then*

$$f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*}) \xrightarrow[x^{(k),*} \to x^* \in S_0]{\epsilon^{(k),*} \to \epsilon^* = 0} f_{\sigma_k}(x^*, 0) = f(x^*) \tag{7}$$

$$\nabla_{(x,\epsilon)} f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*}) \xrightarrow[x^{(k),*} \to x^* \in S_0]{\epsilon^{(k),*} \to \epsilon^* = 0} \nabla_{(x,\epsilon)} f_{\sigma_k}(x^*, 0) = (\nabla f(x^*), 0) \tag{8}$$

The proof of Theorem 3 is similar to Theorem 2.4 of [1], except with the changes listed below.

- Equation (2.16) of [1] should be changed to:

$$\begin{aligned} & \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ x^{(k),*} \to x^* \in S_0}} \frac{\sum_{j=1}^{m} \int_\Omega \left[ \max\left\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j \right\}\right]^2 d\omega}{(\epsilon^{(k),*})^\alpha} \\ = & \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ x^{(k),*} \to x^* \in S_0}} \sum_{j \in J'} \int_\Omega \left[ (\epsilon^{(k),*})^{-\frac{\alpha}{2}} \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^{\gamma - \frac{\alpha}{2}} W_j \right]^2 d\omega \end{aligned} \tag{9}$$

  Here, $J'$ denotes the index set such that for any $j \in J'$, $\max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\} = \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j$.
- Equation (2.17) of [1] should be changed to:

$$\lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ x^{(k),*} \to x^* \in S_0}} \sum_{j \in J'} \int_\Omega \left[ (\epsilon^{(k),*})^{-\frac{\alpha}{2}} \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^{\gamma - \frac{\alpha}{2}} W_j \right]^2 d\omega = 0 \tag{10}$$

- Equation (2.20) of [1] should be changed to:

$$\begin{aligned} & \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ x^{(k),*} \to x^* \in S_0}} \nabla_x f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*}) \\ = & \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ x^{(k),*} \to x^* \in S_0}} \left\{ \frac{\partial f(x^{(k),*})}{\partial x} \right. \\ & \left. + 2(\epsilon^{(k),*})^{-\alpha} \sum_{j=1}^{m} \int_\Omega \max\left\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j \right\} \frac{\partial \phi_j(x^{(k),*}, \omega)}{\partial x} d\omega \right\} \\ = & \nabla_x f(x^*) + \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ x^{(k),*} \to x^* \in S_0}} 2 \sum_{j \in J'} \int_\Omega \left[ (\epsilon^{(k),*})^{-\alpha} \phi_j(x^{(k),*}, \omega) \right. \\ & \left. - (\epsilon^{(k),*})^{\gamma - \alpha} W_j \right] \frac{\partial \phi_j(x^{(k),*}, \omega)}{\partial x} d\omega = \nabla_x f(x^*) \end{aligned} \tag{11}$$

- Equation (2.21) of [1] should be changed to:

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \nabla_\epsilon f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*})\\
=\ &\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \Big\{(\epsilon^{(k),*})^{-\alpha-1}\Big\{-\alpha\sum_{j=1}^{m}\int_\Omega\Big[\max\{0,\phi_j(x^{(k),*},\omega)\\
&\qquad -(\epsilon^{(k),*})^\gamma W_j\}\Big]^2 d\omega\\
&+2\gamma\sum_{j=1}^{m}\int_\Omega \max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\}((-\epsilon^{(k),*})^\gamma W_j)d\omega\Big\}\\
&+\sigma_k\beta(\epsilon^{(k),*})^{\beta-1}\Big\}\\
=\ &\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \Big\{-\alpha\sum_{j\in J'}\int_\Omega\big[\phi_j(x^{(k),*},\omega)(\epsilon^{(k),*})^{-\frac{\alpha+1}{2}}-(\epsilon^{(k),*})^{\gamma-\frac{\alpha+1}{2}}W_j\big]^2 d\omega\\
&+2\gamma\sum_{j\in J'}\int_\Omega\big[\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\big]\big((-\epsilon^{(k),*})^\gamma W_j\big)(\epsilon^{(k),*})^{-\alpha-1}d\omega\Big\}\\
=\ &0
\end{aligned} \tag{12}$$

Theorems 2.5 and 2.6 of [1] are combined as one theorem given below.

**Theorem 4.** *There exists a $k_0 > 0$, such that for any $k \geq k_0$, every local minimizer $(x^{(k),*}, \epsilon^{(k),*})$ of the penalty problem with finite $f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*})$ has the form $(x^*, 0)$ where $x^*$ is a local minimizer of Problem $(P)$.*

*Proof.* On the contrary, we assume that the conclusion is false. Then, there exists a subsequence of $\{(x^{(k),*}, \epsilon^{(k),*})\}$, which is denoted by the original sequence such that for any $k_0 > 0$, there exists a $k' > k_0$ satisfying $\epsilon^{(k'),*} \neq 0$. By Theorem 2, we have

$$\epsilon^{(k),*} \to \epsilon^* = 0,\ x^{(k),*} \to x^* \in S_0,\ \text{as } k \to +\infty$$

Since $\epsilon^{(k),*} \neq 0$ for all $k$, it follows from dividing (2.10) in [1] by $(\epsilon^{(k),*})^{\beta-1}$ that

$$\begin{aligned}
&(\epsilon^{(k),*})^{-\alpha-\beta}\Big\{-\alpha\sum_{j=1}^{m}\int_\Omega\Big[\max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\}\Big]^2 d\omega\\
&+2\gamma\sum_{j=1}^{m}\int_\Omega \max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\}\big((-\epsilon^{(k),*})^\gamma W_j\big)d\omega\Big\}+\sigma_k\beta=0
\end{aligned} \tag{13}$$

This is equivalent to

$$\begin{aligned}
&(\epsilon^{(k),*})^{-\alpha-\beta}\Big\{-\alpha\sum_{j=1}^{m}\int_\Omega\Big[\max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\}\Big]^2 d\omega\\
&+2\gamma\sum_{j=1}^{m}\int_\Omega\Big[\max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\}\big((-\epsilon^{(k),*})^\gamma W_j\big)\\
&+\max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\}\phi_j(x^{(k),*},\omega)\\
&-\max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\}\phi_j(x^{(k),*},\omega)\Big]d\omega\Big\}+\sigma_k\beta=0
\end{aligned} \tag{14}$$

Rearranging (14) yields

$$\begin{aligned}
&(\epsilon^{(k),*})^{-\alpha-\beta}(2\gamma-\alpha)\Big\{\sum_{j=1}^{m}\int_{\Omega}\Big[\max\big\{0,\phi_j(x^{(k),*},\omega) \\
&\qquad\qquad -(\epsilon^{(k),*})^{\gamma}W_j\big\}\Big]^2 d\omega\Big\}+\sigma_k\beta \\
=\ & 2\gamma(\epsilon^{(k),*})^{-\alpha-\beta}\sum_{j=1}^{m}\int_{\Omega}\max\big\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\big\}\phi_j(x^{(k),*},\omega)d\omega
\end{aligned} \tag{15}$$

Letting $k\to+\infty$ in (15) gives

$$2\gamma(\epsilon^{(k),*})^{-\alpha-\beta}\sum_{j=1}^{m}\int_{\Omega}\max\big\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\big\}\phi_j(x^{(k),*},\omega)d\omega\to+\infty \tag{16}$$

Define

$$y^k=(\epsilon^{(k),*})^{-\alpha-\beta}\sum_{j=1}^{m}\int_{\Omega}\max\big\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\big\}d\omega \tag{17}$$

From (16) and (17), we have

$$y^k\to+\infty\ ,\ \text{ as } k\to+\infty \tag{18}$$

Define

$$z^k=y^k/\|y^k\| \tag{19}$$

Clearly

$$\lim_{k\to+\infty}\|z^k\|=\|z^*\|=1 \tag{20}$$

Dividing (2.11) in [1] by $\|y^k\|$ yields

$$\begin{aligned}
&\frac{\frac{\partial f(x^{(k),*})}{\partial x}}{\|y^k\|}+\frac{2(\epsilon^{(k),*})^{-\alpha}}{\|y^k\|}\sum_{j=1}^{m}\int_{\Omega}\max\big\{0,\phi_j(x^{(k),*},\omega) \\
&\qquad\qquad -(\epsilon^{(k),*})^{\gamma}W_j\big\}\frac{\partial\phi_j(x^{(k),*},\omega)}{\partial x}d\omega=0
\end{aligned} \tag{21}$$

Note that $x^{(k),*}\to x^*$ as $k\to+\infty$ and that $\dfrac{\partial f(x)}{\partial x}$ and, for each $j=1,\ \ldots,\ m,\ \phi_j$ and $\dfrac{\partial\phi_j(\cdot\ ,\omega)}{\partial x}$ are continuous in $\mathbb{R}^n$ for each $\omega\in\Omega$, where $\Omega$ is a compact set. Then, it can be shown that there exist constants $\hat{K}$ and $\overline{K}$, independent of $k$, such that, for all $k=1,2,\cdots$,

$$\|\frac{\partial f(x^{(k),*})}{\partial x}\|\le\hat{K} \tag{22}$$

$$\|\frac{\partial\phi_j(x^{(k),*},\omega)}{\partial x}\|\le\overline{K},\ \text{ for } j=1,\cdots,m. \tag{23}$$

By substituting (17) and (19) into (21), we obtain

$$\frac{\frac{\partial f(x^{(k),*})}{\partial x}}{\|y^k\|(\epsilon^{(k),*})^{\beta}} + \frac{2(\epsilon^{(k),*})^{-\alpha-\beta}}{\|y^k\|}\sum_{j=1}^{m}\int_{\Omega}\max\{0,\phi_j(x^{(k),*},\omega) -(\epsilon^{(k),*})^{\gamma}W_j\}\frac{\partial \phi_j(x^{(k),*},\omega)}{\partial x}d\omega = 0 \tag{24}$$

Note that

$$\begin{aligned}\frac{1}{\|y^k\|(\epsilon^{(k),*})^{\beta}} &= \frac{1}{\|(\epsilon^{(k),*})^{-\alpha-\beta}\sum_{j=1}^{m}\max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\}\|(\epsilon^{(k),*})^{\beta}}\\ &= \frac{1}{\|\sum_{j=1}^{m}\max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\}\|(\epsilon^{(k),*})^{-\alpha}}\end{aligned} \tag{25}$$

From Theorem 3, we have $\phi_j(x^{(k),*},\omega) = o((\epsilon^{(k),*})^{\delta})$ and $\gamma > \alpha,\ \delta > \alpha$. Thus

$$\begin{aligned}&\lim_{k\to+\infty} \|\sum_{j=1}^{m}\max\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\}\|(\epsilon^{(k),*})^{-\alpha}\\ =\ &\|\sum_{j=1}^{m}\max\{0,(\epsilon^{*})^{\delta-\alpha}-(\epsilon^{*})^{\gamma-\alpha}W_j\}\|\\ =\ &0\end{aligned} \tag{26}$$

and hence,

$$\lim_{k\to\infty}\frac{1}{\|y^k\|(\epsilon^{(k),*})^{\beta}} \to +\infty. \tag{27}$$

From (22) and (27), it is clear that

$$\frac{|\frac{\partial f(x^{(k),*})}{\partial x}|}{\|y^k\|(\epsilon^{(k),*})^{\beta}} \to +\infty, \quad k\to+\infty \tag{28}$$

On the other hand,

$$\begin{aligned}&\Big|\frac{2(\epsilon^{(k),*})^{-\alpha-\beta}}{\|y^k\|}\sum_{j=1}^{m}\int_{\Omega}\max\{0,\phi_j(x^{(k),*},\omega)(\epsilon^{(k),*})^{\gamma}W_j\}\frac{\partial \phi_j(x^{(k),*},\omega)}{\partial x}d\omega\Big|\\ \le\ &\frac{2(\epsilon^{(k),*})^{-\alpha-\beta}}{\|y^k\|}\sum_{j=1}^{m}\int_{\Omega}\Big|\max\{0,\phi_j(x^{(k),*},\omega)(\epsilon^{(k),*})^{\gamma}W_j\}\frac{\partial \phi_j(x^{(k),*},\omega)}{\partial x}\Big|d\omega\\ =\ &\frac{2(\epsilon^{(k),*})^{-\alpha-\beta}}{\|y^k\|}\sum_{j=1}^{m}\int_{\Omega}\max\{0,\phi_j(x^{(k),*},\omega)(\epsilon^{(k),*})^{\gamma}W_j\}\Big|\frac{\partial \phi_j(x^{(k),*},\omega)}{\partial x}\Big|d\omega\\ \le\ &\frac{2(\epsilon^{(k),*})^{-\alpha-\beta}}{\|y^k\|}\sum_{j=1}^{m}\int_{\Omega}\max\{0,\phi_j(x^{(k),*},\omega)(\epsilon^{(k),*})^{\gamma}W_j\}\overline{K}d\omega\\ =\ &2\overline{K}z^k\end{aligned} \tag{29}$$

where $z^k$ is defined by (19). Clearly, $\|z^k\| = 1$. Thus, it follows from (29) that $2\overline{K}z^k$ is bounded uniformly with respect to $k$. This together with (28) is a contradiction to (24), and hence completing the first part of the proof .

For sufficiently large $k$, every local minimizer $(x^{(k),*}, \epsilon^{(k),*})$ has the form $(x^*, 0)$. It is obvious from Theorem 2 that $x^*$ is a feasible point of Problem $(P)$. This indicates that there is a neighborhood of $x^*$, such that for any feasible $x$ of Problem $(P)$

$$f(x) = f_{\sigma_k}(x, 0) \geq f_{\sigma_k}(x^*, 0) = f(x^*).$$

Therefore, $x^*$ is a local minimizer of Problem $(P)$. This completes the proof. □

## REFERENCES

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*E-mail address*: yuchangjun@126.com
*E-mail address*: k.l.teo@curtin.edu.au
*E-mail address*: zhangls@staff.shu.edu.cn
*E-mail address*: yqbai@shu.edu.cn