# INSIGHTS INTO NUMBER: DEVELOPING FLEXIBLE MENTAL MATHEMATICS.

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*The ability to use flexible strategies when faced with a calculation to do 'in the head' is important to acquire. Many children who rely on remembered facts and standard procedures are limited in their approaches to mental computation. Teaching that develops insights, awareness, and connections appears to give this important flexibility and enable children to make sensible and effective choices of calculating strategies.*

## Introduction

Many children and adults find calculating mentally difficult as they have few strategies to use, rely on inefficient counting or standard methods, and have few, if any, insights into relationships and the structure of the number system. They have possibly learned their mathematics via a combination of remembering facts and reproducing procedures. In many cases they are able to achieve well on tests that require such skills. However, when they have to work in other ways and in other contexts they find they are limited in their approaches by the very things they know. They are unable to be flexible in the way they calculate.

The notions of flexibility in calculating and an understanding of numbers are keys to number sense. In fact, McIntosh, Reys, Reys, Bana and Farrell (1997) define number sense as:

> *A person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful and efficient strategies for managing numerical situations. (p. 3)*

In this paper we argue that a focus on specific strategies in specific lessons does not assist children to develop flexibility with number but rather reduces the teaching of mental computation to an algorithmic approach, where the focus is on a single mental strategy to be taught, remembered, and practised. An alternative to such an approach would see mental strategies embedded in examples of calculations to be drawn out by the teacher and made explicit to the learner. Alongside this would be an emphasis on patterns and relationships in the numbers. The focus on connections within and between numbers gives children the power to use numbers in flexible forms and in flexible ways.

## Developing flexible mental strategies

Thompson (1999, p. 152) presented the following extended model of mental calculation (Figure 1) whereby the establishment of flexible mental strategies consisted of the development and subsequent integration of a range of factors.

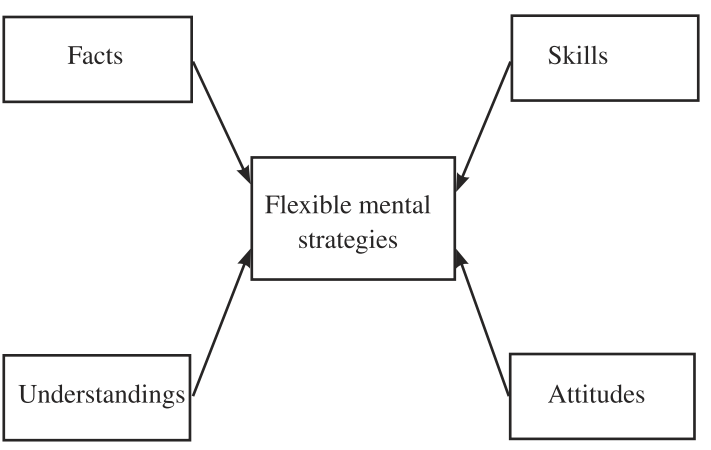



Figure 1 An extended model of mental calculation

The *Facts* category includes knowledge of number facts as well as strategies. *Understandings* refers to properties of the number system such as Commutativity (i.e. 8 x 3 and 3 x 8 will give the same result); and the relationships between the various operations and divisibility rules. *Skills* refers to labour-saving techniques, such as knowing quickly how to multiplying by ten in a range of situations.

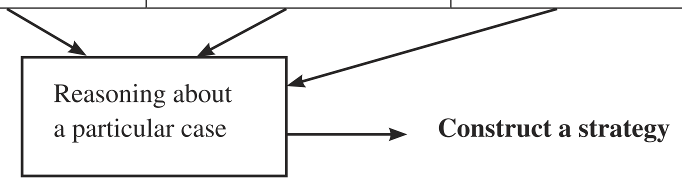
The inclusion of Attitudes in the model is of particular interest. Children need to be confident enough to 'have a go' in order to try various strategies to calculate. The implication for teaching is that mental computation sessions need to be carried out within a supportive classroom environment. A classroom of constant testing via 'quick mentals', 'mad minutes', and activities such as Round the World and Sheriff, does not provide that sort of environment. It develops the negative attitude to mathematics shown by many children, and their exclusion from mathematics learning. Many children will not have a go at a calculation, often due to their previous negative experiences with mental mathematics. Thompson's category relating to attitude is vital in developing effective mental mathematics. The children's attitude must be positive.

## Developing insights into mental calculation

Sugarman (2002) also emphasised the broader version of mental mathematics in the primary classroom context. Contained within Sugarman's diagram (Figure 2) and Thompson's diagram (Figure 1) is the notion espoused by Beishuizen (1997) of mental mathematics *in the head* (the recall of facts) and mental mathematics *with the head* (the use of strategies for calculating). Here is an emphasis on developing awareness or an insight into the workings and the secrets of the number system. It has things to remember but the emphasis is on understanding. It is the opposite of rote learning which accompanies a lack of understanding. In this case there is memorisation, which is committing to memory by children of relationships and number facts. If these are forgotten they can be efficiently recreated. Memorisation assumes understanding.

Figure 2 Contributions to invented methods of calculation

| **What can you do?** | **What do you realise?** | **What can you remember?** |
|---|---|---|
| Developing skills of:<br>• Counting<br>• Partitioning<br>• Transforming | Building awareness of:<br>• The number system<br>• Number relationships<br>• Strategies | Developing recall of facts<br>• Doubles<br>• Number bonds<br>• Multiplication tables |




## Connections, realisations and insights

One of the features of an effective teacher of numeracy (Askew, Brown, Rhodes, Johnson & Wiliam, 1997) is that of helping children make connections, that is children connect their new mathematical knowledge to what they already know or make connections between new pieces of knowledge. A key insight here is for children to realise that *if a fact is known then others can be derived from it.*

Consider the strategy of using a known fact to determine an unknown fact. Children can use what they know to derive new facts, and these in turn may be used to improve their store of known facts. Askew (1998) used the following diagram (Figure 3) to show how this relationship works.

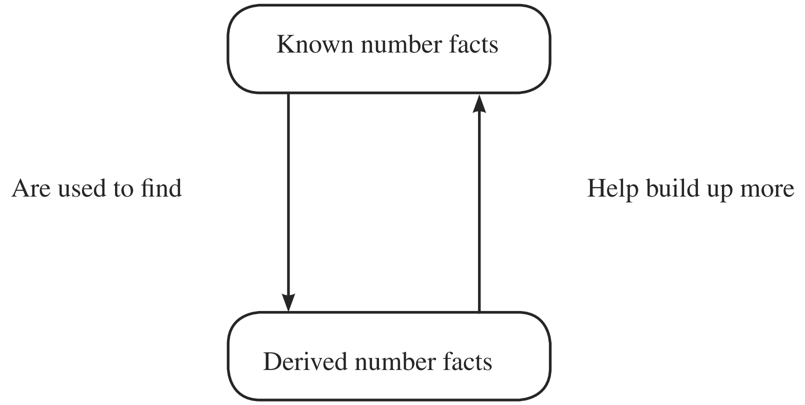



Figure 3 Growing mental facts

This approach might be used by a student to work out 6 x 7 by using the known fact that 6 x 6 is 36 and then adding one more 6 to arrive at the correct answer of 42.

Thompson (1999) argued that the view presented by Askew is rather too simplistic. He noted a more sophisticated viewpoint:

> *It is debatable whether this actually is the case, even for single-digit numbers. It is not necessarily the accumulation of more known facts (important though this may be) that contributes to the expansion of an individual's range of strategies, but rather a more detailed understanding of the workings of our own number system linked to an increased level of confidence to 'have a go'. (p. 151)*

It seems appropriate, if one values flexibility, to use a general mental strategies framework and to draw strategies from the children and then focus on a particular strategy rather than teach a specific lesson on the practice of a specifically taught strategy. Quite clearly some numbers and some calculations lend themselves to particular strategies, or possibily one strategy, but children need to develop this realisation and the ability to choose an appropriate strategy or strategies to use. It is important to encourage the use of an efficient strategy for the particular numbers evident in the calculation. For example, when adding 28 and 27 a student might choose one of several strategies:

20 + 20 + 8 + 7      28 + 20 + 7      28 + 28 -1      27 + 27 + 1.

Which strategy the student choses would depend on the confidence of the child with particular number facts and strategies. What is clear is that the choice to use a particular strategy over another should be left to the child. If, however, the student had chosen to use a counting on in ones strategy, this is clearly inefficient and likely to lead to errors and the student could be encouraged to apply one of the other methods that he or she best understands. A fellow student could be asked to explain his/her method to this student or the rest of the class in order to demonstrate another way to complete the calculation.

Insights into numbers are important to develop as from this knowledge children can begin to 'see' a number in a variety of ways. From these insights children can select the most appropriate way to 'see' a number in a calculation. For example, 29 might be 'seen' as 30 – 1 whereas 100 can be 'seen' in a myriad of ways. The following activity can help with developing a multi-faceted view of a number.

**Tell me about … (72)**

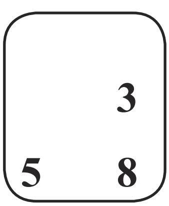
Children in pairs are asked to write down (and later share) as many things as they know about the selected number. The final part of the short lesson will collect and record all the things that the class can tell someone about 72. These facts could be precise, numerical, connected to their lives in some way, or completely weird.

72 is
Six twelves
Almost three quarters of a hundred
An even number
Divisible by 2
The digits add to 9 so it is divisible by 9
It is the age of my Nonna
Karen lives at 72 Hill Street
Half of 144

Over time, children see, hear and write about numbers in different forms and are often more able to connect these facts to other facts and other numbers.

**Connect three numbers**

Children are presented with a small card. On the card are three numbers. Pairs of children have to write different numbers sentences to show how the three numbers are connected. For example, with this card the children might write the following:




3 + 5 = 8     5 + 3 = 8     8 – 5 = 3

**Given and then find**

**1.** Given   3 x 37 = 111, find:

6 x 37     15 x 37     999 ÷ 37     3 x 74

Explain how you used the given information to find the answers.

**2.** Given   $25^2$ = 625, find:

25 x 26     $26^2$     25 x 24     $24^2$

Explain how you used the given information to find the answers.

**3.** Given 12 x 12 = 144, find:

1.2 x 1.2 12 x 1.2 0.12 x 0.12 1.44 ÷ 1.2

Explain how you used the given information to find the answers.

**4.** Given 58 + 77 = 135, find:

68 + 77 58 + 67 135 – 77 135 – 68

145 – 58 258 + 177

Explain how you used the given information to find the answers.

**5.** Given 17 x 43 = 731, find:

17 x 44 18 x 43 18 x 44 16 x 43

Explain how you used the given information to find the answers.

**6.** Given 360 ÷ 9 = 40, find:

360 ÷ 18 360 ÷ 4.5 360 ÷ 45 3600 ÷ 9

Explain how you used the given information to find the answers.

## Routines, landmarks and triggers

The use of routines is particularly powerful in helping children develop number sense and insights into numbers. While there are dangers of overuse and resulting boredom, many children welcome the familiarity of situations and tasks. From a teaching point of view they are useful to reduce the amount of time spent on explaining to the children what has to happen. Familiar activities, such as *Today's Number is,* and *How did you do it?,* can move straight into action as children know exactly what to do.

*Landmark numbers* are numbers that are used a lot by people skilled in mental calculation. They are similar to numbers that can be used to develop insights as noted earlier. Generally, these are numbers that are used to make calculations easy, for example, 10, 100, 12, 24, and 25. Children tend to have lots of information about these numbers and hence can use them in flexible ways. Gradually, by taking part in a range of activities children build their 'sight vocabulary' of such numbers.

*Trigger numbers* are useful for children to recognise as they can be smashed up into parts to make calculations easier or to remind children of a particular calculation strategy. For example, the presence of a 9 suggests a strategy of moving to 10 and later compensating by removing one. Two numbers that are the same or are different by one might trigger the use of a doubling strategy. In the following example the number 25 for many children will trigger the knowledge of 4 x 25 = 100 and its connections.

36 x 25

Sophisticated calculators might realise that 25 is one quarter of 100, therefore if the first number is divisible by 4 this knowledge is then useful in performing the calculation as 36 ÷ 4 x 100 or 9 x 100 = 900.

Or the child could use factors of 36 and reorganising the order of numbers

4 x 9 x 25

9 x 4 x 25 (use of Commutative property)

9 x (4 x 25) (use of Associative property)

9 x 100 = 900

## Metacomputation

One powerful way to help children develop insights into mental calculating methods is to hear the general thinking strategies of an experienced calculator. Modelling thinking when faced with a calculation helps children to think about their own thinking. Here thinking is made explicit. So often novice calculators are not given access to how people approach calculating situations. Modelling general thinking strategies, for example — *What can I do here? What do I know about these numbers? This has got a 9 in it so what strategy can I use?* — is different from teaching a specific procedure for a calculation.

## Teaching approaches

Developing an awareness and insight into the properties and relationships in numbers and the number system takes time. Children gradually build their vocabulary of connections and strategies. This suggests that mental mathematics in the classroom should be undertaken 'little and often'. The use of simple routines and establishing expectations that children will think about and explain their calculation methods in flexible ways will be to the forefront.

The role of the teacher in mental mathematics is vitally important. Mental mathematics in the classroom needs an emphasis on learning and teaching rather than 'testing'. The daily 'ten quick mental' or 'Friday mad minutes' do little to develop a positive attitude to mathematics or the ability to calculate mentally. Careful and insightful planning are the keys to developing good mental calculation abilities in children. Strategies, patterns and relationships need to be carefully embedded in tasks set for children. There needs to be explicit reflection and connection on these strategies, patterns and relationships as part of the overall lesson. A vague and somewhat woolly idea that 'children will discover mathematics for themselves' will not establish flexible mental calculating abilities.

Much can be achieved, alongside careful planning, by the use of materials such as empty number lines, hundred squares, and calculators. Many successful teachers also expect children to make decisions; to choose and explain strategies for calculations rather than reproduce a set, taught procedure.

## Insights into calculating

It appears that some children and adults know quite a lot about calculating while others are unaware of these insights. Possibly, such insights should be made explicit to them as part of their mental mathematics education? A selection is shown here.

| Insight | Example |
|---|---|
| Awkward numbers can be made easier by rounding | 19 becomes 20 or 99 is 100 |
| Subtractions can be done by adding on, especially if the numbers are close | 42-38<br>38 to 40 to 42 |
| Adding can be done in stages using multiples of ten and a hundred as landmarks | 26+48<br>26 to 30 to 70 to 74 |
| The empty number line is a useful way to show a calculating strategy | |
| I can check the size of an answer by rounding the numbers to give an idea of the magnitude of the answer | 32x47<br>is 50x30 = 1500 |
| I can make calculations easier by using 'nice numbers' | 36x25<br>9x4x25<br>9x100 = 900 |
| 100 is a nice number | |
| There is no one correct or proper method for calculating | |
| There are a range of ways for calculating, the strategy will depend on the numbers involved and the purpose of the calculation | |
| Doubling and halving is a useful strategy for adding and multiplying | 6+7 = 6+6+1 = 13<br>6x7 = 6x6 + 6 |
| If a fact is known then others can be worked out from it | |
| Some strategies are more suited to certain calculations | |

## Conclusion

It appears that children are unlikely to develop a range of mental calculating strategies without being exposed to them in explicit ways. This exposure can be achieved in a variety of ways and does not need to have explicit teaching of each strategy in a lock step way to achieve this outcome.

Knowing key facts with understanding enables counting to be avoided. This can lead to children using more efficient methods and avoiding the use of a standard procedure in the head as their only method of calculating.

Letting children into the secrets of mathematics and calculation with their useful insights is important so that children are included in and not excluded from mathematics. More significantly, such insights will help children to achieve success and from this success will come a more positive attitude to mathematics and their personal ability to 'do it'.

Above all sense making by children will begin to flow from the developing insights and the understanding of 'what is going on'. This is vitally important in children establishing efficient, flexible mental mathematics.

## References

Askew, M. (1998). *Teaching primary mathematics: A guide for newly qualified and student teachers.* London: Hodder & Stoughton.

Askew, M., Brown, M., Rhodes, V., Johnson, D. & Wiliam, D. (1997) *Effective teaching of numeracy.* London: King's College.

Beishuizen, M. (1997) Mental arithmetic: Mental recall or mental strategies. *Mathematics Teaching* 160, 16-19.

McIntosh, A., Reys, B., Reys, R., Bana, J. & Farrell, B. (1997). *Number sense in school mathematics: Student performance in four countries*. Perth: MASTEC.

Sugarman, I. (2000). *Children calculating*. The Standards Site: Numeracy.

Thompson, I. (1999). Getting your head around mental calculation. In I. Thompson (Ed). *Issues in teaching numeracy in primary schools*. Buckingham: Open University Press.