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On ϕ_0-stability of a class of singular difference equations

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Abstract

This paper investigates a class of singular difference equations. Using the framework of the theory of singular difference equations and cone-valued Lyapunov functions, some necessary and sufficient conditions on the ϕ_0-stability of a trivial solution of singular difference equations are obtained. Finally, an example is provided to illustrate our results.

MSC: 39A11

Keywords: singular difference equations; cone-valued Lyapunov functions; ϕ_0-stability; uniformly ϕ_0-stability

1 Introduction

The singular difference equations (SDEs), which appear in the Leontiev dynamic model of multisector economy, the Leslie population growth model, singular discrete optimal control problems and so forth, have gained more and more importance in mathematical models of practical areas (see [1] and references cited therein). Anh and Loi [2, 3] studied the solvability of initial-value problems as well as boundary-value problems for SDEs.

It is well known that stability is one of the basic problems in various dynamical systems. Many results on the stability theory of difference equations are presented, for example, by Agarwal [4], Elaydi [5], Halanay and Rasvan [6], Martynjuk [7] and Diblík *et al.* [8]. Recently, Anh and Hoang [9] obtained some necessary and sufficient conditions for the stability properties of SDEs by employing Lyapunov functions. The comparison method, which combines Lyapunov functions and inequalities, is an effective way to discuss the stability of dynamical systems. However, this approach requires that the comparison system satisfies a quasimonotone property which is too restrictive for many applications because this property is not a necessary condition for a desired property like stability of the comparison system. To solve this problem, Lakshmikantham and Leela [10] initiated the method of cone and cone-valued Lyapunov functions and developed the theory of differential inequalities. By employing the method of cone-valued Lyapunov functions, Akpan and Akinyele [11], EL-Sheikh and Soliman [12], Wang and Geng [13] investigated the stability and the ϕ_0-stability of ordinary differential systems, functional differential equations and difference equations, respectively.

However, to the best of our knowledge, there are few results for the ϕ_0-stability of singular difference equations. In this paper, utilizing the framework of the theory of singular difference equations, we give some necessary and sufficient conditions for the ϕ_0-stability of a trivial solution of singular difference equations via cone-valued Lyapunov functions.

2 Preliminaries

The following definitions can be found in reference [10].

Definition 2.1 A proper subset K of R^n is called a cone if

(i) $\lambda K \subseteq K$, $\lambda \geq 0$;
(ii) $K + K \subseteq K$;
(iii) $K = \overline{K}$;
(iv) $K^0 \neq \emptyset$;
(v) $K \cap (-K) = \{0\}$,

where $\overline{K}$ and K^0 denote the closure and interior of K, respectively, and ∂K denotes the boundary of K. The order relation on $\mathbb{R}^n$ induced by the cone K is defined as follows: Let $x, y \in K$, then $x \leq_K y$ iff $y - x \in K$ and $x <_{K^0} y$ iff $y - x \in K^0$.

Definition 2.2 The set $K^* = \{\phi \in R^n, (\phi, x) \geq 0,$ for all $x \in K\}$ is said to be an adjoint cone if it satisfies the properties (i)-(v).

$$x \in K^0 \quad \text{iff} \quad (\phi, x) > 0,$$

and

$$x \in \partial k \quad \text{iff} \quad (\phi, x) = 0 \quad \text{for some } \phi \in K_0^*, K_0 = K - \{0\}.$$

Definition 2.3 A function $g : D \to R^n$, $D \subset R^n$ is said to be quasimonotone relative to K if $x, y \in D$ and $y - x \in \partial K$ implies that there exists $\phi_0 \in K_0^*$ such that

$$(\phi_0, y - x) = 0 \quad \text{and} \quad \big(\phi_0, g(y) - g(x)\big) \geq 0.$$

Definition 2.4 A function $a(r)$ is said to belong to the class $\mathcal{K}$ if $a \in C[R_+, R_+]$, $a(0) = 0$, and $a(r)$ is strictly monotone increasing function in r.

Consider the following SDEs:

$$A_n x_{n+1} + B_n x_n = f_n(x_n), \quad n \geq 0, \tag{2.1}$$

where $A_n, B_n \in R^{m\times m}$ and $f_n : R^m \to R^m$ are given. Throughout this paper, we assume that the matrices A_n are singular, and the corresponding linear homogeneous equations

$$A_n x_{n+1} + B_n x_n = 0, \quad n \geq 0, \tag{2.2}$$

are of index-1 [1–3], *i.e.*, the following hypotheses hold.

(H_1) $\operatorname{rank} A_n = r$, $n \geq 0$,
(H_2) $S_n \cap \ker A_{n-1} = \{0\}$, $n \geq 1$, where $S_n = \{\xi \in R^m : B_n \xi \in \operatorname{im} A_n\}$, $n \geq 0$.

For the next discussion, the following lemma from [9] is needed.

Lemma 2.1 *Suppose that the hypothesis* (H_1) *holds. Then the hypothesis* (H_2) *is equivalent to one of the following statements*:

(i) *the matrix* $G_n := A_n + B_n Q_{n-1,n}$ *is nonsingular*;
(ii) $R^m = S_n \oplus \ker A_{n-1}$.

Let us associate SDEs (2.1) with the initial condition

$$P_{n_0-1}x_{n_0} = P_{n_0-1}\gamma, \quad n \geq 0, \tag{2.3}$$

where γ is an arbitrary vector in R^m and n_0 is a fixed nonnegative integer.

Theorem 2.1 [9] *Let $f_n(x)$ be a Lipschitz continuous function with a sufficiently small Lipschitz coefficient, i.e.,*

$$\|f_n(x) - f_n(\widetilde{x})\| \leq L_n\|x - \widetilde{x}\|, \quad \forall x, \widetilde{x} \in R^m,$$

where

$$\omega_n := L_n\|Q_{n-1,n}G_n^{-1}\|, \quad \forall n \geq 0.$$

Then IVP (2.1), (2.3) *has a unique solution.*

Set $\triangle_n := \{x \in R^m : Q_{n-1}x = Q_{n-1,n}G_n^{-1}[f_n(x) - B_n P_{n-1}x]\}$. If $x = \{x_n\}$ is any solution of IVP (2.1), (2.3), then obviously $x_n \in \triangle_n$ $(n \geq n_0)$.

Definition 2.5 [9] The trivial solution of (2.1) is said to be A-stable (P-stable) if for each $\epsilon > 0$ and any $n_0 \geq 0$, there exists a $\delta = \delta(\epsilon, n_0) \in (0, \epsilon]$ such that

$$\|A_{n_0-1}\gamma\| < \delta \qquad \big(\|P_{n_0-1}\gamma\| < \delta\big) \quad \text{implies} \quad \|x_n(n_0;\gamma)\| < \epsilon, n \geq n_0.$$

Definition 2.6 The trivial solution of (2.1) is said to be

(S_1) A-ϕ_0-stable (P-ϕ_0-stable) if for each $\epsilon > 0$ and any $n_0 \geq 0$, there exists a $\delta = \delta(n_0, \epsilon) \in (0, \epsilon]$ such that for some $\phi_0 \in K_0^*$

$$(\phi_0, A_{n_0-1}\gamma) < \delta\big((\phi_0, P_{n_0-1}\gamma) < \delta\big) \quad \text{implies} \quad \big(\phi_0, x_n(n_0;\gamma)\big) < \epsilon, \quad n \geq n_0.$$

(S_2) A-uniformly ϕ_0-stable (P-uniformly ϕ_0-stable) if δ in (S_1) is independent of n_0.
(S_3) A-asymptotically ϕ_0-stable (P-asymptotically ϕ_0-stable) if for any $n_0 \geq 0$ there exist positive numbers $\delta_0 = \delta_0(n_0)$ and $N = N(n_0, \epsilon)$ such that for some $\phi_0 \in K_0^*$,

$$(\phi_0, A_{n_0-1}\gamma) < \delta_0 \qquad \big((\phi_0, P_{n_0-1}\gamma) < \delta_0\big) \quad \text{implies} \quad \big(\phi_0, x_n(n_0;\gamma)\big) < \epsilon, \quad n \geq n_0 + N.$$

(S_4) A-uniformly asymptotically ϕ_0-stable if δ_0 and N in (S_3) are independent of n_0.

Let K be a cone in R^m, $S_\rho = \{x_n \in R^m, \|A_{n-1}x_n\| < \rho, \rho > 0\}$. $V : Z_+ \times S_\rho \to K$ is continuous in the second variable, we define

$$\Delta V(n, A_{n-1}x_n) := V(n+1, A_n x_{n+1}) - V(n, A_{n-1}x_n),$$

where x_n is any solution of system (2.1).

3 Main results

Lemma 3.1 *The trivial solution of SDEs* (2.1) *is A-uniformly ϕ_0-stable* (*P-uniformly ϕ_0-stable*) *if and only if there exists a function $\psi \in \mathcal{K}$ such that for any solution x_n of SDEs* (2.1) *and some $\phi_0 \in K_0^*$, the following inequality holds*:

$$\begin{aligned} &(\phi_0, x_n) \le \psi\bigl[(\phi_0, A_{n_0-1}x_{n_0})\bigr], \quad n \ge n_0 \\ &\bigl((\phi_0, x_n) \le \psi\bigl[(\phi_0, P_{n_0-1}x_{n_0})\bigr], n \ge n_0\bigr). \end{aligned} \tag{3.1}$$

Proof For each positive ϵ, choose $\delta = \delta(\epsilon) \in (0, \epsilon]$ such that $\psi(\delta) < \epsilon$. If x_n is an arbitrary solution of (2.1) and $(\phi_0, A_{n_0-1}x_{n_0}) < \delta$, then

$$(\phi_0, x_n) \le \psi\bigl[(\phi_0, A_{n_0-1}x_{n_0})\bigr] < \psi(\delta) < \epsilon, \quad n \ge n_0.$$

Then (2.1) is A-uniformly ϕ_0-stable.

Conversely, suppose that the trivial solution of (2.1) is A-uniformly ϕ_0-stable, *i.e.*, for each positive ϵ, there exists a $\delta = \delta(\epsilon) \in (0, \epsilon]$ such that if x_n is any solution of (2.1) which satisfies the inequality $(\phi_0, A_{n_0-1}x_{n_0}) < \delta$, then $(\phi_0, x_n) < \epsilon$ for all $n \ge n_0$. Denote by $\alpha(\epsilon)$ the supremum of for the above $\delta(\epsilon)$. Obviously, if $(\phi_0, A_{n_0-1}x_{n_0}) < \alpha(\epsilon)$ for some n_0, then $(\phi_0, x_n) < \epsilon$ for all $n \ge n_0$. Furthermore, the function $\alpha(\epsilon)$ is positive and increasing, and $\alpha(\epsilon) \le \epsilon$. Considering a function $\beta(\epsilon)$ defined by $\beta(\epsilon) := \frac{1}{\epsilon}\int_0^\epsilon \alpha(t)\,dt$ and $\beta(0) := 0$, it is easy to prove that $\beta \in \mathcal{K}$ and $0 < \beta(\epsilon) < \alpha(\epsilon) \le \epsilon$. Then the inverse of β, denoted by ψ will belong to $\mathcal{K}$. For some $\phi_0 \in K_0^*$, set $\epsilon_n := (\phi_0, x_n)$ and consider two possibilities: (i) If $(\phi_0, x_n) = 0$, then $(\phi_0, x_n) = 0 \le \psi[(\phi_0, A_{n_0-1}x_{n_0})]$; (ii) If for some $(\phi_0, A_{n_0-1}x_{n_0}) < \beta(\epsilon_n)$, in which $\epsilon_n := (\phi_0, x_n) > 0$, then $(\phi_0, x_n) < \epsilon_n = (\phi_0, x_n)$, which is impossible, hence $(\phi_0, A_{n_0-1}x_{n_0}) \ge \beta(\epsilon_n)$, therefore, for some $\phi_0 \in K_0^*$,

$$(\phi_0, x_n) = \epsilon_n \le \beta^{-1}\bigl[(\phi_0, A_{n_0-1}x_{n_0})\bigr] = \psi\bigl[(\phi_0, A_{n_0-1}x_{n_0})\bigr],$$

the proof of Lemma 3.1 is complete. □

Similar to the proof of Lemma 3.1, we have the following.

Lemma 3.2 *The trivial solution of SDEs* (2.1) *is A-ϕ_0-stable* (*P-ϕ_0-stable*) *if and only if there exist functions $\psi_n \in \mathcal{K}$ such that for any solution x_n of* (2.1), *each nonnegative integer n_0 and some $\phi_0 \in K_0^*$, the following inequality holds*:

$$\begin{aligned} &(\phi_0, x_n) \le \psi_{n_0}\bigl[(\phi_0, A_{n_0-1}x_{n_0})\bigr], \quad n \ge n_0 \\ &\bigl((\phi_0, x_n) \le \psi_{n_0}\bigl[(\phi_0, P_{n_0-1}x_{n_0})\bigr], n \ge n_0\bigr). \end{aligned} \tag{3.2}$$

Theorem 3.1 *Assume that*

(i) $V \in C[Z_+ \times S_\rho, K]$, $V(n, 0) = 0$, *$V(n, r)$ is locally Lipschitzian in r relative to K, and for each $(n, r) \in Z_+ \times S_\rho$,*

$$\Delta V(n, A_{n-1}x_n) \le 0;$$

(ii) *$f \in C[K, R^m]$ is quasimonotone in x_n relative to K;*

(iii) $a[(\phi_0, x_n)] \le (\phi_0, V(n, A_{n-1}x_n))$ *for some* $\phi_0 \in K_0^*$ *and* $a \in \mathcal{K}$, $(n,r) \in R_+ \times S_\rho$.
Then the trivial solution of SDEs (2.1) *is* A-ϕ_0-*stable*.

Proof Since $V(n,0) = 0$ and $V(n,r)$ is continuous in r, then given $a_1(\epsilon) > 0$, $a_1 \in \mathcal{K}$, there exists δ_1 such that

$$\|A_{n_0-1}\gamma\| < \delta_1 \quad \text{implies} \quad \left\|V(n_0, A_{n_0-1}\gamma)\right\| < a_1(\epsilon).$$

For some $\phi_0 \in K_0^*$,

$$\|\phi_0\|\|A_{n_0-1}\gamma\| < \|\phi_0\|\delta_1 \quad \text{implies} \quad \|\phi_0\|\left\|V(n_0, A_{n_0-1}\gamma)\right\| < \|\phi_0\|a_1(\epsilon).$$

Thus

$$\left|(\phi_0, A_{n_0-1}\gamma)\right| \le \|\phi_0\|\|A_{n_0-1}\gamma\| < \|\phi_0\|\delta_1$$

implies

$$\left|\left(\phi_0, V(n_0, A_{n_0-1}\gamma)\right)\right| \le \|\phi_0\|\left\|V(n_0, A_{n_0-1}\gamma)\right\| < \|\phi_0\|a_1(\epsilon).$$

It follows that

$$(\phi_0, A_{n_0-1}\gamma) \le \delta \quad \text{implies} \quad \left(\phi_0, V(n_0, A_{n_0-1}\gamma)\right) \le a(\epsilon),$$

where $\|\phi_0\|\delta_1 = \delta$, $\|\phi_0\|a_1(\epsilon) = a(\epsilon)$, $a \in \mathcal{K}$. Let x_n be any solution of (2.1) such that $(\phi_0, A_{n_0-1}\gamma) < \delta$. Then by (i), V is nonincreasing and so

$$V(n, A_{n-1}x_n) \le V(n_0, A_{n_0-1}\gamma), \quad n \ge n_0.$$

Thus $(\phi_0, A_{n_0-1}\gamma) < \delta$ implies

$$a\left[(\phi_0, x_n)\right] \le \left(\phi_0, V(n, A_{n-1}x_n)\right) \le \left(\phi_0, V(n_0, A_{n_0-1}\gamma)\right) < a(\epsilon),$$

i.e.,

$$(\phi_0, A_{n_0-1}\gamma) < \delta \quad \text{implies} \quad (\phi_0, x_n) < \epsilon, \quad n \ge n_0.$$

Then the trivial solution of (2.1) is A-ϕ_0-stable. The proof of Theorem 3.1 is complete. □

Theorem 3.2 *Let the hypotheses of Theorem* 3.1 *be satisfied*, *except the condition* $\Delta V(n, A_{n-1}x_n) \le 0$ *being replaced by*
(iv) $(\phi_0, \Delta V(n, A_{n-1}x_n)) \le -c[(\phi_0, V(n, A_{n-1}x_n))]$, $c \in \mathcal{K}$.
Then the trivial solution of SDEs (2.1) *is* A-*asymptotically* ϕ_0-*stable*.

Proof By Theorem 3.2, the trivial solution of (2.1) is A-ϕ_0-stable. By (iv), $V(n, A_{n-1}x_n)$ is a monotone decreasing function, thus the limit

$$V^* = \lim_{n\to\infty} V(n, A_{n-1}x_n)$$

exists. We prove that $V^* = 0$. Suppose $V^* \neq 0$, then $c(V^*) \neq 0$, $c \in \mathcal{K}$. Since $c(r)$ is a monotone increasing function, then

$$c\big[\big(\phi_0, V(n, A_{n-1}x_n)\big)\big] > c\big[\big(\phi_0, V^*\big)\big],$$

and so from (iv), we get

$$\big(\phi_0, \Delta V(n, A_{n-1}x_n)\big) \le -c\big[\big(\phi_0, V^*\big)\big].$$

Then

$$\big(\phi_0, V(n, A_{n-1}x_n)\big) \le -c\big[\big(\phi_0, V^*\big)\big](n - n_0) + \big(\phi_0, V(n_0, A_{n_0-1}x_{n_0})\big).$$

Thus as $n \to \infty$ and for some $\phi_0 \in K_0^*$, we have $(\phi_0, V(n, A_{n-1}x_n)) \to -\infty$. This contradicts the condition (iii). It follows that $V^* = 0$. Thus

$$\big(\phi_0, V(n, A_{n-1}x_n)\big) \to 0, \quad n \to \infty,$$

and so with (iii)

$$(\phi_0, x_n) \to 0, \quad n \to \infty.$$

Thus for given $\epsilon > 0$, $n_0 \in R_+$, there exist $\delta = \delta(n_0)$ and $N = N(n_0, \epsilon)$ such that

$$(\phi_0, A_{n_0-1}x_{n_0}) < \delta_0 \quad \text{implies} \quad (\phi_0, x_n) < \epsilon, \quad n \ge n_0 + N.$$

Then the trivial solution of (2.1) is A-asymptotically ϕ_0-stable. The proof of Theorem 3.2 is complete. □

Theorem 3.3 *The trivial solution of SDEs* (2.1) *is P-ϕ_0-stable if and only if there exist functions $\psi_n \in \mathcal{K}$ and a Lyapunov function $V \in C[Z_+ \times S_\rho, K]$ such that for some $\phi_0 \in K_0^*$,*

(i) $V(n, 0) = 0$, $n \ge 0$;
(ii) $(\phi_0, y) \le (\phi_0, V(n, P_{n-1}y)) \le \psi_n[(\phi_0, P_{n-1}y)]$, $\forall y \in \triangle_n$, *and some* $\phi_0 \in K_0^*$,
(iii) $\Delta V(n, P_{n-1}y_n) := V(n+1, P_n y_{n+1} - V(n, P_{n-1}y_n)) \le 0$ *for any solution y_n of* (2.1).

Proof *Necessity*. Suppose that the trivial solution of (2.1) is P-ϕ_0-stable, then, according to Lemma 3.2, there exist functions $\psi_n \in \mathcal{K}$ $(n \ge 0)$ such that for any solution x_n of (2.1) and for some $\phi_0 \in K_0^*$,

$$(\phi_0, x_n) \le \psi_{n_0}\big[(\phi_0, P_{n_0-1}x_{n_0})\big]. \tag{3.3}$$

Define the Lyapunov function

$$V(n, \gamma) := \sup_{k \in Z_+} \big\| x_{n+k}(n; \gamma) \big\|; \quad \gamma \in R^m, n \in Z_+,$$

where $x_{n+k} := x_{n+k}(n; \gamma)$ is the unique solution of (2.1) satisfying the initial condition $P_{n-1}x_n = P_{n-1}\gamma$. Moreover, for some $\phi_0 \in K_0^*$, $(\phi_0, V(n, \gamma)) \le \psi_n(\phi_0, P_{n-1}\gamma)$, which implies

$V(n,0) = 0$ and the continuity of function V in the second variable at $\gamma = 0$. For each $y \in \triangle_n$, we have

$$V(n, P_{n-1}y) := \sup_{l \in Z_+} \|x_{n+l}(n; P_{n-1}y)\| \geq \|x_n(n; P_{n-1}y)\|,$$

where $x_k(n; P_{n-1}y)$ denotes the solution of (2.1) satisfying the initial condition $P_{n-1}x_n(n; P_{n-1}y) = P_{n-1}(P_{n-1}y) = P_{n-1}y$. Since $x_n, y \in \triangle_n$, it follows $x_n(n; P_{n-1}y) = y$, hence, for some $\phi_0 \in K_0^*$,

$$\left(\phi_0, V(n, P_{n-1}y)\right) \geq \left(\phi_0, x_n(n; P_{n-1}y)\right) = (\phi_0, y).$$

Further, the inequality (3.3) gives

$$\left(\phi_0, V(n, P_{n-1}y)\right) \leq \psi_n(\phi_0, P_{n-1}y).$$

On the other hand, for an arbitrary solution y_n of (2.1), by the unique solvability of the initial value problem (2.1) and (2.3), we have

$$V(n, P_{n-1}y_n) = \sup_{l \in Z_+} \|x_{n+l}(n; P_{n-1}y)\| = \sup_{l \geq 0} \|y_{n+l}\|.$$

Thus

$$\begin{aligned} V(n+1, P_n y_{n+1}) &= \sup_{l \geq 0} \|y_{n+l+1}\| = \sup_{l \geq 1} \|y_{n+l}\| \\ &\leq \sup_{l \geq 0} \|y_{n+l}\| = V(n, P_{n-1}y), \end{aligned}$$

hence $\Delta V(n, P_{n-1}y_n) \leq 0$. The necessity part is proved.

Sufficiency. Assuming that the trivial solution of (2.1) is not P-ϕ_0-stable, *i.e.*, there exist a positive ϵ_0 and a nonnegative integer n_0 such that for all $\delta \in (0, \epsilon_0]$ and for some $\phi_0 \in K_0^*$, there exists a solution of (2.1) satisfying the inequalities $(\phi_0, P_{n_0-1}x_{n_0}) < \delta$ and $(\phi_0, x_{n_1}) \geq \epsilon_0$ for some $n_1 \geq n_0$.

Since $V(n_0, 0) = 0$ and $V(n_0, \gamma)$ is continuous at $\gamma = 0$, there exists a $\delta_0' = \delta_0'(\epsilon, n_0) > 0$ such that for all $\xi \in R^m$, $\|\xi\| < \delta_0'$, we have $V(n_0, \xi) < \epsilon_0$. Choosing $\delta_0 \leq \min\{\delta_0', \epsilon_0\}$, we can find a solution x_n of (2.1) satisfying $(\phi_0, P_{n_0-1}x_{n_0}) \leq \delta_0$. However, $(\phi_0, x_{n_1}) \geq \epsilon_0$ for some $n_1 \geq n_0$. Since $(\phi_0, P_{n_0-1}x_{n_0}) < \delta_0 \leq \delta_0'$, we get

$$\left(\phi_0, V(n_0, P_{n_0-1}x_{n_0})\right) < \epsilon_0$$

for some $n_1 \geq n_0$. On the other hand, using the properties (iii) of the function V, we find

$$\left(\phi_0, V(n_0, P_{n_0-1}x_{n_0})\right) \geq \left(\phi_0, V(n_1, P_{n_1-1}x_{n_1})\right) \geq (\phi_0, x_{n_1}) \geq \epsilon_0,$$

which leads to a contradiction. The proof of Theorem 3.3 is complete. □

Theorem 3.4 *The trivial solution of SDEs* (2.1) *is P-uniformly* ϕ_0*-stable if and only if there exist functions* $a, b \in \mathcal{K}$ *and a Lyapunov function* $V \in C[Z_+ \times S_\rho, K]$ *such that for some* $\phi_0 \in K_0^*$,

(i) $a[(\phi_0, x)] \le (\phi_0, V(n, P_{n-1}x)) \le b[(\phi_0, P_{n-1}x)]$, $\forall x \in \triangle_n$, $n \ge 0$;
(ii) $\Delta V(n, P_{n-1}x_n) \le 0$ *for any solution* x_n *of* (2.1).

Proof The proof of the necessity part is similar to the corresponding part of Theorem 3.3.

For $\epsilon > 0$, let $\delta = b^{-1}[a(\epsilon)]$ independent of n_0 for $a, b \in \mathcal{K}$, and x_n be any solution of (2.1) such that $(\phi_0, P_{n_0-1}x_{n_0}) < \delta$. Then by (ii), V is nonincreasing and so

$$\big(\phi_0, V(n, P_{n-1}x_n)\big) \le \big(\phi_0, V(n_0, P_{n_0-1}x_{n_0})\big), \quad n \ge n_0.$$

Thus

$$\begin{aligned} a\big[(\phi_0, x_n)\big] &\le \big(\phi_0, V(n, P_{n-1}x_n)\big) \le \big(\phi_0, V(n_0, P_{n_0-1}x_{n_0})\big) \\ &\le b\big[(\phi_0, P_{n_0-1}x_{n_0})\big] < b(\delta) < a(\epsilon), \end{aligned}$$

i.e.,

$$(\phi_0, P_{n_0-1}x_{n_0}) < \delta \quad \text{implies} \quad (\phi_0, x_n) < \epsilon, \quad n \ge n_0.$$

Then the trivial solution of (2.1) is P-uniformly ϕ_0-stable. The proof of Theorem 3.4 is complete. □

4 Example

Consider SDEs (2.1) with the following data:

$$A_n = \begin{pmatrix} n+3 & 0 \\ 0 & 0 \end{pmatrix}; \qquad B_n = \begin{pmatrix} 1 & 0 \\ 0 & n+2 \end{pmatrix}, \quad n \ge -1,$$

and

$$f_n(x) = \frac{1}{n+2}(0,1)^T; \qquad x = (x_1, x_2)^T, \quad n \ge 0.$$

As $\ker A_n = \operatorname{span}\{(0,1)^T\}$, $\operatorname{im} A_n = \operatorname{span}\{(1,0)^T\}$, $n \ge -1$ and $S_n = \operatorname{span}\{(1,0)^T\}$, $n \ge 0$, the hypotheses (H_1), (H_2) are fulfilled, hence SDEs (2.2) is of index-1. Clearly, the canonical projections are

$$P_n = P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \qquad Q_n = Q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

therefore

$$Q_{n-1,n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = Q; \qquad G_n = A_n + B_n Q_{n-1,n} = \begin{pmatrix} n+3 & 0 \\ 0 & n+2 \end{pmatrix},$$

hence $G_n^{-1} = \begin{pmatrix} \frac{1}{n+3} & 0 \\ 0 & \frac{1}{n+2} \end{pmatrix}$. Further, the function $f_n(x)$ is Lipschitz continuous with the Lipschitz coefficient $L_n = (n+2)^{-1}$. Moreover, $f_n(0) = 0$ and $\omega_n := L_n \|Q_{n-1,n}G_n^{-1}\| < 1$. According to Theorem 2.1, IVP (2.1), (2.3) has a unique solution. We have $x \in \triangle_n$ if and only if

$$Q_{n-1}x = Q_{n-1,n}G_n^{-1}\{f_n(x) - B_nP_{n-1}x\},$$

it leads to $x_2 = \frac{x_1}{(n+2)(n+3)}$. Thus

$$\triangle_n = \left\{x = (x_1, x_2)^T : x_2 = \frac{x_1}{(n+2)(n+3)}\right\}, \quad n \geq 0.$$

Let $V(n, \gamma) := 2\|x\|$, we get for each $x \in \triangle_n$,

$$\|x\| = \left(x_1^2 + \frac{x_1^2}{(n+2)^2(n+3)^2}\right)^{\frac{1}{2}} \leq 2|x_1| = 2\|P_{n-1}x\|.$$

Further, $V(n, P_{n-1}x) = 2\|P_{n-1}x\| = 2|x_1|$. Thus, for some $\phi_0 \in K_0^*$,

$$a\big[(\phi_0, x)\big] \leq \big(\phi_0, V(n, P_{n-1}x)\big) \leq b\big[(\phi_0, P_{n-1}x)\big],$$

where $a, b \in \mathcal{K}$ and $a(r) = r$, $b(r) = 2r$. Suppose that x_n is a solution of (2.1) and putting $u_n = P_{n-1}x_n = Px_n$, $v_n = Q_{n-1}x_n = Qx_n$, then we have

$$\begin{aligned}\triangle V(n, P_{n-1}x_n) &= V(n+1, P_nx_{n+1}) - V(n, P_{n-1}x) \\ &= 2\big(\|Px_{n+1}\| - \|Px_n\|\big) = 2\big(\|u_{n+1}\| - \|u_n\|\big).\end{aligned}$$

Using equation (2.8) in [9], we find

$$u_{n+1} = -P_nG_n^{-1}B_nu_n + P_nG_n^{-1}f_n(x_n) = -\frac{1}{n+3}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}u_n,$$

hence $\|u_{n+1}\| - \|u_n\| = -\frac{n+2}{n+3}\|u_n\| \leq \frac{1}{2}\|u_n\|$, then

$$\triangle V(n, P_{n-1}x_n) \leq -\|P_{n-1}x_n\|.$$

According to Theorem 3.4, the trivial solution of (2.1) is P-uniformly ϕ_0-stable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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