Asymptotic Expansions for High-Contrast Linear Elasticity

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Abstract

We study linear elasticity problems with high contrast in the coefficients using asymptotic limits recently introduced. We derive an asymptotic expansion to solve heterogeneous elasticity problems in terms of the contrast in the coefficients. We study the convergence of the expansion in the H^1 norm.
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1. Introduction

There is a growing interest in the computation of solutions of problems governed by partial differential equations with high-contrast coefficients. Solutions to these model problems are multiscale in nature. The solutions to these problems are often approximated using the Finite Element Method

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(FEM), Multiscale Finite Element Method (MsFEM) or alternative forms of these, (cf., Yang and Liu (1997); Wihler (2004); Efendiev and Hou (2009); Gatica et al. (2009); Di Pietro and Nicaise (2013); Xia et al. (2014). and references therein).

Herein, we study linear elasticity problems in heterogeneous media. Our goal is to devise approximate solutions that account for the high contrast in the coefficients. We focus on the dependence of the contrast in the coefficients where the contrast is referred to as the ratio of the jumps in the physical properties. We follow the analysis presented in Calo et al. (2014) that consists of deriving an asymptotic expansion for the solution of the elliptic differential equation in heterogeneous media. Thus, we derive asymptotic expansions to solve linear elasticity problems with high contrast.

The linear elasticity equations model the equilibrium and the local strain of deformable bodies; see Ciarlet (1988, 1997); Love (1944); Sokolnikoff (1956); Kang and Zhong-Ci (1996). The constitutive laws relating stresses and strains depend on the material and the process modeled. For composite materials, physical properties such as the Young's modulus can vary several orders of magnitude and we seek to understand the effects of these variations on the solution. In this setting, the asymptotic expansions that express the solution are useful tools to understand the effects of the high contrast and the interactions between different materials.

We consider the equilibrium equations for a linear elastic material in a smooth domain $D \subset \mathbb{R}^d$. Given $u \in H^1(D)^d$ that represents the displacement field, we denote

$$\epsilon = \epsilon(u) = \left[\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right],$$

where ϵ is the strain tensor which linearly depends on the derivatives of the displacement field u (see Gonzalez and Stuart (2008); Malvern (1969)). We also introduce the stress tensor $\tau(u)$, which depends on the value of strains and is defined as

$$\tau = \tau(u) = 2\mu\epsilon(u) + \lambda \mathrm{tr}\epsilon(u) I_{d\times d}, \tag{1}$$

where $I_{d\times d}$ is the identity matrix in $\mathbb{R}^d$ and $\mathrm{tr}\epsilon(u) = \mathrm{div}(u)$. The Lamé coefficients λ and μ describe the elastic response of an isotropic material, see, e.g., Gonzalez and Stuart (2008); Kang and Zhong-Ci (1996).

We assume that the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is bounded away from 0.5, i.e., the Poisson ratio satisfies $0 < \nu \leq \nu_0 < 0.5$ for some constant value ν_0. The volumetric strain modulus is given by

$$K = \frac{E}{3(1-2\nu)} > 0,$$

and thus $1 - 2\nu > 0$ (see Kang and Zhong-Ci (1996)). Given these assumptions, then $\nu = \nu(x)$ can only have mild variations in D.

We introduce the heterogeneous function $E = E(x)$ that represents the Young's modulus and thus express the shear modulus as

$$\mu(x) = \frac{1}{2}\frac{E(x)}{1+\nu(x)} = \tilde{\mu}(x)E(x),$$

where $\tilde{\mu} = 1/2(1+\nu)$. Thus,

$$\lambda(x) = \frac{1}{2}\frac{E(x)\nu(x)}{(1+\nu(x))(1-2\nu(x))} = \tilde{\lambda}(x)E(x),$$

where we use $\tilde{\lambda} = \frac{\nu}{2(1+\nu)(1-2\nu)}$. The spatial variation of E drives the multiscale response of the solution. We denote

$$\tilde{\tau}(u) = 2\tilde{\mu}\epsilon(u) + \tilde{\lambda}\mathrm{tr}\epsilon(u)I_{d\times d}.$$

Given a vector field f we consider the problem

$$-\operatorname{div}(\tau(u)) = f, \quad \text{in } D, \tag{2}$$

with $u = g$ on ∂D. The tensor τ is defined in (1). We analyze in detail a binary medium $E(x)$ with elastic background and one inclusion (a stiff body) for the case of one inelastic inclusion. The analysis of the case with several highly inelastic inclusions is similar. To parametrize the problem, we consider the background with stiffness 1 and the inclusions with a relative stiffness denoted by η. We derive expansions of the form (see Calo et al. (2014))

$$u_\eta = u_0 + \frac{1}{\eta}u_1 + \frac{1}{\eta^2}u_2 + \cdots . \tag{3}$$

We define each term in the expansion using local problems. We then study the convergence of the expansion in the H^1 norm.

The rest of the paper is organized as follows. In Section 2 we recall the weak formulation and provide an overview of the derivation of the expansion for high contrast inclusions. In Section 3, the convergence for this asymptotic expansion is described. Finally, in Section 4 we state our conclusions and final comments.

2. One interior inclusion problem: problem statement

Let $D \subset \mathbb{R}^d$ be a polygonal domain or a domain with smooth boundary. We consider the following weak formulation of (2). Find $u \in H^1(D)^d$ such that

$$\begin{cases} \mathcal{A}(u,v) = \mathcal{F}(v), & \text{for all } v \in H_0^1(D)^d, \\ u = g, & \text{on } \partial D, \end{cases} \tag{4}$$

where the bilinear form $\mathcal{A}$ and the linear functional $\mathcal{F}$ are defined by

$$\mathcal{A}(u,v)=\int_D 2\tilde{\mu}E\epsilon(u)\cdot\epsilon(v)+\tilde{\lambda}E\mathrm{tr}\epsilon(u)\mathrm{tr}\epsilon(v), \text{ for all } u,v\in H_0^1(D)^d, \quad (5)$$

and

$$\mathcal{F}(v)=\int_D fv, \text{ for all } v\in H_0^1(D)^d, \quad (6)$$

respectively, with $\epsilon(u)\cdot\epsilon(v):=\sum_{i,j=1}^d \epsilon_{ij}(u)\epsilon_{ij}(v)$.

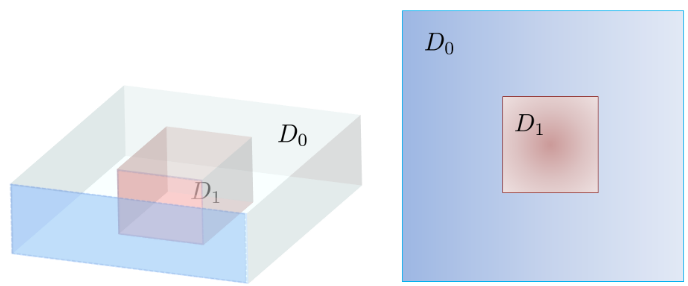

Figure 1: Geometric configuration with one interior inclusion.

The domain D is the disjoint union of a background domain and one inclusion, that is, $D=D_0\cup\overline{D}_1$. We assume that D_0 and D_1 are polygonal domains or domains with smooth boundaries. Let D_0 represent the background domain and the sub-domain D_1 represent inclusion. For simplicity of the presentation we consider only one interior inclusion. Given $w\in H^1(D)^d$ we use the notation $w^{(m)}$, for the restriction of w to the domain D_m, that is

$$w^{(m)}=w|_{D_m}, \quad m=0,1.$$

We also introduce the following notation, given $\Omega\subset D$, we denote by $\mathcal{A}_\Omega$ the bilinear form

$$\mathcal{A}_\Omega(u,v)=\int_\Omega 2\tilde{\mu}\epsilon(u)\cdot\epsilon(v)+\tilde{\lambda}\mathrm{tr}\epsilon(u)\mathrm{tr}\epsilon(v),$$

defined for functions in $u, v \in H^1(\Omega)^d$. If $\Omega \subseteq D_m$, then $\mathcal{A}_\Omega$ does not depend on the Young's modulus $E(x)$, since $E(x)$ is assumed to be defined by piecewise constants. We denote by $\mathcal{R}B(\Omega)$ the subset of *rigid body motions* defined on Ω, for instance, if $d = 2$ we have that

$$\mathcal{R}B(\Omega) = \{(a_1, a_2) + b(x_2, -x_1) : \; a_1, a_2, b \in \mathbb{R}\}, \tag{7}$$

or if $d = 3$ we have

$$\mathcal{R}B(\Omega) = \{(a_1, a_2, a_3) + (b_1, b_2, b_3) \times (x_1, x_2, x_3) : \; a_i, b_i \in \mathbb{R}, i = 1, 2, 3\}. \tag{8}$$

3. One interior inclusion problem: series expansion

We derive and analyze the asymptotic expansion for the case of a single highly elastic inclusion. We follow Calo et al. (2014).

3.1. Derivation

Let E be defined by

$$E(x) = \begin{cases} \eta, & x \in D_1, \\ 1, & x \in D_0 = D \setminus \overline{D}_1, \end{cases} \tag{9}$$

and denote by u_η the solution of the weak formulation (4). We assume that D_1 is compactly included in D ($\overline{D}_1 \subset D$). Since u_η is the solution of (4) with the coefficient (9), we have

$$\mathcal{A}_{D_0}(u_\eta, v) + \eta \mathcal{A}_{D_1}(u_\eta, v) = \mathcal{F}(v), \quad \text{for all } v \in H^1_0(D). \tag{10}$$

We seek to determine $\{u_j\}_{j=0}^\infty \subset H^1(D)^d$ such that

$$u_\eta = u_0 + \frac{1}{\eta} u_1 + \frac{1}{\eta^2} u_2 + \cdots = \sum_{j=0}^{\infty} \eta^{-j} u_j, \tag{11}$$

and such that they satisfy the following Dirichlet boundary conditions

$$u_0 = g \text{ on } \partial D \quad \text{and} \quad u_j = 0 \text{ on } \partial D \text{ for } j \geq 1. \tag{12}$$

We substitute (11) into (10) to obtain that for all $v \in H_0^1(D)$ we have

$$\eta \mathcal{A}_{D_1}(u_0, v) + \sum_{j=0}^{\infty} \eta^{-j} \Big(\mathcal{A}_{D_0}(u_j, v) + \mathcal{A}_{D_1}(u_{j+1}, v) \Big) = \mathcal{F}(v). \tag{13}$$

Now we collect terms with equal powers of η and analyze the resulting subdomain equations.

3.1.1. Term corresponding to η^1

In (13) there is one term corresponding to η to the power 1, thus we obtain the following equation

$$\mathcal{A}_{D_1}(u_0, v) = 0 \text{ for all } v \in H_0^1(D)^d. \tag{14}$$

The problem above corresponds to an elasticity equation posed on D_1 with homogeneous Neumann boundary conditions. Since we assume that $\overline{D}_1 \subset D$, we conclude that $u_0^{(1)}$ is a *rigid body motion*, that is, $u_0^{(1)} \in \mathcal{RB}(D_1)$ where $\mathcal{RB}$ is defined above in (7) and (8).

In the general case, the meaning of this equation depends on the relative position of the inclusion D_1 with respect to the boundary and thus may need to take the boundary data into account.

3.1.2. Terms corresponding to $\eta^0 = 1$

The equation (12) contains three terms corresponding to η to the power 0, which are

$$\mathcal{A}_{D_0}(u_0, v) + \mathcal{A}_{D_1}(u_1, v) = \mathcal{F}(v), \text{ for all } v \in H_0^1(D)^d. \tag{15}$$

Let

$$V_{\mathcal{RB}} = \{v \in H_0^1(D)^d, \text{ such that } v^{(1)} = v|_{D_1} \in \mathcal{RB}(D_1)\}.$$

If we consider $z \in V_{\mathcal{RB}}$ in equation (15) we conclude that u_0 satifies the following problem

$$\mathcal{A}_{D_0}(u_0, z) = \mathcal{F}(z), \qquad \text{for all } z \in V_{\mathcal{RB}}, \tag{16}$$

with $u_0 = g$ on ∂D. The problem (16) is elliptic and has a unique solution (for details see Ciarlet (1997)). To analyze this problem further we proceed as follows. Let $\{\xi_{D_1;\ell}\}_{\ell=1}^{L_d}$ be a basis for the $\mathcal{RB}(D_1)$ space, where L_d is the dimension of the space $\mathcal{RB}$, that is, $L_d = 3$ for $2D$ problems and $L_d = 6$ for $3D$ ones. Then we have that $u_0^{(1)} = \sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}$. We define the harmonic extension of the rigid body motions, $\chi_{D_1;\ell} \in H_0^1(D)^d$ such that

$$\chi_{D_1;\ell}^{(1)} = \xi_{D_1;\ell}, \quad \text{in } D_1,$$

while the harmonic extension of its boundary data in D_0 is given by

$$\begin{aligned} \mathcal{A}_{D_0}(\chi_{D_1;\ell}^{(0)}, z) &= 0, && \text{for all } z \in H_0^1(D_0)^d,\\ \chi_{D_1;\ell}^{(0)} &= \xi_{D_1;\ell}, && \text{on } \partial D_1,\\ \chi_{D_1;\ell}^{(0)} &= 0, && \text{on } \partial D. \end{aligned}$$

Remark 1. *Let w be a harmonic extension to D_0 of its Neumann data on ∂D_0. That is, w satisfies the following problem*

$$\mathcal{A}_{D_0}(w, v) = \int_{\partial D_0} \widetilde{\tau}(w) \cdot n_0 v \quad \text{for all } v \in H^1(D_0)^d,$$

with boundary data $\tilde{\tau}(w)\cdot n_0$ on ∂D_0. Since $\chi_{D_1;\ell} = 0$ in ∂D and $\chi_{D_1;\ell} = \xi_{D_1;\ell}$ on ∂D_1 with $\ell = 1, \ldots, L_d$. We readily have that

$$\mathcal{A}_{D_0}(w, \chi_{D_1;\ell}) = 0\left(\int_{\partial D} \widetilde{\tau}(w) \cdot n_1\right) + \left(\int_{\partial D_1} \widetilde{\tau}(w) \cdot n_0 \xi_{D_1;\ell}\right),$$

and we conclude that for every harmonic function on D_0

$$\mathcal{A}_{D_0}(w, \chi_{D_1;\ell}) = \int_{\partial D_1} \widetilde{\tau}(w) \cdot n_0 \xi_{D_1;\ell}. \qquad (17)$$

In particular, taking $w = \chi_{D_1;\ell}$ *we have*

$$\mathcal{A}_{D_0}(\chi_{D_1;\ell}, \chi_{D_1;\ell}) = \int_{\partial D_1} \widetilde{\tau}(\chi_{D_1;\ell}) \cdot n_0 \chi_{D_1;\ell}. \qquad (18)$$

To obtain an explicit formula for u_0 we use the fact that problem (16) is elliptic and has a unique solution, and the property of the harmonic characteristic functions described in the Remark 1. Thus, We can decompose u_0 into the harmonic extension of its value in D_1, given by $u_0^{(1)} = \sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}$, plus the remainder $u_{0,0} \in H^1(D_0)^d$. Thus, we write

$$u_0 = u_{0,0} + \sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}, \qquad (19)$$

where $u_{0,0} \in H^1(D)$ is defined by $u_{0,0}^{(1)} = 0$ in D_1 and $u_{0,0}^{(0)}$ solves the following Dirichlet problem

$$\begin{aligned} \mathcal{A}_{D_0}(u_{0,0}^{(0)}, y) &= \mathcal{F}(y), \text{ for all } y \in H_0^1(D_0)^d, \\ u_{0,0}^{(0)} &= 0, \text{ on } \partial D_1, \\ u_{0,0}^{(0)} &= g, \text{ on } \partial D. \end{aligned} \qquad (20)$$

From (16) and (19) we get that

$$\sum_{\ell=1}^{L_d} c_{0;\ell}\mathcal{A}_{D_0}(\chi_{D_1;\ell}, \chi_{D_1;m}) = \mathcal{F}(\chi_{D_1;m}) - \mathcal{A}_{D_0}(u_{0,0}, \chi_{D_1;m}), \qquad (21)$$

with $m = 1, \ldots, L_d$. From (21) we obtain the constants $c_{0;\ell}$, $\ell = 1, \ldots, L_d$ by solving a $L_d \times L_d$ linear system. As readily seen, the $L_d \times L_d$ matrix

$$\mathbf{A}_{geom} = [a_{\ell m}]_{\ell,m=1}^{L_d}, \quad \text{where } a_{\ell m} = \mathcal{A}_{D_0}(\chi_{D_1;\ell}, \chi_{D_1;m}). \qquad (22)$$

The matrix $\mathbf{A}_{geom}$ is positive. Given the explicit form of u_0, we use (19) in (15) to find $u_1^{(1)} = u_1|_{D_1}$ from the analysis of (15) we conclude that $u_0^{(0)}$ satisfies the local Dirichlet problem

$$\mathcal{A}_{D_0}(u_0^{(0)}, z) = \int_{D_0} fz, \text{ for all } z \in H_0^1(D_0)^d,$$

with given boundary data ∂D_0 in (20). Equation (15) also represents the transmission conditions across ∂D_1 for the functions $u_0^{(0)}$ and $u_1^{(1)}$. This is easier to see when the forcing f is square integrable. From now on, in order to simplify the presentation, we assume that $f \in L^2(D)$. If $f \in L^2(D)$, we have that $u_0^{(0)}$ and $u_1^{(1)}$ are the only solutions of the problems

$$\mathcal{A}_{D_0}(u_0^{(0)}, z) = \int_{D_0} fz + \int_{\partial D_0 \setminus \partial D} \widetilde{\tau}(u_0^{(0)}) \cdot n_0 z, \quad \text{for all } z \in H^1(D_0)^d,$$

with $z = 0$ on ∂D and $u_0^{(0)} = g$ on ∂D, and

$$\mathcal{A}_{D_1}(u_1^{(1)}, z) = \int_{D_1} fz + \int_{\partial D_1} \widetilde{\tau}(u_1^{(1)}) \cdot n_1 z, \quad \text{for all } z \in H^1(D_1)^d.$$

Replacing these last two equations back into (15) we conclude that

$$\widetilde{\tau}(u_1^{(1)}) \cdot n_1 = -\widetilde{\tau}(u_0^{(0)}) \cdot n_0, \quad \text{on } \partial D_1. \tag{23}$$

Using this interface condition we can obtain $u_1^{(1)}$ in D_1 by writing

$$u_1^{(1)} = \widetilde{u}_1^{(1)} + \sum_{\ell=1}^{L_d} c_{1;\ell} \xi_{D_1;\ell}, \tag{24}$$

where $\widetilde{u}_1^{(1)}$ solves the Neumann problem

$$\mathcal{A}_{D_1}(\widetilde{u}_1^{(1)}, z) = \int_{D_1} fz - \int_{\partial D_1} \widetilde{\tau}(u_0^{(0)}) \cdot n_1 z, \quad \text{for all } z \in H^1(D_1)^d. \tag{25}$$

where the constants $c_{1;\ell}$ are chosen later. Problem (25) needs the following compatibility conditions

$$\int_{D_1} f\xi + \int_{\partial D_1} \widetilde{\tau}(u_0^{(0)}) \cdot n_1 \xi = 0, \quad \text{for all } \xi \in \mathcal{RB},$$

which, using (19) and (23) and noting that $\chi_{D_1;\ell}$ in D_1, reduces to

$$\sum_{\ell=1}^{L_d} c_{0;\ell} \int_{\partial D_1} \widetilde{\tau}(\chi_{D_1;\ell}) \cdot n_1 \chi_{D_1;m} = \int_{D_1} f\chi_{D_1;m} - \int_{\partial D_1} \widetilde{\tau}(\widetilde{u}_{0,0}) \cdot n_1 \chi_{D_1;m} \quad (26)$$

for $m = 1, \ldots, L_d$. This system of L_d equations is the same encountered before in (21). The fact that the two systems are the same follows from the next two integration by parts relations:

(i) according to Remark 1

$$\int_{\partial D_1} \widetilde{\tau}(\chi_{D_1;\ell}) \cdot n_1 \chi_{D_1;m} = \mathcal{A}_{D_0}(\chi_{D_1;\ell}, \chi_{D_1;m}). \quad (27)$$

(ii) we have

$$\int_{\partial D_1} \widetilde{\tau}\left(\widetilde{u}_{0,0} \cdot n_1 \chi_{D_1;m}\right) = \mathcal{A}_{D_0}(u_{0,0}, \chi_{D_1;m}) - \int_{D_0} f\chi_{D_1;m}. \quad (28)$$

By replacing the relations in (27) and (28) into (26) we obtain (21) and conclude that the compatibility condition of problem (25) is satisfied. Next, we discuss how to compute $u_1^{(0)}$ and $\widetilde{u}_1^{(0)}$ to completely define the functions $u_1 \in H^1(D)^d$ and $\widetilde{u}_1 \in H^1(D)^d$. These are presented for general $j \geq 1$ since the construction is independent of j in this range.

3.1.3. Term corresponding to η^{-j} with $j \geq 1$

For powers $1/\eta$ larger or equal to one there are only two terms in the summation that lead to the following system

$$\mathcal{A}_{D_0}(u_j, v) + \mathcal{A}_{D_1}(u_{j+1}, v) = 0, \quad \text{for all } v \in H_0^1(D)^d. \quad (29)$$

This equation represents both the sub-domain problems and the transmission conditions across ∂D_1 for $u_j^{(0)}$ and $u_{j+1}^{(1)}$. Following a similar argument to the one given above, we conclude that $u_j^{(0)}$ is harmonic in D_0 for all $j \geq 1$ and that $u_{j+1}^{(1)}$ is harmonic in D_1 for $j \geq 2$. As before, we have

$$\widetilde{\tau}(u_{j+1}^{(1)}) \cdot n_1 = -\tau(u_j^{(0)}) \cdot n_0. \tag{30}$$

Since $u_j^{(1)}$ in D_1, (e.g., $u_1^{(1)}$ above) is given by the solution of a Neumann problem in D_1. The solution of a Neumann linear elasticity problem is defined up to a rigid body motion. To uniquely determine $u_j^{(1)}$, we write

$$u_j^{(1)} = \widetilde{u}_j^{(1)} + \sum_{\ell=1}^{L_d} c_{j;\ell}\xi_{D_1;\ell}, \tag{31}$$

where $u_j^{(1)}$ is L^2-orthogonal to the rigid body motion of D_1 and the appropriate $c_{j;\ell}$ is determined below. Given $u_j^{(1)}$ in D_1 we find $u_j^{(0)}$ in D_0 by solving a Dirichlet problem with known Dirichlet data, that is,

$$\begin{aligned}
&\mathcal{A}_{D_0}(u_j^{(0)}, z) = 0 \text{ for all } z \in H_0^1(D_0)^d \\
&u_j^{(0)} = u_j^{(1)} \quad \left(= \widetilde{u}_j^{(1)} + \sum_{\ell=1}^{L_d} c_{j;\ell}\xi_{D_1;\ell} \right) \text{ on } \partial D_1 \quad \text{ and } \quad u_j = 0 \text{ on } \partial D.
\end{aligned} \tag{32}$$

We conclude that

$$u_j = \widetilde{u}_j + \sum_{\ell=1}^{L_d} c_{j;\ell}\chi_{D_1;\ell}, \tag{33}$$

where $\widetilde{u}_j^{(0)}$ is defined by (32) replacing $c_{j;\ell}$ by 0. This completes the construction of u_j. Now we proceed to show how to find $u_{j+1}^{(1)}$ in D_1. For this, we use (28) and (29) which lead to the following Neumann problem

$$\mathcal{A}_{D_1}(\widetilde{u}_{j+1}^{(1)}, z) = -\int_{\partial D_1} \widetilde{\tau}(u_j^{(0)}) \cdot n_0 z \quad \text{for all } z \in H^1(D_1)^d. \tag{34}$$

The compatibility condition for this Neumann problem is satisfied if we choose $c_{j;\ell}$ the solution of the $L_d \times L_d$ system

$$\sum_{\ell=1}^{L_d} c_{j;\ell} \int_{\partial D_1} \widetilde{\tau}(\chi_{D_1;\ell} \cdot n_1 \chi_{D_1;m} = -\int_{\partial D_1} \widetilde{\tau}(\widetilde{u}_j) \cdot n_1 \chi_{D_1;m}, \tag{35}$$

with $m = 1, \ldots, L_d$. As pointed out before, see (27), this system can be written as

$$\sum_{\ell=1}^{L_d} c_{j;\ell} \mathcal{A}_{D_0}(\chi_{D_1;\ell}, \chi_{D_1;m}) = -\mathcal{A}_{D_0}(\widetilde{u}_j, \chi_{D_1;m}).$$

In this form we readily see that this $L_d \times L_d$ system matrix is positive definite and therefore solvable. We can choose $u_{j+1}^{(1)}$ in D_1 such that

$$u_{j+1}^{(1)} = \widetilde{u}_{j+1}^{(1)} + \sum_{\ell=1}^{L_d} c_{j+1;\ell} \xi_{D_1;\ell},$$

where $\widetilde{u}_{j+1}^{(1)}$ is properly chosen and, as before

$$\sum_{\ell=1}^{L_d} c_{j+1,\ell} \int_{\partial D_1} \widetilde{\tau}(\chi_{D_1;\ell}) \cdot \chi_{D_1;m} = -\int_{\partial D_1} \widetilde{u}_{j+1} \cdot n_1 \chi_{D_1;m},$$

$m = 1, \ldots, L_d$. Therefore we have the compatibility condition of the Neumann problem to compute $u_{j+2}^{(1)}$. See the equation (34).

3.2. Convergence in $H^1(D)^d$

We study the convergence of the expansion (11) with the Dirichlet data (12). We assume that ∂D and ∂D_1 are sufficiently smooth. We follow the analysis introduced in Calo et al. (2014).

We use standard Sobolev spaces (see for instance, Adams and Fournier (2003)). Given a sub-domain D, we use the $H^1(D)^d$ norm given by

$$\|v\|_{H^1(D)^d}^2 = \|v\|_{L^2(D)^d}^2 + \|\nabla v\|_{L^2(D)^d}^2,$$

and the seminorm

$$|v|^2_{H^1(D)^d} = \|\nabla v\|^2_{L^2(D)^d}.$$

We also use standard trace spaces $H^{1/2}(\partial D)^d$ and the dual space $H^{-1}(D)^d$.

Lemma 2. *Let $\widetilde{w} \in H^1(D)$ be harmonic in D_0 and define*

$$w = \widetilde{w} + \sum_{\ell=1}^{L_d} c_{0;\ell} \xi_{D_1;\ell},$$

where $Y = (c_{0;1}, \ldots, c_{0;L_d})$ is the solution of the L_d-dimensional linear system

$$\mathbf{A}_{geom} Y = -W, \tag{36}$$

where

$$\mathbf{A}_{geom} = [a_{\ell m}]_{\ell,m=1}^{L_d}, \quad \text{with } a_{\ell,m} = \mathcal{A}_{D_0}(\xi_{D_1;\ell}, \xi_{D_1;m}).$$

and

$$W = (\mathcal{A}_{D_0}(\widetilde{w}, \xi_{D_1;1}), \ldots, \mathcal{A}_{D_0}(\widetilde{w}, \xi_{D_1;L_d})), \tag{37}$$

we recall that $L_d = 1, \ldots, \ell$ is the spatial dimension of the subset of rigid body motions $\mathcal{RB}$. Then,

$$\|w\|_{H^1(D)^d} \preceq \|\widetilde{w}\|_{H^1(D)^d},$$

where the hidden constant is the Korn inequality constant of D.

Proof. Since $\sum_{\ell=1}^{L_d} c_{0;\ell} \xi_{D_1;\ell}$ is the Galerkin projection of $\widetilde{w}$ into the space

$\text{Span}\{\xi_{D_1;\ell}\}_{\ell=1}^{L_d}$. From the analysis of Galerkin formulations, we have

$$\begin{aligned}
\mathcal{A}_{D_0}\left(\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}, \sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right) &= Y^T\mathbf{A}_{geom}Y = -Y^TW \\
&= -\sum_{\ell=1}^{L_d} c_{0;\ell}\mathcal{A}_{D_0}(\widetilde{w}, \xi_{D_1;\ell}) \\
&= -\mathcal{A}_{D_0}\left(\widetilde{w}, \sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right) \\
&\leq |\widetilde{w}|_{H^1(D)^d}\left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|_{H^1(D_0)^d},
\end{aligned}$$

by the Korn inequality

$$\begin{aligned}
\left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|^2_{H^1(D_0)^d} &\leq C\mathcal{A}_{D_0}\left(\sum_{0;\ell}^{L_d} c_{0;\ell}\xi_{D_1;\ell}, \sum_{0;\ell}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right) \\
&\leq |\widetilde{w}|_{H^1(D_0)^d}\left|\sum_{\ell=1}^{L_d} C_{0;\ell}\xi_{D_1;\ell}\right|_{H^1(D_0)^d},
\end{aligned}$$

so

$$\left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|_{H^1(D_0)^d} \leq |\widetilde{w}|_{H^1(D_0)^d}.$$

Using the fact above, we get

$$\|w\|_{H^1(D)^d} \leq \|\widetilde{w}\|_{H^1(D)^d} + \left\|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right\|_{H^1(D)^d} \preceq \|\widetilde{w}\|_{H^1(D)^d}.$$

□

For the proof of the convergence of the expansion (11) with the boundary condition (12), we consider the following additional results obtained by applying the Lax-Milgram theorem in Brezis (2010) and the trace theorem in Adams and Fournier (2003).

Lemma 3. *Let u_0 in (19), with $u_{0,0}$ defined in (20), and u_1 be defined by (25) and (32) with $j = 1$. We have that*

$$\|u_0\|_{H^1(D)^d} \preceq \|f\|_{H^{-1}(D)^d} + \|g\|_{H^{1/2}(\partial D)^d}, \tag{38}$$

$$\|\widetilde{u}_1\|_{H^1(D_1)^d} \preceq \|f\|_{H^{-1}(D_1)^d} + \|g\|_{H^{1/2}(\partial D)^d} \tag{39}$$

and

$$\|\widetilde{u}_1\|_{H^1(D_0)^d} \preceq \|\widetilde{u}_1\|_{H^{1/2}(\partial D_1)^d} \preceq \|\widetilde{u}_1\|_{H^1(D_1)^d}. \tag{40}$$

Proof. From the definition of $u_{0,0}$ in (20) we have that

$$\|u_{0,0}\|_{H^1(D_0)^d} \preceq \|f\|_{H^{-1}(D_0)^d} + \|g\|_{H^{1/2}(\partial D)^d},$$

for more details see for instance, Adams and Fournier (2003).

Now, using the Korn inequality for Dirichlet data and the Lax-Milgram theorem we have

$$\|u_0\|_{H^1(D)^d} \preceq |u_0|_{H^1(D)^d} = |u_0|_{H^1(D_0)^d} \leq |u_{0,0}|_{H^1(D_0)^d} + \left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|_{H^1(D_0)^d},$$

and using a similar argument to the one used in Lemma 2, we have that

$$\begin{aligned}\left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|^2_{H^1(D_0)^d} &\leq C\,|u_{0,0}|_{H^1(D_0)^d}\left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|_{H^1(D_0)^d} + \int_D f\left(\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right)\\ &\preceq |u_{0,0}|_{H^1(D_0)^d}\left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|_{H^1(D_0)^d} + \|f\|_{H^{-1}(D)^d}\left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|_{H^1(D_0)^d},\end{aligned}$$

so

$$\left|\sum_{\ell=1}^{L_d} c_{0;\ell}\xi_{D_1;\ell}\right|_{H^1(D_0)^d} \preceq |u_{0,0}|_{H^1(D_0)^d} + \|f\|_{H^{-1}(D)^d}.$$

Using this fact and the definition of $u_{0,0}$, we conclude that

$$\|u_0\|_{H^1(D)^d} \preceq \|f\|_{H^{-1}(D)^d} + \|g\|_{H^{1/2}(\partial D)^d}.$$

This concludes the proof of (38).

Equation (39) uses a similar argument to the one used in the above proof, we use the Korn inequality for Neumann conditions and the trace theorem. Finally, the equation (40) is obtained using Korn inequality for Dirichlet conditions and the trace theorem. Details are not included for the sake of brevity. □

Lemma 4. *Let u_j defined on D_0 by (32) with $c_{j;\ell}$ and u_{j+1} defined on D_1 by (34). For $j \geq 1$ we have that*

$$\|u_{j+1}\|_{H^1(D)^d} \preceq \|u_j\|_{H^1(D_0)^d}.$$

Proof. Let $j \geq 1$. Consider $\widetilde{u}_{j+1}$ defined by the Dirichlet in (32). From the Lemma 2 and combining the Korn inequality for Dirichlet conditions and the trace theorem, we have

$$\|u_{j+1}\|_{H^1(D)^d} \preceq \|\widetilde{u}_{j+1}\|_{H^1(D)^d} \leq C\|\widetilde{u}_{j+1}\|_{H^1(D_1)^d},$$

applying the Korn inequality for the Dirichlet conditions in the last equation we obtain

$$\|\widetilde{u}_{j+1}\|_{H^1(D_1)^d} \preceq \|u_j\|_{H^1(D_0)^d}.$$

Combining these inequalities we have

$$\|u_{j+1}\|_{H^1(D)^d} \preceq \|u_j\|_{H^1(D_0)^d}.$$

This concludes the proof. □

Theorem 5. *There is a constant $C > 0$ such that for every $\eta > C$, the expansion (11) converges (absolutely) in $H^1(D)$. The asymptotic limit u_0 satisfies problem (16) and u_0 can be computed using formula (19).*

Proof. From the Lemma 4 applied repeatedly $j-1$ times, we get that for every $j \geq 2$ there is a constant C such that

$$\begin{aligned} \|u_j\|_{H^1(D)^d} &\leq C\|u_{j-1}\|_{H^1(D_0)^d} \leq C\|u_{j-1}\|_{H^1(D)^d} \\ &\leq \cdots \leq C^{j-1}\|\widetilde{u}_1\|_{H^1(D_0)^d} \end{aligned}$$

and then

$$\left\| \sum_{j=2}^{\infty} \eta^{-j} u_j \right\|_{H^1(D)^d} \leq \frac{\|\widetilde{u}_1\|_{H^1(D_0)^d}}{C} \sum_{j=2}^{\infty} \left(\frac{C}{\eta}\right)^j.$$

The last expansion converges when $\eta > C$. Using (38) and (39) we conclude that there is a constant C_1 such that

$$\left\| \sum_{j=0}^{\infty} \eta^{-j} u_j \right\|_{H^1(D)^d} \preceq C_1 \left(\|f\|_{H^{-1}(D)^d} + \|g\|_{H^{1/2}(\partial D)^d} \right) \sum_{j=0}^{\infty} \left(\frac{C}{\eta}\right)^j.$$

Moreover, the asymptotic limit u_0 satisfies (16). $\square$

Combining Lemmas 2 to 4 we get convergence for the expansion (11) with the boundary condition (12).

Corollary 6. *There are positive constants C and C_1 such that for every $\eta > C$, we have*

$$\left\| u - \sum_{j=0}^{J} \eta^{-j} u_j \right\|_{H^1(D)^d} \leq C_1 \left(\|f\|_H^{-1}(D)^d + \|g\|_{H^{1/2}(D)^d} \right) \sum_{j=J+1}^{\infty} \left(\frac{C}{\eta}\right)^j,$$

for $J \geq 0$.

Remark 7. *The case of several inclusions can be analyzed in similar way and it is not presented here, a description on how to perform this analysis for a scalar problem are given in Calo et al. (2014).*

4. Conclusions

We use asymptotic expansions to study high-contrast linear elasticity problems. In particular, we explain the procedure to compute the terms of the asymptotic expansion for u_η with one stiff inclusion in linear elastic medium. We detail the analysis of the asymptotic power series for one highly inelastic inclusion.

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