Performance comparison of the BIE estimator with the float and fixed GNSS ambiguity estimators

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Abstract. The goal of ambiguity resolution is to make optimal use of the integerness of the ambiguities, and it is the key to high precision GNSS positioning and navigation. However, it should only be applied in case the probability of correct integer ambiguity resolution, i.e. the success rate, is very close to one. In that case, the probability that the fixed baseline will be closer to the true but unknown baseline is larger than that of the float baseline. Clearly, this condition will not be fulfilled for each measurement scenario, and this means that for low success rates a user will prefer the float solution.

However, there exists a baseline estimator that will always be superior to its float and fixed counterparts, albeit that this superiority is measured using a weaker optimality condition. This baseline estimator is the Best Integer Equivariant (BIE) estimator, which is unbiased and of minimum variance within the class of integer equivariant estimators.

In this contribution, the three different estimators are compared. For that purpose, we will focus on the geometry-free GNSS models, either single frequency or dual frequency. The performance of the estimators is compared based on their probability density functions, the variances of the different estimators, and the probabilities that the baseline estimators are within a certain convex region symmetric with respect to the true baseline. This will provide information on whether or not the BIE estimator could be useful in positioning applications.

Keywords. GNSS ambiguity resolution, best integer equivariant unbiased estimator, float and fixed estimator

1 Introduction

A GNSS model generally contains real-valued and integer-valued parameters. The latter are the double difference (DD) carrier phase ambiguities, a. The first group is referred to as the baseline unknowns, b.

The 'float' estimators of the unknown parameters and their variance-covariance (vc-) matrix are obtained after a standard least-squares adjustment, ignoring the integer-constraint on the ambiguity unknowns:

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} ; \quad \begin{pmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{a}\hat{b}} & Q_{\hat{b}} \end{pmatrix} \tag{1}$$

In the next step, the float ambiguities are fixed to integer values, which is referred to as ambiguity resolution:

$$\check{a} = S(\hat{a}) \tag{2}$$

where $S : \mathbb{R}^n \mapsto \mathbb{Z}^n$ is the mapping from the n-dimensional space of real numbers to the n-dimensional space of integers. The optimal result in the sense of maximizing the probability of correct integer estimation (success rate) is obtained using integer least-squares, cf (Teunissen, 1993; Teunissen, 1999). Finally, the float baseline estimator is adjusted by virtue of its correlation with the ambiguities. The result is the so-called 'fixed' baseline estimator:

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \tag{3}$$

Ambiguity resolution should only be applied when there is enough confidence in its results, which means that the success rate should be very close to one. Only then the probability that the fixed baseline is close to the true baseline is higher than the corresponding probability for the float baseline. If this is

not the case, a user will prefer the float solution. So, the choice between float or fixed baseline solution depends on the success rate. However, there exists a baseline estimator that will always be superior to its float and fixed counterparts, albeit that this superiority is measured using a weaker optimality criterion. This baseline estimator is the Best Integer Equivariant (BIE) estimator and it was introduced in (Teunissen, 2003b; Teunissen, 2003a).

2 The BIE estimator

The best integer equivariant (BIE) estimator is based on a new class of estimators. Estimators in this new class only have to fulfill the integer remove-restore property, which requires that $S(\hat{a} - z) + z = S(\hat{a})$. Hence the name integer equivariant (IE) that is assigned to all estimators that belong to this class. The criterion of 'best' that will be used is that the mean squared error (MSE) should be minimal within the IE-class. With this MSE-criterion the best integer equivariant (BIE) estimator is defined as:

$$\hat{\theta}_{\text{BIE}} = \arg \min_{f_\theta \in IE} E\{(f_\theta(y) - \theta)^2\} \tag{4}$$

The minimization is taken over all IE functions $f_\theta(y)$ with $\theta = l_a^T a + l_b^T b$, and $l_a \in \mathbb{R}^n$, $l_b \in \mathbb{R}^p$ are chosen linear functions.

In our GNSS applications we assume that the data are normally distributed. It can be shown that in this case the BIE estimator $\hat{\theta}_{BIE}$ becomes

$$\hat{\theta}_{\text{BIE}} = l_a^T \bar{a} + l_b^T \bar{b} \tag{5}$$

with

$$\begin{cases} \bar{a} = \sum_{z \in \mathbb{Z}^n} z \dfrac{\exp\{-\frac{1}{2}\|\hat{a}-z\|^2_{Q_{\hat{a}}}\}}{\sum_{z \in \mathbb{Z}^n} \exp\{-\frac{1}{2}\|\hat{a}-z\|^2_{Q_{\hat{a}}}\}} \\ \bar{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \bar{a}) \end{cases} \tag{6}$$

This shows that $\bar{a}$ is a weighted sum of all integer vectors and could thus also be written as

$$\bar{a} = \sum_{z \in \mathbb{Z}^n} z w_z(\hat{a}) \tag{7}$$

with $w_z(\hat{a}) \leq 1, \forall z \in \mathbb{Z}^n$ and $\sum_{z \in \mathbb{Z}^n} w_z(\hat{a}) = 1$.

Note that the formal expressions in (6) are identical to their Bayesian counterparts as given in (Betti et al., 1993; Gundlich and Koch, 2002; Gundlich, 2002), but that the distributional properties of the BIE estimator and its Bayesian counterpart of course differ. Since the functional form of the nonBayesian estimator is identical to the Bayesian solution in the

special case of normally distributed data, the theory of BIE-estimation has provided the as yet unknown link with the Bayesian approach of ambiguity resolution.

In (Teunissen, 2003b) it was also shown that

$$\begin{cases} D\{\bar{b}\} \leq D\{\check{b}\} \text{ and } E\{\bar{b}\} = E\{\check{b}\} \\ D\{\bar{b}\} \leq D\{\hat{b}\} \text{ and } E\{\bar{b}\} = E\{\hat{b}\} \end{cases} \tag{8}$$

where $E\{\cdot\}$ is the mathematical expectation operator and $D\{\cdot\}$ the dispersion operator. Eq.(8) shows that the BIE estimator is always 'better' than the float and fixed counterparts in terms of the variance.

3 Comparison of the estimators

Although we know that the BIE estimator outperforms its float and fixed counterparts in terms of precision, it is not clear how large this difference will be under varying measurement scenarios. It is therefore also of interest to compare the three estimators numerically. This also provides the possibility to compare their distributional properties.

Fixed versus BIE ambiguity estimator

Both the fixed and the BIE ambiguity estimator are weighted sums of all integer vectors in $\mathbb{Z}^n$. However, in case of the fixed integer least-squares estimator a weight of 1 is assigned to the integer vector with minimal distance to the float vector, and all other weights are set to zero. In case of the BIE estimator the weights depend on the distance, again measured in the metric of $Q_{\hat{a}}$, of the integer vectors to the float vectors, see Eq.(6). In general the BIE ambiguity estimates will therefore be real-valued.

It follows that in the limits of the precision the following is true:

$$\lim_{\sigma \to \infty} \bar{a} = \hat{a} \quad \text{and} \quad \lim_{\sigma \to 0} \bar{a} = \check{a} \tag{9}$$

with the vc-matrix factored as $Q_{\hat{a}} = \sigma^2 G$. This shows that if the precision is high and thus the success rate is close to one, the BIE estimator will be close to the fixed estimator. On the other hand, if the precision is low the BIE estimator will approximate the float estimator. It is thus especially interesting to know how the BIE estimator performs in the intermediate cases compared to the float and fixed estimators.
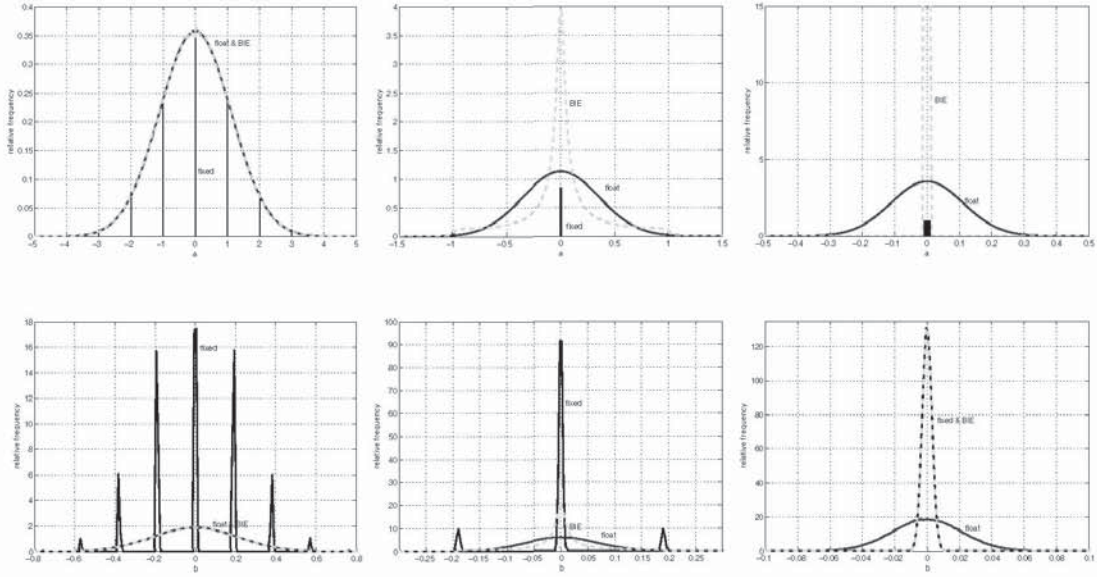
Fig. 1 Examples of distributions of the different estimators. Top: $\hat{a}$ (solid), $\check{a}$ (bars), $\bar{a}$ (dashed); Bottom: $\hat{b}$ (solid), $\check{b}$ (solid, multi-modal), $\bar{b}$ (dashed). Left: 2 epochs; Center: 20 epochs; Right: 200 epochs. Note the different scalings.

Comparison in the one-dimensional case

A random generator was used to generate 500,000 samples of the float range and ambiguity, using the geometry-free single frequency GPS model for k epochs, with vc-matrix:

$$\begin{pmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}_k} \\ Q_{\hat{b}_k\hat{a}} & Q_{\hat{b}_k} \end{pmatrix} = \begin{pmatrix} \frac{\sigma_p^2}{k\lambda^2}(1+\varepsilon) & -\frac{\sigma_p^2}{k\lambda^2} \\ -\frac{\sigma_p^2}{k\lambda^2} & \frac{\sigma_p^2}{1+\varepsilon}\left(\frac{1}{k}+\varepsilon\right) \end{pmatrix}$$

with λ the wavelength of the carrier; σ_p^2 and σ_ϕ^2 are the variances of the DD code and phase observations respectively, and $\varepsilon = \frac{\sigma_\phi^2}{\sigma_p^2}$. For all simulations, the standard deviations were chosen as $\sigma_p = 30$ cm and $\sigma_\phi = 3$ mm. The number of epochs was varied. Note that in the one-dimensional case the fixed ambiguity estimator is obtained by simply rounding the float estimator to the nearest integer.

Figure 1 shows the parameter distributions of all three estimators for different values of k, based on the simulation results. It can be seen that for small k the distribution of the BIE ambiguity and range estimator resembles the normal PDF of the float estimators. For larger k, and thus higher precision, the distribution of the BIE estimators more and more resemble those of the fixed estimators. Note that the multi-modality of the distribution of the BIE range estimator is less pronounced than that of the fixed range estimator.

From Eq.(8) follows that the BIE baseline estimator has smallest variance, but in the limits of the precision the variance will become equal to the variance of the float and fixed estimator. This is illustrated in Figure 2, where the variance ratio of the different estimators is shown as function of k. Also the success rate is shown. Indeed, for small k the variance of the BIE and the float estimator are equal to each other (ratio equals one), and smaller than the variance of the fixed estimator. On the other hand, for large k the variances of the BIE and fixed estimator become equal to each other, and smaller than that of the float estimator. Only after 30 epochs, when the success rate is larger than 0.9, the variance of the fixed baseline estimator becomes lower than the variance of the float estimator.

It can be shown that in the one-dimensional case $\bar{a}$ will always lie in-between $\hat{a}$ and $\check{a}$, so that $|\hat{a} - \check{a}| \geq |\hat{a} - \bar{a}|$, which means that the BIE and the fixed estimator are pulled in the same direction in the one-dimensional case. This is shown in Figure 3 for different precisions (i.e. for different k). The ambiguity residuals are defined as $\check{\epsilon} = \hat{a} - \check{a}$ and $\bar{\epsilon} = \hat{a} - \bar{a}$. This Figure shows that indeed the relations in (9) hold.

Figure 4 shows the probability that the baseline estimators will be within a certain interval 2ϵ that is centered at the true baseline b, again for different precisions. It can be seen that for high success rates indeed the relation $P[\check{b} \in E_b] \geq P[\hat{b} \in E_b]$
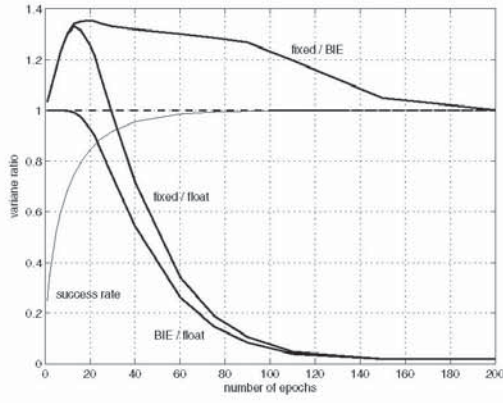
Fig. 2 Variance ratios of: BIE and float estimator; BIE and fixed estimator; fixed and float estimator. Success rate as function of the number of epochs is also shown.
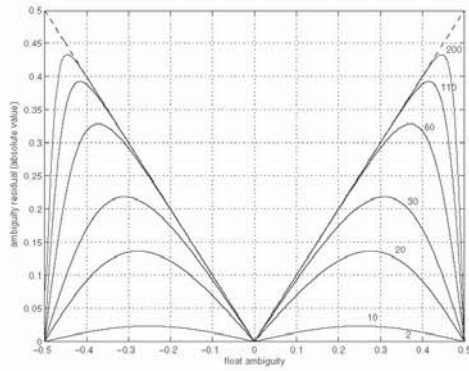
Fig. 3 Absolute values of ambiguity residuals for fixed (dashed) and BIE (solid) estimators for different number of epochs.
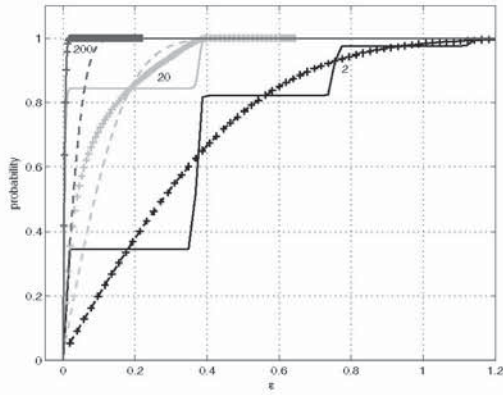
Fig. 4 Probabilities $P[|\hat{b} - b| \leq \epsilon]$ (dashed), $P[|\check{b} - b| \leq \epsilon]$ (solid), $P[|\bar{b} - b| \leq \epsilon]$ (+-signs) for different number of epochs.

k	$P(\check{a} = a)$	P_1	P_2	P_3
1	0.2490	0.5571	0.6049	0.4429
2	0.3465	0.5828	0.5871	0.4172
5	0.5221	0.6349	0.6419	0.3652
10	0.6846	0.7071	0.7163	0.2961
20	0.8436	0.8196	0.8257	0.2395
30	0.9180	0.8847	0.8867	0.3006
60	0.9860	0.9368	0.9368	0.4575
75	0.9940	0.9393	0.9393	0.4804
110	0.9991	0.9332	0.9332	0.4965
200	1.0000	0.9129	0.9129	0.5040

Table 1 Probabilities $P_1 = P(|\check{b} - b| \leq |\hat{b} - b|)$, $P_2 = P(|\bar{b} - b| \leq |\hat{b} - b|)$, and $P_3 = P(|\bar{b} - b| \leq |\check{b} - b|)$.

is true, with E_b a convex region centered at b, which means that the probability that the fixed baseline will be closer to the true but unknown baseline is larger than that of the float baseline, cf. (Teunissen, 2003b). However, for lower success rates some probability mass for $\check{b}$ can be located far from b because of the multi-modal distribution. Ideally, the probability should be high for small ϵ and reach its maximum as soon as possible. For lower success rates, the float and fixed estimators will only fulfill one of these conditions. The probability for the BIE estimator always falls more or less in-between those of the float and fixed estimators, or is equal to one of these probabilities.

The probabilities shown in Figure 4 are determined by counting the number of solutions that fall within a certain interval, but it is also interesting to compare the estimators on a sample by sample basis. In order to do so, one could determine for each sample which of the three estimators is closest to the true b, and then count for each estimator how often it was better than the other estimators.

In Table 1 the probabilities $P(|\check{b} - b| \leq |\hat{b} - b|)$, $P(|\bar{b}-b| \leq |\hat{b}-b|)$, and $P(|\bar{b}-b| \leq |\check{b}-b|)$ are given for different number of epochs. Note that if the ambiguities are fixed correctly, that does not automatically imply that $\check{b}$ is better than $\hat{b}$ because of the probability distribution of $\hat{b}$. It follows that the probability that $\bar{b}$ is better than the corresponding $\hat{b}$ is larger or equal to the probability that $\check{b}$ is better than $\hat{b}$. That is because the ambiguity residuals that are used to compute the fixed and BIE baseline estimator, see eqs.(3) and (6), have the same sign, and $|\hat{a} - \bar{a}| \leq |\hat{a} - \check{a}|$ as was shown in Figure 3. If the $\hat{b}$ is already close to the true solution, it is possible that $\bar{b}$ is closer to b, but that $\check{b}$ is pulled 'over' the true solution so that it has the opposite sign as $\hat{b}$ and the distance to b is larger. Note
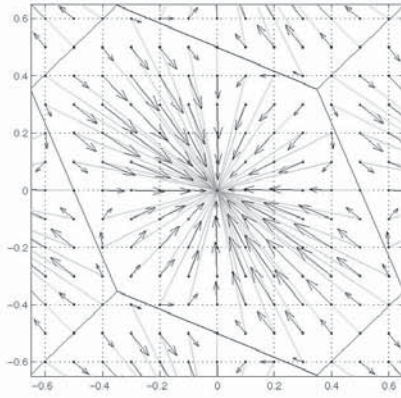
Fig. 5 For different float ambiguity vectors the corresponding BIE and fixed estimator are determined. The arrows point from $\hat{a}$ to corresponding $\bar{a}$. $\sigma_p = 0.8$m, $\sigma_\phi = 8$mm.

that this does not necessarily hold true for the higher dimensional case ($n > 1$). For $k > 1$ it turns out that if $\check{b}$ is better than $\hat{b}$, then also $\bar{b}$ will better than $\hat{b}$.

Comparison in the two-dimensional case

A similar approach was followed to generate samples for the geometry-free dual-frequency GPS model for one satellite-pair. The double difference standard deviations were chosen such that success rates between 0.3 and 0.99 were obtained, see Table 2.

Figure 5 shows an example of the BIE and fixed ambiguity estimates that correspond to certain float ambiguity vectors. The hexagons are the pull-in regions; all float solutions that fall in a specific pull-in region are fixed to the corresponding integer grid point in the center of this region in case of integer least-squares. It can be seen that the BIE estimator is also pulled in approximately the same direction, but the ambiguity residuals are always much smaller for this example. If the float solution falls close to an integer grid point, the BIE estimator is pulled in that direction. On the other hand, if the float solution falls close to the boundary of a pull-in region, which means that the distance to at least two integer grid points is approximately the same, the ambiguity residual is small and the BIE estimator is pulled in another direction.

Figure 6 shows a scatter plot of simulated $\hat{a}$ and corresponding $\bar{a}$. For higher precisions, the 'star'-shape of the distribution of $\bar{a}$ becomes even stronger; for lower precisions the distribution becomes more ellipse-shaped like that of $\hat{a}$.

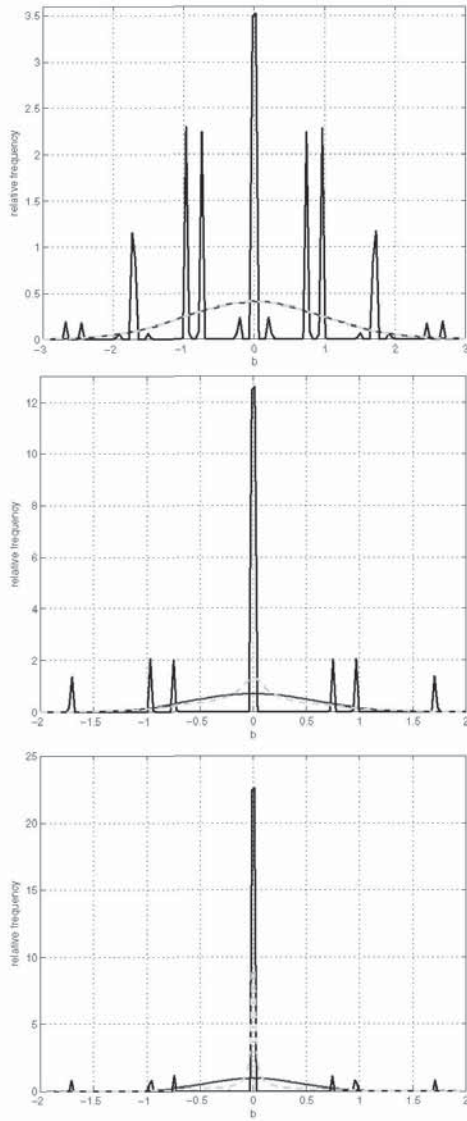The distribution of the three baseline estimators is shown for different precisions. The probabilities
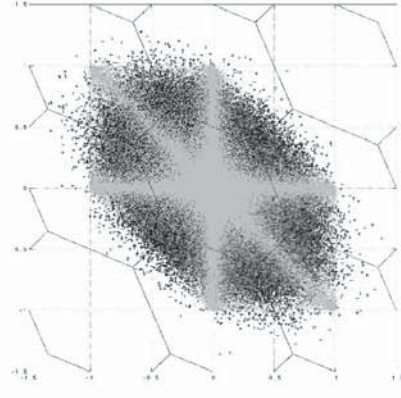
Fig. 6 Scatter plot of float ambiguities (black) and BIE ambiguity estimates (grey). $\boldsymbol{\sigma_p} = 0.6$m, $\boldsymbol{\sigma_\phi} = 6$mm.
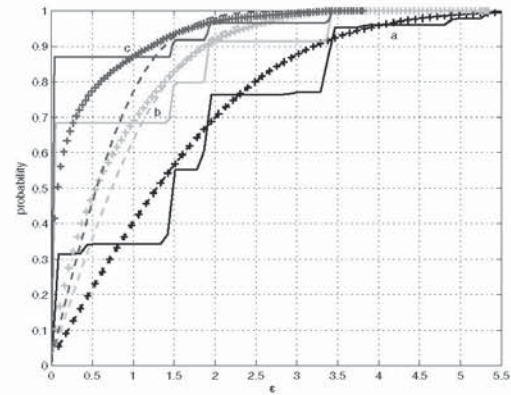
Fig. 7 $P[|\hat{b} - b| \leq \epsilon]$ (dashed), $P[|\check{b} - b| \leq \epsilon]$ (solid), $P[|\bar{b} - b| \leq \epsilon]$ (+-signs). a) $\sigma_p = 1.4$m, $\sigma_\phi = 14$mm; b) $\sigma_p = 0.8$m, $\sigma_\phi = 8$mm; c) $\sigma_p = 0.6$m, $\sigma_\phi = 6$mm.

that the baseline estimators will be within a certain interval 2ϵ that is centered at the true baseline b is shown in Figure 7.

Table 2 shows the probabilities $P(|\check{b}-b| \leq |\hat{b}-b|)$, $P(|\bar{b} - b| \leq |\hat{b} - b|)$, and $P(|\bar{b} - b| \leq |\check{b} - b|)$. This Table only shows the probabilities that the estimators will perform better than the other estimators. However, this does not say how much better, or how much worse in the instances that they do not perform better. Therefore, one should also consider the probabilities as shown in Figure 7. From the Table it follows namely that $P(|\check{b} - b| \leq |\hat{b} - b|)$ is larger than $P(|\bar{b} - b| \leq |\hat{b} - b|)$. On the other hand, Figure 7 shows that if $\check{b}$ is not close to b, it will immediately be far from the true solution (because of the step-wise function), whereas for the BIE estimator that is not necessarily the case.

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Fig. 8 Examples of distributions of the different baseline estimators: $\hat{b}$ (solid), $\check{b}$ (solid, multi-modal), $\bar{b}$ (dashed). Top: $\sigma_p = 1.4$m, $\sigma_\phi = 14$mm; Center: $\sigma_p = 0.8$m, $\sigma_\phi = 8$mm; Bottom: $\sigma_p = 0.6$m, $\sigma_\phi = 6$mm.

σ	P_s	P_1	P_2	P_3
1.4	0.3119	0.5256	0.5209	0.4744
0.8	0.6749	0.7237	0.7076	0.2827
0.6	0.8591	0.8745	0.8593	0.2050
0.4	0.9853	0.9834	0.9812	0.4336

Table 2 Probabilities $P_s = P(\check{a} = a)$, $P_1 = P(|\check{b} - b| \leq |\hat{b} - b|)$, $P_2 = P(|\bar{b} - b| \leq |\hat{b} - b|)$, and $P_3 = P(|\bar{b} - b| \leq |\check{b} - b|)$. Double difference code and phase standard deviations respectively: $\sigma_p = \sigma$m and $\sigma_\phi = \sigma$cm.

4 Concluding remarks

The key to high-precision GNSS positioning is to make use of the constraint that the unknown double difference ambiguities are integer-valued. In practice therefore the ambiguities are fixed using an appropriate ambiguity resolution method. Only if the probability of correct integer estimation, referred to as the success rate, is very high the fixed ambiguities are then used to find the corresponding fixed baseline solution. If, on the other hand, the success rate is not considered high enough, one has to stick with the float solution and collect more data before a fixed solution can be obtained.

In this paper it has been shown, that the BIE estimator might be useful to circumvent this problem. This estimator is always 'best' in the sense that it minimizes the mean squared errors and at the same it will give almost the same results as the fixed estimor when the precision is very high, and it will give almost the same results as the float solution when the precision is very low.

The next steps are to implement BIE estimation such that it can be used for real GPS applications and to see how it performs then.

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