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Inverse Optimal Control of Linear Distributed Parameter Systems

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Abstract

A constructive method is developed to design inverse optimal controllers for a class of linear distributed parameter systems (DPSs). Inverse optimality guarantees that the cost functional to be minimized is meaningful in the sense that the symmetric and positive definite weighting kernel matrix on the states is chosen after the control design instead of being specified at the start of the control design. Inverse optimal design enables that the Riccati nonlinear partial differential equation (PDE) can be simplified to a Bernoulli PDE, which can be solved analytically. The control design is based on a new Green matrix formula, a new unique and bounded solution of a linear PDE, and an analytical solution of a Bernoulli PDE. Both distributed and finite control problems are addressed. An example is given.

Keywords: Distributed parameter systems, Inverse optimal control, Bernoulli PDE, Riccati PDE, Finite controls

1 Introduction

Optimal control of DPSs, i.e., systems governed by PDEs, has been under development since 1960s [3], [19], [18], [17], [14], [12], [2], [4], [11], and can be roughly classified into two main approaches.

In the first approach referred to as the modal control one, the PDEs are discretized to obtain lumped-parameter systems described in terms of modal coordinates, i.e., systems of ordinary differential equations (ODEs), to which the classical control design methods [1], [8], [10] can be applied. The modal control approach, see [15], [16], [6], can only control a certain number of modes of a distributed parameter system, and has difficulty in computing appropriate gain matrices.

The second approach applies semigroup theory to represent PDEs as ODEs in Hilbert spaces. From here the classical optimal control results are extended into infinite-dimensional systems [18], [4], [12] [2]. This approach eventually results in operator Riccati equations, which have similarities to the results presented here. The operator Riccati equations, which are equivalent to Riccati nonlinear PDEs in the Euclidean space, are nonlinear and are to be solved backward in time.

Difficulties arisen in solving the Riccati nonlinear PDEs and two-point-boundary value (TPBV) problems, which are resulted from the classical design of optimal controllers for DPSs, motivate the approach of designing inverse optimal controllers. The difference between the direct and the inverse optimal control problems is that the former designs a controller that minimizes a given cost functional, while the latter seeks a controller that minimizes a "meaningful" cost functional, which is a part of the solution of the Bernoulli PDE. The proposed inverse optimal control design in this paper utilizes the recent inverse optimal filter design for linear DPSs proposed by the author [5], and is related to the development of inverse optimal controllers for systems governed by nonlinear ODEs in [13], [9]. The inverse approach in [13], [9] uses a control Lyapunov function, which is a solution of Hamilton-Jacobi-Bellman with a meaningful cost, for systems governed by nonlinear ODEs obtained by solving a stabilization problem.

The rest of the paper is organized as follows. The control problem is formulated in Section 2. Section 3 contains preliminary results including a matrix Green's formula, a formula for differentiation of a generalized inverse matrix, a derivation of the unique and bounded solution of a linear PDE, and an analytical solution of a Bernoulli nonlinear PDE in terms of the system Green function. In Section 4, the calculus of variations approach is used to derive the inverse optimal controllers. The distributed and finite control designs are presented in Sections 4 and 5, respectively. An example is given Section 6. Proofs are given in Appendices A and B.

Notations: For a symmetric and positive definite matrix $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{y})$, the notation $\boldsymbol{A}^{+}(\boldsymbol{x},\boldsymbol{y})$ denotes its generalized inverse such that $\int_D \boldsymbol{A}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{A}^{+}(\boldsymbol{y},\boldsymbol{x}')d\boldsymbol{y} = \boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}')$ with $\boldsymbol{I}$ being the identity matrix, and $\delta(\boldsymbol{x}-\boldsymbol{x}')$ being the Dirac delta function of $(\boldsymbol{x}-\boldsymbol{x}')$. For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of the same size, the notation $<\boldsymbol{a},\boldsymbol{b}>$ denotes their inner product. For a matrix operator $\boldsymbol{A}_{\boldsymbol{x}}$, the notation $\boldsymbol{A}^{*}_{\boldsymbol{x}}$ denotes its adjoint.

2 Problem formulation

Let D be a open bounded set in Euclidean n-space E^n with piecewise smooth boundary S, and let t denote time defined on an interval $T=[t_0,t_f]$ with $t_f>t_0$. In this paper, we consider the following class of linear DPSs:

$$\begin{aligned}
&\frac{\partial \boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t) + \boldsymbol{B}_d(\boldsymbol{x},t)\boldsymbol{u}_d(\boldsymbol{x},t), \quad \forall\,\boldsymbol{x}\in D,\\
&\boldsymbol{\chi}(\boldsymbol{x},t_0) = \boldsymbol{\chi}_0(\boldsymbol{x}), \quad \forall\,\boldsymbol{x}\in D,\\
&\boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\chi}(\boldsymbol{\xi},t) = \boldsymbol{B}_b(\boldsymbol{\xi},t)\boldsymbol{u}_b(\boldsymbol{\xi},t), \quad \forall\,\boldsymbol{\xi}\in S,
\end{aligned} \tag{1}$$

defined for $t \in T$, where $\boldsymbol{x} = \mathrm{col}(x_1, ..., x_n) \in D$ is the n-dimensional spatial coordinate vector; $\boldsymbol{\chi}(\boldsymbol{x}, t) = \mathrm{col}(\chi_1(\boldsymbol{x}, t), ..., \chi_r(\boldsymbol{x}, t))$ is the r-dimensional vector state; $\boldsymbol{B}_d(\boldsymbol{x}, t)$ and $\boldsymbol{B}_b(\boldsymbol{\xi}, t)$ are $r \times k$ and $r \times l$ matrix functions, respectively; $\boldsymbol{u}_d(\boldsymbol{x}, t) \in \mathbb{U}_d$ is the k-dimensional vector control input distributed over the interior; $\boldsymbol{u}_b(\boldsymbol{\xi}, t) \in \mathbb{U}_b$ is the l-dimensional vector control input distributed over the boundary. We assume that the permissible control spaces $\mathbb{U}_d$ and $\mathbb{U}_b$ are open, and make the following assumption on the operators $\boldsymbol{A_x}$ and $\boldsymbol{\beta_\xi}$, and the matrices $\boldsymbol{B_d}(\boldsymbol{x}, t)$ and $\boldsymbol{B_b}(\boldsymbol{\xi}, t)$.

Assumption 2.1

1. *The matrix operators $\boldsymbol{A_x}$ and $\boldsymbol{\beta_\xi}$ are given by*

$$\begin{aligned} \boldsymbol{A_x}[\bullet] &= \sum_{i,j=1}^{n} \boldsymbol{A}_{ij}(\boldsymbol{x}, t) \frac{\partial^2 [\bullet]}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \boldsymbol{B}_i(\boldsymbol{x}, t) \frac{\partial [\bullet]}{\partial x_i} + \boldsymbol{C}(\boldsymbol{x}, t)[\bullet], \\ \boldsymbol{\beta_\xi}[\bullet] &= \sum_{j=1}^{n} \boldsymbol{A}_j(\boldsymbol{\xi}, t) \frac{\partial [\bullet]}{\partial x_j} + \boldsymbol{F}(\boldsymbol{\xi}, t)[\bullet], \end{aligned} \tag{2}$$

where the $\boldsymbol{A}_{ij}(\boldsymbol{x}, t)$, $\boldsymbol{B}_i(\boldsymbol{x}, t)$, $\boldsymbol{C}(\boldsymbol{x}, t)$, and $\boldsymbol{F}(\boldsymbol{\xi}, t)$ are $r \times r$ matrices, and

$$\boldsymbol{A}_j(\boldsymbol{\xi}, t) = \sum_{i=1}^{n} \boldsymbol{A}_{ij}(\boldsymbol{\xi}, t) \cos(\boldsymbol{n_\xi}, x_i), \tag{3}$$

with $\boldsymbol{n_\xi}$ being the outward normal to the boundary S at the point $\boldsymbol{\xi} \in S$, and $(\boldsymbol{n_\xi}, x_i)$ being the angle between the outward normal $\boldsymbol{n_\xi}$ and the x_i-axis. Furthermore, the matrix $\boldsymbol{A}_{ij}(\boldsymbol{x}, t)$ is symmetric, i.e., $\boldsymbol{A}_{ij}(\boldsymbol{x}, t) = \boldsymbol{A}_{ji}(\boldsymbol{x}, t)$.

2. *There exist symmetric and positive definite matrices $\boldsymbol{R}_d(\boldsymbol{x}, \boldsymbol{y}, t)$ and $\boldsymbol{R}_b(\boldsymbol{\xi}, \boldsymbol{\gamma}, t)$ such that the matrices*

$$\begin{aligned} \bar{\boldsymbol{R}}_d(\boldsymbol{x}, \boldsymbol{y}, t) &= \boldsymbol{B}_d(\boldsymbol{x}, t) \boldsymbol{R}_d^{+}(\boldsymbol{x}, \boldsymbol{y}, t) \boldsymbol{B}_d^{T}(\boldsymbol{y}, t), \\ \bar{\boldsymbol{R}}_b(\boldsymbol{\xi}, \boldsymbol{\gamma}, t) &= \boldsymbol{B}_b(\boldsymbol{\xi}, t) \boldsymbol{R}_b^{+}(\boldsymbol{\xi}, \boldsymbol{\gamma}, t) \boldsymbol{B}_b^{T}(\boldsymbol{\gamma}, t), \end{aligned} \tag{4}$$

are bounded and symmetric and positive definite.

3. *Let $\boldsymbol{A_x^*}$ and $\boldsymbol{\beta_\xi^*}$ be the adjoint operators of $\boldsymbol{A_x}$ and $\boldsymbol{\beta_\xi}$, i.e.,*

$$\begin{aligned} \boldsymbol{A_x^*}[\bullet] &= \sum_{i,j=1}^{n} \frac{\partial^2 (\boldsymbol{A}_{ij}(\boldsymbol{x}, t)[\bullet])}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (\boldsymbol{B}_i(\boldsymbol{x}, t)[\bullet])}{\partial x_i} + \boldsymbol{C}(\boldsymbol{x}, t)[\bullet], \\ \boldsymbol{\beta_\xi^*}[\bullet] &= \sum_{j=1}^{n} \boldsymbol{A}_j(\boldsymbol{\xi}, t) \frac{\partial [\bullet]}{\partial x_j} - \boldsymbol{K}(\boldsymbol{\xi}, t)[\bullet] + \boldsymbol{F}(\boldsymbol{\xi}, t)[\bullet], \end{aligned} \tag{5}$$

with

$$\boldsymbol{K}(\boldsymbol{\xi}, t) = \sum_{i=1}^{n} \Big[\boldsymbol{B}_i(\boldsymbol{\xi}, t) - \sum_{j=1}^{n} \frac{\partial \boldsymbol{A}_{ij}(\boldsymbol{\xi}, t)}{\partial x_j} \Big] \cos(\boldsymbol{n_\xi}, x_i). \tag{6}$$

There exists a matrix operator $\boldsymbol{L}_{\boldsymbol{x}}$ such that the system

$$\begin{aligned}&\frac{\partial \boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t}=-\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t)\bar{\boldsymbol{A}}^*_{\boldsymbol{y}}-[\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}]^T\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t),\\&\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{y},t_0)=\boldsymbol{Z}_0(\boldsymbol{x},\boldsymbol{y}),\\&\boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{Z}(\boldsymbol{\xi},\boldsymbol{y},t)=0\end{aligned}\tag{7}$$

is exponentially stable at the origin, where

$$\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}[\bullet]=\boldsymbol{A}^*_{\boldsymbol{x}}[\bullet]+\boldsymbol{L}_{\boldsymbol{x}}[\bullet].\tag{8}$$

Moreover, the matrix $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$ defined by

$$\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)=\boldsymbol{L}_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)+\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{L}^T_{\boldsymbol{y}},\tag{9}$$

is symmetric and positive definite for a symmetric and positive definite matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$.

In this paper, we consider the following control objective

Control Objective 2.2 *Design admissible control pair $\boldsymbol{u}_d\in\mathbb{U}_d$ and $\boldsymbol{u}_b\in\mathbb{U}_b$ so as to minimize the following cost functional:*

$$J=\int_{t_0}^{t_f}Ldt+J_f,\tag{10}$$

where

$$\begin{aligned}L=&\frac{1}{2}\int_{D^2}\big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}+\frac{1}{2}\int_{D^2}\big\langle\boldsymbol{u}_d(\boldsymbol{x},t),\boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{u}_d(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\\&+\frac{1}{2}\int_{S^2}\big\langle\boldsymbol{u}_b(\boldsymbol{\xi},t),\boldsymbol{R}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{u}_b(\boldsymbol{\gamma},t)\big\rangle dS_{\boldsymbol{\xi}}dS_{\boldsymbol{\gamma}},\\J_f=&\frac{1}{2}\int_{D^2}\big\langle\boldsymbol{\chi}(\boldsymbol{x},t_f),\boldsymbol{Q}_f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\end{aligned}\tag{11}$$

with $dS_{\boldsymbol{\xi}}$ being the surface element of S at the point $\boldsymbol{\xi}$; $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{Q}_f(\boldsymbol{x},\boldsymbol{y})$ are symmetric and nonnegative definite matrices; and $\boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{R}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)$ are symmetric positive definite matrices defined in Assumption 2.1.

3 Preliminaries

This section presents important preliminary results to be used in the control design.

3.1 Matrix Green's Formula

Lemma 3.1 *Consider the matrix differential operators $\boldsymbol{A_x}$ and $\boldsymbol{\beta_\xi}$ defined in (2), and their adjoint operators $\boldsymbol{A}^*_{\boldsymbol{x}}$ and $\boldsymbol{\beta}^*_{\boldsymbol{\xi}}$ defined in (5). Then*

$$\int_D \Big[\Big\langle \boldsymbol{A_x}\boldsymbol{\gamma}(\boldsymbol{x},t), \boldsymbol{\lambda}(\boldsymbol{x},t)\Big\rangle - \Big\langle \boldsymbol{\gamma}(\boldsymbol{x},t), \boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{\lambda}(\boldsymbol{x},t)\Big\rangle\Big] d\boldsymbol{x} = \int_S \Big[\Big\langle \boldsymbol{\beta_\xi}\boldsymbol{\gamma}(\boldsymbol{\xi},t), \boldsymbol{\lambda}(\boldsymbol{\xi},t)\Big\rangle - \Big\langle \boldsymbol{\gamma}(\boldsymbol{\xi},t), \boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{\lambda}(\boldsymbol{\xi},t)\Big\rangle\Big] dS_{\boldsymbol{\xi}}, \tag{12}$$

where $d\boldsymbol{x} = dx_1 dx_2 ... dx_n$ and $dS_{\boldsymbol{\xi}}$ is the surface element of S at the point $\boldsymbol{\xi}$.

Proof. See Proof of Lemma 3.1 in [5].

3.2 Time derivative of the generalized inverse of a matrix

Lemma 3.2 *Let $\boldsymbol{A}^+(\boldsymbol{x},\boldsymbol{y},t)$ denote the generalized inverse of a matrix $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{y},t)$, i.e,*

$$\int_D \boldsymbol{A}(\boldsymbol{x}',\boldsymbol{y}',t)\boldsymbol{A}^+(\boldsymbol{y}',\boldsymbol{y},t)d\boldsymbol{y}' = \boldsymbol{I}\delta(\boldsymbol{x}'-\boldsymbol{y}). \tag{13}$$

The time derivative of the generalized inverse of $\boldsymbol{A}^+(\boldsymbol{x},\boldsymbol{y},t)$ is given by

$$\frac{\partial \boldsymbol{A}^+(\boldsymbol{x},\boldsymbol{y},t)}{\partial t} = -\int_{D^2} \boldsymbol{A}^+(\boldsymbol{x},\boldsymbol{x}',t)\frac{\partial \boldsymbol{A}(\boldsymbol{x}',\boldsymbol{y}',t)}{\partial t}\boldsymbol{A}^+(\boldsymbol{y}',\boldsymbol{y},t)d\boldsymbol{x}'d\boldsymbol{y}'. \tag{14}$$

Proof. See Proof of Lemma 3.2 in [5].

3.3 Existence and uniqueness of the bounded solution of a linear partial differential equation

Lemma 3.3 *Consider the following matrix linear PDE:*

$$\begin{aligned} &\frac{\partial \boldsymbol{N}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t} = \boldsymbol{A_x}\boldsymbol{N}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{N}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{A}^T_{\boldsymbol{y}} + \boldsymbol{R}(\boldsymbol{x},\boldsymbol{y},t), \\ &\boldsymbol{N}(\boldsymbol{x},\boldsymbol{y},t_0) = \boldsymbol{N}_0(\boldsymbol{x},\boldsymbol{y}), \\ &\boldsymbol{\beta_\xi}\boldsymbol{N}(\boldsymbol{\xi},\boldsymbol{y},t) = 0, \quad \boldsymbol{\xi} \in S, \boldsymbol{y} \in D, \end{aligned} \tag{15}$$

where the operators $\boldsymbol{A_x}$ and $\boldsymbol{\beta_\xi}$ are defined in (2); $\boldsymbol{R}(\boldsymbol{x},\boldsymbol{y},t) \in L^1(L^\infty(D\times D);0,\infty)$ and $\boldsymbol{N}_0(\boldsymbol{x},\boldsymbol{y}) \in L^\infty(D\times D)$ are bounded matrices. Then there exists a unique bounded solution $\boldsymbol{N}(\boldsymbol{x},\boldsymbol{y},t)$ of (15) and $\boldsymbol{N}(\boldsymbol{x},\boldsymbol{y},t)$ is given by

$$\boldsymbol{N}(\boldsymbol{x},\boldsymbol{y},t) = \int_{D^2} \boldsymbol{G}(\boldsymbol{x},t;\boldsymbol{x}',t_0)\boldsymbol{N}_0(\boldsymbol{x}',\boldsymbol{y}')\boldsymbol{G}^T(\boldsymbol{y},t;\boldsymbol{y}',t_0)d\boldsymbol{x}'d\boldsymbol{y}' + \int_{t_0}^t\int_{D^2} \boldsymbol{G}(\boldsymbol{x},t-\tau+t_0;\boldsymbol{x}',t_0)\boldsymbol{R}(\boldsymbol{x}',\boldsymbol{y}',\tau)\boldsymbol{G}^T(\boldsymbol{y},t-\tau+t_0;\boldsymbol{y}',t_0)d\boldsymbol{x}'d\boldsymbol{y}'d\tau, \tag{16}$$

where the Green function $\boldsymbol{G}(\boldsymbol{x}, t; \boldsymbol{x}', t_0)$ *being satisfied*

$$\begin{aligned} &\frac{\partial \boldsymbol{G}(\boldsymbol{x}, t; \boldsymbol{x}', t_0)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}} \boldsymbol{G}(\boldsymbol{x}, t; \boldsymbol{x}', t_0), \\ &\boldsymbol{G}(\boldsymbol{x}, t_0; \boldsymbol{x}', t_0) = \boldsymbol{I}\delta(\boldsymbol{x} - \boldsymbol{x}'), \\ &\boldsymbol{\beta}_{\boldsymbol{\xi}} \boldsymbol{G}(\boldsymbol{\xi}, t; \boldsymbol{x}', t_0) = 0. \end{aligned} \tag{17}$$

Proof. Proof of this lemma follows the same lines as in Proof of Lemma 3.3 in [5]. Hence, it is omitted here.

3.4 Analytical solution of Bernoulli nonlinear PDE

Lemma 3.4 *Consider the following Bernoulli nonlinear PDE*

$$\begin{aligned} \frac{\partial \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)}{\partial t} &= -\bar{\boldsymbol{A}}^*_{\boldsymbol{x}} \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t) + \int_{D^2} \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{x}', t) \bar{\boldsymbol{R}}_d(\boldsymbol{x}', \boldsymbol{x}'', t) \boldsymbol{P}(\boldsymbol{x}'', \boldsymbol{y}, t) d\boldsymbol{x}' d\boldsymbol{x}'' \\ &\quad - \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)[\bar{\boldsymbol{A}}^*_{\boldsymbol{y}}]^T + \int_{S^2} \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\xi}, t) \bar{\boldsymbol{R}}_b(\boldsymbol{\xi}, \boldsymbol{\gamma}, t) \boldsymbol{P}(\boldsymbol{\gamma}, \boldsymbol{y}, t) dS_{\boldsymbol{\xi}} dS_{\boldsymbol{\gamma}}, \\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t_f) &= \boldsymbol{P}_f(\boldsymbol{x}, \boldsymbol{y}); \quad (\boldsymbol{x}, \boldsymbol{y}) \in D, \\ \boldsymbol{\beta}^*_{\boldsymbol{\xi}} \boldsymbol{P}(\boldsymbol{\xi}, \boldsymbol{y}, t) &= 0; \quad \boldsymbol{\xi} \in S, \boldsymbol{y} \in D \end{aligned} \tag{18}$$

where $\bar{\boldsymbol{R}}_d(\boldsymbol{x}, \boldsymbol{y}, t)$*,* $\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}, \boldsymbol{\gamma}, t)$ *and* $\boldsymbol{P}_f(\boldsymbol{x}, \boldsymbol{y})$ *are symmetric and positive definite matrices, and* $\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}$ *is defined in (8). Then, there exists a unique bounded solution* $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ *of (18), and* $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ *is given by*

$$\begin{aligned} &\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t) = \bar{\boldsymbol{P}}(\boldsymbol{x}, \boldsymbol{y}, \bar{t}), \ \bar{t} = -t, \\ &\bar{\boldsymbol{P}}(\boldsymbol{x}, \boldsymbol{y}, \bar{t}) = \int_{D^2} \boldsymbol{G}(\boldsymbol{x}, \bar{t}; \boldsymbol{x}', -t_f) \Big[\boldsymbol{P}^+_f(\boldsymbol{x}', \boldsymbol{y}') + \boldsymbol{M}(\boldsymbol{x}', \boldsymbol{y}', \bar{t}) \Big]^+ \boldsymbol{G}^T(\boldsymbol{y}, \bar{t}; \boldsymbol{y}', -t_f) d\boldsymbol{x}' d\boldsymbol{y}', \end{aligned} \tag{19}$$

where

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \bar{t}) = &\int_{-t_f}^{\bar{t}} \int_{D^2} \boldsymbol{G}^T(\boldsymbol{\eta}, \tau; \boldsymbol{x}, -t_f) \bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{\eta}, \boldsymbol{\alpha}, \tau) \boldsymbol{G}(\boldsymbol{\alpha}, \tau; \boldsymbol{y}, -t_f) d\boldsymbol{\eta}\, d\boldsymbol{\alpha}\, d\tau + \\ &\int_{-t_f}^{\bar{t}} \int_{S^2} \boldsymbol{G}^T(\boldsymbol{\xi}, \tau; \boldsymbol{x}, -t_f) \bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{\xi}, \boldsymbol{\gamma}, \tau) \boldsymbol{G}(\boldsymbol{\gamma}, \tau; \boldsymbol{y}, -t_f) dS_{\boldsymbol{\eta}}\, dS_{\boldsymbol{\gamma}}\, d\tau \end{aligned} \tag{20}$$

with $\bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{\eta}, \boldsymbol{\alpha}, \bar{t}) = \bar{\boldsymbol{R}}_d(\boldsymbol{\eta}, \boldsymbol{\alpha}, t)$*,* $\bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{\xi}, \boldsymbol{\gamma}, \bar{t}) = \bar{\boldsymbol{R}}_b(\boldsymbol{\xi}, \boldsymbol{\gamma}, t)$*, and the Green function* $\boldsymbol{G}(\boldsymbol{x}, \bar{t}; \boldsymbol{x}', \bar{t}')$ *being satisfied*

$$\begin{aligned} &\frac{\partial \boldsymbol{G}(\boldsymbol{x}, \bar{t}; \boldsymbol{x}', \bar{t}')}{\partial \bar{t}} = \bar{\boldsymbol{A}}^*_{\boldsymbol{x}} \boldsymbol{G}(\boldsymbol{x}, \bar{t}; \boldsymbol{x}', \bar{t}'), \\ &\boldsymbol{G}(\boldsymbol{x}, \bar{t}'; \boldsymbol{x}', \bar{t}') = \boldsymbol{I}\delta(\boldsymbol{x} - \boldsymbol{x}'), \\ &\boldsymbol{\beta}^*_{\boldsymbol{\xi}} \boldsymbol{G}(\boldsymbol{\xi}, \bar{t}; \boldsymbol{x}', \bar{t}') = 0. \end{aligned} \tag{21}$$

Proof. See Appendix A.

4 Inverse Optimal Control Design

4.1 Derivation of control inputs $\boldsymbol{u}_d(\boldsymbol{x}, t)$ and $\boldsymbol{u}_b(\boldsymbol{\xi}, t)$

Since the control objective is to minimize the cost functional (10) subject to the constraints (1), we introduce the Lagrange multiplier $\boldsymbol{\lambda}(\boldsymbol{w}, t)$ with $\boldsymbol{w} \in D$ or $\boldsymbol{w} \in S$, and construct the following extended functional

$$\begin{aligned} L_1 =& L + \int_D \Big\langle \boldsymbol{\lambda}(\boldsymbol{x}, t), -\frac{\partial \boldsymbol{\chi}(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{A}_{\boldsymbol{x}} \boldsymbol{\chi}(\boldsymbol{x}, t) + \boldsymbol{B}_d(\boldsymbol{x}, t)\boldsymbol{u}_d(\boldsymbol{x}, t)\Big\rangle d\boldsymbol{x} + \\ & \int_S \Big\langle \boldsymbol{\lambda}(\boldsymbol{\xi}, t), -\boldsymbol{\beta}_{\boldsymbol{\xi}} \boldsymbol{\chi}(\boldsymbol{\xi}, t) + \boldsymbol{B}_b(\boldsymbol{\xi}, t)\boldsymbol{u}_b(\boldsymbol{\xi}, t)\Big\rangle dS_{\boldsymbol{\xi}} \end{aligned} \tag{22}$$

to remove the constraints (1). Therefore, the problem of minimizing the cost functional (10) subject to the constraints (1) is equivalent to the one of minimizing:

$$J_1 = \int_{t_0}^{t_f} L_1 dt + J_f \tag{23}$$

without any constraints. Deriving and setting the weak variation δJ_1, see Appendix B, with respect to $\boldsymbol{\chi}(\boldsymbol{x}, t)$, $\boldsymbol{\lambda}(\boldsymbol{x}, t)$, $\boldsymbol{u}_d(\boldsymbol{x}, t)$ and $\boldsymbol{u}_b(\boldsymbol{\xi}, t)$ to zero result in the following necessary conditions referred to as the set of Euler-Lagrange equations:

$$\begin{aligned} & \frac{\partial \boldsymbol{\chi}(\boldsymbol{x}, t)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}} \boldsymbol{\chi}(\boldsymbol{x}, t) + \boldsymbol{B}_d(\boldsymbol{x}, t)\boldsymbol{u}_d(\boldsymbol{x}, t), \\ & \boldsymbol{\chi}(\boldsymbol{x}, t_0) = \boldsymbol{\chi}_0(\boldsymbol{x}), \\ & \boldsymbol{\beta}_{\boldsymbol{\xi}} \boldsymbol{\chi}(\boldsymbol{\xi}, t) = \boldsymbol{B}_b(\boldsymbol{\xi}, t)\boldsymbol{u}_b(\boldsymbol{\xi}, t), \\ & \frac{\partial \boldsymbol{\lambda}(\boldsymbol{x}, t)}{\partial t} = -\boldsymbol{A}^*_{\boldsymbol{x}} \boldsymbol{\lambda}(\boldsymbol{x}, t) - \int_D \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{y}, t)\boldsymbol{\chi}(\boldsymbol{y}, t) d\boldsymbol{y}, \\ & \boldsymbol{\beta}^*_{\boldsymbol{\xi}} \boldsymbol{\lambda}(\boldsymbol{\xi}, t) = 0, \\ & \boldsymbol{\lambda}(\boldsymbol{x}, t_f) = \int_D \boldsymbol{Q}_f(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{\chi}(\boldsymbol{y}, t_f) d\boldsymbol{y}, \\ & \boldsymbol{u}_d(\boldsymbol{x}, t) = -\int_D \boldsymbol{R}^+_d(\boldsymbol{x}, \boldsymbol{y}, t)\boldsymbol{B}^T_d(\boldsymbol{y}, t)\boldsymbol{\lambda}(\boldsymbol{y}, t) d\boldsymbol{y}, \\ & \boldsymbol{u}_b(\boldsymbol{\xi}, t) = -\int_S \boldsymbol{R}^+_b(\boldsymbol{\xi}, \boldsymbol{\gamma}, t)\boldsymbol{B}^T_b(\boldsymbol{\gamma}, t)\boldsymbol{\lambda}(\boldsymbol{\gamma}, t) dS_{\boldsymbol{\gamma}}. \end{aligned} \tag{24}$$

It is noted that the whole system (24) is not of two initial-value problems but constitutes a single, TPBV problem in T.

4.2 Solution of Euler-Lagrange equations

In this subsection, we will derive the solution of the set of Euler-Lagrange equations (24). As such, we introduce the following coordinate transformation

$$\boldsymbol{\lambda}(\boldsymbol{x}, t) = \int_D \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)\boldsymbol{\chi}(\boldsymbol{y}, t) d\boldsymbol{y}, \tag{25}$$

where $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is symmetric and positive semidefinite to be determined so that $\boldsymbol{\lambda}(\boldsymbol{x},t)$ satisfies the third, fourth and fifth equations of (24). As such, we first choose the boundary and terminal conditions for $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ as:

$$\begin{aligned}&\boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{P}(\boldsymbol{\xi},\boldsymbol{y},t)=0, \quad \boldsymbol{y}\in D,\, \boldsymbol{\xi}\in S,\\ &\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t_f)=\boldsymbol{Q}_f(\boldsymbol{x},\boldsymbol{y}).\end{aligned} \tag{26}$$

It is seen from (26) that $\boldsymbol{\lambda}(\boldsymbol{x},\boldsymbol{y},t)$ satisfies the boundary and terminal conditions specified by the fourth and fifth equations of (24). Next, we determine $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ so that $\boldsymbol{\lambda}(\boldsymbol{x},t)$ satisfies the third equation of (24). Differentiating both sides of (25) with respect to t along the solutions of the first equation of (24) results in

$$\frac{\partial\boldsymbol{\lambda}(\boldsymbol{x},t)}{\partial t}=\int_D\frac{\partial\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t}\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y}+\int_D\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\Big[\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{\chi}(\boldsymbol{y},t)+\boldsymbol{B}_d(\boldsymbol{y},t)\boldsymbol{u}_d(\boldsymbol{y},t)\Big]d\boldsymbol{y}. \tag{27}$$

Comparing (27) with the third equation of (24) and using (25) give

$$\begin{aligned}\int_D\frac{\partial\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t}\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y}=&-\int_D\boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y}-\int_D\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y}\\&+\int_D\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{B}_d(\boldsymbol{y},t)\boldsymbol{u}_d(\boldsymbol{y},t)d\boldsymbol{y}-\int_D\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y}.\end{aligned} \tag{28}$$

Taking the inner product (28) with $\boldsymbol{\chi}(\boldsymbol{x},t)$ then integrating over D yield

$$\begin{aligned}&\int_{D^2}\Big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\frac{\partial\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t}\boldsymbol{\chi}(\boldsymbol{y},t)\Big\rangle d\boldsymbol{x}d\boldsymbol{y}=-\int_{D^2}\big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}-\\&\int_{D^2}\big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}+\int_{D^2}\big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{B}_d(\boldsymbol{y},t)\boldsymbol{u}_d(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\\&-\int_{D^2}\big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}.\end{aligned} \tag{29}$$

Applying Lemma 3.1 to the second integral term in the right hand side of (29) gives

$$\begin{aligned}\int_{D^2}\big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}=&\int_{D^2}\big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\big[\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)[\boldsymbol{A}^*_{\boldsymbol{y}}]^T\big]\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\\&+\int_D\int_S\big\langle\boldsymbol{\chi}(\boldsymbol{x},t),\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi},t)\boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\chi}(\boldsymbol{\xi},t)\big\rangle d\boldsymbol{x}dS_{\boldsymbol{\xi}},\end{aligned} \tag{30}$$

where we have used the boundary condition (26) with a note that $\boldsymbol{P}(\boldsymbol{\xi},\boldsymbol{x},t)=\boldsymbol{P}^T(\boldsymbol{x},\boldsymbol{\xi},t)$ since $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is symmetric. Substituting (30) into (29) with $\boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\chi}(\boldsymbol{x},t)=\boldsymbol{B}_b(\boldsymbol{\xi},t)\boldsymbol{u}_b(\boldsymbol{\xi},t)$, see the third equation of (1), and substituting $\boldsymbol{\lambda}(\boldsymbol{x},t)$ in (25) into the expressions of the control inputs $\boldsymbol{u}_d(\boldsymbol{x},t)$ and $\boldsymbol{u}_b(\boldsymbol{x},t)$ defined in the last two equations of (24), which are then substituted into the last two integral terms in the right hand side of (29), results in

$$\int_{D^2}\left\langle \boldsymbol{\chi}(\boldsymbol{x},t),\frac{\partial \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t}\boldsymbol{\chi}(\boldsymbol{y},t)\right\rangle d\boldsymbol{x}d\boldsymbol{y} = -\int_{D^2}\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\rangle d\boldsymbol{x}d\boldsymbol{y}-$$
$$\int_{D^2}\langle \boldsymbol{\chi}(\boldsymbol{x},t), [\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)[\boldsymbol{A}^*_{\boldsymbol{y}}]^T]\boldsymbol{\chi}(\boldsymbol{y},t)\rangle d\boldsymbol{x}d\boldsymbol{y} - \int_{D^2}\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\rangle d\boldsymbol{x}d\boldsymbol{y}+$$
$$\int_{D^2}\langle \boldsymbol{\chi}(\boldsymbol{x},t), [\int_{D^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}',t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}',\boldsymbol{x}'',t)\boldsymbol{P}(\boldsymbol{x}'',\boldsymbol{y},t)d\boldsymbol{x}'d\boldsymbol{x}'']\boldsymbol{\chi}(\boldsymbol{y},t)\rangle d\boldsymbol{x}d\boldsymbol{y}+$$
$$\int_{D^2}\langle \boldsymbol{\chi}(\boldsymbol{x},t), [\int_{S^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi},t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{P}(\boldsymbol{\gamma},\boldsymbol{y},t)dS_{\boldsymbol{\xi}}dS_{\boldsymbol{\gamma}}]\boldsymbol{\chi}(\boldsymbol{y},t)\rangle d\boldsymbol{x}d\boldsymbol{y}. \tag{31}$$

Since $\boldsymbol{\chi}(\boldsymbol{x},t)$ is the nontrivial solution of (24), the equation (31) is equivalent to

$$\begin{aligned}\frac{\partial \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t} =& -\boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) - \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)[\boldsymbol{A}^*_{\boldsymbol{y}}]^T - \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)+\\ &\int_{D^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}',t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}',\boldsymbol{x}'',t)\boldsymbol{P}(\boldsymbol{x}'',\boldsymbol{y},t)d\boldsymbol{x}'d\boldsymbol{x}''+\\ &\int_{S^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi},t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{P}(\boldsymbol{\gamma},\boldsymbol{y},t)dS_{\boldsymbol{\xi}}dS_{\boldsymbol{\gamma}},\end{aligned} \tag{32}$$

which is a Riccati nonlinear PDE. Since solving the above Riccati nonlinear PDE is extremely difficult in general, we propose a special choice of the matrix $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$ to simplify the Riccati nonlinear PDE (32) to a Bernoulli nonlinear PDE, of which an analytical solution can be found. As such, we choose the matrix $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$ as

$$\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{L}_x\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{L}^T_y, \tag{33}$$

where $\boldsymbol{L}_x$ is defined in Lemma 3.4. Substituting (33) into (32) results in

$$\begin{aligned}\frac{\partial \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t} =& -\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) + \int_{D^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}',t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}',\boldsymbol{x}'',t)\boldsymbol{P}(\boldsymbol{x}'',\boldsymbol{y},t)d\boldsymbol{x}'d\boldsymbol{x}''\\ &-\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)[\bar{\boldsymbol{A}}^*_{\boldsymbol{y}}]^T + \int_{S^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi},t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{P}(\boldsymbol{\gamma},\boldsymbol{y},t)dS_{\boldsymbol{\xi}}dS_{\boldsymbol{\gamma}},\end{aligned} \tag{34}$$

where the matrix operator $\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}$ is defined in (8). It is now seen that (34) is a Bernoulli nonlinear PDE. The unique and bounded solution $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ of (34) is given in (19). The control inputs $\boldsymbol{u}_d(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{u}_b(\boldsymbol{\xi},t)$ are found by substituting $\boldsymbol{\lambda}(\boldsymbol{x},t)$ given in (25) into the last two equations of (24), i.e.,

$$\begin{aligned}\boldsymbol{u}_d(\boldsymbol{x},t) &= -\int_{D^2}\boldsymbol{R}^+_d(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{B}^T_d(\boldsymbol{y},t)\boldsymbol{P}(\boldsymbol{y},\boldsymbol{y}',t)\boldsymbol{\chi}(\boldsymbol{y}',t)d\boldsymbol{y}d\boldsymbol{y}',\\ \boldsymbol{u}_b(\boldsymbol{\xi},t) &= -\int_D\int_S\boldsymbol{R}^+_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{B}^T_b(\boldsymbol{\gamma},t)\boldsymbol{P}(\boldsymbol{\gamma},\boldsymbol{y}',t)\boldsymbol{\chi}(\boldsymbol{y}',t)d\boldsymbol{y}'dS_{\boldsymbol{\gamma}}.\end{aligned} \tag{35}$$

The control design has been completed. The closed loop system consisting of substituting the control inputs $\boldsymbol{u}_d(\boldsymbol{x},t)$ and $\boldsymbol{u}_b(\boldsymbol{\xi},t)$ in (35) into (1) and $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$

generated by (34) with the boundary and terminal conditions in (26) is

$$
\begin{aligned}
&\frac{\partial \boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t) - \int_{D^2}\bar{\boldsymbol{R}}_d(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y}',t)\boldsymbol{\chi}(\boldsymbol{y}',t)d\boldsymbol{y}d\boldsymbol{y}', \quad \forall\, \boldsymbol{x}\in D,\\
&\boldsymbol{\chi}(\boldsymbol{x},t_0) = \boldsymbol{\chi}_0(\boldsymbol{x}), \quad \forall\, \boldsymbol{x}\in D,\\
&\boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\chi}(\boldsymbol{\xi},t) = -\int_D\int_S \bar{\boldsymbol{R}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{P}(\boldsymbol{\gamma},\boldsymbol{y}',t)\boldsymbol{\chi}(\boldsymbol{y}',t)d\boldsymbol{y}'dS_{\boldsymbol{\gamma}}, \quad \forall\, \boldsymbol{\xi}\in S,\\
&\frac{\partial \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t} = -\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) + \int_{D^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}',t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}',\boldsymbol{x}'',t)\boldsymbol{P}(\boldsymbol{x}'',\boldsymbol{y},t)d\boldsymbol{x}'d\boldsymbol{x}''\\
&\qquad\qquad - \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)[\bar{\boldsymbol{A}}^*_{\boldsymbol{y}}]^T + \int_{S^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi},t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{P}(\boldsymbol{\gamma},\boldsymbol{y},t)dS_{\boldsymbol{\xi}}dS_{\boldsymbol{\gamma}},\\
&\boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{P}(\boldsymbol{\xi},\boldsymbol{y},t) = 0, \quad \boldsymbol{y}\in D,\, \boldsymbol{\xi}\in S,\\
&\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t_f) = \boldsymbol{Q}_f(\boldsymbol{x},\boldsymbol{y}).
\end{aligned} \tag{36}
$$

4.3 Lyapunov stability analysis

In this subsection, we analyze stability of the closed loop system (36) using the Lyapunov direct method [7] by considering the following Lyapunov functional candidate

$$
V = \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}, \tag{37}
$$

where $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is the solution of the fourth equation of (36). Differentiating both sides of (37) with respect to t along the solutions of (36) yields

$$
\begin{aligned}
\frac{dV}{dt} =& \int_{D^2}\big\langle \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y} + \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\\
&- \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \big(\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)[\bar{\boldsymbol{A}}^*_{\boldsymbol{y}}]^T\big)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\\
&- \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y} + \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y},
\end{aligned} \tag{38}
$$

where $\boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t)$ are given by

$$
\begin{aligned}
\boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t) &= \int_{D^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}',t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}',\boldsymbol{x}'',t)\boldsymbol{P}(\boldsymbol{x}'',\boldsymbol{y},t)d\boldsymbol{x}'d\boldsymbol{x}'',\\
\boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t) &= \int_{S^2}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi},t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{P}(\boldsymbol{\gamma},\boldsymbol{y},t)dS_{\boldsymbol{\xi}}dS_{\boldsymbol{\gamma}}.
\end{aligned} \tag{39}
$$

Now applying Lemma 3.1 to the first three integral terms in the right hand side of (37) with a note that $\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}$ is defined in (8) and $\boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{P}(\boldsymbol{\xi},\boldsymbol{y},t) = 0$ yields

$$
\frac{dV}{dt} = -\int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), (\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t))\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}, \tag{40}
$$

where the positive definite matrix $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$ is defined in (Q.equation.assumption). Since the matrices $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$, $\boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t)$, and $\boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t)$ are symmetric and positive definite, from (37) and (40) we conclude that $\boldsymbol{\chi}(\boldsymbol{x},t)$ is exponentially stable in L_2 norm at the origin.

5 Finite number of controls

We assume that there are M_d controllers at fixed points $\boldsymbol{x}_1, ..., \boldsymbol{x}_{M_d}$ of the domain D and M_b controllers at fixed points $\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_{M_b}$ of the boundary S. Thus, by setting

$$\begin{aligned}
\boldsymbol{B}_d(\boldsymbol{x},t)\boldsymbol{u}_d(\boldsymbol{x},t) &= \sum_{i=1}^{M_d} \boldsymbol{B}_d(\boldsymbol{x}_i,t)\boldsymbol{u}_d(\boldsymbol{x}_i,t)\boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}_i),\\
\boldsymbol{B}_b(\boldsymbol{\xi},t)\boldsymbol{u}_b(\boldsymbol{\xi},t) &= \sum_{i=1}^{M_b} \boldsymbol{B}_b(\boldsymbol{\xi}_i,t)\boldsymbol{u}_b(\boldsymbol{\xi}_i,t)\boldsymbol{I}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_i),
\end{aligned} \tag{41}$$

in the distributed parameter systems (1), we see that the distributed parameter systems (1) are written as follows:

$$\begin{aligned}
&\frac{\partial \boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t) + \sum_{i=1}^{M_d} \boldsymbol{B}_d(\boldsymbol{x}_i,t)\boldsymbol{u}_d(\boldsymbol{x}_i,t)\boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}_i), \quad \forall \boldsymbol{x} \in D,\\
&\boldsymbol{\chi}(\boldsymbol{x},t_0) = \boldsymbol{\chi}_0(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in D,\\
&\boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\chi}(\boldsymbol{\xi},t) = \sum_{i=1}^{M_b} \boldsymbol{B}_b(\boldsymbol{\xi}_i,t)\boldsymbol{u}_b(\boldsymbol{\xi}_i,t)\boldsymbol{I}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_i), \quad \forall \boldsymbol{\xi} \in S.
\end{aligned} \tag{42}$$

Moreover, by setting

$$\begin{aligned}
\int_{D^2}\big\langle \boldsymbol{u}_d(\boldsymbol{x},t), \boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{u}_d(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y} =&\int_{D^2}\sum_{i,j=1}^{Md}\Big[\big\langle \boldsymbol{u}_d(\boldsymbol{x}_i,t), \boldsymbol{R}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)\boldsymbol{u}_d(\boldsymbol{y}_j,t)\big\rangle\Big]\\
&\times \boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}_i)\boldsymbol{I}\delta(\boldsymbol{y}-\boldsymbol{y}_i)d\boldsymbol{x}d\boldsymbol{y}\\
\int_{S^2}\big\langle \boldsymbol{u}_b(\boldsymbol{\xi},t), \boldsymbol{R}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{u}_b(\boldsymbol{\gamma},t)\big\rangle dS_{\boldsymbol{x}}dS_{\boldsymbol{\gamma}} =&\int_{S^2}\sum_{i,j=1}^{Mb}\Big[\big\langle \boldsymbol{u}_b(\boldsymbol{\xi}_i,t), \boldsymbol{R}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,t)\boldsymbol{u}_b(\boldsymbol{\gamma}_j,t)\big\rangle\Big]\\
&\times \boldsymbol{I}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_i)\boldsymbol{I}\delta(\boldsymbol{\gamma}-\boldsymbol{\gamma}_j)dS_{\boldsymbol{x}}dS_{\boldsymbol{\gamma}}
\end{aligned} \tag{43}$$

in the cost functional L defined in (11), the cost functional L is rewritten as

$$\begin{aligned}
L =&\frac{1}{2}\int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}+\\
&\frac{1}{2}\int_{D^2}\sum_{i,j=1}^{Md}\Big[\big\langle \boldsymbol{u}_d(\boldsymbol{x}_i,t), \boldsymbol{R}_d(\boldsymbol{x}_i,\boldsymbol{y}_j,t)\boldsymbol{u}_d(\boldsymbol{y}_j,t)\big\rangle\Big]\boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}_i)\boldsymbol{I}\delta(\boldsymbol{y}-\boldsymbol{y}_i)d\boldsymbol{x}d\boldsymbol{y}+\\
&\frac{1}{2}\int_{S^2}\sum_{i,j=1}^{Mb}\Big[\big\langle \boldsymbol{u}_b(\boldsymbol{\xi}_i,t), \boldsymbol{R}_b(\boldsymbol{\xi}_i,\boldsymbol{\gamma}_j,t)\boldsymbol{u}_b(\boldsymbol{\gamma}_j,t)\big\rangle\Big]\boldsymbol{I}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_i)\boldsymbol{I}\delta(\boldsymbol{\gamma}-\boldsymbol{\gamma}_i)dS_{\boldsymbol{x}}dS_{\boldsymbol{\gamma}}.
\end{aligned} \tag{44}$$

Now we can use the same procedure in the previous section to obtain the following control expressions

$$\sum_{i,j=1}^{Md} \boldsymbol{R}_d(\boldsymbol{x}_i, \boldsymbol{x}_j, t)\boldsymbol{u}_d(\boldsymbol{x}_j, t) = -\sum_{i=1}^{M_d} \boldsymbol{B}_d^T(\boldsymbol{x}_i, t)\boldsymbol{\lambda}(\boldsymbol{x}_i, t),\ \boldsymbol{\lambda}(\boldsymbol{x}_i, t) = \int_D \boldsymbol{P}(\boldsymbol{x}_i, \boldsymbol{x}, t)\boldsymbol{\chi}(\boldsymbol{x}, t)d\boldsymbol{x},$$
$$\sum_{i,j=1}^{Mb} \boldsymbol{R}_b(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j, t)\boldsymbol{u}_b(\boldsymbol{\xi}_j, t) = -\sum_{i=1}^{M_b} \boldsymbol{B}_b^T(\boldsymbol{\xi}_i, t)\boldsymbol{\lambda}(\boldsymbol{\xi}_i, t),\ \boldsymbol{\lambda}(\boldsymbol{\xi}_i, t) = \int_D \boldsymbol{P}(\boldsymbol{\xi}_i, \boldsymbol{x}, t)\boldsymbol{\chi}(\boldsymbol{x}, t)d\boldsymbol{x}. \tag{45}$$

Therefore, it is sufficient that we choose

$$\begin{aligned} \sum_{j=1}^{Md} \boldsymbol{R}_d(\boldsymbol{x}_i, \boldsymbol{x}_j, t)\boldsymbol{u}_d(\boldsymbol{x}_j, t) &= -\boldsymbol{B}_d^T(\boldsymbol{x}_i, t)\int_D \boldsymbol{P}(\boldsymbol{x}_i, \boldsymbol{x}, t)\boldsymbol{\chi}(\boldsymbol{x}, t)d\boldsymbol{x}, \\ \sum_{j=1}^{Mb} \boldsymbol{R}_b(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j, t)\boldsymbol{u}_b(\boldsymbol{\xi}_j, t) &= -\boldsymbol{B}_b^T(\boldsymbol{\xi}_i, t)\int_D \boldsymbol{P}(\boldsymbol{\xi}_i, \boldsymbol{x}, t)\boldsymbol{\chi}(\boldsymbol{x}, t)d\boldsymbol{x}. \end{aligned} \tag{46}$$

Then we can write (46) as

$$\begin{aligned} \boldsymbol{u}_d(\boldsymbol{x}_i, t) &= -\sum_{j=1}^{M_d} \boldsymbol{R}_d^{-1}(\boldsymbol{x}_i, \boldsymbol{x}_j, t)\boldsymbol{B}_d^T(\boldsymbol{x}_j, t)\int_D \boldsymbol{P}(\boldsymbol{x}_j, \boldsymbol{x}, t)\boldsymbol{\chi}(\boldsymbol{x}, t)d\boldsymbol{x}, \\ \boldsymbol{u}_b(\boldsymbol{\xi}_i, t) &= -\sum_{j=1}^{M_b} \boldsymbol{R}_b^{-1}(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j, t)\boldsymbol{B}_b^T(\boldsymbol{\xi}_j, t)\int_D \boldsymbol{P}(\boldsymbol{\xi}_j, \boldsymbol{x}, t)\boldsymbol{\chi}(\boldsymbol{x}, t)d\boldsymbol{x}. \end{aligned} \tag{47}$$

To find the differential equation for $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$, we carry out the same analysis as from (32) to (34) to obtain

$$\begin{aligned} \frac{\partial \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)}{\partial t} =& -\bar{\boldsymbol{A}}_{\boldsymbol{x}}^* \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t) + \sum_{i,j=1}^{M_d} \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{x}_i', t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}_i', \boldsymbol{x}_j'', t)\boldsymbol{P}(\boldsymbol{x}_j'', \boldsymbol{y}, t) \\ & - \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)[\bar{\boldsymbol{A}}_{\boldsymbol{y}}^*]^T + \sum_{i,j=1}^{M_b} \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\xi}_i, t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j, t)\boldsymbol{P}(\boldsymbol{\xi}_j, \boldsymbol{y}, t), \end{aligned} \tag{48}$$

where

$$\begin{aligned} \bar{\boldsymbol{R}}_d(\boldsymbol{x}_i, \boldsymbol{x}_j, t) &= \boldsymbol{B}_d(\boldsymbol{x}_i, t)\boldsymbol{R}_d^{-1}(\boldsymbol{x}_i, \boldsymbol{x}_j, t)\boldsymbol{B}_d^T(\boldsymbol{x}_j, t) \\ \bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j, t) &= \boldsymbol{B}_b(\boldsymbol{\xi}_i, t)\boldsymbol{R}_b^{-1}(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j, t)\boldsymbol{B}_b^T(\boldsymbol{\xi}_j, t). \end{aligned} \tag{49}$$

The boundary and terminal conditions for $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, t)$ are the same as in (26). The analytical solution of (48) is the same as (19) except for $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \bar{t})$ given by

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \bar{t}) =& \int_{-t_f}^{\bar{t}} \sum_{i,j=1}^{M_d} \boldsymbol{G}^T(\boldsymbol{\eta}_i, \tau; \boldsymbol{x}, -t_f)\bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{\eta}_i, \boldsymbol{\alpha}_j, \tau)\boldsymbol{G}(\boldsymbol{\alpha}_j, \tau; \boldsymbol{y}, -t_f) + \\ & \int_{-t_f}^{\bar{t}} \sum_{i,j=1}^{M_b} \boldsymbol{G}^T(\boldsymbol{\xi}_i, \tau; \boldsymbol{x}, -t_f)\bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{\xi}_i, \boldsymbol{\gamma}_j, \tau)\boldsymbol{G}(\boldsymbol{\gamma}_j, \tau; \boldsymbol{y}, -t_f). \end{aligned} \tag{50}$$

The closed loop system consisting of substituting the control inputs $\boldsymbol{u}_d(\boldsymbol{x}_i,t)$ and $\boldsymbol{u}_b(\boldsymbol{\xi}_i,t)$ in (47) into (1) and $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ generated by (48) with the boundary and terminal conditions in (26) is

$$\begin{aligned}
&\frac{\partial \boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t) - \sum_{i,j=1}^{M_d} \bar{\boldsymbol{R}}_d(\boldsymbol{x}_i,\boldsymbol{x}_j,t)\int_D \boldsymbol{P}(\boldsymbol{x}_j,\boldsymbol{\alpha},t)\boldsymbol{\chi}(\boldsymbol{\alpha},t)d\boldsymbol{\alpha},\ \forall\,\boldsymbol{x}\in D,\\
&\boldsymbol{\chi}(\boldsymbol{x},t_0) = \boldsymbol{\chi}_0(\boldsymbol{x}), \quad \forall\,\boldsymbol{x}\in D,\\
&\boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\chi}(\boldsymbol{\xi},t) = -\sum_{i,j=1}^{M_b} \bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i,\boldsymbol{\xi}_j,t)\int_D \boldsymbol{P}(\boldsymbol{\xi}_j,\boldsymbol{\alpha},t)\boldsymbol{\chi}(\boldsymbol{\alpha},t)d\boldsymbol{\alpha}, \quad \forall\,\boldsymbol{\xi}\in S,\\
&\frac{\partial \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)}{\partial t} = -\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) + \sum_{i,j=1}^{M_d}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}'_i,t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}'_i,\boldsymbol{x}''_j,t)\boldsymbol{P}(\boldsymbol{x}''_j,\boldsymbol{y},t)\\
&\qquad\qquad - \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)[\bar{\boldsymbol{A}}^*_{\boldsymbol{y}}]^T + \sum_{i,j=1}^{M_b}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi}_i,t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i,\boldsymbol{\xi}_j,t)\boldsymbol{P}(\boldsymbol{\xi}_j,\boldsymbol{y},t),\\
&\boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{P}(\boldsymbol{\xi},\boldsymbol{y},t) = 0, \quad \boldsymbol{y}\in D,\ \boldsymbol{\xi}\in S,\\
&\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t_f) = \boldsymbol{Q}_f(\boldsymbol{x},\boldsymbol{y}).
\end{aligned} \tag{51}$$

5.1 Lyapunov stability analysis

In this subsection, we analyze stability of the closed loop system (51) using the Lyapunov direct method [7] by considering the following Lyapunov functional candidate

$$V = \int_{D^2}\Big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\Big\rangle d\boldsymbol{x}d\boldsymbol{y}, \tag{52}$$

where the symmetric and positive definite matrix $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ is the solution of the fourth equation of the closed loop system (51). Differentiating both sides of (52) with respect to t along the solutions of (51) yields

$$\begin{aligned}
\frac{dV}{dt} =& \int_{D^2}\big\langle \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y} + \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\\
&- \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \big(\bar{\boldsymbol{A}}^*_{\boldsymbol{x}}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)[\bar{\boldsymbol{A}}^*_{\boldsymbol{y}}]^T\big)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\\
&- \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y} + \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), \boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y}\\
=& - \int_{D^2}\big\langle \boldsymbol{\chi}(\boldsymbol{x},t), (\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t))\boldsymbol{\chi}(\boldsymbol{y},t)\big\rangle d\boldsymbol{x}d\boldsymbol{y},
\end{aligned} \tag{53}$$

where $\boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t)$ are given by

$$\begin{aligned}
\boldsymbol{K}_{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y},t) &= \sum_{i,j=1}^{M_d}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}'_i,t)\bar{\boldsymbol{R}}_d(\boldsymbol{x}'_i,\boldsymbol{x}''_j,t)\boldsymbol{P}(\boldsymbol{x}''_j,\boldsymbol{y},t)\\
\boldsymbol{K}_{\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y},t) &= \sum_{i,j=1}^{M_b}\boldsymbol{P}(\boldsymbol{x},\boldsymbol{\xi}_i,t)\bar{\boldsymbol{R}}_b(\boldsymbol{\xi}_i,\boldsymbol{\xi}_j,t)\boldsymbol{P}(\boldsymbol{\xi}_j,\boldsymbol{y},t).
\end{aligned} \tag{54}$$

Since the matrices $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$, $\boldsymbol{K_d}(\boldsymbol{x},\boldsymbol{y},t)$, and $\boldsymbol{K_b}(\boldsymbol{x},\boldsymbol{y},t)$ are symmetric and positive definite, from (37) and (53) we can see that V is positive definite functional of $\boldsymbol{\chi}(\boldsymbol{x},t)$, $\boldsymbol{x}\in D$ and that $\frac{dV}{dt}\leq 0$. Therefore, V is not increasing for all $t\in T$. Moreover, from (37) and (53) we conclude that $\boldsymbol{\chi}(\boldsymbol{x},t)$ is exponentially stable in L_2 norm at the origin. It is noted that exponential stability in this case is not as strong as in the case of distributed control as analyzed in Subsection 4.3 unless $M_d\to\infty$ and $M_b\to\infty$. This can be seen from the fact that the solution $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ of the fourth equation of (51) is given in (19) with the matrix $\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ defined in (50) which is not strictly positive definite in the sense that at $\boldsymbol{x}\neq\boldsymbol{x}_i$, $i=1,...,M_d$ and $\boldsymbol{\xi}\neq\boldsymbol{\xi}_i$, $i=1,...,M_b$ the matrix $\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ might be zero. This is not surprising since a finite number of controls is considered in this case.

Remark 5.1 *From the optimal control design in the previous section and the observer design in this section, we observe the followings:*

1. *The equation (34) and the equation (50) in [5] for the case of spatial continuous control inputs and spatial continuous measurements are of the same form.*
2. *The equation of (48) and the equation (60) in [5] for the case of finite control inputs and discrete measurements are also of the same form.*

The above observation allows us to make an important remark about duality between the optimal control design and the optimal observer design: Solving the optimal control design problem is equivalent to solving the optimal observer design problem. This can be more clearly seen by looking at mappings: $t\to -t$ *and* $\boldsymbol{A_x}\to\boldsymbol{A_x^*}$ *and* $\boldsymbol{\beta_\xi}\to\boldsymbol{\beta_\xi^*}$ *from the optimal control design to the optimal observer design.*

6 An example

Since the most difficult task of the proposed control design in the previous sections is to choose the operator $\boldsymbol{L}_x$ so that the matrix $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)$ determined by (9) is positive definite, we present an example to illustrate on how to choose this operator. As such, we consider the following one-dimensional heat equation:

$$\begin{aligned}&\frac{\partial\chi(x,t)}{\partial t}=a\frac{\partial^2\chi(x,t)}{\partial x^2}+c\chi(x,t)+u_d(x,t),\quad x\in(0,1),\\&\chi(x,t_0)=\chi_0(x),\\&\frac{\partial\chi(x,t)}{\partial x}\Big|_{x=0}=u_b(0,t),\quad\frac{\partial\chi(x,t)}{\partial x}\Big|_{x=1}=u_b(1,t)\end{aligned}\tag{55}$$

where a and c are constants. From (55), we have $A_x[\bullet]=A_x^*[\bullet]=a\frac{\partial^2[\bullet]}{\partial x^2}+c[\bullet]$. Thus, we choose the operator $L_x[\bullet]$ as

$$L_x[\bullet]=(-\bar{a}-a)\frac{\partial^2[\bullet]}{\partial x^2}+(\bar{c}-c)[\bullet]\tag{56}$$

where $\bar{a}$ and $\bar{c}$ are positive constants satisfying

$$\bar{a} + a > 0,\ \bar{c} - c > 0. \tag{57}$$

According to (8), we have the operator $\bar{A}_x^*$ given by

$$\bar{A}_x^*[\bullet] = -\bar{a}\frac{\partial^2[\bullet]}{\partial x^2} + \bar{c}[\bullet]. \tag{58}$$

Thus, we have $z(x,y,t)$-system, see (7), given by

$$\begin{aligned}
&\frac{\partial z(x,,y,t)}{\partial t} = \bar{a}\frac{\partial^2(x,t)}{\partial x^2} + \bar{a}\frac{\partial^2(x,t)}{\partial y^2} - 2\bar{c}z(x,y,t), \quad x \in (0,1),\\
&z(x,y,t_0) = z_0(x,y),\\
&\frac{\partial z(x,y,t)}{\partial x}\Big|_{x=0} = 0, \quad \frac{\partial z(x,,y,t)}{\partial x}\Big|_{x=1} = 0.
\end{aligned} \tag{59}$$

By Lemma 3.3, the unique solution of (59) is

$$z(x,y,t) = \int_0^1\int_0^1 G(x,t;x',t_0)z_0(x',y')G^T(y,t;y',t_0)dx'dy' \tag{60}$$

where the Green function $G(x,t;x',t_0)$ is the solution of

$$\begin{aligned}
&\frac{\partial G(x,t;x',t_0)}{\partial t} = \bar{a}\frac{\partial^2 G(x,t;x',t_0)}{\partial x^2} - \bar{c}G(x,t;x',t_0),\\
&G(x,t_0;x',t_0) = \delta(x-x'),\\
&\frac{\partial G(x,t;x',t_0)}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial G(x,t;x',t_0)}{\partial x}\Big|_{x=1} = 0.
\end{aligned} \tag{61}$$

A calculation shows that the solution of (61) is

$$G(x,t;x',t_0) = e^{-\bar{c}(t-t_0)} + 2\sum_{n=1}^{\infty} e^{-(n^2\pi^2\bar{a}+\bar{c})(t-t_0)}\cos(n\pi x)\cos(n\pi x'), \tag{62}$$

which exponentially converges to zero as t tends to infinity. This in turn implies from (60) that the Green function $G(x,t;x',t_0)$ exponentially converges to zero. Next, we show that the matrix $Q(x,y,t)$ defined in (9) with L_x given by (56) is symmetric and positive definite. As such, the matrix $Q(x,y,t)$ is given y

$$Q(x,y,t) = -(\bar{a}+a)\frac{\partial^2 P(x,y,t)}{\partial x^2} - (\bar{a}+a)\frac{\partial^2 P(x,y,t)}{\partial y^2} - 2(\bar{c}-c)P(x,y,t), \tag{63}$$

where $P(x,y,t)$ is given in (19) with the Green function $G(x,t;x',t')$ being the solution of the following system

$$\begin{aligned}
&\frac{\partial G(x,t;x',t')}{\partial t} = -\bar{a}\frac{\partial^2 G(x,t;x',t')}{\partial x^2} + \bar{c}G(x,t;x',t'),\\
&G(x,t';x',t') = \delta(x-x'),\\
&\frac{\partial G(x,t;x',t')}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial G(x,t;x',t')}{\partial x}\Big|_{x=1} = 0.
\end{aligned} \tag{64}$$

A calculation shows that the solution of (64) is

$$G(x,t;x',t') = e^{\bar{c}(t-t')} + 2\sum_{n=1}^{\infty} e^{(n^2\pi^2\bar{a}+\bar{c})(t-t')}\cos(n\pi x)\cos(n\pi x'). \tag{65}$$

Substituting $G(x,t;x',t')$ given in (65) into $P(x,y,t)$ given in (19), then into (63) shows that the matrix $Q(x,y,t)$ is symmetric and positive definite as the constants $\bar{a}$ and $\bar{c}$ were chosen such that the conditions (57) hold.

7 Conclusions

In this paper, we used the calculus of variation approach to propose a constructive method to design inverse optimal controllers for a class of linear distributed parameter systems. The most shining point of the paper is the introduction of inverse optimality concept that relaxes difficulties in solving the Riccati nonlinear PDEs. Both spatial continuous and discrete control inputs were addressed. The proposed control design in this paper can be used for solving other control and filter design problems for DPSs.

A Proof of Lemma 3.4

To prove Lemma 3.4, we first verify that $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ given in (19) satisfies the Riccati nonlinear PDE (18). Then, we prove that the solution $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ given in (19) is unique and bounded.

A.1 Verification of the solution $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$

We proceed by proving that $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ satisfies the terminal condition, i.e., the second equation in (18), that $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ satisfies the boundary condition i.e., the third equation in (18), and that $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ satisfies the first equation (18).

First, to prove that $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ satisfies the terminal condition in (18), from (20) and (21), we have

$$\begin{aligned} &\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},-t_f) = 0, \\ &\boldsymbol{G}(\boldsymbol{x},-t_f;\boldsymbol{x}',-t_f) = \boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}'). \end{aligned} \tag{66}$$

Substituting (66) into the third equation of (19) at $\bar{t} = -t_f$ yields

$$\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},-t_f) = \int_{D^2} \boldsymbol{I}\delta(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{P}_f(\boldsymbol{x'},\boldsymbol{y'})\boldsymbol{I}\delta(\boldsymbol{y}-\boldsymbol{y}')d\boldsymbol{x}'d\boldsymbol{y}' = \boldsymbol{P}_f(\boldsymbol{x},\boldsymbol{y}). \tag{67}$$

From the first two equations of (19), we have $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t_f) = \bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},-t_f)$. Hence $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t_f) = \boldsymbol{P}_f(\boldsymbol{x},\boldsymbol{y})$.

Second, to prove that $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ satisfies the boundary condition in (18), from the third equation of (19), we have

$$\begin{aligned}\boldsymbol{\beta}_{\boldsymbol{\xi}}^*\bar{\boldsymbol{P}}(\boldsymbol{\xi},\boldsymbol{y},\bar{t}) &= \int_{D^2}\boldsymbol{\beta}_{\boldsymbol{\xi}}^*\boldsymbol{G}(\boldsymbol{\xi},\bar{t};\boldsymbol{x}',-t_f)\big[\boldsymbol{P}_f^+(\boldsymbol{x}',\boldsymbol{y}')+\boldsymbol{M}(\boldsymbol{x}',\boldsymbol{y}',\bar{t})\big]^+\boldsymbol{G}^T(\boldsymbol{y},\bar{t};\boldsymbol{y}',-t_f)d\boldsymbol{x}'d\boldsymbol{y}'\\ &= 0\end{aligned} \tag{68}$$

where we have used the third equation of (21). On the other hand, from the first equation of (19) we have $\boldsymbol{\beta}_{\boldsymbol{\xi}}^*\bar{\boldsymbol{P}}(\boldsymbol{\xi},\boldsymbol{y},-\bar{t}) = \boldsymbol{\beta}_{\boldsymbol{\xi}}^*\bar{\boldsymbol{P}}(\boldsymbol{\xi},\boldsymbol{y},t)$. Hence $\boldsymbol{\beta}_{\boldsymbol{\xi}}^*\bar{\boldsymbol{P}}(\boldsymbol{\xi},\boldsymbol{y},t) = 0$.

Third, to prove that $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ satisfies the first equation of (18), we differentiating both sides of the third equation of (19) with respect to $\bar{t}$ to obtain

$$\begin{aligned}\frac{\partial\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})}{\partial\bar{t}} &= \int_{D^2}\bar{\boldsymbol{A}}_{\boldsymbol{x}}^*\boldsymbol{G}(\boldsymbol{x},\bar{t};\boldsymbol{x}',-t_f)\big[\boldsymbol{P}_f^+(\boldsymbol{x}',\boldsymbol{y}')+\boldsymbol{M}(\boldsymbol{x}',\boldsymbol{y}',\bar{t})\big]^+\boldsymbol{G}^T(\boldsymbol{y},\bar{t};\boldsymbol{y}',-t_f)d\boldsymbol{x}'d\boldsymbol{y}'\\ &+ \int_{D^2}\boldsymbol{G}(\boldsymbol{x},\bar{t};\boldsymbol{x}',-t_f)\big[\boldsymbol{P}_f^+(\boldsymbol{x}',\boldsymbol{y}')+\boldsymbol{M}(\boldsymbol{x}',\boldsymbol{y}',\bar{t})\big]^+[\bar{\boldsymbol{A}}_{\boldsymbol{y}}^*\boldsymbol{G}(\boldsymbol{y},\bar{t};\boldsymbol{y}',-t_f)]^T d\boldsymbol{x}'d\boldsymbol{y}'+\Omega\\ &= \bar{\boldsymbol{A}}_{\boldsymbol{x}}^*\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})+\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})[\bar{\boldsymbol{A}}_{\boldsymbol{y}}^*]^T+\Omega,\end{aligned} \tag{69}$$

where we have used the first equation of (21), and

$$\Omega = \int_{D^2}\boldsymbol{G}(\boldsymbol{x},\bar{t};\boldsymbol{x}',-t_f)\frac{\partial\big[\boldsymbol{P}_f^+(\boldsymbol{x}',\boldsymbol{y}')+\boldsymbol{M}(\boldsymbol{x}',\boldsymbol{y}',\bar{t})\big]^+}{\partial\bar{t}}\boldsymbol{G}^T(\boldsymbol{y},\bar{t};\boldsymbol{y}',-t_f)d\boldsymbol{x}'d\boldsymbol{y}'. \tag{70}$$

Applying Lemma 3.2 to (70) arrives at

$$\begin{aligned}\Omega = -\int_{D^4}&\boldsymbol{G}(\boldsymbol{x},\bar{t};\boldsymbol{x}',-t_f)\big[\boldsymbol{P}_f^+(\boldsymbol{x}',\boldsymbol{x}'')+\boldsymbol{M}(\boldsymbol{x}',\boldsymbol{x}'',\bar{t})\big]^+\frac{\partial\boldsymbol{M}(\boldsymbol{x}'',\boldsymbol{y}'',\bar{t})}{\partial\bar{t}}\times\\ &\big[\boldsymbol{P}_f^+(\boldsymbol{y}'',\boldsymbol{y}')+\boldsymbol{M}(\boldsymbol{y}'',\boldsymbol{y}',\bar{t})\big]^+\boldsymbol{G}^T(\boldsymbol{y},\bar{t};\boldsymbol{y}',-t_f)d\boldsymbol{x}'d\boldsymbol{y}'d\boldsymbol{x}''d\boldsymbol{y}''\end{aligned} \tag{71}$$

Using the expression of $\boldsymbol{M}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ defined in (20) gives

$$\begin{aligned}\frac{\partial\boldsymbol{M}(\boldsymbol{x}'',\boldsymbol{y}'',\bar{t})}{\partial\bar{t}} &= \int_{D^2}\boldsymbol{G}^T(\boldsymbol{\eta},\bar{t};\boldsymbol{x},-t_f)\bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{\eta},\boldsymbol{\alpha},\bar{t})\boldsymbol{G}(\boldsymbol{\alpha},\bar{t};\boldsymbol{y},-t_f)d\boldsymbol{\eta}\,d\boldsymbol{\alpha}+\\ &\int_{S^2}\boldsymbol{G}^T(\boldsymbol{\xi},\bar{t};\boldsymbol{x},-t_f)\bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},\bar{t})\boldsymbol{G}(\boldsymbol{\gamma},\bar{t};\boldsymbol{y},-t_f)dS_{\boldsymbol{\xi}}\,dS_{\boldsymbol{\gamma}},\end{aligned} \tag{72}$$

which is substituted into (71) and using the third equation of (18), we have

$$\begin{aligned}\Omega = &-\int_{D^2}\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{\eta},\bar{t})\bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{\eta},\boldsymbol{\alpha},\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{\alpha},\boldsymbol{y},\bar{t})d\boldsymbol{\eta}\,d\boldsymbol{\alpha}\\ &-\int_{S^2}\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{\xi},\bar{t})\bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{\gamma},\boldsymbol{y},\bar{t})dS_{\boldsymbol{\xi}}\,dS_{\boldsymbol{\gamma}}.\end{aligned} \tag{73}$$

Substituting (73) into (69) yields

$$\begin{aligned}\frac{\partial\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})}{\partial\bar{t}} &= \bar{\boldsymbol{A}}_{\boldsymbol{x}}^*\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})-\int_{D^2}\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{\eta},\bar{t})\bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{\eta},\boldsymbol{\alpha},\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{\alpha},\boldsymbol{y},\bar{t})d\boldsymbol{\eta}\,d\boldsymbol{\alpha}\\ &+\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})[\bar{\boldsymbol{A}}_{\boldsymbol{y}}^*]^T-\int_{S^2}\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{\xi},\bar{t})\bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{\gamma},\boldsymbol{y},\bar{t})dS_{\boldsymbol{\xi}}\,dS_{\boldsymbol{\gamma}}.\end{aligned} \tag{74}$$

Since $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) = \bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ and $t = -\bar{t}$, see the first two equations of (19), the equation (74) is equivalent to the first equation of (18).

A.2 Proof of uniqueness of $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$

Since $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t) = \bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ and $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t_f)=\boldsymbol{P}_f(\boldsymbol{x},\boldsymbol{y})$ is symmetric and positive definite, $\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t}_f)$ is symmetric and positive definite. On the other hand, it is seen from (74) that a trivial solution is $\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t}) = 0$ for all $(\boldsymbol{x},\boldsymbol{y}) \in D \times D$ and $\bar{t} \geq -t_f$. Moreover, since $\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t}_f)$ is symmetric and positive definite, we focus on a non-zero solution of (74). As such, we let $\bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ be the generalized inverse of $\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})$. An application of Lemma 3.2 results in differentiation of $\bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ with respect to $\bar{t}$ along the solutions of (74):

$$\begin{aligned}
&\frac{\partial \bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t})}{\partial \bar{t}} = -\int_{D^2} \bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{x}',\bar{t})\bar{\boldsymbol{A}}^{*}_{\boldsymbol{x}'}\bar{\boldsymbol{P}}(\boldsymbol{x}',\boldsymbol{y}',\bar{t})\bar{\boldsymbol{P}}^{+}(\boldsymbol{y}',\boldsymbol{y},\bar{t})d\boldsymbol{x}'d\boldsymbol{y}' - \\
&\int_{D^2} \bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{x}',\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{x}',\boldsymbol{y}',\bar{t})[\bar{\boldsymbol{A}}^{*}_{\boldsymbol{y}'}]^T\bar{\boldsymbol{P}}^{+}(\boldsymbol{y}',\boldsymbol{y},\bar{t})d\boldsymbol{x}'d\boldsymbol{y}' + \\
&\int_{D^4} \bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{x}',\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{x}',\boldsymbol{\eta},\bar{t})\bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{\eta},\boldsymbol{\alpha},\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{\alpha},\boldsymbol{y}',\bar{t})\bar{\boldsymbol{P}}^{+}(\boldsymbol{y}',\boldsymbol{y},\bar{t})d\boldsymbol{x}'d\boldsymbol{y}'d\boldsymbol{\eta}\, d\boldsymbol{\alpha} + \\
&\int_{D^2}\int_{S^2} \bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{x}',\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{x}',\boldsymbol{\xi},\bar{t})\bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{\xi},\boldsymbol{\gamma},\bar{t})\bar{\boldsymbol{P}}(\boldsymbol{\gamma},\boldsymbol{y}',\bar{t})\bar{\boldsymbol{P}}^{+}(\boldsymbol{y}',\boldsymbol{y},\bar{t})d\boldsymbol{x}'d\boldsymbol{y}'dS_{\boldsymbol{\xi}}\, dS_{\boldsymbol{\gamma}} \\
&= -\bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t})\bar{\boldsymbol{A}}^{*}_{\boldsymbol{y}} - [\bar{\boldsymbol{A}}^{*}_{\boldsymbol{x}}]^T\bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t}) + \bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{x},\boldsymbol{y},\bar{t}) + \bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{x},\boldsymbol{y},\bar{t}).
\end{aligned} \tag{75}$$

An application of Lemma 3.3 to (75) shows that there exists a unique solution $\bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t})$, which is given in the third equation of (19). Finally, boundedness of the unique solution $\bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ is bounded follows from the item 3) in Assumption 2.1 and (75) since $\bar{\bar{\boldsymbol{R}}}_d(\boldsymbol{x},\boldsymbol{y},\bar{t}) + \bar{\bar{\boldsymbol{R}}}_b(\boldsymbol{x},\boldsymbol{y},\bar{t}) = \bar{\boldsymbol{R}}_d(\boldsymbol{x},\boldsymbol{y},t) + \bar{\boldsymbol{R}}_b(\boldsymbol{x},\boldsymbol{y},t)$, which is bounded by the item 2) of Assumption 2.1. Uniqueness and boundedness of $\bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ imply those of $\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ or $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$ since $\bar{\boldsymbol{P}}^{+}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ is the generalized inverse of $\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t})$ and $\bar{\boldsymbol{P}}(\boldsymbol{x},\boldsymbol{y},\bar{t}) = \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y},t)$. □

B Derivation of (24)

Let us introduce an Euler or weak variation $\delta\boldsymbol{\chi}(\boldsymbol{x},t) = \varepsilon\boldsymbol{\varrho}(\boldsymbol{x},t)$, where $\boldsymbol{\varrho}(\boldsymbol{x},t)$ is an arbitrary twice continuously differentiable function. The necessary conditions for the cost functional J_1 to be minimized are found by setting the weak variation δJ_1 of the functional J_1 to be equal to zero. Now assuming that the end points t_0 and t_f are not fixed so as when the parameter ε varies so do the end points t_0 and t_f, i.e., $t_0 = t_0(\varepsilon)$ and $t_f = t_f(\varepsilon)$. As such, the weak variation δJ_1 is given by

$$\begin{aligned}
\delta J_1 =& \left[\frac{\partial}{\partial\varepsilon}\int_{t_0(\varepsilon)}^{t_f(\varepsilon)} L_1 dt\right]\Bigg|_{\varepsilon\to 0} + \delta J_f \\
=& \left[L_1\big|_{t=t_f}\frac{dt_f(\varepsilon)}{d\varepsilon}\right]\Bigg|_{\varepsilon\to 0} - \left[L_1\big|_{t=t_0}\frac{dt_0(\varepsilon)}{d\varepsilon}\right]\Bigg|_{\varepsilon\to 0} + \int_{t_0(\varepsilon)}^{t_f(\varepsilon)}\left[\frac{\partial L_1}{\partial\varepsilon}\right]\Bigg|_{\varepsilon\to 0} dt + \delta J_f,
\end{aligned} \tag{76}$$

where we have used Leibnitz's differentiation rule. On the other hand, a calculation shows that

$$\left[L_1\big|_{t=t_f}\frac{dt_f(\varepsilon)}{d\varepsilon}\right]\Bigg|_{\varepsilon\to 0} - \left[L_1\big|_{t=t_0}\frac{dt_0(\varepsilon)}{d\varepsilon}\right]\Bigg|_{\varepsilon\to 0} = L_1\delta t\big|_{t_0}^{t_f},$$

$$
\begin{aligned}
\int_{t_0(\varepsilon)}^{t_f(\varepsilon)} \left[\frac{\partial L_1}{\partial \varepsilon}\right]\Bigg|_{\varepsilon\to 0} dt = &\int_{t_0}^{t_f}\int_D\Big\langle \frac{\delta L_1}{\delta \boldsymbol{\chi}(\boldsymbol{x},t)} - \frac{\partial}{\partial t}\left[\frac{\delta L_1}{\delta(\partial \boldsymbol{\chi}(\boldsymbol{x},t)/\partial t)}\right], \boldsymbol{\varrho}(\boldsymbol{x},t)\Big\rangle d\boldsymbol{x}dt \\
&+ \int_S\Big\langle \frac{\delta L_1}{\delta \boldsymbol{\chi}(\boldsymbol{\xi},t)}, \boldsymbol{\varrho}(\boldsymbol{\xi},t)\Big\rangle dS_{\boldsymbol{\xi}} - \left[\int_D\Big\langle \frac{\delta L_1}{\delta(\partial \boldsymbol{\chi}(\boldsymbol{x},t)/\partial t)}, \frac{\partial \boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t}\Big\rangle d\boldsymbol{x}\right]\delta t\Bigg|_{t_0}^{t_f} \\
&+ \left[\int_D\Big\langle \frac{\delta L_1}{\delta(\partial \boldsymbol{\chi}(\boldsymbol{x},t)/\partial t)}, \delta\boldsymbol{\chi}(\boldsymbol{x},t)\Big\rangle d\boldsymbol{x}\right]\Bigg|_{t_0}^{t_f},
\end{aligned}
\tag{77}
$$

$$
\delta J_f = \int_D \left\langle \frac{\delta J_f}{\delta \boldsymbol{\chi}(\boldsymbol{x},t_f)}, \delta\boldsymbol{\chi}(\boldsymbol{x},t_f)\right\rangle d\boldsymbol{x}.
$$

We calculate δL_1 as follows:

$$
\begin{aligned}
\delta L_1 &= \int_D \frac{\delta L_1}{\delta\varepsilon}\Bigg|_{\varepsilon\to 0} d\boldsymbol{x} + \int_S \frac{\delta L_1}{\delta\varepsilon}\Bigg|_{\varepsilon\to 0} dS_{\boldsymbol{\xi}} \\
&= \int_D \langle \int_D \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y}, \boldsymbol{\varrho}(\boldsymbol{x},t)\rangle d\boldsymbol{x} + \int_D \langle \boldsymbol{\lambda}(\boldsymbol{x},t), \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\varrho}(\boldsymbol{x},t)\rangle d\boldsymbol{x} \\
&\quad - \int_S \langle \boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\varrho}(\boldsymbol{\xi},t), \boldsymbol{\lambda}(\boldsymbol{\xi},t)\rangle dS_{\boldsymbol{\xi}}
\end{aligned}
\tag{78}
$$

From Lemma 3.1, we have

$$
\begin{aligned}
&\int_D \langle \boldsymbol{\lambda}(\boldsymbol{x},t), \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\varrho}(\boldsymbol{x},t)\rangle d\boldsymbol{x} - \int_S \langle \boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\varrho}(\boldsymbol{\xi},t), \boldsymbol{\lambda}(\boldsymbol{\xi},t)\rangle dS_{\boldsymbol{\xi}} = \\
&\qquad \int_D \langle \boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{\lambda}(\boldsymbol{x},t), \boldsymbol{\varrho}(\boldsymbol{x},t)\rangle d\boldsymbol{x} - \int_S \langle \boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{\lambda}(\boldsymbol{\xi},t), \boldsymbol{\varrho}(\boldsymbol{\xi},t)\rangle dS_{\boldsymbol{\xi}}
\end{aligned}
\tag{79}
$$

where $\boldsymbol{A}^*_{\boldsymbol{x}}$ and $\boldsymbol{\beta}^*_{\boldsymbol{\xi}}$ are given in (5). Substituting (79) into (78) results in

$$
\begin{aligned}
\delta L_1 = &\int_D \langle \int_D \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y}, \boldsymbol{\varrho}(\boldsymbol{x},t)\rangle d\boldsymbol{x} + \\
&\int_D \langle \boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{\lambda}(\boldsymbol{x},t), \boldsymbol{\varrho}(\boldsymbol{x},t)\rangle d\boldsymbol{x} - \int_S \langle \boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{\lambda}(\boldsymbol{\xi},t), \boldsymbol{\varrho}(\boldsymbol{\xi},t)\rangle dS_{\boldsymbol{\xi}}.
\end{aligned}
\tag{80}
$$

Thus, we have

$$
\begin{aligned}
\frac{\delta L_1}{\delta \boldsymbol{\chi}(\boldsymbol{x},t)} &= \boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{\lambda}(\boldsymbol{x},t) + \int_D \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y}, \\
\frac{\delta L_1}{\delta \boldsymbol{\chi}(\boldsymbol{\xi},t)} &= -\boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{\lambda}(\boldsymbol{\xi},t).
\end{aligned}
\tag{81}
$$

Since $\boldsymbol{\varrho}(\boldsymbol{x},t)$ is arbitrary and $\boldsymbol{\varrho}(\boldsymbol{x},t_0) = 0$ and $\boldsymbol{\varrho}(\boldsymbol{x},t_f) = 0$, setting $\delta J_1 = 0$ gives

$$
\begin{aligned}
&\frac{\delta L_1}{\delta \boldsymbol{\chi}(\boldsymbol{x},t)} - \frac{\partial}{\partial t}\left[\frac{\delta L_1}{\delta(\partial \boldsymbol{\chi}(\boldsymbol{x},t)/\partial t)}\right] = 0, \ \frac{\delta L_1}{\delta \boldsymbol{\chi}(\boldsymbol{\xi},t)} = 0, \\
&\left[L_1 - \int_D\Big\langle \frac{\delta L_1}{\delta(\partial \boldsymbol{\chi}(\boldsymbol{x},t)/\partial t)}, \frac{\partial \boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t}\Big\rangle d\boldsymbol{x}\right]\delta t\Bigg|_{t_0}^{t_f} + \left[\int_D\Big\langle \frac{\delta L_1}{\delta(\partial \boldsymbol{\chi}(\boldsymbol{x},t)/\partial t)}, \delta\boldsymbol{\chi}(\boldsymbol{x},t)\Big\rangle d\boldsymbol{x}\right]\Bigg|_{t_0}^{t_f} \\
&\qquad + \int_D \left\langle \frac{\delta J_f}{\delta \boldsymbol{\chi}(\boldsymbol{x},t_f)}, \delta\boldsymbol{\chi}(\boldsymbol{x},t_f)\right\rangle d\boldsymbol{x} = 0.
\end{aligned}
\tag{82}
$$

From the expressions of L_1 and J_f, see (23) and (11), and t_0, t_f and $\boldsymbol{\chi}(\boldsymbol{x},t_0)$ are fixed for our current problem, the equations in (82) with the use of (81) are equivalent to

$$\begin{aligned}
&\frac{\partial \boldsymbol{\lambda}(\boldsymbol{x},t)}{\partial t} = -\boldsymbol{A}^*_{\boldsymbol{x}}\boldsymbol{\lambda}(\boldsymbol{x},t) - \int_D \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\chi}(\boldsymbol{y},t)d\boldsymbol{y},\\
&\boldsymbol{\beta}^*_{\boldsymbol{\xi}}\boldsymbol{\lambda}(\boldsymbol{\xi},t) = 0,\\
&\boldsymbol{\lambda}(\boldsymbol{x},t_f) = \int_D \boldsymbol{Q}_f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\chi}(\boldsymbol{y},t_f)d\boldsymbol{y}.
\end{aligned} \tag{83}$$

Similarly, we can derive the weak variation δJ_1 with respect to $\boldsymbol{\lambda}(\boldsymbol{x},t)$, $\boldsymbol{u}_d(\boldsymbol{x},t)$ and $\boldsymbol{u}_b(\boldsymbol{\xi},t)$ to obtain

$$\begin{aligned}
&\frac{\delta L_1}{\delta \boldsymbol{\lambda}(\boldsymbol{x},t)} - \frac{\partial}{\partial t}\left[\frac{\delta L_1}{\delta(\partial \boldsymbol{\lambda}(\boldsymbol{x},t)/\partial t)}\right] = 0 \;\Rightarrow\; \frac{\partial \boldsymbol{\chi}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{\chi}(\boldsymbol{x},t) + \boldsymbol{B}_d(\boldsymbol{x},t)\boldsymbol{u}_d(\boldsymbol{x},t),\\
&\frac{\delta L_1}{\delta \boldsymbol{\lambda}(\boldsymbol{\xi},t)} = 0 \qquad\qquad\qquad\qquad\qquad \Rightarrow\; \boldsymbol{\beta}_{\boldsymbol{\xi}}\boldsymbol{\chi}(\boldsymbol{\xi},t) = \boldsymbol{B}_b(\boldsymbol{\xi},t)\boldsymbol{u}_b(\boldsymbol{\xi},t),\\
&\frac{\delta L_1}{\delta \boldsymbol{u}_d(\boldsymbol{x},t)} - \frac{\partial}{\partial t}\left[\frac{\delta L_1}{\delta(\partial \boldsymbol{u}_d(\boldsymbol{x},t)/\partial t)}\right] = 0 \;\Rightarrow\; \boldsymbol{u}_d(\boldsymbol{x},t) = -\int_D \boldsymbol{R}^+_d(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{B}^T_d(\boldsymbol{y},t)\boldsymbol{\lambda}(\boldsymbol{y},t)d\boldsymbol{y},\\
&\frac{\delta L_1}{\delta \boldsymbol{u}_b(\boldsymbol{\xi},t)} - \frac{\partial}{\partial t}\left[\frac{\delta L_1}{\delta(\partial \boldsymbol{u}_b(\boldsymbol{\xi},t)/\partial t)}\right] = 0 \;\Rightarrow\; \boldsymbol{u}_b(\boldsymbol{\xi},t) = -\int_S \boldsymbol{R}^+_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)\boldsymbol{B}^T_b(\boldsymbol{\gamma},t)\boldsymbol{\lambda}(\boldsymbol{\gamma},t)dS_{\boldsymbol{\gamma}},
\end{aligned} \tag{84}$$

where $\boldsymbol{R}^+_d(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{R}^+_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)$ are the generalized inverses of $\boldsymbol{R}_d(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{R}_b(\boldsymbol{\xi},\boldsymbol{\gamma},t)$, respectively. The equations (83) and (84) together with the initial condition $\boldsymbol{\chi}(\boldsymbol{x},t_0)$ in (1) result in the set of Euler-Lagrange equations (24).

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