# The $SC^1$ Property of an Expected Residual Function Arising from Stochastic Complementarity Problems

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**Abstract.** The stochastic nonlinear complementarity problem has been recently reformulated as an expected residual minimization problem which minimizes an expected residual function defined by an NCP function. In this paper, we show that the expected residual function defined by the *Fischer-Burmeister* function is an $SC^1$ function.

**Keywords:** Stochastic complementarity problems; NCP function; Expected residual function; $SC^1$ property

## 1 Introduction

Let $(\Omega, \mathcal{F}, \rho)$ be a probability space, where $\Omega$ is a subset of $\Re^m$ and $\rho$ is a standard probability measure (see [11]). We assume that $\rho$ is known. We consider the following stochastic nonlinear complementarity problem (SNCP): Find an $x \in \Re^n$ such that

$$x \geq 0, \; F(x, \omega) \geq 0, \; x^T F(x, \omega) = 0, \tag{1}$$

where $F : \Re^n \times \Omega \to \Re^n$ is continuously differentiable with respect to $x$ for any fixed $\omega \in \Omega$. When $F(x, \omega) := M(\omega)x + q(\omega)$, the above SNCP reduces to the following stochastic linear

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complementarity problem (SLCP) [2]:

$$x \geq 0, \ F(x, \omega) := M(\omega)x + q(\omega) \geq 0, \ x^T F(x, \omega) = 0, \tag{2}$$

where $M(\omega) \in \Re^{n \times n}$ and $q(\omega) \in \Re^n$ for $\omega \in \Omega$.

Stochastic complementarity problems have been studied in [2, 3, 8]. In particular, Chen and Fukushima [2] have recently proposed a new deterministic formulation for (1) which is to find a vector $x \in \Re^n_+$ that minimizes an *expected residual function*:

$$\min_{x \in \Re^n_+} E\left\{\|\Phi(x, \omega)\|^2\right\}, \tag{3}$$

where $E$ stands for the expectation and the function $\Phi$ is defined as follows:

$$\Phi(x, \omega) := \begin{pmatrix} \phi(F_1(x, \omega), x_1) \\ \phi(F_2(x, \omega), x_2) \\ \vdots \\ \phi(F_n(x, \omega), x_n) \end{pmatrix}. \tag{4}$$

Here, $\phi : \Re^2 \to \Re$ is an NCP function which has the property

$$\phi(a, b) = 0 \Longleftrightarrow a \geq 0, b \geq 0, ab = 0.$$

Over the last decade, various NCP functions have been studied for solving complementarity problems [7]. Among them, the Fischer-Burmeister (FB) function is one of the frequently used NCP functions, which is defined as follows:

$$\phi_{\mathrm{FB}}(a, b) = \sqrt{a^2 + b^2} - a - b.$$

Let

$$G_{\mathrm{FB}}(x) := E\left\{\|\Phi_{\mathrm{FB}}(x, \omega)\|^2\right\} = \int_{\Omega} \|\Phi_{\mathrm{FB}}(x, \omega)\|^2 \rho(\omega) d\omega. \tag{5}$$

Here, the integral is the Lebesgue integral [11] with respect to the probability measure $\rho$. When $\Omega$ has finitely many elements, the integral is a sum. Throughout this paper, we assume that for any $x \in \Re^n$,$G_{\mathrm{FB}}(x)$ is well-defined and finite valued. In what follows, we let

$$\Psi_{\mathrm{FB},\omega}(x) := \|\Phi_{\mathrm{FB}}(x, \omega)\|^2 = \sum_{i=1}^{n} \left(\phi_{\mathrm{FB}}(F_i(x, \omega), x_i)\right)^2. \tag{6}$$

The expected residual minimization (ERM) formulation (3) for the SNCP was first proposed by Chen and Fukushima [2] and some properties of the ERM problem (3) for the SLCP have been studied in [2, 3]. It is showed in [3] that the ERM method may produce robust solutions with minimum sensitivity, high reliability, and low risk in violation of feasibility.

The main aim of this paper is to study the $SC^1$ property of $G_{\rm FB}$. A function $f : \Re^n \to \Re$ is said to be an $SC^1$ *function* if $f$ is continuously differentiable and its gradient is semismooth. The $SC^1$ property is a crucial condition for superlinear convergence of the SQP method for solving once, but not twice, continuously differentiable nonlinear programming problems [5, 9, 12, 13]. In this paper we show that $G_{\rm FB}$ defined as in (5) is an $SC^1$ function (see Theorem 1). Based on these properties, like [6, 9, 12], superlinear convergence of SQP method for (3) can be established if we can compute $G_{\rm FB}$ and $\nabla G_{\rm FB}$ numerically. Clearly, when $\Omega$ has finitely many elements, $G_{\rm FB}$ and $\nabla G_{\rm FB}$ can be computed numerically. In [2, 3], Chen and Fukushima also gave some examples for which $G_{\rm FB}$ and $\nabla G_{\rm FB}$ can be computed numerically. When $G_{\rm FB}$ cannot be evaluated exactly, the SQP method cannot be used to solve (3). In this case, in [2, 3], a sample average approximation (SAA) is used to solve (3) approximately.

We now give some background material. Semismoothness was originally introduced by Mifflin [10] for functionals. In [15], Qi and Sun extended the definition of semismooth functions to vector valued functions. The function $H : \Re^n \to \Re^m$ is said to be semismooth at $x \in \Re^n$, if

$$\lim_{\substack{Q\in\partial H(x+th') \\ h'\to h,\, t\downarrow 0}} \{Qh'\}$$

exists for any $h \in \Re^n$, where $\partial H(x)$ is the generalized Jacobian of $H$ at $x$ in the sense of Clarke [4]. A function $H$ is said to be a semismooth function if it is semismooth everywhere on $\Re^n$.

We consider an integral function $\Upsilon : X \to \Re$, defined by

$$\Upsilon(x) := \int_\Omega f(x,\omega)\rho(\omega)d\omega, \tag{7}$$

where $f : X \times \Omega \to \Re$, $X$ is an open subset of $\Re^n$. For $\omega \in \Omega$, we let $f_\omega(\cdot) := f(\cdot,\omega)$. Since $\rho$ is a probability measure, $\rho$ is also a finite measure defined on $(\Omega, \mathcal{F})$. Hence, we have the following result.

**Proposition 1** *[14] Let $\bar{x} \in X$. Suppose that*

*(i) there exist a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ and an integrable function $\kappa : \Omega \to \Re_+$ such that for a.e. $\omega \in \Omega$,*

$$|f(x,\omega) - f(\bar{x},\omega)| \le \kappa(\omega)\|x - \bar{x}\|, \quad \text{for all } x \in \mathcal{N}(\bar{x});$$

*(ii) for a.e. $\omega \in \Omega$, $f_\omega(\cdot)$ is semismooth at $\bar{x}$;*

*(iii) there exist a neighborhood $\mathcal{N}(0)$ of $0 \in \Re^n$ and an integrable function $\gamma : \Omega \to \Re_+$ such that for a.e. $\omega \in \Omega$,*

$$\frac{|f'_\omega(\bar{x}+h;h) - f'_\omega(\bar{x};h)|}{\|h\|} \le \gamma(\omega), \quad \text{for all } h \in \mathcal{N}(0).$$

*Then $\Upsilon(\cdot)$ is semismooth at $\bar{x}$.*

## 2 The $SC^1$ Property of $G_{\text{FB}}$

In this section, we discuss the $SC^1$ property of $G_{\text{FB}}$ defined in (5). We first make the following assumptions:

**(A1)** Let $\bar{x} \in \Re^n$. There exist a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ and an integrable function $g(\omega)$, such that for any $x \in \mathcal{N}(\bar{x})$,

$$\sum_{i=1}^n \left\{ |F_i(x,\omega)|^2 + \|\nabla_x F_i(x,\omega)\|^2 \right\} \le g(\omega), \quad a.e.\ \ \omega \in \Omega. \tag{8}$$

**Remark 1.** (i) It is clear that (A1) is equivalent to the condition that $\|F(x,\omega)\|^2$ and $\sum_{i=1}^n \|\nabla_x F_i(x,\omega)\|^2$ are dominated uniformly on $\mathcal{N}(\bar{x})$ by an integrable function, respectively, that is, there exist integrable functions $\kappa_1(\omega)$ and $\kappa_2(\omega)$ such that for any $x \in \mathcal{N}(\bar{x})$,

$$\|F(x,\omega)\|^2 \le \kappa_1(\omega) \tag{9}$$

and

$$\sum_{i=1}^n \|\nabla_x F_i(x,\omega)\|^2 \le \kappa_2(\omega). \tag{10}$$

Moreover, it is easy to see by (10), there exists an integrable function $\kappa(\omega)$ such that

$$\|F(x,\omega) - F(\bar{x},\omega)\| \le \kappa(\omega)\|x - \bar{x}\|,$$

which means that $F(\cdot,\omega)$ is a Lipschitz function with Lipschitz constant $\kappa(\omega)$. On the other hand, it is easy to prove that if $\|F(\bar{x},\omega)\|^2$ is integrable and (10) is satisfied, then $\|F(x,\omega)\|^2$ is dominated uniformly on $\mathcal{N}(\bar{x})$ by an integrable function.

(ii) If $\Omega \subset \Re^m$ is compact and $\nabla_x F(\cdot,\cdot)$ is continuous with respect to $x$ and $\omega$, then we can easily prove that (A1) holds.

(iii) Suppose that $F(x,\omega) = M(\omega)x + q(\omega)$, where $M(\omega) \in R^{n\times n}$ and $q(\omega) \in R^n$ for $\omega \in \Omega$. We can prove that the condition (A1) is equivalent to the following condition [8]:

$$E\left(\|M(\omega)^T M(\omega)\|\right) < +\infty, \quad E\left(\|q(\omega)\|^2\right) < +\infty. \tag{11}$$

**(A2)** Let $\bar{x} \in \Re^n$. There exist a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ and a function $\beta(\omega)$ satisfying

$$\int_\Omega (\beta(\omega))^2 \rho(\omega) d\omega < +\infty,$$

such that for every $i = 1, \cdots, n$ and any $x, x' \in \mathcal{N}(\bar{x})$,

$$\|\nabla_x F_i(x, \omega) - \nabla_x F_i(x', \omega)\| \leq \beta(\omega)\|x - x'\|. \tag{12}$$

**Theorem 1** *Let $\bar{x} \in \Re^n$. Suppose that assumption (A1) holds. Then $G_{\mathrm{FB}}$ is continuously differentiable at $\bar{x}$. Furthermore, if in addition (A2) holds and for a.e. $\omega \in \Omega$, every $F_i(\cdot, \omega)$ is $SC^1$ at $\bar{x}$, then $G_{\mathrm{FB}}$ is $SC^1$ at $\bar{x}$.*

**Proof.** It follows from (5) that

$$G_{\mathrm{FB}}(x) = \sum_{i=1}^n g_i(x),$$

where

$$g_i(x) = \int_\Omega \varphi_i(x, \omega) \rho(\omega) d\omega,$$

and

$$\varphi_i(x, \omega) = 2\left(x_i^2 + F_i^2(x, \omega) + x_i F_i(x, \omega)\right) - 2(x_i + F_i(x, \omega))\|(x_i, F_i(x, \omega))\|.$$

By simple computation, we have

$$\nabla g_i(x) = \int_\Omega \nabla_x \varphi_i(x, \omega) \rho(\omega) d\omega,$$

where

$$\nabla_x \varphi_i(x, \omega) = 4x_i e_i + 4F_i(x, \omega)\nabla_x F_i(x, \omega) + 2F_i(x, \omega)e_i + 2x_i \nabla_x F_i(x, \omega) - 2\nabla_x q_i(x, \omega),$$

and

$$q_i(x, \omega) = (x_i + F_i(x, \omega))\|(x_i, F_i(x, \omega))\|.$$

For any $x$ and $\omega \in \Omega$, we denote $F_i(x, \omega)$ and $\nabla_x F_i(x, \omega)$ by $F_i$ and $\nabla F_i$, respectively. For any $\omega \in \Omega$, it is easy to check that $q_i(x, \omega)$ is continuously differentiable. For $(x_i, F_i) = (0, 0)$, we have

$$\nabla q_i(x, \omega) = 0.$$

For $(x_i, F_i) \neq (0, 0)$, we have

$$\nabla q_i(x, \omega) = (e_i + \nabla F_i)\|(x_i, F_i)\| + (x_i + F_i)\left[\frac{x_i}{\|(x_i, F_i)\|} e_i + \frac{F_i}{\|(x_i, F_i)\|} \nabla F_i\right].$$

The above function can be written as a sum of functions $r_1$, $r_2$ and $r_3$, where

$$\begin{aligned}
r_1(x,\omega) &= (e_i+\nabla F_i)\|(x_i,F_i)\|,\\
r_2(x,\omega) &= \begin{cases} \dfrac{(x_i+F_i)x_ie_i}{\|(x_i,F_i)\|}, & \text{if } (x_i,F_i)\neq(0,0)\\ 0, & \text{if } (x_i,F_i)=(0,0),\end{cases}\\
r_3(x,\omega) &= \begin{cases} \dfrac{(x_i+F_i)F_i\nabla F_i}{\|(x_i,F_i)\|}, & \text{if } (x_i,F_i)\neq(0,0)\\ 0, & \text{if } (x_i,F_i)=(0,0).\end{cases}
\end{aligned}$$

Then, we have

$$\nabla_x\varphi_i(x,\omega)=4x_ie_i+4F_i\nabla F_i+2F_ie_i+2x_i\nabla F_i-2(e_i+\nabla F_i)\|(x_i,F_i)\|-2r_2(x,\omega)-2r_3(x,\omega).$$

Thus, we can write $\nabla g_i(x)$ into

$$\nabla g_i(x)=\int_\Omega \nabla_x\varphi_i(x,\omega)\rho(\omega)d\omega=2f_1(x)-2f_2(x)-2f_3(x),$$

where

$$f_1(x) := \int_\Omega [2x_ie_i+2F_i\nabla F_i+F_ie_i+x_i\nabla F_i-(e_i+\nabla F_i)\|(x_i,F_i)\|]\,\rho(\omega)d\omega, \tag{13}$$

$$f_2(x) := \int_\Omega r_2(x,\omega)\rho(\omega)d\omega, \tag{14}$$

$$f_3(x) := \int_\Omega r_3(x,\omega)\rho(\omega)d\omega. \tag{15}$$

To prove that $G_{\text{FB}}$ is continuously differentiable at $\bar{x}$ under the condition (A1), we need to show that for $i=1,2,...,n$, the functions $f_1(x), f_2(x)$ and $f_3(x)$ are continuous at $\bar{x}$. We now prove that $f_3(\cdot)$ is continuous at $\bar{x}$, the proofs of other two functions being continuous are analogous.

It follows from (A1) that for any $x\in\mathcal{N}(\bar{x})$,

$$|r_3(x,\omega)|\le\|F_i\|\|\nabla F_i\|\le g(\omega),$$

for any $\omega\in\Omega$. Thus, by the Lebesgue Dominated Convergence theorem, we obtain

$$\lim_{x\to\bar{x}} f_3(x)=\int_\Omega \lim_{x\to\bar{x}} r_3(x,\omega)\rho(\omega)d\omega=\int_\Omega r_3(\bar{x},\omega)\rho(\omega)d\omega=f_3(\bar{x}).$$

This means that $f_3$ is continuous at $\bar{x}$.

In order to prove that $G_{\text{FB}}$ is $SC^1$, we only need to show that for $i=1,2,...,n$, the functions $f_1(x), f_2(x)$ and $f_3(x)$ are semismooth. We now prove that $f_3(\cdot)$ is semismooth, the proofs of other two functions being semismooth are analogous. We denote the $j$-th component of $f_3$ and $\nabla_x F_i(x,\omega)$ by $f_{j,3}$ and $(\nabla_x F_i(x,\omega))_j$ respectively, and denote

$$f_{j,3}(x) = \int_\Omega r_{j,3}(x,\omega)\rho(\omega)d\omega,$$

where

$$r_{j,3}(x,\omega) = \begin{cases} \dfrac{(x_i+F_i)F_i(\nabla_x F_i(x,\omega))_j}{\|(x_i,F_i)\|}, & \text{if } (x_i,F_i) \neq (0,0) \\ 0, & \text{if } (x_i,F_i) = (0,0). \end{cases}$$

Consider the function defined by

$$\eta(a,b) := \begin{cases} \dfrac{(a+b)b}{\|(a,b)\|}, & \text{if } (a,b) \neq (0,0) \\ 0, & \text{if } (a,b) = (0,0). \end{cases}$$

By simple inequality analysis we obtain that for any $(a,b),(c,d) \in \Re^n$,

$$|\eta(a,b) - \eta(c,d)| \le \left(4 + 2\sqrt{2}\right) \|(a,b) - (c,d)\|. \tag{16}$$

We denote $F_i(x,\omega)$, $\nabla_x F_i(x,\omega)$, $F_i(\bar{x},\omega)$ and $\nabla_x F_i(\bar{x},\omega)$ by $F_i$, $\nabla F_i$, $\bar{F}_i$ and $\nabla\bar{F}_i$, respectively. Thus, by (16) and (A2), we have

$$\begin{aligned} & |r_{j,3}(x,\omega) - r_{j,3}(\bar{x},\omega)| \\ \le\ & \left| \frac{(x_i+F_i)F_i}{\|(x_i,F_i)\|} - \frac{(\bar{x}_i+\bar{F}_i)\bar{F}_i}{\|(\bar{x}_i,\bar{F}_i)\|} \right| |(\nabla F_i)_j| + \frac{|(\bar{x}_i+\bar{F}_i)\bar{F}_i|}{\|(\bar{x}_i,\bar{F}_i)\|} |(\nabla F_i)_j - (\nabla\bar{F}_i)_j| \\ \le\ & (4+2\sqrt{2})\left(1 + \|\nabla_x F_i(\hat{x},\omega)\|\right)|(\nabla F_i)_j|\|x-\bar{x}\| + 2|\bar{F}_i||(\nabla F_i)_j - (\nabla\bar{F}_i)_j| \\ \le\ & (4+2\sqrt{2})\left(1+\sqrt{g(\omega)}\right)\sqrt{g(\omega)}\|x-\bar{x}\| + 2\sqrt{g(\omega)}\beta(\omega)\|x-\bar{x}\|, \end{aligned} \tag{17}$$

where $\hat{x}$ (it is dependent on $\omega$) is in the open segment connecting $x$ and $\bar{x}$, and the last inequality comes from (A1) and (A2). It follows from (17) that the condition (i) in Proposition 1 holds for the function $r_{j,3}(x,\omega)$.

It has been proved in [6] that $\eta$ is semismooth. Since the composite of semismooth functions is semismooth, we have that $r_{j,3}(x,\omega)$ is semismooth with respect to $x$ for a.e. $\omega \in \Omega$. This means that the condition (ii) in Proposition 1 holds for the function $r_{j,3}(x,\omega)$.

For every $\omega \in \Omega$, we let

$$r_{j,3,\omega}(x) = r_{j,3}(x,\omega)$$

and

$$p_{j,\omega}(x) = (\nabla_x F_i(x,\omega))_j.$$

For any $(x,\omega) \in \Re^n \times \Omega$ and $h \in \Re^n$, if $(x_i, F_i) = (0,0)$ and $\nabla F_i^T h = 0$, we have

$$r'_{j,3,\omega}(x;h) = 0. \tag{18}$$

If $(x_i, F_i) = (0,0)$ and $\nabla F_i^T h \neq 0$, we have

$$r'_{j,3,\omega}(x;h) = \frac{(h_i + \nabla F_i^T h)(\nabla F_i)_j \nabla F_i^T h}{\|(h_i, \nabla F_i^T h)\|}. \tag{19}$$

If $(x_i, F_i) \neq (0,0)$, we have

$$\begin{aligned} r'_{j,3,\omega}(x;h) \quad = \quad & \frac{[(e_i + \nabla F_i)F_i(\nabla F_i)_j + (x_i + F_i)\nabla F_i(\nabla F_i)_j]^T h}{\|(x_i, F_i)\|} \\ & -\frac{[(F_i^2 + x_i F_i)(\nabla F_i)_j(x_i e_i + F_i \nabla F_i)]^T h}{\|(x_i, F_i)\|^3} \\ & +\frac{(x_i + F_i)F_i p'_{j,\omega}(x;h)}{\|(x_i, F_i)\|} \end{aligned} \tag{20}$$

By (A2), we have that there exists a neighborhood $\mathcal{N}'(\bar{x})$ of $\bar{x}$ such that for any $x \in \mathcal{N}'(\bar{x})$

$$\frac{|p'_{j,\omega}(x;h)|}{\|h\|} \leq \beta(\omega).$$

Consequently, by (18)-(20) and (A1), for $\|h\|$ small enough, we have that

$$\begin{aligned} & \frac{|r'_{j,3,\omega}(x+h;h) - r'_{j,3,\omega}(x;h)|}{\|h\|} \\ \leq \quad & \frac{|r'_{j,3,\omega}(x+h;h)| + |r'_{j,3,\omega}(x;h)|}{\|h\|} \\ \leq \quad & 2\left[3\left(1 + \sqrt{g(\omega)}\right)\sqrt{g(\omega)} + 2g(\omega) + 2\sqrt{g(\omega)}\beta(\omega)\right] + 4g(\omega) \\ = \quad & 6\sqrt{g(\omega)} + 14g(\omega) + 4\sqrt{g(\omega)}\beta(\omega). \end{aligned} \tag{21}$$

This means the condition (iii) in Proposition 1 holds for the function $r_{j,3}(x,\omega)$. It follows from Proposition 1 that $f_{j,3}(x)$ is semismooth. Hence, we obtain the desired result and complete the proof of the theorem. ∎

From Theorem 1, we have the following theorem.

**Theorem 2** *Suppose that assumptions (A1) and (A2) hold for every $\bar{x} \in \Re^n$ and for a.e. $\omega \in \Omega$, every $F_i(\cdot,\omega)$ is $SC^1$ with respect to $x$. Then $G_{\rm FB}$ is $SC^1$.*

If $F(x,\omega) = M(\omega)x + q(\omega)$, then assumption (A2) is implied by assumption (A1). Hence, from above theorem we have the following corollary.

**Corollary 1** *Suppose that $F(x,\omega) = M(\omega)x + q(\omega)$, where $M(\omega) \in \Re^{n\times n}$ and $q(\omega) \in \Re^n$ for $\omega \in \Omega$, and condition (11) holds. Then $G_{\rm FB}$ is $SC^1$.*

**Remark 2.** In [2, 3], it is pointed out that for some SNCPs the ERM problem defined by the FB function may have no solutions while the ERM problems defined by the min function and the penalized FB NCP-function [1] have solutions. We can prove that under the conditions of Theorem 1 the expected residual function defined by the penalized FB NCP-function is an $SC^1$ function as well by following from Proposition 1 [1] and the same arguments used for Theorem 1. However, the expected residual function defined by the min function does not have this nice property under the conditions of Theorem 1.

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