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Research Article

L^p Estimates for Weak Solutions to Nonlinear Degenerate Parabolic Systems

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This paper is devoted to the L^p estimates for weak solutions to nonlinear degenerate parabolic systems related to Hörmander's vector fields. The reverse Hölder inequalities for degenerate parabolic system under the controllable growth conditions and natural growth conditions are established, respectively, and an important multiplicative inequality is proved; finally, we obtain the L^p estimates for the weak solutions by combining the results of Gianazza and the Caccioppoli inequality.

1. Introduction

The L^p estimates of the weak solutions to elliptic and parabolic systems under the controllable growth conditions and natural growth conditions in the Euclidean space $\mathbb{R}^n$ were well studied (see [1–3]). Roughly speaking, if the weak solution is $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ with $\Omega \subset \mathbb{R}^n$ being an open set, then there exists an index $p > 2$ such that $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. This result plays an important role in the study of the Hölder partial regularities. However, the index p cannot be arbitrarily large. Folland established the local L^p regularity and local Schauder estimates of the sub-Laplace operator structured on the vector fields of homogeneous group (satisfying Hömander's rank condition) in [4]. Rothschild and Stein [5] proved the local L^p estimates of the square sum operators structured on the smooth Hörmander's vector fields, through establishing the lifting and approximating theory of Hörmander's vector fields and the local theory of the second-order invariant differential operator on the homogeneous group. Then, the estimates of weak solutions to the linear and nonlinear subelliptic equations and degenerate parabolic equations stemmed from the noncommutative vector fields that caused extensive concern; see [6–13]. In particular, Gianazza obtained in [11] the L^p estimates of the derivatives of the weak solutions to nonlinear diagonal equations related to the Hörmander vector fields. Recently, the L^p estimates of weak solutions to nonlinear subelliptic systems related to Hörmander's vector fields under the superquadratic controllable growth condition and natural growth condition were studied in [14] and optimal partial regularity for subelliptic systems related to Hörmander's vector fields was obtained in [15]. There are also some papers related on regularity for weak solutions to the quasilinear degenerate elliptic systems and degenerate elliptic systems with VMO coefficients were studied in [16, 17]. Those works motivated us to study the interesting question: can we get the L^p estimates of weak solutions to the nonlinear degenerate parabolic systems structured on Hörmander's vector fields?

Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set. We consider the nonlinear degenerate parabolic systems related to Hömander's condition

$$\partial_t u^\alpha - \sum_{i=1}^{k} X_i^* A_i^\alpha (t, \xi, u, Xu) = B^\alpha (t, \xi, u, Xu), \qquad \alpha = 1, \dots, N, \tag{1}$$

under controllable growth conditions and natural growth conditions, where the controllable growth conditions are

$$A_i^\alpha(t,\xi,u,P)P_i^\alpha \geq \lambda|P|^2 - \Lambda|u|^r - f^2, \quad f \in L^{s_1}(\Omega),$$
$$\left|A_i^\alpha(t,\xi,u,P)\right| \leq \Lambda\left(|P| + |u|^{r/2} + f_i^\alpha\right),$$
$$0 \leq f_i^\alpha \in L^{s_2}(\Omega), \quad (2)$$
$$\left|B^\alpha(t,\xi,u,P)\right| \leq \Lambda\left(|P|^{2(1-1/r)} + |u|^{r-1} + f^\alpha\right),$$
$$0 \leq f^\alpha \in L^{s_3}(\Omega),$$

with $s_1 > 2$, $s_2 > 2$, $s_3 > r/(r-1)$, and $r = 2^* = 2(Q+2)/Q$ being a Sobolev critical exponent; and the natural growth conditions are

$$A_i^\alpha(t,\xi,u,P)P_i^\alpha \geq \lambda|P|^2 - f^2, \quad f \in L^{s_1}(\Omega), \quad (3)$$

$$\left|A_i^\alpha(t,\xi,u,P)\right| \leq \Lambda|P| + f_i^\alpha, \quad 0 \leq f_i^\alpha \in L^{s_2}(\Omega), \quad (4)$$

$$\left|B^\alpha(t,\xi,u,P)\right| \leq \Lambda|P|^2 + f^\alpha, \quad 0 \leq f^\alpha \in L^{s_3'}(\Omega), \quad (5)$$

with $s_3' > 1$.

This paper is organized as follows: in Section 2, we introduce some notation, definitions, and basic facts. In Section 3, we state our main results. Section 4 is devoted to some important lemmas, including the Sobolev embedding theorem, the Poincaré inequalities, and reverse Hölder's inequalities for the parabolic case. In particular, we establish an important multiplicative inequality by using the Fefferman-Phong inequality. We prove the main results in Section 5.

2. Preliminary

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, and path-connected set and $\{X_i\}_{i=1}^k$ a family of C^∞ real-valued vector fields defined in a neighborhood of the closure $\overline{\Omega}$ of Ω. For a multi-index $\alpha = (i_1, \ldots, i_k)$, we denote by X_α the commutator

$$\left[X_{i_1}, \left[X_{i_2}, \ldots, \left[X_{i_{k-1}}, X_{i_k}\right]\right]\right], \quad (6)$$

of length $l = |\alpha|$. Throughout this paper, we suppose that the vector fields $X = \{X_1, \ldots, X_k\}$ satisfy Hömander's condition: there exists some positive integer s such that $\{X_\alpha\}_{|\alpha|\leq s}$ span the tangent space of $\mathbb{R}^n$ at each point of Ω; that is,

$$\operatorname{rank} \operatorname{Lie}\left[X_1, \ldots, X_k\right] \equiv n. \quad (7)$$

Assume that an admissible path γ is a Lipschitz curve $\gamma : [a,b] \mapsto \Omega$ such that there exist $c_i(t)$, $a \leq t \leq b$ satisfying $\sum_{i=1}^k c_i(t)^2 \leq 1$ and $\gamma'(t) = \sum_{i=1}^k c_i(t)X_i(\gamma(t))$. Then we define a metric associated to $\{X_1, \ldots, X_k\}$ on Ω

$$\varrho_{cc}(\xi,\eta) =: \min\{b \geq 0 : \exists\gamma : [0,b] \mapsto \Omega, \text{ s.t. } \gamma(0) = \xi, \ \gamma(b) = \eta\}. \quad (8)$$

We refer to it as Carnot-Carathéodory metric, and we call it C-C metric for short. It is showed in [18] that ϱ_{cc} is a distance.

We can define the C-C ball and the C-C sphere as

$$B_R(\xi) = B(\xi,R) = \{\eta : \varrho_{cc}(\xi,\eta) < R\}, \quad (9)$$

$$\partial B_R(\xi) = \partial B(\xi,R) = \{\eta : \varrho_{cc}(\xi,\eta) = R\}, \quad (10)$$

respectively, from (8). In [18], the authors proved that the Lebesgue measure satisfies the doubling property related to the C-C ball: given a bounded set $\Omega \subset \mathbb{R}^n$, there are positive constants C_1 and $R_0 > 0$ such that

$$\left|B_{2R}(\xi)\right| \leq C_1\left|B_R(\xi)\right|, \quad (11)$$

for $\xi \in \Omega$ and $0 < R < R_0$. Let $Q = \log_2 C_1$; the number Q acts as a dimension and is called the local homogeneous dimension related to Ω and the system $\{X_i\}_{i=1}^k$.

We use (ξ, t) to denote a point in $\Omega \times \mathbb{R}$ and $\xi \in \Omega \subset \mathbb{R}^n$. We define the parabolic Carnot- Carathéodory distance between the two points in $\Omega \times \mathbb{R}$ as

$$d_P\left((\xi,t),(\xi',t')\right) = \sqrt{\varrho_{cc}(\xi,\xi')^2 + |t-t'|}. \quad (12)$$

We use the notation $Q_R(\xi,t) = B_R(\xi) \times (t-R^2, t)$ for parabolic cylinders in $\Omega \times \mathbb{R}$, where $B_R(\xi)$ is defined in (9).

For an open set $\Omega \subset \mathbb{R}^n$, we set $\Omega_R(\xi) = B_R(\xi) \cap \Omega$. For an open set $U \subset \mathbb{R}^{n+1}$, we let $U_R(\xi,t) = U \cap Q_R(\xi,t)$. We write $U(t_0)$ for the set of all points (ξ, t_0) in U and $I(U)$ for the set of all t such that $U(t)$ is nonempty. We also write $U := U_T^S$ for the cylinder $(S,T) \times \Omega$, where $-\infty < S < T < \infty$ and U_T for U when $S = 0$. We denote by $W_2^{0,1}(U)$ the Hilbert space with the inner product

$$\langle u, v\rangle_{W_2^{0,1}(U)} := \int_U uv + \sum_{k=1}^d X_k u X_k v. \quad (13)$$

Let $W_2^{1,1}(U)$ be subspace of $W_2^{0,1}(U)$ such that $\partial_t u \in L_2(U)$ and $V_2(U)$ are to be the Banach space consisting of all elements of $W_2^{0,1}(U)$ having a finite norm $\|u\|_{V_2(U)} := |||u|||_U$, with $\|u\|_U^2 = \|Xu\|_{L_2(U)}^2 + \sup_{t\in I(U)}\|u(t,\cdot)\|_{L_2(U(t))}^2$. Set $V_{0,2}(U)$ to be the set of all functions u in $V_2(U)$ vanishes on the lateral boundary $\partial_l U := (S,T) \times \partial\Omega$ of U.

From the well-known Sobolev-like embedding theorem (see [19], II,3), we have

$$\|u\|_{L_r(U)} \leq N(Q)\|u\|_U, \quad \forall u \in V_{0,2}(U), \ r = \frac{2(Q+2)}{Q}. \quad (14)$$

We denote $H_p^{-1}(U)$ to be the space consisting of all functions u satisfying

$$\inf\left\{\|\mathbf{F}\|_{L_p(U)} + \|h\|_{L_p(U)} \mid u = \operatorname{div}_X \mathbf{F} + h\right\} < \infty, \quad (15)$$

with $\operatorname{div}_X \mathbf{F} = \sum_{i=1}^k X_i F_i$. It is easy to see that $H_p^{-1}(U)$ is a Banach space. For any $u \in H_p^{-1}(U)$, we can define the norm,

$$\|u\|_{H_p^{-1}(U)} = \inf\left\{\|\mathbf{F}\|_{L_p(U)} + \|h\|_{L_p(U)} \mid u = \operatorname{div}_X \mathbf{F} + h\right\}, \quad (16)$$

and define $\mathscr{H}^1_p(U) = \{u\colon u, Xu \in L_p(U),\ \partial_t u \in H_p^{-1}(U)\}$. Note that $\mathscr{H}^1_2(U) \subset V_2(U)$.

3. Main Results

We call $u \in V_{0,2}(U_T)$ the weak solution of the systems (1) if

$$\begin{aligned}&\int_\Omega u^\alpha(\xi,t)\,\varphi^\alpha(\xi,t)\,d\xi\Big|_0^t\\&\quad+\int_\Omega\left[-u^\alpha\partial_t\varphi^\alpha + A_i^\alpha(t,\xi,u,Xu)\,X_i\varphi^\alpha\right]d\xi\,dt\\&= \int_0^t\int_\Omega B^\alpha(t,\xi,u,Xu)\,\varphi^\alpha d\xi\,dt,\end{aligned}\tag{17}$$

for any test functions $\varphi \in W^{1,1}_{0,2}(U_T)$ and $t \in [0,T]$.

The main results in this paper are as follows.

Theorem 1. *(A) Assume that A_i^α, B^α satisfy (2), $u \in V_{0,2}(U_T)$ is a weak solution to systems (1), and $f \in L_2(U_T)$, $f_i^\alpha \in L_2(U_T)$, and $f^\alpha \in L_{r/(r-1)}(U_T)$. Then for any $(\xi_0,t_0) \in \mathbb{R}^{n+1}$ and $0 < R \le 1$ with $B_{R/2}(\xi_0) \subset B_R(\xi_0) \subset \Omega$ and $t_0 \ge R^2$, one has*

$$\begin{aligned}&⨍_{Q_{R/2}(\xi_0,t_0)}\left(|Xu|^2+|u|^r\right)d\xi\,dt\\&\le C\left[⨍_{Q_R(\xi_0,t_0)}\left(|Xu|^q+|u|^{rq/2}\right)d\xi\,dt\right]^{2/q}\\&\quad+C⨍_{Q_R(\xi_0,t_0)}\left(|f|^2+\left|\overline{f}\right|^2+|F|^2\right)d\xi\,dt+C\\&\quad\cdot\sup_{t\in(t_0-R^2,t_0)}\left(\int_{B_R(\xi_0)}|u|^2\,d\xi\right)^{2/Q}\\&\quad\cdot\left(⨍_{Q_R(\xi_0,t_0)}|Xu|^2\,d\xi\,dt\right),\end{aligned}\tag{18}$$

where $\overline{f} = (f_i^\alpha)$, $F = |f^\alpha|^{(1/2)(r/(r-1))}$, $C = C(n,\lambda,\Lambda,\|u\|_{V_2}, \|f\|_{L_2}, \|f^\alpha\|_{L_{r/(r-1)}}, \|f_i^\alpha\|_{L_2})$, $q = 2(Q+2)/(Q+4) \in (1,2)$, and $1/r + 1/q = 1$.

(B) Let $u \in V_{0,2}(U_T)$ be a weak solution of (1), $f, f_i^\alpha \in L_\sigma(U_T)$, $\sigma \in (2,\infty)$; $f^\alpha \in L_\tau(U_T)$, $\tau \in (r/(r-1),\infty)$. Then there exists $p > 2$ such that $u \in \mathscr{H}^1_p(U_T)$. Moreover, the inequality,

$$\|u\|_{L_{rp/2}((\varepsilon,T)\times\Omega)} + \|Xu\|_{L_p((\varepsilon,T)\times\Omega)} \le C,\tag{19}$$

is valid for any $0 < \varepsilon < T$, where $C = C(n,\lambda,\Lambda,\sigma,\tau,u, \|f\|_{L_\sigma(U_T)}, \|f_i^\alpha\|_{L_\sigma(U_T)}, \|f^\alpha\|_{L_\tau(U_T)}, \varepsilon, T, |\Omega|)$, $p = p(n,\lambda,\Lambda,\sigma,\tau,u,\|f\|_{L_\sigma(U_T)}, \|f_i^\alpha\|_{L_\sigma(U_T)}, \|f^\alpha\|_{L_\tau(U_T)})$.

Theorem 2. *(C) Suppose that A_i^α, B^α satisfy the natural growth conditions (3)–(5); $u \in V_{0,2}(U_T) \cap L_\infty(U_T)$ is the weak solution of systems (1), $\sup_{\Omega\times(t-R^2,t)}|u| = M$, $2\Lambda M < \lambda$ (λ is the same as in (3)), and $f \in L_2(U_T)$, $f_i^\alpha \in L_2(U_T)$, $f^\alpha \in L_{r/(r-1)}(U_T)$. Then for any $(\xi_0,t_0) \in \mathbb{R}^{n+1}$ and $0 < R \le 1$ with $B_{R/2}(\xi_0) \subset B_R(\xi_0) \subset \Omega$ and $t_0 \ge R^2$, one has*

$$\begin{aligned}&⨍_{Q_{R/2}(\xi_0,t_0)}|Xu|^2\,d\xi\,dt\\&\le C\left(⨍_{Q_R(\xi_0,t_0)}|Xu|^q\,d\xi\,dt\right)^{2/q}\\&\quad+C⨍_{Q_R(\xi_0,t_0)}\left(|f|^2+\left|\overline{f}\right|^2+|F|^2\right)d\xi\,dt\\&\quad+C\sup_{t\in(t_0-R^2,t_0)}\left(\int_{B_R(\xi_0)}|u|^2\,d\xi\right)^{2/Q}\\&\quad\cdot\left(⨍_{Q_R(\xi_0,t_0)}|Xu|^2\,d\xi\,dt\right),\end{aligned}\tag{20}$$

for $R < R_0$, where $\overline{f} = (f_i^\alpha)$, $F = |f^\alpha|^{r/2(r-1)}$, $C = C(n,\lambda,\Lambda,\|u\|_{V_2},\|f\|_{L_2},\|f^\alpha\|_{L_{r/(r-1)}},\|f_i^\alpha\|_{L_2})$. R_0 depend on M and the L_2 norm for Xu.

(D) Let $u \in V_{0,2}(U_T)$ be the weak solution of (1) and $\sup_{\Omega\times(t-R^2,t)}|u| = M$, $2\Lambda M < \lambda$, $f, f_i^\alpha \in L_\sigma(U_T)$, $\sigma \in (2,\infty)$; $f^\alpha \in L_\tau(U_T)$, $\tau \in (r/(r-1),\infty)$. Then there exists $p > 2$ such that $u \in \mathscr{H}^1_p(U_T)$. Moreover, the inequality,

$$\|Xu\|_{L_p((\varepsilon,T)\times\Omega)} \le C,\tag{21}$$

is valid for any $0 < \varepsilon < T$, where $C = C(n,\lambda,\Lambda,\sigma,\tau,u, \|f\|_{L_\sigma(U_T)}, \|f_i^\alpha\|_{L_\sigma(U_T)}, \|f^\alpha\|_{L_\tau(U_T)}, \varepsilon, T, |\Omega|, M)$, and $p = p(n,\lambda,\Lambda,\sigma,\tau,u,\|f\|_{L_\sigma(U_T)}, \|f_i^\alpha\|_{L_\sigma(U_T)}, \|f^\alpha\|_{L_\tau(U_T)}, M)$.

4. Some Lemmas

Lu established Pioncaré's inequality in [20]; we can similarly derive the following lemma by replacing the C-C ball B_R with the parabolic cylinder Q_R.

Lemma 3. *There exist C and R such that for any $u \in W_q^{0,1}(Q_R(\xi_0,t_0))$, $1 < q < Q$, $1 \le p \le qQ/(Q-q)$*

$$\begin{aligned}&\left(⨍_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^p\,d\xi\,dt\right)^{1/p}\\&\le CR\left(⨍_{Q_R(\xi_0,t_0)}|Xu|^q\,d\xi\,dt\right)^{1/q},\end{aligned}\tag{22}$$

where $\overline{u}(t) := (1/C_1)⨍_{Q_R(\xi_0,t_0)}u\eta^4 d\xi\,dt$; η is the cut-off function related to ξ.

From (22), we can get the following Sobolev-Poincaré type inequality immediately.

Corollary 4. *Let $u \in W_q^{0,1}(Q_R(\xi_0, t_0))$, $1 < q < Q$, then $u \in L^p(Q_R(\xi_0, t_0))$, $p = Qq/(Q-q)$, and one has*

$$\begin{aligned} &\left(\int_{Q_R(\xi_0,t_0)} |u - \overline{u}(t)|^p \, d\xi \, dt\right)^{1/p} \\ &\quad \le C\left(\int_{Q_R(\xi_0,t_0)} |Xu|^q \, d\xi \, dt\right)^{1/q}. \end{aligned} \tag{23}$$

Next, we recall Fefferman-Phong's inequality (see [21]) to get the multiplicative inequality which plays an important role in this paper.

Theorem 5 (Fefferman-Phong's inequality). *Let $\Omega \subset \mathbb{R}^n$ be an open set and Q a homogeneous dimension. Set $R_0 > 0$; suppose that $1 < p < Q$, $1 < s \le Q/p$, $\beta \ge 1$, and $V \in L^{s,ps}(\beta B)$, where $B = B_R(\xi_0)$, $\xi_0 \in \Omega$, $0 < R < R_0$, and $\beta B \subset \Omega$. Then there exists $C = C(\Omega, X) > 0$ such that for any $u \in L^{1,p}(\beta B)$*

$$\int_B |u - u_B|^p \, |V| \, d\xi \le C \, \|V\|_{L^{s,ps}}(\beta B) \int_{\beta B} |Xu| \, d\xi, \tag{24}$$

where $u_B = (1/|B|) \int_B u \, d\xi$ and $u \in L^{p,\mu}(\Omega)$ $(\mu > 0)$ implies $u \in L^p_{\mathrm{loc}}(\Omega)$ and

$$\sup_{\xi \in \Omega, 0 < r < \dim \Omega} \left(\frac{r^\mu}{|B_r(\xi) \cap \Omega|} \int_{B_r(\xi) \cap \Omega} |u|^p \, d\xi\right)^{1/p} < \infty. \tag{25}$$

Lemma 6 (multiplicative inequality for p). *Let $\Omega \subset \mathbb{R}^n$ be an open set and Q a homogeneous dimension. Let $R_0 > 0$, $1 < p < Q$, $\beta \ge 1$, $\xi_0 \in \Omega$, $0 < R < R_0$, and $\beta B \subset \Omega$, where $B = B_R(\xi_0)$. Then there exists $C = C(\Omega, X) > 0$ such that for any $u \in W^{1,p}(\beta B)$*

$$\begin{aligned} &\|u\|_{L_r(B)} \\ &\quad \le C \, \|u\|_{L_p(\beta B)}^{p/(Q+p)} \left(\|Xu\|_{L_p(B_R)} + R^{-1} \|u\|_{L_p(B_R)}\right)^{Q/(Q+p)}, \end{aligned} \tag{26}$$

where $r = p(Q+p)/Q$.

Proof. Since $u \in W^{1,p}(\beta B)$, we have $u \in L^{pQ/(Q-p)}$ by Sobolev's embedding theorem; then

$$\int_B |u - u_B|^r \, d\xi = \int_B |u - u_B|^p \, |u - u_B|^{p^2/Q} \, d\xi. \tag{27}$$

Let $|u - u_B|^{p^2/Q} = V$; then $V \in L^{(Q^2/(Q-p))(1/p)}(\beta B)$. Let $s = Q/p$; one has $V \in L^{(Q/(Q-p))(Q/p)}(\beta B) \subset L^{Q/p}(\beta B) = L^{Q/p,Q}(\beta B)$.

By Theorem 5, we obtain

$$\begin{aligned} &\int_B |u - u_B|^p \, |V| \, d\xi \le C \, \|V\|_{L^{Q/p}}(\beta B) \int_{\beta B} |Xu| \, d\xi, \\ &\|V\|_{L^{Q/p}(\beta B)} = \left\| |u - u_B|^{p^2/Q} \right\|_{L^{Q/p}(\beta B)} \\ &\quad = \left\| |u - u_B| \right\|_{L^p(\beta B)}^{p^2/Q}. \end{aligned} \tag{28}$$

Therefore

$$\begin{aligned} &\|u - u_B\|_{L^r(B)} \\ &\quad \le C\left(\left\||u - u_B|\right\|_{L^p(\beta B)}^{p^2/Q} \int_{\beta B} |Xu| \, d\xi\right)^{Q/p(Q+p)} \\ &\quad = \left\||u - u_B|\right\|_{L^p(\beta B)}^{p/(Q+p)} \|Xu\|_{L^p(\beta B)}^{Q/(Q+p)}, \end{aligned} \tag{29}$$

and then

$$\begin{aligned} &\|u\|_{L^r}(B) \le C\left(\left\||u - u_B|\right\|_{L^p(\beta B)}^{p^2/Q} \int_{\beta B} |Xu| \, d\xi\right)^{Q/p(Q+p)} \\ &\quad \le C \left\||u - u_B|\right\|_{L^p(\beta B)}^{p/(Q+p)} \|Xu\|_{L^p(\beta B)}^{Q/(Q+p)} + \|u_B\|_L^r(B) \\ &\quad \le C\left(\left\||u_B|\right\|_{L^p(\beta B)}^{p/(Q+p)} + \left\||u|\right\|_{L^p(\beta B)}^{p/(Q+p)}\right) \|Xu\|_{L^p(\beta B)}^{Q/(Q+p)} \\ &\quad + \|u_B\|_L^r(B), \\ &\|u_B\|_{L^r(B)} = \left(\frac{1}{|B|} \int_B u \, d\xi\right) |B|^{1/r} \le \frac{1}{|B|}\left(\int_B u \, d\xi\right)^{1/p} \\ &\quad \cdot |B|^{(p-1)/p} \, |B|^{1/r} \le \|u\|_{L^p} \, \frac{R^{(p-1)Q/p + Q^2/p(Q+p)}}{R^Q} \\ &\quad = \|u\|_{L^p} \left(R^{-1}\right)^{Q/(p+Q)}, \\ &\left\||u_B|\right\|_{L^p(\beta B)}^{p/(Q+p)} = \left(\frac{1}{|B|} \int_B u \, d\xi\right)^{p/(Q+p)} |B|^{1/Q+p} \\ &\quad \le \frac{1}{|B|^{p/(Q+p)}}\left(\left(\int_B u \, d\xi\right)^{1/p} |B|^{(p-1)/p}\right)^{p/(Q+p)} \\ &\quad \cdot |B|^{1/(Q+p)} = \|u\|_{L^p(\beta B)}^{p/(p+Q)}. \end{aligned} \tag{30}$$

□

When $p = 2$, we get the multiplicative inequality that we will use in this paper.

Lemma 7. *For any $R > 0$ and $u \in V_{0,2}(Q_R)$, it holds that*

$$\begin{aligned} &\|u\|_{L_r(B_R)} \le C(d) \, \|u\|_{L_2(B_R)}^{2/(Q+2)} \\ &\qquad \cdot \left(\|Xu\|_{L_2(B_R)} + R^{-1} \|u\|_{L_2(B_R)}\right)^{Q/(Q+2)}. \end{aligned} \tag{31}$$

Gianazza [11] showed a reverse Hölder inequality on homogeneous spaces. Since the ball $B_R(\xi_0)$ which is induced by the vector fields is a homogeneous space, it is obvious that the cylinder $Q_R(\xi_0, t_0) = B_R(\xi_0) \times (t - R^2, t)$ is also a homogeneous space. Therefore, we just need to replace the ball $B_R(\xi_0)$ in [11] with the parabolic cylinder $Q_R(\xi_0, t_0)$. Similarly, we can get

Lemma 8. *Let g and h be nonnegative functions and satisfy the following.*

(1) $g \in L_\gamma(Q)$, $\gamma > 1$; $h \in L_\beta(Q)$, $\beta > \gamma$.

(2) *Assume that* $\forall (\xi_0, t_0) \in Q$, $Q_R(\xi_0, t_0) \subset Q$ *and* $\forall R : 0 < R < (1/2)\operatorname{dist}((\xi_0, t_0), \partial Q) \wedge R_0$; *one has*

$$\fint_{Q_R(\xi_0,t_0)} g^\gamma d\xi\, dt \le C_0 \left[\fint_{Q_{8R}(\xi_0,t_0)} g\, d\xi\, dt\right]^\gamma + \fint_{Q_{8R}(\xi_0,t_0)} h^\gamma d\xi\, dt + \theta \fint_{Q_{8R}(\xi_0,t_0)} g^\gamma d\xi\, dt, \tag{32}$$

where the constants R_0, θ, and C_0 are positive, $\theta \in (0,1)$. Then there exist $\varepsilon > 0$ and $C > 0$, such that $g \in L^p_{\mathrm{loc}}(Q)$, $\forall p \in [\gamma, \gamma + \varepsilon)$. Moreover, for $Q_R(\xi_0, t_0) \subset Q$, $R < R_0$

$$\left[\fint_{Q_R(\xi_0,t_0)} g^p d\xi\, dt\right]^{1/p} \le C \left[\fint_{Q_{8R}(\xi_0,t_0)} g^\gamma d\xi\, dt\right]^{1/\gamma} + C \left[\fint_{Q_{8R}(\xi_0,t_0)} h^p d\xi\, dt\right]^{1/p}, \tag{33}$$

is valid, where C and ε depend on $C_0, \theta, \gamma, \beta, K (\ge 1)$ and n.

We will use the following cut-off function. Let $\eta(\xi) \in C_0^\infty(B_R(\xi_0))$, $\zeta(t) \in C_0^\infty(-1, 1)$ be cut-off functions and satisfy $0 \le \eta, \zeta \le 1$ (on $B_R(\xi_0)$), $\eta \equiv 1$ (on $B_{R/2}(\xi_0)$), and $\zeta \equiv 1$ (on $(-1/2, 1/2)$), $|X\eta| \le C/R$. Note that $\int_{B_R(\xi_0)} \eta^4 d\xi = C_1 |B_R(\xi_0)|$, where C_1 is independent of R.

Lemma 9 (Caccioppoli inequality). *Let $u \in V_{0,2}(U_T)$ be the weak solutions of systems (1) and $f \in L_2(U_T)$, $f_i^\alpha \in L_2(U_T)$, and $f^\alpha \in L_{r/(r-1)}(U_T)$. Then for any $\xi_0 \in \mathbb{R}^d$, $0 < R \le R_0$, and $R^2 \le t_0 \le T$*

$$\begin{aligned} &\sup_{t \in (t_0 - R^2, t_0)} \int_{\Omega_R(\xi_0)} u^2 (\xi \cdot t)\, \eta^4 \zeta^2 d\xi + \lambda \int_{U_R(\xi_0,t_0)} |Xu|^2 \eta^4 \zeta^2 \\ &\le CR^{-2} \int_{U_R(\xi_0,t_0)} |u|^2 \eta^2 \zeta + C \int_{U_R(\xi_0,t_0)} |u|^r \\ &\quad + C \int_{U_R(\xi_0,t_0)} \left(f^2 + \left|f_i^\alpha\right|^2 + \left|f^\alpha\right|^{r/(r-1)}\right), \end{aligned} \tag{34}$$

where $C = C(n, \lambda, \Lambda)$.

Proof. Take the test function $\phi = u(\xi, t)\eta^4\zeta^2$. Substituting ϕ in (17), we have

$$\begin{aligned} &\frac{1}{2} \int_{\Omega_R(\xi_0)} u^2 \eta^4 \zeta^2 d\xi + \int_{t_0 - R^2}^{t} \int_{\Omega_R(\xi_0)} A_i^\alpha (t, \xi, u, Xu)\, X_i^* \left(u\eta^4\zeta^2\right) d\xi\, dt \\ &= \int_{t_0 - R^2}^{t} \int_{\Omega_R(\xi_0)} B^\alpha (t, \xi, u, Xu)\, u\eta^4\zeta^2 d\xi\, dt + \int_{t_0 - R^2}^{t} \int_{\Omega_R(\xi_0)} u^2 \eta^4 \zeta \partial_t \zeta\, d\xi\, dt. \end{aligned} \tag{35}$$

By using the controllable growth conditions (2)

$$\begin{aligned} &\frac{1}{4} \sup_{t \in (t_0 - R^2, t_0)} \int_{\Omega_R(\xi_0)} u^2 \eta^4 \zeta^2 d\xi + \frac{\lambda}{2} \int_{Q_R(\xi_0,t_0)} |Xu|^2 \eta^4 \zeta^2 d\xi\, dt \\ &\le \frac{\Lambda}{2} \int_{U_R(\xi_0,t_0)} |u|^r \eta^4 \zeta^2 d\xi\, dt + \frac{1}{2} \int_{U_R(\xi_0,t_0)} f^2 \eta^4 \zeta^2 d\xi\, dt \\ &\quad + 4\Lambda \int_{U_R(\xi_0,t_0)} |Xu|\, |u|\, \eta^3 \left|X\eta\right| \zeta^2 d\xi\, dt + 4\Lambda \int_{U_R(\xi_0,t_0)} |u|^{r/2} |u|\, \eta^3 \left|X\eta\right| \zeta^2 d\xi\, dt \\ &\quad + 4\Lambda \int_{U_R(\xi_0,t_0)} \left|f_i^\alpha\right| |u|\, \eta^3 \left|X\eta\right| \zeta^2 d\xi\, dt + \int_{U_R(\xi_0,t_0)} B^\alpha (t, \xi, u, Xu)\, |u|\, \eta^4 \zeta^2 d\xi\, dt \\ &\quad + \int_{U_R(\xi_0,t_0)} u^2 \eta^4 \zeta \left|\partial_t \zeta\right| d\xi\, dt := G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + G_7. \end{aligned} \tag{36}$$

We estimate the following terms by Young's inequality:

$$\begin{aligned} G_3 &= 4\Lambda \int_{U_R(\xi_0,t_0)} |Xu|\, |u|\, \eta^3 \left|X\eta\right| \zeta^2 \\ &\le \frac{\lambda}{16} \int_{U_R(\xi_0,t_0)} |Xu|^2 \eta^4 \zeta^2 + C \int_{U_R(\xi_0,t_0)} |u|^2 \left|X\eta\right|^2 \eta^2 \zeta^2 \end{aligned}$$

$$\leq \frac{\lambda}{16}\int_{U_R(\xi_0,t_0)} |Xu|^2\eta^4\zeta^2 + CR^{-2}\int_{U_R(\xi_0,t_0)} |u|^2\eta^2\zeta^2,$$

$$\begin{aligned}
G_4 &= 4\Lambda\int_{U_R(\xi_0,t_0)} |u|^{r/2}|u|\eta^3|X\eta|\zeta^2 \\
&\leq \frac{\lambda}{16}\int_{U_R(\xi_0,t_0)} |u|^r\eta^4\zeta^2 + C\int_{U_R(\xi_0,t_0)} |u|^2|X\eta|^2\eta^2\zeta^2 \\
&\leq \frac{\lambda}{16}\int_{U_R(\xi_0,t_0)} |u|^r\eta^4\zeta^2 + CR^{-2}\int_{U_R(\xi_0,t_0)} |u|^2\eta^2\zeta^2, \\
G_5 &= 4\Lambda\int_{U_R(\xi_0,t_0)} f_i^\alpha|u|\eta^3|X\eta|\zeta^2 \\
&\leq \frac{\lambda}{16}\int_{U_R(\xi_0,t_0)} |f_i^\alpha|^2\eta^4\zeta^2 + C\int_{Q_R(\xi_0,t_0)} |u|^2|X\eta|^2\eta^2\zeta^2 \\
&\leq \frac{\lambda}{16}\int_{U_R(\xi_0,t_0)} |f_i^\alpha|^2\eta^4\zeta^2 + CR^{-2}\int_{U_R(\xi_0,t_0)} |u|^2\eta^2\zeta^2, \\
G_6 &= \int_{U_R(\xi_0,t_0)} B^\alpha(t,\xi,u,Xu)|u|\eta^4\zeta^2 \\
&\leq C\int_{U_R(\xi_0,t_0)} |Xu|^{2(1-1/r)}|u|\eta^4\zeta^2 + C\int_{U_R(\xi_0,t_0)} |u|^{r-1}|u|\eta^4\zeta^2 + C\int_{Q_R(\xi_0,t_0)} |f^\alpha||u|\eta^4\zeta^2 \\
&\leq \frac{\lambda}{16}\int_{U_R(\xi_0,t_0)} |Xu|^2\eta^4\zeta^2 + C\int_{U_R(\xi_0,t_0)} |u|^r + C\int_{Q_R(\xi_0,t_0)} |f^\alpha|^{r/(r-1)}.
\end{aligned} \tag{37}$$

Combining the above estimations and $\zeta^2 \leq \zeta$, then we get (34). □

Corollary 10. *Under the same assumptions as in Lemma 9, we have*

$$\sup_{t\in(t_0-R^2/4,t_0)} \int_{\Omega_{R/2}(\xi_0)} u^2 d\xi \longrightarrow 0, \tag{38}$$

as $R \to 0$, uniformly in ξ_0 and t_0.

Proof. By Hölder's inequality

$$R^{-2}\int_{U_R(\xi_0,t_0)} |u|^2 \leq C\left(\int_{U_R(\xi_0,t_0)} |u|^r\right)^{2/r}. \tag{39}$$

From this inequality and (34) we have

$$\begin{aligned}
&\sup_{t\in(t_0-R^2/4,t_0)} \int_{\Omega_{R/2}(\xi_0)} u^2 d\xi \\
&\leq C\int_{U_R(\xi_0,t_0)} \left(|u|^r + f^2 + |f_i^\alpha|^2 + |f^\alpha|^{r/(r-1)}\right) + C\left(\int_{U_R(\xi_0,t_0)} |u|^r\right)^{2/r}.
\end{aligned} \tag{40}$$

Then f, f^α, f_i^α, u, the assumption of g, and (14), as well as the absolute continuity of Lebesgue, integrals imply (38). □

5. The Proof of the Main Results

5.1. The Proof of Theorem 1. Take the test function $\varphi = \eta(\xi)^4\zeta(t)^2(u - \bar{u}(t))$, where

$$\begin{aligned}
\bar{u}(t) &= \left(\int_{B_R(\xi_0)} \eta^4 d\xi\right)^{-1}\int_{B_R(\xi_0)} u\eta^4 d\xi \\
&= \frac{1}{C_1}⨍_{B_R(\xi_0)} u\eta^4 d\xi,
\end{aligned} \tag{41}$$

and then

$$\begin{aligned}
\int_{B_R(\xi_0)} \bar{u}(t)\eta^4 d\xi &= C_1|B_R(\xi_0)|\frac{1}{C_1}⨍_{B_R(\xi_0)} u\eta^4 d\xi \\
&= \int_{B_R(\xi_0)} u\eta^4 d\xi,
\end{aligned} \tag{42}$$

so we have

$$\begin{aligned}
&\int_{B_R(\xi_0)} \partial_t\bar{u}(t)(u-\bar{u}(t))\eta^4\zeta^2 d\xi \\
&= \partial_t\bar{u}(t)\zeta^2\int_{B_R(\xi_0)} (u-\bar{u}(t))\eta^4 d\xi \\
&= \partial_t\bar{u}(t)\zeta^2\int_{B_R(\xi_0)} \left(u\eta^4 - \bar{u}(t)\eta^4\right) d\xi = 0,
\end{aligned} \tag{43}$$

and then

$$\begin{aligned}
&\int_{B_R(\xi_0)} \partial_t u(u-\bar{u}(t))\eta^4\zeta^2 d\xi \\
&= \int_{B_R(\xi_0)} (\partial_t u - \partial_t\bar{u}(t))(u-\bar{u}(t))\eta^4\zeta^2 d\xi.
\end{aligned} \tag{44}$$

Substituting φ in (17), we have

$$\begin{aligned}
&\frac{1}{2}\int_{B_R(\xi_0)}(u-\overline{u}(t))^2\eta^4\zeta^2 d\xi \\
&\quad+\int_{t_0-R^2}^{t}\int_{B_R(\xi_0)}A_i^\alpha(t,\xi,u,Xu) \\
&\quad\cdot X_i^*\left((u-\overline{u}(t))\eta^4\zeta^2\right)d\xi\,dt \\
&\quad=\int_{t_0-R^2}^{t}\int_{B_R(\xi_0)}B^\alpha(t,\xi,u,Xu)(u-\overline{u}(t))\eta^4\zeta^2 d\xi\,dt \\
&\quad+\int_{t_0-R^2}^{t}\int_{B_R(\xi_0)}(u-\overline{u}(t))^2\eta^4\zeta\partial_t\zeta\,d\xi\,dt.
\end{aligned}\tag{45}$$

By the controllable growth conditions (2)

$$\begin{aligned}
&\frac{1}{4}\sup_{t\in(t_0-R^2,t_0)}\int_{B_R(\xi_0)}(u-\overline{u}(t))^2\eta^4\zeta^2 d\xi \\
&\quad+\frac{\lambda}{2}\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 d\xi\,dt \\
&\quad\le\frac{\Lambda}{2}\int_{Q_R(\xi_0,t_0)}|u|^r\eta^4\zeta^2 d\xi\,dt \\
&\quad+\frac{1}{2}\int_{Q_R(\xi_0,t_0)}f^2\eta^4\zeta^2 d\xi\,dt \\
&\quad+4\Lambda\int_{Q_R(\xi_0,t_0)}|Xu|\,|u-\overline{u}(t)|\,\eta^3\left|X\eta\right|\zeta^2 d\xi\,dt \\
&\quad+4\Lambda\int_{Q_R(\xi_0,t_0)}|u|^{r/2}\,|u-\overline{u}(t)|\,\eta^3\left|X\eta\right|\zeta^2 d\xi\,dt \\
&\quad+4\Lambda\int_{Q_R(\xi_0,t_0)}\left|f_i^\alpha\right|\,|u-\overline{u}(t)|\,\eta^3\left|X\eta\right|\zeta^2 d\xi\,dt \\
&\quad+\int_{Q_R(\xi_0,t_0)}B^\alpha(t,\xi,u,Xu)\,|u-\overline{u}(t)|\,\eta^4\zeta^2 d\xi\,dt \\
&\quad+\int_{Q_R(\xi_0,t_0)}(u-\overline{u}(t))^2\eta^4\zeta\left|\partial_t\zeta\right|d\xi\,dt := I_1+I_2 \\
&\quad+I_3+I_4+I_5+I_6+I_7.
\end{aligned}\tag{46}$$

We estimate each term by using Young's inequality

$$\begin{aligned}
I_3&=4\Lambda\int_{Q_R(\xi_0,t_0)}|Xu|\,|u-\overline{u}(t)|\,\eta^3\left|X\eta\right|\zeta^2 \\
&\le\frac{\lambda}{16}\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 \\
&\quad+C\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\left|X\eta\right|^2\eta^2\zeta^2 \\
&\le\frac{\lambda}{16}\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 \\
&\quad+CR^{-2}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^2\zeta^2, \\
I_4&=4\Lambda\int_{Q_R(\xi_0,t_0)}|u|^{r/2}\,|u-\overline{u}(t)|\,\eta^3\left|X\eta\right|\zeta^2 \\
&\le\frac{\lambda}{16}\int_{Q_R(\xi_0,t_0)}|u|^r\eta^4\zeta^2 \\
&\quad+C\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\left|X\eta\right|^2\eta^2\zeta^2 \\
&\le\frac{\lambda}{16}\int_{Q_R(\xi_0,t_0)}|u|^r\eta^4\zeta^2 \\
&\quad+CR^{-2}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^2\zeta^2, \\
I_5&=4\Lambda\int_{Q_R(\xi_0,t_0)}f_i^\alpha\,|u-\overline{u}(t)|\,\eta^3\left|X\eta\right|\zeta^2 \\
&\le\frac{\lambda}{16}\int_{Q_R(\xi_0,t_0)}\left|f_i^\alpha\right|^2\eta^4\zeta^2 \\
&\quad+C\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\left|X\eta\right|^2\eta^2\zeta^2 \\
&\le\frac{\lambda}{16}\int_{Q_R(\xi_0,t_0)}\left|f_i^\alpha\right|^2\eta^4\zeta^2 \\
&\quad+CR^{-2}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^2\zeta^2, \\
I_6&=\int_{Q_R(\xi_0,t_0)}B^\alpha(t,\xi,u,Xu)\,|u-\overline{u}(t)|\,\eta^4\zeta^2 \\
&\le C\int_{Q_R(\xi_0,t_0)}|Xu|^{2(1-1/r)}\,|u-\overline{u}(t)|\,\eta^4\zeta^2 \\
&\quad+C\int_{Q_R(\xi_0,t_0)}|u|^{r-1}\,|u-\overline{u}(t)|\,\eta^4\zeta^2 \\
&\quad+C\int_{Q_R(\xi_0,t_0)}\left|f^\alpha\right|\,|u-\overline{u}(t)|\,\eta^4\zeta^2 \\
&\le\frac{\lambda}{16}\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2+C\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^r \\
&\quad+C\int_{Q_R(\xi_0,t_0)}|u|^r+C\int_{Q_R(\xi_0,t_0)}\left|f^\alpha\right|^{r/(r-1)}.
\end{aligned}\tag{47}$$

Therefore

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)}\int_{B_R(\xi_0)}(u-\overline{u}(t))^2\eta^4\zeta^2 d\xi \\
&\quad+\lambda\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 d\xi\,dt
\end{aligned}$$

$$\begin{aligned}
&\leq CR^{-2}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^2\zeta^2\\
&+C\int_{Q_R(\xi_0,t_0)}\left(|u-\overline{u}(t)|^r+|u|^r\right)\\
&+C\int_{Q_R(\xi_0,t_0)}\left(f^2+\left|f_i^\alpha\right|^2+\left|f^\alpha\right|^{r/(r-1)}\right)\\
&:=C_0\left(J_1+J_2+J_3\right).
\end{aligned} \tag{48}$$

Note that

$$\begin{aligned}
\int_{B_R(\xi_0)}|u-\overline{u}(t)|^2d\xi&=\int_{B_R(\xi_0)}\Bigg|u(\xi,t)\\
&-\frac{1}{C_1}⨍_{B_R(\xi_0)}u\left(\xi',t\right)\eta^4\left(\xi'\right)d\xi'\Bigg|^2d\xi\\
&=\int_{B_R(\xi_0)}\Bigg|\frac{1}{C_1}⨍_{B_R(\xi_0)}u(\xi,t)\eta^4\left(\xi'\right)d\xi'\\
&-\frac{1}{C_1}⨍_{B_R(\xi_0)}u\left(\xi',t\right)\eta^4\left(\xi'\right)d\xi'\Bigg|^2d\xi\\
&\leq C_2\int_{B_R(\xi_0)}⨍_{B_R(\xi_0)}\left|u(\xi,t)-u\left(\xi',t\right)\right|d\xi'd\xi.
\end{aligned} \tag{49}$$

So we can use Young's inequality and Sobolev-Poincaré inequality (23) to estimate J_1:

$$\begin{aligned}
J_1&\leq CR^{-2}\left(R^2\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^4\zeta^2\right.\\
&\left.+C\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\right)\leq\frac{\lambda}{16C_0}\\
&\cdot\sup_{t\in(t_0-R^2,t_0)}\int_{B_R(\xi_0)}|u-\overline{u}(t)|^2\eta^4\zeta^2\\
&+C\left(R^{-q}\int_{Q_R(\xi_0,t_0)}|Xu|^q\right)^{2/q},
\end{aligned} \tag{50}$$

where

$$q=\frac{2(Q+2)}{Q+4}\in(1,2). \tag{51}$$

Next, we will estimate J_2. By the triangle inequality

$$\begin{aligned}
J_2&\leq C\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^r\eta^4\zeta^2\\
&+C\int_{Q_R(\xi_0,t_0)}|\overline{u}(t)|^r\eta^4\zeta^2:=J_{21}+J_{22}.
\end{aligned} \tag{52}$$

By Poincaré inequality (22) and Lemma 7

$$\begin{aligned}
&\int_{t_0-R^2}^{t_0}\int_{B_R(\xi_0)}|u-\overline{u}(t)|^rd\xi\,dt\\
&\leq C\int_{t_0-R^2}^{t_0}\left(\int_{B_R(\xi_0)}|u-\overline{u}(t)|^2d\xi\right)^{2/Q}\\
&\cdot\left[\int_{B_R(\xi_0)}|Xu|^2+R^{-2}\int_{B_R(\xi_0)}|u-\overline{u}(t)|^2d\xi\right]dt\\
&\leq C\int_{t_0-R^2}^{t_0}\left(\int_{B_R(\xi_0)}|u-\overline{u}(t)|^2d\xi\right)^{2/Q}\\
&\cdot\int_{B_R(\xi_0)}|Xu|^2d\xi\,dt,
\end{aligned} \tag{53}$$

then we can estimate J_{21}:

$$\begin{aligned}
J_{21}&\leq C\sup_{t\in(t_0-R^2,t_0)}\left(\int_{B_R(\xi_0)}|u-\overline{u}(t)|^2d\xi\right)^{2/Q}\\
&\cdot\int_{Q_R(\xi_0,t_0)}|Xu|^2\leq C\sup_{t\in(t_0-R^2,t_0)}\left(\int_{B_R(\xi_0)}|u|^2d\xi\right)^{2/Q}\\
&\cdot\int_{Q_R(\xi_0,t_0)}|Xu|^2.
\end{aligned} \tag{54}$$

For J_{22}, by the triangle inequality

$$\begin{aligned}
J_{22}&\leq C\int_{Q_R(\xi_0,t_0)}|\overline{u}(t)|^r\\
&\leq C\int_{Q_R(\xi_0,t_0)}|\overline{u}(t)-c|^r+C\int_{Q_R(\xi_0,t_0)}c^r\\
&\leq C\int_{Q_R(\xi_0,t_0)}|\overline{u}(t)-c|^r+CR^2\left|B_R(\xi_0)\right|c^r\\
&:=J_{221}+J_{222},
\end{aligned} \tag{55}$$

where

$$c=⨍_{t_0-R^2}^{t_0}\overline{u}(t)\,dt. \tag{56}$$

By $rq/2>1$ and Hölder inequality, we have

$$\begin{aligned}
J_{222}&=CR^2\left|B_R(\xi_0)\right|\left(⨍_{t_0-R^2}^{t_0}\overline{u}(t)\,dt\right)^r\\
&=CR^2\left|B_R(\xi_0)\right|\left(⨍_{t_0-R^2}^{t_0}\left(\frac{1}{C_1}⨍_{B_R(\xi_0)}u\eta^4d\xi\right)dt\right)^r\\
&=CR^2\left|B_R(\xi_0)\right|\left(⨍_{Q_R(\xi_0,t_0)}|u|\right)^r\\
&\leq CR^2\left|B_R(\xi_0)\right|\left(⨍_{Q_R(\xi_0,t_0)}|u|^{rq/2}\right)^{2/q}.
\end{aligned} \tag{57}$$

To estimate J_{221}, we use the Poincaré inequality in t

$$\begin{aligned}
J_{221} &\le CR^2 \left|B_R(\xi_0)\right| \left(\int_{t_0-R^2}^{t_0} \left|\partial_t \overline{u}\right| dt\right)^r \\
&\le CR^2 \left|B_R(\xi_0)\right| \left(\int_{t_0-R^2}^{t_0} \left|B_R(\xi_0)\right|^{-1}\right. \\
&\cdot \left.\left|\int_{B_R(\xi_0)} \partial_t u \eta^4 d\xi\right| dt\right)^r = CR^2 \left|B_R(\xi_0)\right| \\
&\cdot \left(\int_{t_0-R^2}^{t_0} \left|B_R(\xi_0)\right|^{-1}\right. \\
&\cdot \int_{B_R(\xi_0)} \left|\left(X_i^* A_i^\alpha(t,\xi,u,Xu) + B^\alpha(t,\xi,u,Xu)\right)\right. \\
&\cdot \left.\left.\eta^4 d\xi\right| dt\right)^r \coloneqq CR^2 \left|B_R(\xi_0)\right| K^r,
\end{aligned} \tag{58}$$

integrating by parts for K, combining with the controllable growth conditions

$$\begin{aligned}
K &\le CR^2 ⨍_{Q_R(\xi_0,t_0)} \left|A_i^\alpha\right| \eta^3 \left|X\eta\right| \\
&\quad + CR^2 ⨍_{Q_R(\xi_0,t_0)} \left|B^\alpha\right| \eta^4 \\
&\le CR ⨍_{Q_R(\xi_0,t_0)} \left(|Xu| + |u|^{r/2} + \left|f_i^\alpha\right|\right) \\
&\quad + CR^2 ⨍_{Q_R(\xi_0,t_0)} \left(|Xu|^{2(1-1/r)} + |u|^{r-1} + \left|f^\alpha\right|\right) \\
&\coloneqq C\left(K_1 + K_2 + K_3 + K_4 + K_5 + K_6\right).
\end{aligned} \tag{59}$$

It remains to estimate each term by the Hölder inequality

$$\begin{aligned}
K_1 &= R ⨍_{Q_R(\xi_0,t_0)} |Xu| \le R \left(⨍_{Q_R(\xi_0,t_0)} |Xu|^{(2/r)(rq/2)}\right)^{2/rq} \\
&\cdot \left(⨍_{Q_R(\xi_0,t_0)} |Xu|^{((r-2)/r)(2r/(r-2))}\right)^{(r-2)/2r} \\
&= \left(⨍_{Q_R(\xi_0,t_0)} |Xu|^q\right)^{2/rq} \left(\int_{Q_R(\xi_0,t_0)} |Xu|^2\right)^{(r-2)/2r}, \\
K_2 &= R ⨍_{Q_R(\xi_0,t_0)} |u|^{r/2} \le R \left(⨍_{Q_R(\xi_0,t_0)} |u|^{rq/2}\right)^{2/rq} \\
&\cdot \left(⨍_{Q_R(\xi_0,t_0)} |u|^{((r-2)/2)(2r/(r-2))}\right)^{(r-2)/2r} \\
&= \left(⨍_{Q_R(\xi_0,t_0)} |u|^{rq/2}\right)^{2/rq} \left(\int_{Q_R(\xi_0,t_0)} |u|^r\right)^{(r-2)/2r}, \\
K_3 &= R ⨍_{Q_R(\xi_0,t_0)} \left|f_i^\alpha\right| \le R \left(⨍_{Q_R(\xi_0,t_0)} \left|f_i^\alpha\right|^2\right)^{1/2} \\
&\le R \left[\left(⨍_{Q_R(\xi_0,t_0)} \left|f_i^\alpha\right|^{(4/r)(r/2)}\right)\right. \\
&\cdot \left.\left(⨍_{Q_R(\xi_0,t_0)} \left|f_i^\alpha\right|^{(2(r-2)/r)(r/(r-2))}\right)\right]^{1/2} \\
&= \left(⨍_{Q_R(\xi_0,t_0)} \left|f_i^\alpha\right|^2\right)^{1/r} \left(\int_{Q_R(\xi_0,t_0)} \left|f_i^\alpha\right|^2\right)^{(r-2)/2r}, \\
K_4 &= R^2 ⨍_{Q_R(\xi_0,t_0)} |Xu|^{2(1-1/r)} \\
&\le R^2 \left(⨍_{Q_R(\xi_0,t_0)} |Xu|^{(2/r)(rq/2)}\right)^{2/rq} \\
&\cdot \left(⨍_{Q_R(\xi_0,t_0)} |Xu|^{(2(r-2)/r)(r/(r-2))}\right)^{(r-2)/r} \\
&= \left(⨍_{Q_R(\xi_0,t_0)} |Xu|^q\right)^{2/rq} \left(\int_{Q_R(\xi_0,t_0)} |Xu|^2\right)^{(r-2)/r}, \\
K_5 &= R^2 ⨍_{Q_R(\xi_0,t_0)} |u|^{r-1} \le R^2 \left(⨍_{Q_R(\xi_0,t_0)} |u|^{rq/2}\right)^{2/rq} \\
&\cdot \left(⨍_{Q_R(\xi_0,t_0)} |u|^{(r-2)(r/(r-2))}\right)^{(r-2)/r} \\
&= \left(⨍_{Q_R(\xi_0,t_0)} |u|^{rq/2}\right)^{2/rq} \left(\int_{Q_R(\xi_0,t_0)} |u|^r\right)^{(r-2)/r}, \\
K_6 &= R^2 ⨍_{Q_R(\xi_0,t_0)} \left|f^\alpha\right| \le R^2 \left(⨍_{Q_R(\xi_0,t_0)} \left|f^\alpha\right|^{r/(r-1)}\right)^{(r-1)/r} \\
&\le R^2 \left[\left(⨍_{Q_R(\xi_0,t_0)} \left|f^\alpha\right|^{(r/(r-1)^2)(r-1)}\right)^{1/(r-1)}\right. \\
&\cdot \left.\left(⨍_{Q_R(\xi_0,t_0)} \left|f^\alpha\right|^{(r(r-2)/(r-1)^2)((r-1)/(r-2))}\right)^{(r-2)/(r-1)}\right]^{(r-1)/r} \\
&= \left(⨍_{Q_R(\xi_0,t_0)} \left|f^\alpha\right|^{r/(r-1)}\right)^{1/r} \left(\int_{Q_R(\xi_0,t_0)} \left|f^\alpha\right|^{r/(r-1)}\right)^{(r-2)/r}.
\end{aligned} \tag{60}$$

By f, f^α, f_i^α, u, the estimation of g, and (14), we get

$$\int_{Q_R(\xi_0,t_0)} |Xu|^2,$$

$$\int_{Q_R(\xi_0,t_0)} |f|^2,$$

$$\int_{Q_R(\xi_0,t_0)} |f_i^\alpha|^2,$$
$$\int_{Q_R(\xi_0,t_0)} |f^\alpha|^{(r-1)/r},$$
$$\int_{Q_R(\xi_0,t_0)} |u|^r, \tag{61}$$

that are uniformly bounded. Combining all the above inequalities

$$\begin{aligned}
&\lambda \fint_{Q_R(\xi_0,t_0)} |Xu|^2 \eta^4\zeta^2 \\
&\le C\left(\fint_{Q_R(\xi_0,t_0)} |Xu|^q + |u|^{rq/2}\right)^{2/q} \\
&\quad + C\fint_{Q_R(\xi_0,t_0)} \left(|f|^2 + |f_i^\alpha|^2 + |f^\alpha|^{r/(r-1)}\right) \\
&\quad + C\sup_{t\in(t_0-R^2,t_0)} \left(\int_{B_R(\xi_0)} |u|^2 \,d\xi\right)^{2/Q} \fint_{Q_R(\xi_0,t_0)} |Xu|^2,
\end{aligned} \tag{62}$$

where $C = C(n, \lambda, \Lambda, \|u\|_{U_2}, \|f\|_{L_2}, \|f^\alpha\|_{L_{r/(r-1)}}, \|f_i^\alpha\|_{L_2})$. (A) is obtained by this estimation and the estimate of J_2.

Next we will prove (B): firstly, we extend u to $Q_T := (0,T)\times\mathbb{R}^n$ such that if $\xi \in \mathbb{R}^n \setminus \Omega$, then $u(\xi,t) = 0$. It is obvious that $u \in V_2(Q)$. Similarly, we extend f, f^α, and f_i^α and let $R < R_0/4$.

Set $(\xi_0,t_0) \in Q_T$ such that $Q_{4R}(\xi_0,t_0) \subset Q_T$. We consider the case $B_{4R}(\xi_0) \subset \Omega$. Following the results in (A)

$$\begin{aligned}
&\fint_{Q_{R/2}(\xi_0,t_0)} \left(|Xu|^2 + |u|^r\right) d\xi\,dt \\
&\quad \le C\left[\fint_{Q_{4R}(\xi_0,t_0)} \left(|Xu|^q + |u|^{rq/2}\right) d\xi\,dt\right]^{2/q} \\
&\quad + C\fint_{Q_{4R}(\xi_0,t_0)} \left(|f|^2 + \left|\overline{f}\right|^2 + |F|^2\right) d\xi\,dt \\
&\quad + C\sup_{t\in(t_0-R^2,t_0)} \left(\int_{B_{4R}(\xi_0)} |u|^2\,d\xi\right)^{2/Q} \\
&\quad \cdot \left(\fint_{Q_{4R}(\xi_0,t_0)} |Xu|^2\,d\xi\,dt\right),
\end{aligned} \tag{63}$$

where $\overline{f} = (f_i^\alpha)$, $F = |f^\alpha|^{(1/2)(r/(r-1))}$, and $C = C(n, \lambda, \Lambda, \|u\|_{V_2}, \|f\|_{L_2}, \|f^\alpha\|_{L_{r/(r-1)}}, \|f_i^\alpha\|_{L_2})$.

By Corollary 10, there exists $R_0' = R_0'(u, \|f\|_{L_2}, \|f^\alpha\|_{L_{r/(r-1)}}, \|f_i^\alpha\|_{L_2}) > 0$ small enough, such that for all $0 < R \le R_0'$

$$C\sup_{t\in(t_0-R^2,t_0)} \left(\int_{B_R(\xi_0)} |u|^2\,d\xi\right)^{2/Q} < 1. \tag{64}$$

For $p \in (2, \min\{\sigma(2(r-1)/r)\tau\}]$ and $Q_{4R}(\xi_0', t_0') \subset Q_T$, we take $\gamma = 2/q > 1$ in Lemma 8, $g = |Xu|^q + |u|^{rq/2}$, $h = |f|^q + |f_i^\alpha|^q + |F|^q$; then

$$\begin{aligned}
&\fint_{Q_{R/2}(\xi_0',t_0')} \left(|Xu|^p + |u|^{rp/2}\right) d\xi\,dt \\
&\quad \le C\left[\fint_{Q_{4R}(\xi_0',t_0')} \left(|Xu|^2 + |u|^r\right) d\xi\,dt\right]^{p/2} \\
&\quad + C\fint_{Q_{4R}(\xi_0',t_0')} \left(|f|^p + \left|\overline{f}\right|^p + |F|^p\right) d\xi\,dt,
\end{aligned} \tag{65}$$

where p and C depend on $n, \lambda, \Lambda, \|u\|_{V_2}, \|f\|_{L_2}, \|f^\alpha\|_{L_{r/(r-1)}}, \|f_i^\alpha\|_{L_2}$. Take the suitable parabolic cylinder $Q_{R/2}(\xi_0', t_0')$ to cover $(\varepsilon, T)\times\Omega$ such that $Q_{4R}(\xi_0', t_0') \subset Q_T$; the result is obtained.

5.2. The Proof of Theorem 2. In the similar way to the proof of Theorem 1, we get (45). By the natural growth conditions (3)–(5), we have

$$\begin{aligned}
&\frac{1}{4}\sup_{t\in(t_0-R^2,t_0)} \int_{B_R(\xi_0)} (u - \overline{u}(t))^2 \eta^4\zeta^2 d\xi \\
&\quad + \frac{\lambda}{2}\int_{Q_R(\xi_0,t_0)} |Xu|^2 \eta^4\zeta^2 d\xi\,dt \\
&\quad \le \frac{1}{2}\int_{Q_R(\xi_0,t_0)} f^2\eta^4\zeta^2 d\xi\,dt \\
&\quad + \frac{\Lambda}{2}\int_{Q_R(\xi_0,t_0)} |Xu|\,|u - \overline{u}(t)|\,\eta^3 \left|X\eta\right| \zeta^2 d\xi\,dt \\
&\quad + \frac{1}{2}\int_{Q_R(\xi_0,t_0)} \left|f_i^\alpha\right| |u - \overline{u}(t)|\,\eta^3 \left|X\eta\right| \zeta^2 d\xi\,dt \\
&\quad + \frac{\Lambda}{2}\int_{Q_R(\xi_0,t_0)} |Xu|^2\,|u - \overline{u}(t)|\,\eta^4\zeta^2 d\xi\,dt \\
&\quad + \frac{1}{2}\int_{Q_R(\xi_0,t_0)} \left|f^\alpha\right| |u - \overline{u}(t)|\,\eta^4\zeta^2 d\xi\,dt \\
&\quad + \int_{Q_R(\xi_0,t_0)} (u - \overline{u}(t))^2 \eta^4\zeta \left|\partial_t\zeta\right| d\xi\,dt =: R_1 + R_2 \\
&\quad + R_3 + R_4 + R_5 + R_6.
\end{aligned} \tag{66}$$

We estimate each term by Young inequality and Sobolev-Poincaré inequality

$$\begin{aligned}
R_2 &\leq \frac{\varepsilon\Lambda}{2}\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 \\
&\quad + \frac{C\Lambda}{\varepsilon}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2|X\eta|^2\eta^2\zeta^2 \\
&\leq \frac{\varepsilon\Lambda}{2}\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 \\
&\quad + \frac{C\Lambda}{\varepsilon}R^{-2}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^2\zeta^2 \\
&\leq \frac{\varepsilon\Lambda}{2}\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 \\
&\quad + \frac{C\Lambda}{\varepsilon}R^{-2}\left(\int_{Q_R(\xi_0,t_0)}|Xu|^{4/r}\right)^{r/4}, \\
R_3 &\leq C\int_{Q_R(\xi_0,t_0)}\left|f_i^\alpha\right|^2\eta^4\zeta^2 \\
&\quad + C\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2|X\eta|^2\eta^2\zeta^2 \\
&\leq C\int_{Q_R(\xi_0,t_0)}\left|f_i^\alpha\right|^2\eta^4\zeta^2 \\
&\quad + CR^{-2}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^2\zeta^2 \\
&\leq C\int_{Q_R(\xi_0,t_0)}\left|f_i^\alpha\right|^2\eta^4\zeta^2 \\
&\quad + CR^{-2}\left(\int_{Q_R(\xi_0,t_0)}|Xu|^{4/r}\right)^{r/4}, \\
R_4 &= \frac{\Lambda}{2}\int_{Q_R(\xi_0,t_0)}|Xu|^2(u-\overline{u}(t))\eta^4\zeta^2 \\
&\leq \Lambda M\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2, \\
R_5 &\leq C\int_{Q_R(\xi_0,t_0)}\left|f^\alpha\right|^{r/(r-1)}\eta^4\zeta^2 \\
&\quad + C\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^r.
\end{aligned} \tag{67}$$

Therefore

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)}\int_{B_R(\xi_0)}(u-\overline{u}(t))^2\eta^4\zeta^2 d\xi \\
&\quad + (\lambda-2\Lambda M-\varepsilon\Lambda)\int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 d\xi\, dt \\
&\leq \frac{C}{\varepsilon}\Lambda R^{-2}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^2\zeta^2 \\
&\quad + C\int_{Q_R(\xi_0,t_0)}\left(|u-\overline{u}(t)|^r\right) \\
&\quad + C\int_{Q_R(\xi_0,t_0)}\left(f^2+\left|f_i^\alpha\right|^2+\left|f^\alpha\right|^{r/(r-1)}\right).
\end{aligned} \tag{68}$$

Since $\lambda - 2\Lambda M > 0$, we take $\varepsilon = (\lambda - 2\Lambda M)/3\Lambda$; then

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)}\int_{B_R(\xi_0)}(u-\overline{u}(t))^2\eta^4\zeta^2 d\xi \\
&\quad + \int_{Q_R(\xi_0,t_0)}|Xu|^2\eta^4\zeta^2 d\xi\, dt \\
&\leq CR^{-2}\int_{Q_R(\xi_0,t_0)}|u-\overline{u}(t)|^2\eta^2\zeta^2 \\
&\quad + C\int_{Q_R(\xi_0,t_0)}\left(|u-\overline{u}(t)|^r\right) \\
&\quad + C\int_{Q_R(\xi_0,t_0)}\left(f^2+\left|f_i^\alpha\right|^2+\left|f^\alpha\right|^{r/(r-1)}\right) \\
&:= C_0\left(S_1+S_2+S_3\right).
\end{aligned} \tag{69}$$

Following the estimate (50) in Theorem 1, one has

$$\begin{aligned}
S_1 &\leq \frac{\lambda}{16C_0}\sup_{t\in(t_0-R^2,t_0)}\int_{B_R(\xi_0)}|u-\overline{u}(t)|^2\eta^4\zeta^2 \\
&\quad + C\left(R^{-q}\int_{Q_R(\xi_0,t_0)}|Xu|^q\right)^{2/q}.
\end{aligned} \tag{70}$$

Similar to the estimate of J_{21} in Theorem 1

$$S_2 \leq C\sup_{t\in(t_0-R^2,t_0)}\left(\int_{B_R(\xi_0)}|u|^2 d\xi\right)^{2/Q}\int_{Q_R(\xi_0,t_0)}|Xu|^2. \tag{71}$$

By $f, f^\alpha, f_i^\alpha, u$, the assumption of g, and (14), we know that

$$\begin{aligned}
&\int_{Q_R(\xi_0,t_0)}|Xu|^2, \\
&\int_{Q_R(\xi_0,t_0)}|f|^2, \\
&\int_{Q_R(\xi_0,t_0)}\left|f_i^\alpha\right|^2, \\
&\int_{Q_R(\xi_0,t_0)}\left|f^\alpha\right|^{(r-1)/r}, \\
&\int_{Q_R(\xi_0,t_0)}|u|^r.
\end{aligned} \tag{72}$$

They are uniformly bounded. Combining all the estimations above, we have

$$\begin{aligned}
&⨍_{Q_R(\xi_0,t_0)} |Xu|^2 \eta^4 \zeta^2 \\
&\le C \left(⨍_{Q_R(\xi_0,t_0)} |Xu|^q \right)^{2/q} \\
&\quad + C ⨍_{Q_R(\xi_0,t_0)} \left(|f|^2 + |f_i^\alpha|^2 + |f^\alpha|^{r/(r-1)} \right) \\
&\quad + C \sup_{t\in(t_0-R^2,t_0)} \left(\int_{B_R(\xi_0)} |u|^2 \, d\xi \right)^{2/d} ⨍_{Q_R(\xi_0,t_0)} |Xu|^2 ,
\end{aligned} \tag{73}$$

where $C = C(n, \lambda, \Lambda, \|u\|_{U_2}, \|f\|_{L_2}, \|f^\alpha\|_{L_{r/(r-1)}}, \|f_i^\alpha\|_{L_2})$. Then (C) is derived.

Now we prove (D). Similar to (B), we just need to take $\gamma = 2/q > 1$, $g = |Xu|^q$, $h = |f|^q + |f_i^\alpha|^q + |F|^q$ in Lemma 8; then

$$\begin{aligned}
&⨍_{Q_{R/2}(\xi_0',t_0')} |Xu|^p \, d\xi \, dt \\
&\le C \left[⨍_{Q_{4R}(\xi_0',t_0')} |Xu|^2 \, d\xi \, dt \right]^{p/2} \\
&\quad + C ⨍_{Q_{4R}(\xi_0',t_0')} \left(|f|^p + |\overline{f}|^p + |F|^p \right) d\xi \, dt,
\end{aligned} \tag{74}$$

where p and C depend on $n, \lambda, \Lambda, \|u\|_{V_2}, \|f\|_{L_2}, \|f^\alpha\|_{L_{r/(r-1)}}$, $\|f_i^\alpha\|_{L_2}, M$. The proof is complete by using suitable parabolic cylinder $Q_{R/2}(\xi_0', t_0')$ to cover $(\varepsilon, T) \times \Omega$ such that $Q_{4R}(\xi_0', t_0') \subset Q_T$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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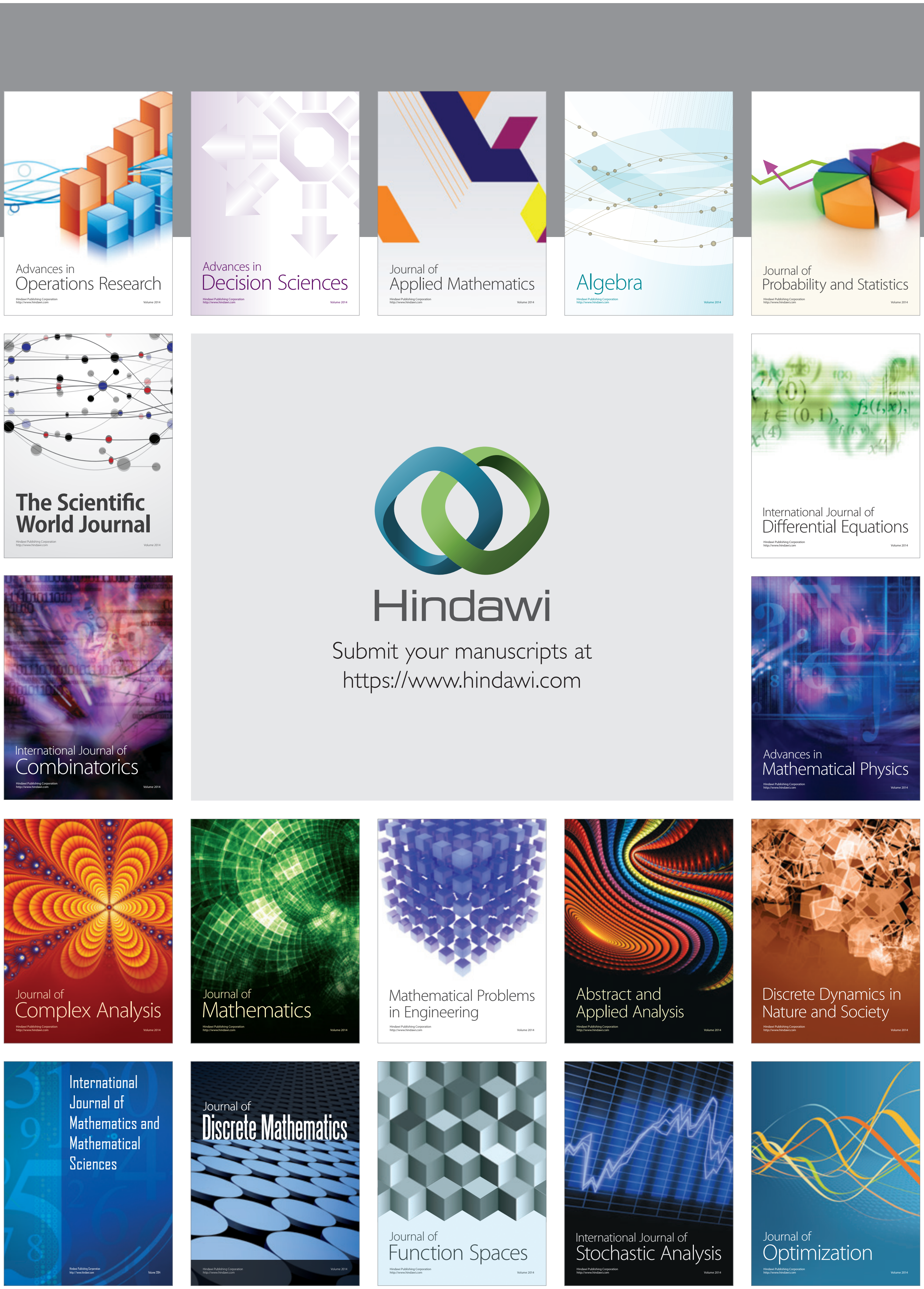
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