Convergence analysis of a monotonic penalty method for American option pricing

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\begin{abstract}
This paper is devoted to study the convergence analysis of a monotonic penalty method for pricing American options. A monotonic penalty method is first proposed to solve the complementarity problem arising from the valuation of American options, which produces a nonlinear degenerated parabolic PDE with Black–Scholes operator. Based on the variational theory, the solvability and convergence properties of this penalty approach are established in a proper infinite dimensional space. Moreover, the convergence rate of the combination of two power penalty functions is obtained.
\end{abstract}

1. Introduction

The valuation and hedging of financial option contracts are important subject matters, both in mathematics and financial management. European options and American options are the two major types of options. The holders of European options have the right to exercise the contracts only at their maturity dates. In the celebrated paper [2], explicit pricing formulas are derived for European options. On the other hand, American options, which represent most traded options, can be exercised at any time up to the maturity dates. However, no closed-form solutions exist for American options of their valuation. The main reason is that its solution depends on the strategy for exercising the options and the values of the options.

Formally, American option pricing problem is of free boundary nature, which is often formulated as a differential complementarity problem or variational inequalities. Currently, there are several methods to solve this kind of problems. The explicit lattice method [4,15] is simple and computationally inexpensive. But some disadvantages are also obvious, such as the lack of accuracy of the results obtained. Thus, the use of these results in the real financial market could have great adverse consequences. Projected successive over relaxation (PSOR) method [3,11] is also commonly used. In general, this method is fast and robust for many kinds of American option pricing problems. However, its convergence rate depends crucially on the choice of the relaxation parameter and it exhibits exponential solution-time behavior as the number of space discretization points increases. The linear programming method [6] is very suitable for pricing a single-factor American option. However, it is not well equipped to handle sparse matrix systems, especially in the case of multi-asset options.

It is well known that variational inequalities or linear complementarity problems can be solved by penalty methods (cf. [1,7,9]). Recently, the penalty method for pricing American options was also presented in [19,8,14,16,17]. An advantage of penalty method is that it is simple to implement and can make full use of the existing softwares to handle the sparse
matrix structure. Moreover, it is suitable for any type of discretization, for any dimension, and for any unstructured meshes. It also works for multiple-connected problems, and problems with nonlinearity, such as uncertain volatility models, drift-dominated problems, transaction cost models and jump diffusion models [15]. On the contrary, a major disadvantage is that the solution obtained by penalty method only satisfies approximately the complementarity conditions. However, the error can be controlled by adjusting the penalty parameter.

In [19,8], the quadratic \((l_2)\) and linear \((l_1)\) penalty methods were used to pricing American options. It was shown that the \((l_2)\) penalty method is smooth and the \((l_1)\) penalty method is linear and semismooth. Hence, these two penalty methods are easy to handle and implement. However, when penalty parameter gets large, computational difficulties can be encountered by these methods. To overcome this difficulty, the lower penalty method \((l_k, 0 < k < 1)\) has been developed to solve American option pricing problems (cf. [14,16]). It was shown that the \(l_k\) penalty method requires weaker conditions for exact penalized representation and possesses a smaller penalty parameter when compared with \(l_1\) and \(l_2\) penalty methods [13]. Moreover, the rate of convergence of the lower order penalty method is much faster than those of quadratic and linear penalty methods. Unfortunately, due to the strong nonsmoothness of the lower order penalty method, the choice of the penalty parameters is extremely sensitive.

To balance the behaviors of \(l_1, l_2\) and the lower order penalty method, some new type penalty methods should be developed. Hence, in this paper we first unify and generalize all of these penalty methods and propose a monotonic penalty method. Then, a special monotonic penalty method—the combination of two power penalty methods—is developed to match the solution obtained by penalty method only satisfies approximately the complementarity conditions. However, the error can be controlled by adjusting the penalty parameter.

The organization of this paper is as follows. In Section 2, we introduce the American option pricing model and its equivalent formulations: differential complementarity problem and variational inequalities. Section 3 gives a monotonic penalty approach to the complementarity problem. Also, its solvability is presented. In Section 4, the convergence analysis of the monotonic penalty method is given. In Section 5, two special monotonic penalty functions are studied in detail. Their convergence rates are also established in this section.

Before proceeding, some standard notation is to be used in the paper. For an open set \(S \subset \mathbb{R}\) and \(1 \leq p \leq \infty\), let \(L^p(S) = \{v: (\int_S |v(x)| dx)^{1/p} < \infty\}\) denote the space of all \(p\)-power integrable functions on \(S\). We use the \(\|\cdot\|_{L^p(S)}\) to denote the norm on \(L^p(S)\). With \(m = 1, 2, \ldots,\) and \(p = 2\), we let \(H^m(S)\) denote the usual Sobolev space over the domain \(S\) defined by \(H^m(S) = \{v: v \in L^2(S), \frac{\partial^m v}{\partial x^m} \in L^2(S), 0 \leq |\alpha| \leq m, \}\), where \(\alpha\) is a positive integer. Its norm \(\|\cdot\|_{H^m(S)}\) is defined by \(\|v\|_{H^m(S)} = (\int_S \sum_{|\alpha| \leq m} |(\partial^\alpha v)|^2 dx)^{1/2}\). We put \(H^0_0(S) = \{v: v \in H^m(S), \v_{|\partial S} = 0\}\), where \(\partial S\) is the boundary of \(S\). Finally, for any Hilbert space \(W(S)\), the norm of \(L^p(0, T; W(S))\) is denoted by

\[
\|v\|_{L^p(0, T; W(S))} = \left( \int_0^T \|v(\cdot, t)\|_{W(S)}^p dt \right)^{1/p}.
\]

Obviously, \(L^p(0, T; L^p(S)) = L^p(S \times (0, T))\).

To handle the degeneracy in the Black-Scholes equation, we need to introduce a weighted Sobolev space. In the case of one-dimensional space, we define the weighted Sobolev space \(H^1_{0,\omega}(S)\) as

\[
H^1_{0,\omega}(S) = \{v: v \in L^2(S), \v_{x}\v_{x} / \partial x \in L^2(S), \text{ and } v_{|\partial S} = 0\},
\]

with the norm

\[
\|v\|_{H^1_{0,\omega}(S)} = \left( \int_S \left( v^2 + x^2 (\v_{x})^2 \right) dx \right)^{1/2}.
\]

It is also easy to prove that the pair \((H^1_{0,\omega}(S), (\cdot, \cdot)_{H^1_{0,\omega}(S)})\) is a Hilbert space by defining a weighted inner product on \(H^1_{0,\omega}(S)\) with

\[
(v, v)_{H^1_{0,\omega}(S)} = (v, v)_{L^2(S)} + (x\v_{x} / \partial x, x\v_{x} / \partial x)_{L^2(S)}.
\]
For clarity, we will often simply write $v(\cdot, t)$ as $v(t)$ when we regard $v(\cdot, t)$ as an element of $H^1_{0, \text{eq}}(S)$. From time to time, we will also suppress the independent time variable $t$ when it causes no confusion in doing so.

2. Mathematic model

Consider an asset with price $x$ which satisfies the following stochastic differential equation

$$dx = \mu x dt + \sigma x dW,$$

where $W$ is a standard Brownian motion, $\mu$ is the drift rate, $\sigma$ denotes a deterministic local volatility. Let $V(x, t)$ denote the value of a standard American put option, $T$ the expiry time, and $K$ the striking price. It is well known, under the non-arbitrage assumption, that the American option pricing problem can be formally stated as a linear differential complementarity problem as follows:

$$
\begin{align*}
&LV(x, t) \geq 0, \\
&V(x, t) - V^*(x) \geq 0, \\
&LV(x, t) \cdot (V(x, t) - V^*(x)) = 0,
\end{align*}
$$

(1)

a.e. in $\Omega = I \times (0, T)$, where $I = (0, X) \subset \mathbb{R}$ is the variable range of the underlying asset price. Realistically, we should choose $X \gg K$. In (1),

$$LV := -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - r x \frac{\partial V}{\partial x} + r V$$

denotes the Black-Scholes differential operator, $r$ is the risk-free interest rate, $V^*(x) = \max\{K - x, 0\}$ is the payoff function. The final condition $V(x, t)$ at $t = T$ is given by

$$V(x, T) = V^*(x).$$

(2)

Additionally, the boundary conditions are

$$V(0, t) = K, \quad V(X, t) = 0.$$  

(3)

The system of (1)-(3) is the original American option pricing model.

By introducing a new variable

$$u(x, t) := e^{\beta t} (V_0(x) - V(x, t)),$$

where

$$V_0(x) = (1 - x/X)K,$$

we first transform (1)-(3) into the following equivalent standard form satisfying homogeneous Dirichlet boundary conditions.

Problem 1.

$$
\begin{align*}
&Lu(x, t) \leq f(x, t), \\
&u(x, t) - u^*(x, t) \leq 0, \\
&(Lu(x, t) - f(x, t)) \cdot (u(x, t) - u^*(x, t)) = 0,
\end{align*}
$$

(6)

a.e. in $\Omega$, where

$$Lu := -\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left[ a x^2 \frac{\partial u}{\partial x} + bu \right] + cu$$

is the self-adjoint form with

$$a = \frac{1}{2}\sigma^2, \quad b = r - \sigma^2, \quad c = r + b + \beta, \quad f(x, t) = e^{\beta t} LV_0(x).$$

(7)

The payoff function becomes

$$u^*(x, t) = e^{\beta t} (V^*(x) - V_0(x)),$$

(8)

and the new boundary conditions are

$$u(0, t) = u(X, t) = 0, \quad t \in [0, T).$$

(9)
It is a standard result that the linear complementarity problem (6)–(9) can be reformulated as the following variational inequalities.

**Problem 2.** Find \( u(t) \in \mathcal{K} \), such that, for all \( v \in \mathcal{K} \),

\[
\left( -\frac{\partial u(t)}{\partial t}, v - u(t) \right) + A(u(t), v - u(t); t) \geq (f(t), v - u(t)) \tag{10}
\]
a.e. in \((0, T)\), where \( A(\cdot, \cdot; t) \) is a bilinear form defined by

\[
A(u, v; t) := \left( \alpha x^2 \frac{\partial u}{\partial x} + bxu, \frac{\partial v}{\partial x} \right) + (cu, v), \quad u, v \in H^1_{0,\text{eo}}(I),
\]
and

\[
\mathcal{K} = \{ v \in H^1_{0,\text{eo}}(I) : v \leq u^* \}
\]
is a convex and closed subset of \( H^1_{0,\text{eo}}(I) \).

For Problem 2, we establish the following unique solvability result.

**Lemma 3.** Variational inequality (10) has a unique solution.

**Proof.** In fact, by virtue of the coerciveness of the operator \( A(\cdot, \cdot; t) \), the conclusion is a consequence of Theorem 2.3 in [1], in which the unique solvability for a parabolic variational inequality problem is established. In view of the definition of the weighted Sobolev space \( H^1_{0,\text{eo}}(I) \), the coerciveness of the operator \( A(\cdot, \cdot; t) \) can be shown as follows:

\[
A(v, v; t) = \alpha x^2 \frac{\partial v}{\partial x} + bxv, \frac{\partial v}{\partial x} + (cv, v) = \left( \alpha x^2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} \right) + \left( r + b + \frac{b}{2} \right) v, v
\]

which follows from (4) and (7).

3. The monotonic penalty approach

In this section, we propose a monotonic penalty approach to the complementarity problem (6)–(9). First, we give the following definition of a monotonic operator. A function \( \rho(\cdot) : L^2(\Omega) \rightarrow L^2(\Omega) \) is called monotonic, if for any \( u, v \in L^2(\Omega) \), it holds

\[
(\rho(u) - \rho(v), u - v) \geq 0.
\]

Now, with this definition, a monotonic penalty approach to Problem 1 is stated as follows.

**Problem 4.**

\[
\mathcal{L} u_\lambda(x, t) + \lambda \rho(u_\lambda(x, t)) = f(x, t), \quad (x, t) \in \Omega,
\]
where \( \lambda > 0 \) is the penalty parameter and \( \rho(\cdot) \) is a continuous, monotonic penalty function subject to

\[
\begin{cases}
\rho(u_\lambda) > 0, & \text{if } u_\lambda(t) \notin \mathcal{K}, \\
\rho(u_\lambda) = 0, & \text{if } u_\lambda(t) \in \mathcal{K}.
\end{cases}
\]

Clearly, the variational form corresponding to Problem 4 is

**Problem 5.** Find \( u_\lambda(t) \in H^1_{0,\text{eo}}(I) \) such that, for all \( v \in H^1_{0,\text{eo}}(I) \),

\[
\left( -\frac{\partial u_\lambda(t)}{\partial t}, v \right) + A(u_\lambda(t), v; t) + \lambda (\rho(u_\lambda(t)), v) = (f(t), v) \tag{12}
\]
a.e. in \((0, T)\) and \( u_\lambda(x, T) = u^*(x, T) \).
For Problem 4, we have the following unique solvability result.

**Theorem 6.** Suppose that \( \rho(\cdot) : L^2(\Omega) \to L^2(\Omega) \) satisfies the following conditions:

1. \( \rho \) is monotonic in \( L^2(\Omega) \).
2. \( \rho \) is continuous.

Then, Problem 4 has a unique solution.

**Proof.** First, note that \( f(x, t) = e^{\beta t} L V_0(x) \) is sufficiently smooth in \( (x, t) \), since \( V_0(x) \) is defined as (5). We now prove this theorem by showing that the variational form of the nonlinear operator on the left-hand side of (11) is strictly monotone and continuous. In fact, for any \( v_1(t), v_2(t) \in H^{1, \text{ref}}(I) \) a.e. in \( (0, T) \) with the final condition being equal to \( u^*(x, T) \) at \( t = T \), it follows from the integration by parts that

\[
(\mathcal{L}(v_1 - v_2), v_1 - v_2) + \lambda(\rho(v_1) - \rho(v_2), v_1 - v_2) = \left( -\frac{\partial(v_1 - v_2)}{\partial \tau}, v_1 - v_2 \right) + A(v_1 - v_2, v_1 - v_2; t) + \lambda(\rho(v_1) - \rho(v_2), v_1 - v_2). 
\]

(13)

Since \( \rho(v) \) is a monotonic function, it is non-decreasing in \( v \). Thus,

\[
\lambda(\rho(v_1) - \rho(v_2), v_1 - v_2) = \int_0^X (\rho(v_1) - \rho(v_2))(v_1 - v_2) \, dx \geq 0.
\]

(14)

Denote \( e(\tau) = v_1(\tau) - v_2(\tau) \). Integrating both sides of (13) from 0 to \( T \), and using the above inequality and the coerciveness of operator \( A \), we have

\[
\int_0^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau = \int_0^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau + \int_0^T \left( \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) + \int_0^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau \right.
\]

\[
\geq C \int_0^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau \geq C \int_0^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau.
\]

(15)

where \( C > 0 \) is a generic constant. However, for any \( t \in (0, T) \), integrating by parts gives

\[
\int_t^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau = (e(t), e(t)) = -\int_t^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau,
\]

since \( e(T) = 0 \). From this it follows that

\[
\int_t^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau = \frac{1}{2} (e(t), e(t)) \geq 0.
\]

(15)

Therefore, from (14) and (15), we have

\[
\int_0^T \left( \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) + \int_0^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau \right.
\]

\[
\geq C \int_0^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau \geq C \int_0^T \left( -\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) \, d\tau.
\]

This implies that the operator on the left-hand side of (11) is strictly monotone.

Moreover, for any \( v, w \in L^2(0, T; H^{1, \text{ref}}(I)) \), it is easy to show by a standard argument that

\[
\int_0^T |A(v(t), w(t); t)| \, dt \leq C \| v \|_{L^2(0, T; H^{1, \text{ref}}(I))} \| w \|_{L^2(0, T; H^{1, \text{ref}}(I))},
\]

which means that the operator \( A(v, w; t) \) is continuous in both \( v \) and \( w \). Also, it is obvious that \( \rho(v, w) \) is continuous in both \( v \) and \( w \). Therefore, using a standard result (see, for example, p. 37 in [10]), we can conclude that Problem 4 is uniquely solvable. \( \square \)
4. Convergence analysis

The regularity results of the solution to the penalized problems have been studied extensively in several monographs such as [5,1,12]. In brief, under the assumption that $u_\lambda(x,t)$ and $f(x,t)$ are sufficiently smooth, we have the following regularity results:

$$\frac{\partial u_\lambda(x,t)}{\partial t}, u_\lambda(x,t) \in L^2(0,T;H^1_{0,m}(I)) \cap L^\infty(0,T;L^2(I)).$$

and

$$\rho(u_\lambda(x,t)), f(x,t) \in L^\infty(0,T;L^2(I)).$$

On this basis, we have the following convergence result.

**Theorem 7.** Let $u$ and $u_\lambda$ be the solution to Problem 2 and Problem 4, respectively. Then,

$$\lim_{\lambda \to \infty} \left( \|u_\lambda - u\|_{L^\infty(0,T;L^2(I))} + \|u_\lambda - u\|_{L^2(0,T;H^1_{0,m}(I))} \right) = 0.$$

**Proof.** The proof is divided into three parts. First, we obtain a priori estimates for $\{u_\lambda\}$, then the weak convergence of $\{u_\lambda\}$, and finally the strong convergence of $\{u_\lambda\}$.

(I) A priori estimate for $\{u_\lambda\}$.

Let $v_0(t) \in K$, then $\rho(v_0(t)) = 0$. Setting $v(t) = u_\lambda(t) - v_0(t)$, we have

$$-\frac{\partial u_\lambda(t)}{\partial t}, u_\lambda(t) - v_0(t) + A(u_\lambda(t), u_\lambda(t) - v_0(t); t) + \lambda(\rho(u_\lambda(t)) - \rho(v_0(t)), u_\lambda(t) - v_0(t)) = (f(t), u_\lambda(t) - v_0(t)).$$

Since $\rho$ is monotonic, it follows that $(\rho(u_\lambda(t)) - \rho(v_0(t)), u_\lambda(t) - v_0(t)) \geq 0$. Thus, we get

$$-\frac{\partial u_\lambda(t)}{\partial t}, u_\lambda(t) - v_0(t) + A(u_\lambda(t), u_\lambda(t) - v_0(t); t) \leq (f(t), u_\lambda(t) - v_0(t)),

and hence

$$-\frac{1}{2} \frac{d}{dt} |u_\lambda(t) - v_0(t)|^2 + A(u_\lambda(t), u_\lambda(t); t) \leq (f(t), u_\lambda(t) - v_0(t)) + A(u_\lambda(t), v_0(t); t).$$

Therefore,

$$-\frac{1}{2} \frac{d}{dt} |u_\lambda(t) - v_0(t)|^2 + \alpha \left\| u_\lambda(t) \right\|^2_{H^1_{0,m}(I)} \leq c |u_\lambda(t)|^2 + c \left\| u_\lambda(t) \right\|^2_{L^2(I)} + c \left\| f(t) \right\|^2_{L^2(I)} |u_\lambda(t) - v_0(t)|_{L^2(I)}.$$

Since $v_0(t)$ is a bounded element of $L^2(I)$, we have

$$-\frac{d}{dt} |u_\lambda(t) - v_0(t)|^2 + 2\alpha \left\| u_\lambda(t) \right\|^2_{H^1_{0,m}(I)} \leq c |u_\lambda(t) - v_0(t)|^2 + \alpha \left\| u_\lambda(t) \right\|^2_{H^1_{0,m}(I)} + c \left( 1 + \left\| f(t) \right\|^2_{L^2(I)} \right).$$

(16)

Integrating both sides of (16) from $t$ to $T$, we obtain

$$|u_\lambda(t) - v_0(t)|^2 + \alpha \int_t^T \left\| u_\lambda(\tau) \right\|^2_{H^1_{0,m}(I)} d\tau \leq c \int_t^T |u_\lambda(\tau) - v_0(\tau)|^2 d\tau + c(T - t) + \left( \int_t^T \left\| f(\tau) \right\|^2_{L^2(I)} d\tau \right) + |u_\lambda(T) - v_0(T)|^2.$$

(17)

By defining $\eta(t) = |u_\lambda(t) - v_0(t)|^2$, the above inequality can be expressed as

$$\begin{aligned}
\eta(t) &\leq c \int_t^T \eta(\tau) d\tau + d; \\
d &\leq cT + \left( \int_0^T \left\| f(\tau) \right\|^2_{L^2(I)} d\tau \right) + |u_\lambda(T) - v_0(T)|^2.
\end{aligned}$$
By virtue of Gronwall’s inequality, the above inequalities imply \( \eta(t) \leq d \exp(\omega t) \), i.e.

\[
|u_\lambda(t) - v_0(t)|^2 \leq d \exp(\omega t).
\]

On this basis, we deduce that

\[
\|u_\lambda(x, t)\|_{L^\infty(0, T; L^2(I))} \leq C \quad (C \text{ independent of } \lambda \text{ and } u_\lambda).
\]  (18)

Thus, from (17) and (18), we have the following estimates

\[
\|u_\lambda\|_{L^\infty(0, T; L^2(I))} + \|u_\lambda\|_{L^2(0, T; H^1_0(I))} \leq C,
\]  (19)

where \( C \) is a constant independent of \( \lambda \) and \( u_\lambda \).

(II) Weak convergence of \( \{u_\lambda\} \).

(19) implies that \( \{u_\lambda\} \) is uniformly bounded in the space \( L^2(0, T; H^1_0(I)) \cap L^\infty(0, T; L^2(I)) \). Therefore, there exists a subsequence of \( \{u_\lambda\} \), still denote it by \( \{u_\lambda\} \), such that

\[
\lim_{\lambda \to \infty} u_\lambda = \bar{u}, \quad \text{weakly in } L^2(0, T; H^1_0(I)) \cap L^\infty(0, T; L^2(I)).
\]

Our next task is to show that \( \bar{u} \) is a solution to Problem 2.

From (12), we have

\[
\int_0^T \left( \rho(u_\lambda(t), v(t)) \right) dt = \frac{1}{\lambda} \left[ \int_0^T (f(t), v(t)) dt - \int_0^T \left( \frac{\partial u_\lambda(t)}{\partial \tau}, v(t) \right) dt - \int_0^T A(u_\lambda(t), v(t); t) dt \right].
\]

Thus,

\[
\left\| \rho(u_\lambda) \right\|_{L^\infty(0, T; L^2(I))} = O(1/\lambda).
\]

Therefore, as \( \lambda \to \infty \), we have \( u_\lambda \to \bar{u} \), and \( \left\| \rho(u_\lambda) \right\|_{L^\infty(0, T; L^2(I))} \to 0 \). Hence,

\[
\left\| \rho(\bar{u}) \right\|_{L^\infty(0, T; L^2(I))} = 0.
\]

So, we obtain \( \rho(\bar{u}) = 0 \). By the definition of \( \rho(\bar{u}) \) in Problem 4, we see that \( \bar{u}(t) \in K \). In addition, \( \bar{u}(x, t) \in L^2(0, T; L^2(I)) \).

For any \( v(t) \in K \), we have \( \rho(v(t)) = 0 \). Replacing \( v(t) \) with \( v(t) - u_\lambda(t) \) in (12), we obtain

\[
\left( -\frac{\partial u_\lambda(t)}{\partial t}, v(t) - u_\lambda(t) \right) + A(u_\lambda(t), v(t) - u_\lambda(t); t) + \lambda \left( \rho(u_\lambda(t)) - \rho(v(t)) \right), v(t) - u_\lambda(t) = (f(t), v(t) - u_\lambda(t)).
\]

It follows from the monotonicity of \( \rho(\cdot) \) that for any \( v(t) \in K \),

\[
\rho(u_\lambda(t)) - \rho(v(t)), v(t) - u_\lambda(t) \leq 0,
\]

and hence

\[
\left( -\frac{\partial u_\lambda(t)}{\partial t}, v(t) - u_\lambda(t) \right) + A(u_\lambda(t), v(t) - u_\lambda(t); t) - (f(t), v(t) - u_\lambda(t)) \geq 0.
\]

Equivalently,

\[
\left( -\frac{\partial u_\lambda(t)}{\partial t}, v(t) \right) + A(u_\lambda(t), v(t); t) - (f(t), v(t)) \geq \left( -\frac{\partial u_\lambda(t)}{\partial t}, u_\lambda(t) \right) + A(u_\lambda(t), u_\lambda(t); t)  \quad \text{a.e. in } (0, T).
\]  (20)

Integrating both sides of (20) from \( t \) to \( T \), we obtain

\[
\int_t^T \left( -\frac{\partial u_\lambda(t)}{\partial t}, v(t) \right) dt + \int_t^T A(u_\lambda(t), v(t); t) dt - \int_t^T (f(t), v(t)) dt \geq \int_t^T \left( -\frac{\partial u_\lambda(t)}{\partial t}, u_\lambda(t) \right) dt + \int_t^T A(u_\lambda(t), u_\lambda(t); t) dt
\]

\[
= (u_\lambda(t), u_\lambda(t)) + \int_t^T A(u_\lambda(t), u_\lambda(t); t) dt.
\]  (21)
It follows from the properties of weak convergence that
\[
\liminf_{\lambda \to \infty} \left( u_\lambda(t), u_\lambda(t) \right) + \int_t^T A(u_\lambda(\tau), u_\lambda(\tau); \tau) d\tau \geq \left( \bar{u}(t), \bar{u}(t) \right) + \int_t^T A(\bar{u}(\tau), \bar{u}(\tau); \tau) d\tau.
\] (22)

From (21) and (22), we have
\[
\liminf_{\lambda \to \infty} \left[ \int_t^T \left( -\frac{\partial u_\lambda(\tau)}{\partial \tau}, v(\tau) \right) d\tau + \int_t^T A(u_\lambda(\tau), v(\tau); \tau) d\tau - \int_t^T (f(\tau), v(\tau)) d\tau \right]
\geq \int_t^T \left( -\frac{\partial \bar{u}(\tau)}{\partial \tau}, \bar{u}(\tau) \right) d\tau + \int_t^T A(\bar{u}(\tau), \bar{u}(\tau); \tau) d\tau.
\]

Therefore,
\[
\int_t^T \left( -\frac{\partial \bar{u}(\tau)}{\partial \tau}, v(\tau) \right) d\tau + \int_t^T A(\bar{u}(\tau), v(\tau); \tau) d\tau - \int_t^T (f(\tau), v(\tau)) d\tau \geq \int_t^T \left( -\frac{\partial \bar{u}(\tau)}{\partial \tau}, \bar{u}(\tau) \right) d\tau + \int_t^T A(\bar{u}(\tau), \bar{u}(\tau); \tau) d\tau,
\]
i.e.
\[
\int_t^T \left( -\frac{\partial \bar{u}(\tau)}{\partial \tau}, v(\tau) - \bar{u}(\tau) \right) d\tau + \int_t^T A(\bar{u}(\tau), v(\tau) - \bar{u}(\tau); \tau) d\tau \geq \int_t^T (f(\tau), v(\tau) - \bar{u}(\tau)) d\tau
\]
for all \( t \in (0, T) \), which is equivalent to
\[
\left( -\frac{\partial \bar{u}(t)}{\partial t}, v(t) - \bar{u}(t) \right) + A(\bar{u}(t), v(t) - \bar{u}(t); t) \geq (f(t), v(t) - \bar{u}(t))
\]
a.e. in \((0, T)\), for all \( v(t) \in K \). This shows that \( \bar{u} = u \), and that the whole sequence \( u_\lambda \) converges weakly to \( u \).

**III** Strong convergence of \( \{u_\lambda\} \).

Setting \( v(t) = u_\lambda(t) - u(t) \) in (12), we obtain
\[
\left( -\frac{\partial u_\lambda(t)}{\partial t}, u_\lambda(t) - u(t) \right) + A(u_\lambda(t), u_\lambda(t) - u(t); t) + \lambda \left( \rho(u_\lambda(t)) - \rho(u(t)), u_\lambda(t) - u(t) \right) = (f(t), u_\lambda(t) - u(t))
\]
a.e. in \((0, T)\). Using the monotonic property of \( \rho(\cdot) \), we have
\[
\left( -\frac{\partial u(t)}{\partial t}, u_\lambda(t) - u(t) \right) + A(u_\lambda(t), u_\lambda(t) - u(t); t) \leq (f(t), u_\lambda(t) - u(t))
\] (23)
a.e. in \((0, T)\). Reformulating (23) yields
\[
\left( -\frac{\partial (u_\lambda(t) - u(t))}{\partial t}, u_\lambda(t) - u(t) \right) + A(u_\lambda(t) - u(t), u_\lambda(t) - u(t); t) \\
\leq (f(t), u_\lambda(t) - u(t)) + \left( -\frac{\partial u(t)}{\partial t}, u_\lambda(t) - u(t) \right) + A(u(t), u_\lambda(t) - u(t); t).
\] (24)

Integrating both sides of (24) from \( t \) to \( T \) and then using the coerciveness property of the operator \( A \), we obtain
\[
\left| u_\lambda(t) - u(t) \right|^2 + \alpha \int_t^T \left| u_\lambda(\tau) - u(\tau) \right|^2 \|u_\lambda\|_{H^1(\Omega)} d\tau \leq c \left[ \int_t^T \left| f(\tau) \right|^2 \|f\|_{L^2(\Omega)} d\tau \right]^{\frac{1}{2}} \left[ \int_t^T \left| u_\lambda(\tau) - u(\tau) \right|^2 \|u_\lambda\|_{L^2(\Omega)} d\tau \right]^{\frac{1}{2}}
\]
\[
+ c \left[ \int_t^T \left| u(\tau) \right|^2 \|f\|_{L^2(\Omega)} d\tau \right]^{\frac{1}{2}} \left[ \int_t^T \left| u_\lambda(\tau) - u(\tau) \right|^2 \|u_\lambda\|_{L^2(\Omega)} d\tau \right]^{\frac{1}{2}}
\]
\[
+ c \left[ \int_t^T \left| \frac{\partial u}{\partial t} \right|^2 \|f\|_{L^2(\Omega)} d\tau \right]^{\frac{1}{2}} \left[ \int_t^T \left| u_\lambda(\tau) - u(\tau) \right|^2 \|u_\lambda\|_{L^2(\Omega)} d\tau \right]^{\frac{1}{2}}
\]
\[
\leq C \left[ \int_t^T \left| u_\lambda(\tau) - u(\tau) \right|^2 \|u_\lambda\|_{L^2(\Omega)} d\tau \right]^{\frac{1}{2}},
\] (25)
for all \( t \in (0, T) \). Thus, it follows from (25) that
\[
\lim_{\lambda \to \infty} \left( \|u_\lambda - u\|_{L^\infty(0, T; L^2(I))} + \|u_\lambda - u\|_{L^2(0, T; H^1_{0,m}(I))} \right) = 0.
\]

5. Some special monotonic penalty methods

The class of monotonic penalty methods is very rich. \( l_2, l_1 \) and \( l_k \) \((0 < k < 1)\) are the most frequently used monotonic penalty methods. These penalty methods for American option pricing have been extensively studied in [14,8,19]. In this section, we shall propose two new monotonic penalty methods, i.e. ‘valley at zero’ penalty method and a combination of two power penalty methods.

1. ‘Valley at zero’ penalty method

\[
\rho(u) = \begin{cases} 
1, & \text{if } u \geq 1, \\
u, & \text{if } 0 \leq u \leq 1, \\
0, & \text{if } u \leq 0.
\end{cases}
\] (26)

For a detailed study of ‘valley at zero’ penalty functions, see [18]. Obviously, it is a monotonic penalty function. When \( u \leq 1 \) this penalty method is identical to the \( l_1 \) penalty method.

2. A combination of two power penalty methods

\[
c_{k,m}(u) = \left[ \max(u, 0) \right]^k + \left[ \max(u, 0) \right]^m, \quad 0 < k < 1 < m < \infty.
\] (27)

It is clear that the combined function \( c_{k,m}(u) \) given by (27) is also monotonic. The motivation for this new kind of penalty functions is to make full use of the advantages of the higher order and lower order penalty methods and overcome their difficulties as mentioned in the introduction. The mechanism of this combined penalty method is that if \( u \) is not a ‘good’ initial guess, then a higher order penalty term \((m > 1)\) plays a dominant role in the behavior of \( u \), controlling it to converge to near zero as quickly as possible. Then, the problem will behave as a well-defined initial value problem, hence asymptotically the lower order penalty term \((0 < k < 1)\) will play the dominant role. The combination of these two power penalty methods possesses a good convergence behavior with a desirable convergence rate. In the following, we establish a convergence rate of the penalty method (27).

**Lemma 8.** Let \( u_\lambda \in L^p(\Omega) \) be the solution to Problem 5. If the combined penalty function \( c_{k,m}(u) \) is used, then there exists a positive constant \( C \), independent of \( u_\lambda \) and \( \lambda \), such that

\[
\|u_\lambda - u^*\|_{L^p(\Omega)} \leq \frac{C}{\lambda^{1/(p-1)}},
\] (28)

\[
\|u_\lambda - u^*\|_{L^\infty(0, T; L^2(I))} + \|u_\lambda - u^*\|_{L^\infty(0, T; H^1_{0,m}(I))} \leq \frac{C}{\lambda^{1/(2p-2)}},
\] (29)

where \( p = 1 + \frac{m+k}{2} \).

**Proof.** Assume that \( C \) is a generic positive constant, independent of \( u_\lambda \) and \( \lambda \). To simplify the notation, we let \( \psi(\cdot, t) = u_\lambda(\cdot, t) - u^*(\cdot, t) \) \(= H_{0,m}(I) \) for almost all \( t \in (0, T) \), where \( u_\lambda(\cdot, t) - u^*(\cdot, t) = \max(u(\cdot, t) - u^*(\cdot, t), 0) \). Now, setting \( v(t) = \psi(\cdot, t) \) in (12), and replacing \( \rho(u) \) with \( c_{k,m}(u) = [\max(u, 0)]^k + [\max(u, 0)]^m \), we have

\[
\left( -\frac{\partial u_\lambda(t)}{\partial t}, \psi(t) \right) + A(u_\lambda(t), \psi(t); t) + \lambda(\psi^m(t) + \psi^k(t), \psi(t)) = (f(t), \psi(t))
\] (30)

a.e. in \((0, T)\), since
\[
c_{k,m}(u) = \left[ \max(u, 0) \right]^k + \left[ \max(u, 0) \right]^m = \max[u^k + u^m, 0] = u^k + u^m, \quad \text{when } u \geq 0.
\]

Taking \(-\left( \frac{\partial u^*(t)}{\partial t}, \psi(t) \right) + A(u^*(t), \psi(t); t)\) away from both sides of (30) gives

\[
\left( -\frac{\partial (u_\lambda(t) - u^*(t))}{\partial t}, \psi(t) \right) + A(u_\lambda(t) - u^*(t), \psi(t); t) + \lambda(\psi^m(t) + \psi^k(t), \psi(t))
\] (31)

\[
= (f(t), \psi(t)) + \left( \frac{\partial u^*(t)}{\partial t}, \psi(t) \right) - A(u^*(t), \psi(t); t).
\]

Integrating both sides of (31) from \( t \) to \( T \) and using the coerciveness property of the operator \( A \) and Hölder’s inequality, we get
This implies that
\[ \frac{1}{2} \langle \varphi(t), \varphi(t) \rangle + \int_t^T \| \varphi(\tau) \|^2_{H^1_0(I)} d\tau + \lambda \int_t^T (\varphi^m(\tau) + \varphi^k(\tau), \varphi(\tau)) d\tau \]
\[ \leq \int_t^T (f(\tau), \varphi(\tau)) d\tau - \beta \int_t^T e^{\beta \tau} (V_0(\tau) - V^*, \varphi(\tau)) d\tau - \int_t^T A(u^*(\tau), \varphi(\tau); \tau) d\tau \]
\[ \leq C \left( \int_t^T \| \varphi(\tau) \|^p_{L^p(I)} d\tau \right)^{1/p} + \beta \int_t^T e^{\beta \tau} (V_0(\tau) - V^*, \varphi(\tau)) d\tau - \int_t^T A(u^*(\tau), \varphi(\tau); \tau) d\tau. \]
(32)

Noting that if \( a, b \geq 0 \) then \( a + b \geq 2\sqrt{ab} \), we have
\[ \frac{1}{2} \langle \varphi(t), \varphi(t) \rangle + \int_t^T \| \varphi(\tau) \|^2_{H^1_0(I)} d\tau + \lambda \int_t^T \| \varphi(\tau) \|^p_{L^p(I)} d\tau \]
\[ \leq \frac{1}{2} \langle \varphi(t), \varphi(t) \rangle + \int_t^T \| \varphi(\tau) \|^2_{H^1_0(I)} d\tau + \lambda \int_t^T (\varphi^m(\tau) + \varphi^k(\tau), \varphi(\tau)) d\tau. \]
(33)

where \( p = \frac{m+k}{2} \).

From (32) and (33), it follows that
\[ \frac{1}{2} \langle \varphi(t), \varphi(t) \rangle + \int_t^T \| \varphi(\tau) \|^2_{H^1_0(I)} d\tau + \lambda \int_t^T \| \varphi(\tau) \|^p_{L^p(I)} d\tau \leq C \left( \int_t^T \| \varphi(\tau) \|^p_{L^p(I)} d\tau \right)^{1/p}. \]
(34)

This implies that
\[ \lambda \int_t^T \| \varphi(\tau) \|^p_{L^p(I)} d\tau \leq C \left( \int_t^T \| \varphi(\tau) \|^p_{L^p(I)} d\tau \right)^{1/p}. \]

From this, it follows that
\[ \left( \int_t^T \| \varphi(\tau) \|^p_{L^p(I)} d\tau \right)^{1/p} \leq \frac{C}{\lambda^{1/(p-1)}}, \quad \text{where} \quad p = 1 + \frac{m+k}{2}. \]
(35)

Then, from (34) and (35), we have
\[ \frac{1}{2} \langle \varphi(t), \varphi(t) \rangle + \int_t^T \| \varphi(\tau) \|^2_{H^1_0(I)} d\tau \leq C \left( \int_t^T \| \varphi(\tau) \|^p_{L^p(I)} d\tau \right)^{1/p} \leq \frac{C}{\lambda^{1/(p-1)}}, \]
and hence
\[ (\varphi(t), \varphi(t))^{1/2} + \left( \int_t^T \| \varphi(\tau) \|^2_{H^1_0(I)} d\tau \right)^{1/2} \leq \frac{C}{\lambda^{1/(2p-2)}}, \quad \text{for all} \ t \in (0, T). \]

Clearly, by replacing \( \varphi(\cdot, t) \) with \( [u_{\lambda}(\cdot, t) - u^*(\cdot, t)]_+ \), we obtain readily (28) and (29).

On the basis of Lemma 8, we obtain Theorem 9 given below.

**Theorem 9.** Assume that the assumptions of Lemma 8 are satisfied, then for the combined penalty function \( c_k,m(u) \) defined by (27), it holds that as \( \lambda \to \infty \),
\[ \| u_{\lambda} - u \|_{L^\infty(0,T;L^2(I))} + \| u_{\lambda} - u \|_{L^2(0,T;H^1_0(I))} \leq \frac{C}{\lambda^{1/(m+k)}}, \]
(36)

In particular, when \( k = 1/2 \) and \( m = 2 \),
\[ \| u_{\lambda} - u \|_{L^\infty(0,T;L^2(I))} + \| u_{\lambda} - u \|_{L^2(0,T;H^1_0(I))} \leq \frac{C}{\lambda^{2/5}}, \]
as \( \lambda \to \infty \).
Proof. We still use the notation of Lemma 8. Setting \( v = \min(v, 0) \) and \( R_\lambda = u - u^* + [u_\lambda - u^*]_- \), it follows that when \( \alpha > 0 \),

\[
  u - u_\lambda = R_\lambda - \varphi, \quad \text{and} \quad (\varphi^\alpha, [u_\lambda - u^*]_-) = [u_\lambda - u^*]_+^\alpha [u_\lambda - u^*]_- \equiv 0. \tag{37}
\]

Set \( v = u - R_\lambda \) in (10) and \( v = R_\lambda \) in (12). Then by replacing \( \rho(u) \) with \( c_{k,m}(u) = [\max u, 0] + [\max[0, u]]^m \), we obtain

\[
\begin{align*}
  \left(-\frac{\partial u}{\partial t}, -R_\lambda\right) + A(u, -R_\lambda; t) &\geq (f, -R_\lambda), \tag{38} \\
  \left(-\frac{\partial u_\lambda}{\partial t}, R_\lambda\right) + A(u_\lambda, R_\lambda; t) + \lambda (\varphi^m + \varphi^{1/k}, R_\lambda) &= (f, R_\lambda). \tag{39}
\end{align*}
\]

Combining (38) and (39) gives

\[
\left(-\frac{\partial (u - u_\lambda)}{\partial t}, R_\lambda\right) + A(u_\lambda - u, R_\lambda; t) + \lambda (\varphi^m + \varphi^{1/k}, R_\lambda) \geq 0.
\]

It follows from \( u \leq u^* \) that

\[
(\varphi^m + \varphi^{1/k}, R_\lambda) = (\varphi^m + \varphi^{1/k}, u - u^*) + (\varphi^m + \varphi^{1/k}, [u_\lambda - u^*]_-) = (\varphi^m + \varphi^{1/k}, u - u^*) \leq 0.
\]

Thus,

\[
\left(-\frac{\partial (u(t) - u_\lambda(t))}{\partial t}, R_\lambda(t)\right) + A(u(t) - u_\lambda(t), R_\lambda(t); t) \leq 0. \tag{40}
\]

From (37) and (40), we get

\[
\left(-\frac{\partial R_\lambda(t)}{\partial t}, R_\lambda(t)\right) + A(R_\lambda(t), R_\lambda(t); t) \leq \left(-\frac{\partial \varphi(t)}{\partial t}, R_\lambda(t)\right) + A(\varphi(t), R_\lambda(t); t).
\]

Integrating both sides of the above from \( t \) to \( T \) and then using Cauchy–Schwarz inequality, we obtain

\[
\begin{align*}
  (R_\lambda(t), R_\lambda(t)) + \int_t^T A(R_\lambda(\tau), R_\lambda(\tau); \tau) \, d\tau &\leq \int_t^T \left(-\frac{\partial \varphi(\tau)}{\partial \tau}, R_\lambda(\tau)\right) \, d\tau + \int_t^T A(\varphi(\tau), R_\lambda(\tau); \tau) \, d\tau \\
  &\leq (\varphi(t), R_\lambda(t)) + \int_t^T \left(\varphi(\tau), \frac{\partial R_\lambda(\tau)}{\partial \tau}\right) \, d\tau + \int_t^T A(\varphi(\tau), R_\lambda(\tau); \tau) \, d\tau \\
  &\leq (\varphi(t), R_\lambda(t)) + \int_t^T \left(\varphi(\tau), \frac{\partial R_\lambda(\tau)}{\partial \tau}\right) \, d\tau + \int_t^T A(\varphi(\tau), R_\lambda(\tau); \tau) \, d\tau \\
  &\leq \|\varphi\|_{L^1_{\infty}(0,T;L^2(\Omega))} \|R_\lambda\|_{L^1_{\infty}(0,T;L^2(\Omega))} + C_1 \|\varphi\|_{L^1_{p,q}(0,T;H^{1/2}_0(\Omega))} \|R_\lambda\|_{L^1_{p,q}(0,T;H^{1/2}_0(\Omega))} \\
  &\quad + C_2 \|\varphi\|_{L^1_{p,q}(\Omega)} \left\|\frac{\partial u}{\partial t}\right\|_{L^1_{p,q}(\Omega)} + \|V_0 - V^*\|_{L^1_{p,q}(\Omega)} \\
  &\leq \|\varphi\|_{L^1_{\infty}(0,T;L^2(\Omega))} \|R_\lambda\|_{L^1_{\infty}(0,T;L^2(\Omega))} + C_1 \|\varphi\|_{L^1_{p,q}(0,T;H^{1/2}_0(\Omega))} \|R_\lambda\|_{L^1_{p,q}(0,T;H^{1/2}_0(\Omega))} + C_2 \lambda^{1/(p-1)},
\end{align*}
\]

where \( p = 1 + \frac{m+1}{2} \), and \( 1/p + 1/q = 1 \). Using the coerciveness property of the operator \( A \) and (29), we obtain

\[
\left(\|R_\lambda\|_{L^1_{\infty}(0,T;L^2(\Omega))} + \|R_\lambda\|_{L^1_{p,q}(0,T;H^{1/2}_0(\Omega))}\right)^2 \leq \frac{C}{\lambda^{1/(p-1)}},
\]

i.e.

\[
\|R_\lambda\|_{L^1_{\infty}(0,T;L^2(\Omega))} + \|R_\lambda\|_{L^1_{p,q}(0,T;H^{1/2}_0(\Omega))} \leq \frac{C}{\lambda^{1/(2p-2)}}.
\]
By the triangle inequality, we finally have
\[
\|u - u_\lambda\|_{L^\infty(0,T;L^2(I))} + \|u - u_\lambda\|_{L^2(0,T;H^1_0(u))} \\
\leq \|R_j\|_{L^\infty(0,T;L^2(I))} + \|R_j\|_{L^2(0,T;H^1_0(u))} + \|\varphi\|_{L^\infty(0,T;L^2(I))} + \|\varphi\|_{L^2(0,T;H^1_0(u))} \\
\leq \frac{C}{\lambda^{1/(2p-2)}}.
\]
This is the estimate (36).

Setting \( k = 2 \) and \( m = 2 \), and hence \( p = 9/4 \), we obtain
\[
\|u - u_\lambda\|_{L^\infty(0,T;L^2(I))} + \|u_\lambda - u\|_{L^2(0,T;H^1_0(u))} \leq \frac{C}{\lambda^{1/5}}.
\]

**Remark 10.** From the proof of Theorem 9, we see that by setting \( k = 1/m \) we obtain the convergence rate of the lower order penalty method \( l_k \), that is, \( O(\lambda^{-1/2k}) \). Likewise, we can get the convergence rates of \( I_2 \) and \( l_1 \) penalty methods by setting \( m = 1/k = 2 \) and \( m = k = 1 \), respectively. In view of this point, Theorem 9 actually gives a unified convergence rate of all the above penalty methods considered in [14] for American option pricing and in [7] for variational inequalities.

### 6. Conclusions

We have studied the monotonic penalty method for pricing American options. By using the equivalence of LCP and variational inequalities, the solvability and convergence properties of the monotonic penalty method were established. We have shown that the solution to the monotonic penalized nonlinear equation converges to that of the original LCP. The unified convergence rate of some monotonic penalty methods was obtained. Specifically, a combined penalty method was under detailed investigation.

### Acknowledgments

The authors would like to thank two anonymous referees for their helpful comments and suggestions toward the improvement of this paper.

### References


