Constrained optimisation in granular network flows: Games with a loaded dice

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Abstract. Flows in real world networks are rarely the outcome of unconditional random allocations as, say, the roll of a dice. Think, for example, of force transmission through a contact network in a quasistatically deforming granular material. Forces 'flow' through this network in a highly conditional manner. How much force is transmitted between two contacting particles is always conditional not only on all the other forces acting between the particles in question but also on those acting on the other particles in the system. Broadly, we are interested in the nature and extent to which flows through a contact network favour certain pathways over others, and how the mechanisms that govern such biased flows for a given imposed loading history determine the future evolution of the contact network. Our first step is to solve a selection of fundamental combinatorial optimisation problems on the contact network from the perspective of force transmission. Here we report on solutions to the Maximum Flow Minimum Cost Problem for a weighted contact network where the weights assigned to the links of the contact network are varied according to their contact types. We found that those pathways through which the maximum flow of force is transmitted, in the direction of the maximum principal stress, at minimum cost – pass through the great majority of the force chains. Although the majority of the contacts in these pathways are elastic, the plastic contacts bear an undue influence on the minimum cost.

Keywords: Granular materials; Network optimisation; Force chains; Force transmission
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INTRODUCTION

Flow networks are extremely common in everyday life (e.g. traffic flows, internet, telecommunications, airline routes and scheduling, fluid or gas pipelines) [1]. Consequently, the optimisation of flows through networks holds tremendous significance and broad utility in practice; this includes finding the shortest path between two points, scheduling a timetable under a given set of constraints, designing routes to minimise an associated cost when delivering commodities from the factory to customers, to name just a few examples [1]. This study forms part of a broader program that seeks to examine traditional granular mechanics problems (e.g. force transmission, energy flow, fluid flow through deforming particulate systems) through the lens of combinatorial optimisation and optimal control theory. This perspective, novel to granular media mechanics and physics, has shown promise. In [2], a weighted proximity network is constructed based on the contacts and relative displacement between particles. A maximum flow problem was formulated, and the bottleneck of force transmission was identified to lie inside the shear band. In [3], a maximum flow minimum cost problem of an unweighted contact network has been employed to similarly examine force transmission in dense granular materials.

In this paper, we will solve the maximum flow minimum cost problem for a *weighted contact network* where weights assigned to interparticle contacts are varied according to their contact types. Special attention will be paid to a set of contacts, called the *maximum flow minimum cost pathway* (MFMCP), that always transmit nonzero flow for all solutions of the maximum flow minimum cost problem. Our aim is to quantify force transmission through the weighted contact network at a cost associated with the energy dissipated at the contacts. The solution to this problem has been studied to explore the links between contact topology, force transmission and energy dissipation.

FLOW NETWORK

We construct flow networks from the evolving contact networks of a dense granular material of N grains under quasistatic loading. As the major load bearing contacts, i.e. those transmitting above the global average normal contact forces, are generally aligned in the direction of the maximum (most compressive) principal stress axis, we illustrate the maximum flow minimum cost problem by considering pathways through the contact network which optimise force transmission in this direction from source to sink. Here, we choose the top wall as the source and the bottom wall as the sink; switching the assignment, i.e. top wall as sink and bottom wall as source, will not affect the final result of this analysis.

Let $\mathscr{N} = \{1,\ldots,N,s,t\}$ be a set of nodes where s represents the source, t represents the sink and all other nodes are intermediate nodes. For each interparticle contact, we will define two equal and opposite directed edges, i.e, if particles i and j are in contact, then the directed edges (i,j) and (j,i) will be included in the set of edges $\mathscr{A}$. Furthermore, let $(s,i) \in \mathscr{A}$ if particle i contacts the top wall and let $(i,t) \in \mathscr{A}$ if particle i contacts the bottom wall. Here, all interparticle contacts are treated equally, each of a capacity of one unit. Contacts between a particle and a wall are assigned infinite capacities, meaning that we allow the source to supply as much flow as possible and the sink to consume as much flow as possible. Thus, for each pair $(i,j) \in \mathscr{A}$, the capacity of the directed edge (i,j), u_{ij}, is given by:

$$u_{ij} = \begin{cases} \infty, & \text{if } i = s \text{ or } j = t, \\ 1, & \text{otherwise.} \end{cases}$$

With the digraph $(\mathscr{N},\mathscr{A})$ and capacities u_{ij}, $(\mathscr{N},\mathscr{A},u,s,t)$ is called a *flow network* with *source* s and *sink* t. Thus, at each strain state, we can construct a *flow network* from the contact network.

An ordered set of values $f = \{f_{ij} : (i,j) \in \mathscr{A}\}$ is a *feasible flow vector* on $(\mathscr{N},\mathscr{A},u,s,t)$ if it satisfies the following constraints:

$$0 \le f_{ij} \le u_{ij}, \qquad (i,j) \in \mathscr{A}, \tag{1}$$

$$\sum_{j:\,(i,j)\in\mathscr{A}} f_{ij} - \sum_{j:\,(j,i)\in\mathscr{A}} f_{ji} = 0, \qquad i \in \mathscr{N} \setminus \{s,t\}. \tag{2}$$

The *capacity constraint* (1) requires that each edge carries a non-negative amount of flow which cannot exceed the capacity of the edge, while the *conservation constraint* (2) means that flows are preserved: the amounts of flow into and out of each node are equal, except for the source and the sink.

Let $\mathscr{F}$ denote the set of all feasible flow vectors on $(\mathscr{N},\mathscr{A},u,s,t)$. For each $f \in \mathscr{F}$, define

$$\text{val}(f) = \sum_{j:\,(s,j)\in\mathscr{A}} f_{sj} - \sum_{j:\,(j,s)\in\mathscr{A}} f_{js}.$$

Then $\text{val}(f)$ is the amount of flow from source to sink. The flow value through interparticle contact between particles i and j can be simply defined as $|f_{ij} - f_{ji}|$. Thus, as mentioned earlier, the force transmission direction chosen (from top to bottom walls versus bottom to top walls) will not alter the final result.

MAXIMUM FLOW MINIMUM COST PROBLEM

We now formulate the maximum flow minimum cost problem. First we solve the maximum flow problem, defined as follows:

Problem 1 *Find an* $f \in \mathscr{F}$ *to maximise* $\text{val}(f)$ *over* $\mathscr{F}$.

Problem 1 can be solved by Ford-Fulkerson method [1]. Let $F_{\max} = \max_{f\in\mathscr{F}} \text{val}(f)$. Note that solution to Problem 1 may be non-unique. Let $\mathscr{M}_{\max}$ denote the set of solutions.

We are interested in flow pathways (i.e. a set of edges or contacts in the contact network) that can transmit the maximum flow $F_{\max}$ — at minimum cost — in the direction of the maximum principal stress. This leads to the following maximum flow minimum cost problem which assigns a scalar weight to a member edge according to a prescribed cost of transmitting a unit of flow through that edge.

Problem 2 *Find an* $f \in \mathscr{M}_{\max}$ *to minimise the cost function*

$$E(f) = \sum_{(i,j)\in\mathscr{A}} c_{ij} f_{ij},$$

where c_{ij} *denotes the cost per unit of flow through edge* (i,j)*, over* $\mathscr{M}_{\max}$.

Problem 2 was proposed by Edmonds [4]. Furthermore, Problems 1 and 2 can be solved together using the algorithm developed in [4].

Note that the solution to Problem 2 may also be non-unique. Let $\mathscr{M}_{\min}$ denote the set of solutions to Problem 2. For each $f \in \mathscr{M}_{\min}$, define

$$\mathscr{F}_f = \{(i,j) \in \mathscr{A} : f_{ij} > 0\}.$$

$\mathscr{F}_f$ is the set of edges that are used to transmit non-zero flow for flow vector f. Since the solution to Problem 2 may be non-unique, there is no point in investigating $\mathscr{F}_f$ for a solution f. Instead, a more informative set is $\mathscr{R}$, the set of edges that are always utilised in the transmission of non-zero flows for all solutions of Problem 2:

$$\mathscr{R} = \bigcap_{f\in\mathscr{M}_{\min}} \mathscr{F}_f.$$

We call $\mathscr{R}$ the *maximum flow minimum cost pathway* (MFMCP).

RESULTS

To determine the efficacy of the above network optimisation analysis in the characterisation of granular material behaviour, we consider the systems recently examined in [5]. Due to space limitations, we refer readers to that paper for details of the discrete element simulation which supplied the data on contact network and contact types used here. The sample is a densely packed, polydisperse assembly of 5098 spherical particles, subjected to plane strain (biaxial) compression, under constant confining pressure. Relative motion at the particle-particle and particle-wall contacts are resisted by forces represented by various combinations of a linear spring, a dashpot and a friction slider. Normal and tangential resistive forces, as well as a moment (rolling resistance) act at each contact, in accordance with [6]. A spring-dashpot-slider combination defines the resistive force between contacting particles, namely,

$$f_n = k^n \Delta u_n + b^n \Delta v_n, \tag{3}$$

$$f_t = \begin{cases} k^t \Delta u_t + b^t \Delta v_t, \text{ if } k^t |\Delta u_t| < \mu^t |f_n|, & \text{(4a)} \\ \text{sign}(\Delta u_t) \mu^t |f_n|, \text{ otherwise}, & \text{(4b)} \end{cases}$$

where f_n and f_t are the normal and tangential components of contact force, k^n and k^t are the spring stiffness coefficients; b^n and b^t are the viscous damping coefficients; and μ^t is the Coulomb friction coefficient. The rolling resistance or contact moment I, defined in an analogous fashion to Coulomb's law, is expressed as

$$I = \begin{cases} k^r \Delta\alpha + b^r \Delta\dot{\alpha}, \text{ if } k^r |\Delta\alpha| < \mu^r R_{\min} |f_n|, & \text{(5a)} \\ \text{sign}(\Delta\alpha) \mu^r R_{\min} |f_n|, \text{ otherwise}, & \text{(5b)} \end{cases}$$

where $R_{\min}$ denotes the smaller of the radii of the two contacting particles, k^r and b^r are the spring stiffness and viscous damping coefficients, respectively; and μ^r is the friction coefficient. The remaining quantities in the above relations are: the relative normal and tangential displacements and relative rotation denoted, respectively, by Δu_n, Δu_t and $\Delta\alpha$, and the relative normal and tangential translational and rotational velocities denoted, respectively, by Δv_n, Δv_t and $\Delta\dot{\alpha}$.

Four modes of contact can be distinguished depending on the magnitude of the contact forces and moments: full-stick, sliding, rolling and slide-and-roll contacts. The contact is full-stick or elastic if (4a) and (5a) are satisfied. The conditions for the three modes of plastic contact are: (4b) and (5a) for a sliding contact, (4a) and (5b) for a rolling contact and (4b) and (5b) for a slide-and-roll contact. We assign a simple cost function, reflecting the energy dissipation at plastic contacts. Edge weights are positive integers bigger than the baseline cost of 1 assigned to the elastic contact. For each pair $(i, j) \in \mathscr{A}$, the cost c_{ij} is given as follows:

$$c_{ij} = \begin{cases} 1, & \text{if } i = s, \text{ or } j = t, \text{ or it is full-stick}, \\ 2, & \text{if it is sliding or rolling}, \\ 3, & \text{if it is slide-and-roll}. \end{cases} \tag{6}$$

For each strain state, we solve the maximum flow minimum cost problem as mentioned earlier. Keep in mind that the maximum flow (i.e. the maximum number of units of flow) is computed first for the *unweighted* contact network. The higher the maximum flow, the more effective is the network as a transmission medium. Thus we can use maximum flow to quantify the influence of contact topology (or fabric) on the capacity of the contact network as a transmission medium for force, for the case when no bias is introduced and all edges (contacts) of the network are treated equally. The strain evolution of the stress ratio and maximum flow is shown in Figure 1. Three distinct regimes are evident in the maximum flow. During the initial stages of the strain-hardening regime, the maximum flow is observed to be near invariant and at its highest throughout loading history. The maximum flow then progressively weakens in the lead up to peak stress ratio and right through to the end of the strain-softening regime. In the large strain 'steady critical state' regime, although relatively large fluctuations which dip between axial strains 0.0542 and 0.0835 can be observed, the maximum flow appears to be more or less constant at the start (axial strains from 0.04 to 0.0542) and at large axial strains (beyond 0.0835). Thus, using information on the contact topology alone, the evolution of maximum flow suggests that force propagation through the contact network in the direction of maximum principal stress – i.e. that direction along which force chains form – degrades with loading history. This trend is consistent with the three distinct regimes originally uncovered in [7] from local non-affine deformation at the scale of a particle and its first ring of neighbours.

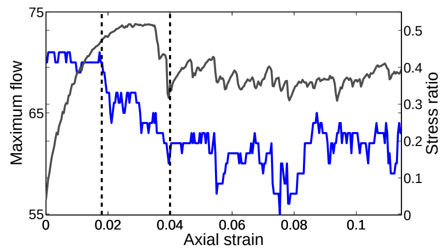

FIGURE 1. Strain evolution of maximum flow (blue) and stress ratio (black).

We now turn to the minimum cost. Let f^* be a solution of Problem 2. Then $E(f^*)/F_{\max}$ means the average cost for one unit of flow propagating from the top wall to the bottom wall. Note that the distance between top and bottom walls is decreasing with strain; in this system, the shortest path length from the top wall to the bottom wall is also decreasing. The average cost divided by the shortest path length from the source (top wall) to the sink (bottom wall) also captures the three distinct regimes observed in the strain evolution of the maximum flow: see Figure 2. Furthermore, this cost per path length tracks the relative population of plastic contacts throughout the loading history, suggesting that the plastic contacts in these pathways play a dominant role on cost more than

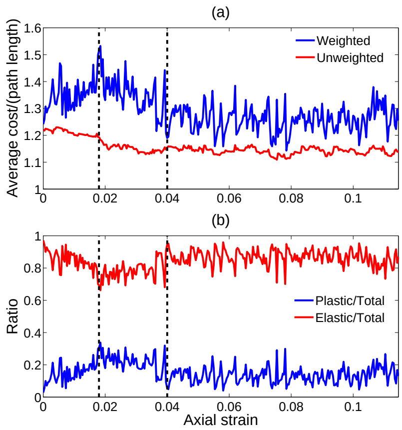

FIGURE 2. Strain evolution of (a) average cost/(path length) and (b) the ratio of the number of contacts of a given type (i.e. elastic, plastic) to total number of contacts. In the weighted contact network, the cost c_{ij} is defined by (6) and in the unweighted contact network, the cost $c_{ij} = 1$, $(i, j) \in \mathscr{A}$.

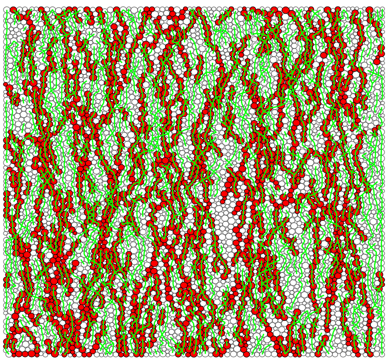

FIGURE 3. MFMCP for weighted contact network at $\varepsilon_{22} = 0.0342$. Red shaded particles are force chains and green lines represent the edges in MFMCP.

the elastic contacts. The influence of the plastic contacts on the average cost per path length is significant: the correlation between the relative population of plastic contacts and the average cost per path length, as measured by the Pearson's product moment coefficient, is 0.88. For comparison, Figure 2 shows that the strain evolution of the cost of transmitting the maximum flow through the unweighted contact network, i.e. $c_{ij} = 1$ for all (i, j) in $\mathscr{A}$, exhibits a qualitatively similar trend to that of the maximum flow.

We also conducted a preliminary study of the set of edges in MFMCP over three consecutive strain states at the peak stress ratio – when the system's force chain network is at its strongest. Recall MFMCP are the edges that are always utilised in those pathways through which the maximum flow is transmitted in the direction of the maximum principal stress at minimum cost. Figure 3 shows the MFMCP at $\varepsilon_{22} = 0.0342$. Most of contacts in MFMCPs are elastic (79-86%) in both the weighted and unweighted networks. Also the MFMCPs from the weighted network pass through most (90-91%) of the force chain particles as determined using the force chain algorithm in [8]; the same can be said of the MFMCPs from the unweighted network, although the percentages are slightly less (83-86%). Interestingly, even though the common or intersection set between the MFMCP and the plastic contacts, at least for peak stress ratio, comprises only a small fraction from either set (below 21%), we find that the strain evolution of the average cost per path length correlates strongly with the relative population of plastic contacts in the system (recall the Pearson's product moment coefficient is 0.88).

CONCLUSION

We solved a fundamental combinatorial optimisation problem, the Maximum Flow Minimum Cost Problem, for the weighted contact network at each strain state of a dense granular assembly under a quasistatic biaxial compression with constant confining pressure. Our objective was to understand how the material 'selects' those pathways through which the maximum flow of force is transmitted, in the direction of the maximum principal stress, at minimum cost. Contact topology is found to have a significant but not an exclusive influence on the choice of these pathways. Moreover, although the elastic contacts form the great majority in these pathways, it is the minority group of plastic contacts which dispropotionately influences the minimum cost. In particular, we observed a very high correlation between the relative population of plastic contacts in the system and the average cost per path length, throughout loading history. A preliminary study of those edges/contacts that are always utilised in these pathways during peak stress ratio showed that the chosen pathways pass through the great majority of force chain particles.

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