

FIRST ORDER BSPDEs IN HIGHER DIMENSION FOR OPTIMAL CONTROL PROBLEMS*

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Abstract. This paper studies the first order backward stochastic partial differential equations suggested earlier for a case of multidimensional state domain with a boundary. These equations represent analogues of Hamilton–Jacobi–Bellman equations and allow one to construct the value function for stochastic optimal control problems with unspecified dynamics where the underlying processes do not necessarily satisfy stochastic differential equations of a known kind with a given structure. The problems considered arise in financial modeling.

Key words. stochastic optimal control, Hamilton–Jacobi–Bellman equations, backward SPDEs, first order BSPDEs

AMS subject classifications. 91G80, 93E20, 91G10

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1. Introduction. In Bender and Dokuchaev [6, 7] some special first order backward stochastic partial differential equations (BSPDEs) were suggested and studied for the case of a one dimensional state variable. These equations were used for optimal value functions for pricing problems for swing options with very mild assumptions on the underlying payoff process. The present paper extends these results on multidimensional case.

Stochastic partial differential equations (SPDEs) are well studied in the literature, including the case of forward and backward equations; see, e.g., [54, 1, 14, 15, 53, 55, 50, 3, 15, 47, 16, 37, 42, 48, 31, 13, 20, 49, 34] and the bibliographies therein.

BSPDEs represent versions of the so-called Bismut–Peng equations, where the diffusion term is not given a priori but needs to be found; see, e.g., [51, 55, 18, 19, 21, 22, 23, 24, 25, 26, 28, 29, 40, 46] and the bibliographies therein.

Some additional conditions on the coercivity are usually imposed in the literature; see, e.g., condition (0.4) in [53, Chapter 4]. Without these conditions, a parabolic type SPDE is regarded as degenerate. These degenerate equations of the second order (with a relaxed coercivity condition) were also widely studied; see the bibliographies in [29, 30, 25, 26]. For the degenerate backward SPDEs in the whole space, i.e., without boundaries, regularity results were obtained in [46, 40, 28, 29]. Some special first order forward SPDEs without boundary were considered in [36] and [41] by the method of characteristics, and in [38]. The methods developed in these works cannot be applied in the case of a domain with boundary because of regularity issues that prevent using an approximation of the differential operator by a nondegenerate one. It turns out that the theory of degenerate SPDEs in domains is much harder than in the whole space and was, to the best of our knowledge, not addressed yet in the existing literature, except for first attempts in [6, 7] and in [25]. The present paper also considers a problem of this kind.

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It is common to use BSPDEs as stochastic analogues of the backward Kolmogorov–Fokker–Planck equations or related Hamilton–Jacobi–Bellman (HJB) equations known for controlled Markov diffusion type processes. In non-Markovian control problems, the backward HJB equations have to be replaced by corresponding backward SPDEs; this was first observed by [51] in a setting with backward SPDEs of the second order equations such that the matrix of the higher order coefficients is positive definite.

In [6, 7], new special first order BSPDEs were introduced. They represented analogues of HJB equations for some non-Markovian stochastic optimal control problems associated with pricing of swing options in continuous time. These equations are not exactly differential, since their solutions can be discontinuous in time, and they allow very mild conditions on the underlying driving stochastic processes with unspecified dynamics. More precisely, the method does not have to assume a particular evolution law of the underlying process; the underlying processes do not necessarily satisfy stochastic differential equations of a known kind with a given structure. In particular, the first order BSPDEs describe the value function even in the situation where the underlying price process cannot be described via a stochastic equation ever described in the literature. The numerical solution requires just to calculate certain conditional expectations of the functions of the process without using its evolution law. (See the discussion in section 4.) It can be also noted that these equations are not the same as the first order deterministic HJB equations known in the deterministic optimal control.

In the present paper, we extend the approach suggested in [6, 7] and derive some first order BSPDEs for multidimensional domains using a different proof. Again, the suggested BSPDEs represent analogues of HJB equations for some non-Markovian stochastic optimal control problems associated with applications in financial modeling. The paper establishes the existence of solutions for these equations and the fact the value functions for underlying control problems satisfy these equations. Some numerical methods are discussed.

2. Problem setting. Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $\Omega = \{\omega\}$. Let $X(t) = (X_1(t), \dots, X_n(t))^T$ be a current random process with the values in \mathbf{R}^n such that $X(t)$ is RCLL (right continuous with left limits) process, $\mathbf{E} \sup_{t \in [0, T]} |X(t)|^2 < +\infty$, and $X_i(t) \geq 0$ a.s. for all t and i .

Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by $X(t)$.

We emphasize that an evolution equation for X is not specified, similarly to the setting from [6, 7]. For instance, we do not assume that $X(t)$ is solution of an Itô's equation with particular drift and diffusion coefficient, or of any other equation such as jump-diffusion equation, etc. The case where dynamics of X is described by one of these equations is not excluded; however, one does not need the structure and parameters of these equations for our analysis. This is an unusual setting for stochastic control and the theory of HJB equations. As is discussed in section 4 below, one has to know conditional distributions of $X(t_{k+1})$ given \mathcal{F}_{t_k} and a sampling sequence $\{t_k\} \subset [0, T]$ for numerical implementation of the analytical results.

Let a positive integer n be given.

For $x = (x_1, \dots, x_n)^T \in \mathbf{R}^n$ and $y = (y_1, \dots, y_n)^T \in \mathbf{R}^n$, we write $x \preceq y$ if and only if $x_i \leq y_i$ for all i .

Let $\Gamma_0 \subset \mathbf{R}^n$ be a closed convex conic set such that if $x \in \mathbf{R}^n$, $x \preceq \hat{x}$ and $\hat{x} \in \Gamma_0$ then $x \in \Gamma_0$, and such that, for any $x = (x_1, \dots, x_n)^T \in \Gamma_0$ and any $j \in \{1, \dots, n\}$, there exists $M > 0$ such that $x = (x_1, \dots, x_j + M, \dots, x_n)^T \notin \Gamma_0$. Let $\hat{g} \in \mathbf{R}^n$ be a given vector with positive components, and let $\Gamma = \{y \in \mathbf{R}^n : y = x + \hat{g}, x \in \Gamma_0\}$.

Example 2.1. In particular, the following choices of Γ are admissible.

- (i) $\Gamma = \{x = (x_1, \dots, x_n)^\top \in \mathbf{R}^n : x_i \leq \alpha_i, i = 1, \dots, n\}$, where $\alpha_i > 0$ are given.
- (ii) $\Gamma = \{x \in \mathbf{R}^n : x^\top a \leq 1\}$, where $a \in \mathbf{R}^n$ is a given vectors with positive components.
- (iii) $\Gamma = \{x = (x_1, \dots, x_n)^\top \in \mathbf{R}^n : x^\top a_j \leq 1, j = 1, \dots, m\}$, where $m > 0$, and where $a_j \in \mathbf{R}^n$ are given nonzero vectors with nonnegative components, such that, for any $i \in \{1, \dots, n\}$, there exists j such that i th component of a_j is positive.

Let $K \subset \mathbf{R}^n$ be a given convex set.

For $y \in \Gamma$, let $U(t, y)$ be the set of processes $u(s) = (u_1(s), \dots, u_n(s)) : [t, T] \times \Omega \rightarrow K$ being adapted to \mathcal{F}_t and such that $y + \int_t^T u(s) ds \in \Gamma$ a.s.

Let a function $f(x, u, t) : \mathbf{R}^n \times K \times [0, T] \rightarrow \mathbf{R}$ be given.

We consider two cases:

- (i) $f(x, u, t) = u^\top x$ and $K = [0, L]^n$ for some given $L > 0$.
- (ii) $f(x, u, t) = u^\top x - u^\top G u$, where G is a symmetric positive-definite matrix, and $K = \mathbf{R}^n$.

These conditions seems very special; however, they cover many important optimization problems arising in mathematical finance. The special case where $G = 0$ and $n = 1$ covers the pricing problem for swing options considered in [6, 7].

2.1. Optimal control problem. For given $y \in \Gamma$ and $t < T$, we consider the problem

$$(2.1) \quad \text{Maximize } \mathbf{E}F(u, t) \quad \text{over } u \in U(y, t).$$

Here

$$(2.2) \quad F(u, t) = \int_t^T f(X(s), u(s), s) ds.$$

Let y be a \mathcal{F}_t -measurable random variable with values in $[0, 1]$, and let

$$(2.3) \quad J(t, y) = \text{ess sup}_{u \in U(t, y)} \mathbf{E}_t F(u, t).$$

In other words, this is the value function for problem (2.1). Here $\mathbf{E}_t = \mathbf{E}\{\cdot | \mathcal{F}_t\}$; we use this notation for brevity.

Applications in finance. In [6, 7], it was shown that the above introduced optimization problem (2.1) with $n = 1$, $K = [0, L]$, and $G = 0$ gives a solution for pricing problem for swing options with underlying payoff process $X(t)$. In this setting, $u(t)$ is the exercise rate selected by the option holder. The swing option holder wishes to maximize the expected cumulative payoff by selection of the distribution in time $u(t)$ of the exercise rights. In addition, it was shown that this problem can be used for approximation of prices for multidimensional American options, with the choice of $L \rightarrow +\infty$. Similarly, problem (2.1) with $n > 1$ can be interpreted as a pricing problem for swing options on consumptions of n different types of energy. The choice of $G \neq 0$ can be used to model the settings where there are no hard constraints on the rates of exercises, with a penalty for excessive rates instead. Again, these problems can be used for approximation of the classical solutions for the pricing problems for multidimensional American options. For this, we have can consider either $G \rightarrow 0$ and $K = \mathbf{R}^n$ or $G \neq 0$, $K = [0, L]^n$, and $L \rightarrow +\infty$.

Another possible setting is where the swing option holder has to maximize the expected cumulative payoff but where there is a preferable vector of exercise rates $\tilde{u}(t)$; it could be a currently observable random process defined by dynamically changing technological or market conditions. Let $X(t)$ be the current vector of the underlying payoffs. The swing option holder wished to maximize

$$\int_0^T [X(t)^\top u(t) - (u(t) - \tilde{u}(t))^\top G(u(t) - \tilde{u}(t))] dt.$$

The term $(u(t) - \tilde{u}(t))^\top G(u(t) - \tilde{u}(t))$ represents some penalty for mismatching the preferable rate $\tilde{u}(t)$. This is equivalent to the maximization of

$$\int_0^T [X(t)^\top u(t) + 2\tilde{u}(t)^\top Gu(t) - u(t)^\top Gu(t)] dt = \int_0^T [\tilde{X}(t)^\top u(t) - u(t)^\top Gu(t)] dt,$$

where $\tilde{X}(t) = X(t) + 2G\tilde{u}(t)$. This is a special case of the problem introduced above.

Some related setting for optimal energy trading was developed in [27].

Existence of optimal u and properties of J .

LEMMA 2.1. *For every pair (t, Y) , where t is a stopping time and Y is a \mathcal{F}_t -measurable random vector with values in Γ , there is an optimal control $\bar{u} \in U(t, Y)$.*

Let $u^{t,y}(s)$ be an optimal control for (2.3) (or one of the optimal controls).

LEMMA 2.2.

- (i) *For $t \in [0, T]$ and $\bar{y} \in \Gamma$, $\sup_{y \in \Gamma: \bar{y} \preceq y, t \in [0, T]} \text{ess sup}_\omega |J(t, y)| \leq \text{const}$.*
- (ii) *For any $t \in [0, T]$ and $y, \hat{y} \in \Gamma$ such that $y \preceq \hat{y}$, we have that $J(t, \hat{y}) \leq J(t, y)$ a.s.*
- (iii) *For any $t \in [0, T]$ and $j \in \{1, \dots, n\}$, $J(t, y)$ is almost surely Lipschitz in y uniformly in any bounded subset of $[0, T] \times \Gamma$.*

LEMMA 2.3. *The function $J(t, y)$ is concave in $y \in \Gamma$, a.s. for all $t \in [0, T]$.*

By Lemma 2.3, the left-hand partial derivatives $D_{y_j}^- J(t, y)$ and the right-hand partial derivatives $D_{y_j}^+ J(t, y)$ exist and are uniquely defined.

3. The main result. We denote by $D_y^\pm J(s, y)$ the vector columns $\{D_{y_i}^\pm J(s, y)\}_{i=1}^n$ in \mathbf{R}^n . We denote $(x)_+ = \max(x, 0)$.

Let $\partial\Gamma$ be the boundary of Γ .

THEOREM 3.1. *The value function J satisfies the following first order BSPDE in (t, y) :*

$$(3.1) \quad J(t, y) = \mathbf{E} \left\{ \int_t^T \sup_{v \in K} (f(X(s), v, t) + v^\top D_y^+ J(s, y)) ds \middle| \mathcal{F}_t \right\}, \quad t < T, \quad y \in \Gamma,$$

$$J(t, y)|_{y \in \partial\Gamma} = 0.$$

In particular, if $f(t, x, u) = u^\top X$ and $K = [0, L]^n$, then the equation has the form

$$J(t, y) = \mathbf{L} \mathbf{E} \left\{ \int_t^T \left(\sum_{i=1}^n (X_i(s) + D_{y_i}^+ J(s, y))_+ \right) ds \middle| \mathcal{F}_t \right\}, \quad t < T, \quad y \in \Gamma,$$

$$J(t, y)|_{y \in \partial\Gamma} = 0.$$

If $K = \mathbf{R}^n$ and $f(t, x, u) = u^\top X - u^\top Gu$, then the equation has the form

$$J(t, y) = \frac{1}{4} \mathbf{E} \left\{ \int_t^T (X(s) + D_y^+ J(s, y))^\top G^{-1} (X(s) + D_y^+ J(s, y)) ds \middle| \mathcal{F}_t \right\}, \quad t < T, \quad y \in \Gamma,$$

$$J(t, y)|_{y \in \partial\Gamma} = 0.$$

Remark 3.1. Theorem 3.1 and Lemma 2.1 imply the existence of a solution of problem (3.2). However, it does not state its uniqueness. For a special case of $n = 1$ and $G = 0$, the uniqueness was established in [7]; this task required significant analytical efforts. We leave the problem of uniqueness of the solution for $n > 1$ for further research.

4. On numerical implementation. For Markov models, corresponding HJB equations can be solved using finite difference. Unfortunately, this approach may not be effective for a large number of parameters describing the dynamics of the Markov model. The features of first order BSPDEs (3.2) allow one to use some alternative methods described below.

On numerical feasibility for an unspecified dynamics law for $X(t)$. Equation (3.2) can be solved after discretization backward in t using the first order finite differences in t and y , with calculation of the conditional expectation on each step by the Monte-Carlo method. This approach allows one to use the following attractive feature of (3.2): the solution of this equation can be calculated even for models where the dynamics law for $X(t)$ is not specified. For this, one needs to know, for a time-discretization sequence $\{t_k\}$, the conditional distributions of $X(t_{k+1})$ given \mathcal{F}_{t_k} .

The benefit is that we do not need a hypothesis about the dynamics of X or the equation for its evolution, i.e., if it is an Itô's equation, etc. Moreover, there are models where the information about the distribution of $X(t_{k+1})$ is more accessible and reliable than the information about the dynamics law. In general, the dynamics law for X is more difficult to establish since it is not robust with respect to the variations of the probability distribution as can be seen from the following example.

Example 4.1. Let X be a Wiener process. This process can be approximated by pathwise absolutely continuous processes $X_\varepsilon(t)$ that will be, therefore, statistically indistinguishable from X and yet will have a very different dynamics law, without the amazing features of the Itô's processes.

If the dynamics of $X(t)$ are described by a particular equation (such as an Itô's equation), then the parameters of the equation will define the resulting conditional expectation but does not have to be used directly.

Estimation of J using pathwise optimization. In a case where the dynamics law of $X(t)$ is assumed to be known, there is a possibility to estimate J using so-called pathwise optimal control.

Up to the end of this section, we assume that $\{\mathcal{F}_t\}$ is the filtration generated by a Wiener process $W(t)$ taking the values in \mathbf{R}^n . We assume that $X(t)$ is an RCLL stochastic process adapted to \mathcal{F}_t .

Consider a linear normed space $\bar{\mathcal{X}} = L_2([0, T]; L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n))$. Let \mathcal{X} be the closed subspace obtained as the closure of the set of all progressively measurable with respect to $\{\mathcal{F}_t\}$ processes from $\bar{\mathcal{X}}$.

For $y \in \Gamma$, let $\bar{U}(t, y) \subset \bar{\mathcal{X}}$ be the set of processes $u(s) = (u_1(s), \dots, u_n(s)) : [t, T] \times \Omega \rightarrow K$ being \mathcal{F}_T -measurable for all t and such that $y + \int_t^T u(s) ds \in \Gamma$ a.s.

Assume that we are given $y_0 \in \mathbf{R}^m$. Let us consider an optimal control problem

$$(4.1) \quad \text{Maximize } \mathbf{E}F(u, 0) \quad \text{over } u \in U(y_0, 0).$$

Let $y_u^{(0)} = u$, $y_u^k(t) = \int_0^t y_u^{(k-1)}(s)ds$, $k = 1, 2, 3, \dots$, where $y_u(t) = \int_0^t u(s)ds$.

Consider a linear normed space $\bar{V} = L_2([0, T]; L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^{n \times n}))$. Let V be the closure of the set of all progressively measurable with respect to $\{\mathcal{F}_t\}$ processes from \bar{V} .

LEMMA 4.1. *Let $u \in \bar{U}(0, y_0)$. The following statements are equivalent:*

- (i) $u \in U(0, y_0)$.
- (ii) *There exists $k > 1$ such that $y_u^{(k)} \in \mathcal{X}$.*
- (iii) *For any $k \geq 0$, $y_u^{(k)} \in \mathcal{X}$.*
- (iv) *For any $k \geq 0$ and any $v \in V$,*

$$\mathbf{E} \int_0^T \mu(t)^\top y_u^{(k)}(t) dt = \mathbf{E} \int_0^T \mu^{(k)}(t)^\top u(t) dt = 0.$$

Here $M(t) = \int_0^t v(s)dW(s)$, $\mu(t) = M(T) - M(t)$, $\mu^{(0)} = \mu$, $\mu^{(k)}(t) = -\int_t^T \mu^{(k-1)}(s)ds$, $k = 1, 2, 3, \dots$

Let $k \in \{0, 1, 2, \dots\}$ be selected.

For $v \in V$, let $M(t) = \int_0^t v(s)dW(s)$. Clearly, $M(T) \in L_2(\Omega, \mathcal{F}_T, \mathbf{P})$ and $M(t) = \mathbf{E}_t M(T)$. Set $\mu(t) = M(T) - M(t)$. For $u \in \bar{U}(0, y_0)$, $v \in V$, and $\mu^{(k)} = \mu^{(k)}(\cdot, v)$ defined as above, introduce Lagrangian

$$\mathcal{L}(u, v) = F(u, 0) + \mathbf{E} \int_0^T \mu^{(k)}(t)^\top u(t) dt.$$

The following theorem is similar to Theorem 5.1 from [27]. A related result is presented in Theorem 5.1 in [7].

THEOREM 4.1.

$$(4.2) \quad \sup_{u \in U(0, y_0)} \mathbf{E}F(u, 0) = \sup_{u \in \bar{U}(0, y_0)} \inf_{v \in V} \mathcal{L}(u, v) = \inf_{v \in V} \sup_{u \in \bar{U}(0, y_0)} \mathcal{L}(u, v).$$

It can be noted that Theorem 4.1 does not establish the existence of a saddle point. However, it can be used to estimate the value $\sup_{u \in \bar{U}} \mathbf{E}F(u, 0)$ using Monte-Carlo simulation of $\mu^{(k)}$ and a pathwise solution of the problem $\sup_{u \in \bar{U}} \mathcal{L}(u, v)$ in the spirit of the methods developed in [17, 10, 52, 2, 39, 4, 9, 6, 7, 5]; this supremum can be found using pathwise optimization in the class of *anticipating* controls $u \in \bar{U}$ that do not have to be adapted. An advantage of this approach is that it seeks only the solution starting from a particular $y(0)$, whereas the HJB approach described above requires calculating the solution from all starting points. The papers mentioned here suggest running Monte-Carlo over a set of martingales that are considered to be independent variables for the Lagrangian. In the term of Theorem 4.1, this means maximization over the set $v \in V$. Unfortunately, this set is quite wide. On the other hand, the optimal martingale has a very particular dependence on the underlying stochastic process and optimal value function, in the cases of some known explicit solutions. For example, for a related problem considered in [7], the corresponding optimal martingale was found to be $\mu(t) = D_y J(t, Y(t))$, where $Y(t)$ was an optimal state process, and $J(t, y)$ was the optimal value function for the problem satisfying a

first order BSPDE being an analogue of the HJB equation (Theorem 5.1 in [7]). This shows that a sequence of randomly generated martingales may not attend a close proximity of the optimal martingale in a reasonable time. Theorem 4.1 allows one to replace simulation of martingales by simulation of more special processes $\mu^{(k)}(t)$ that are $k - 1$ times pathwise differentiable, with absolutely continuous derivative $d^{k-1}\mu^{(k)}(t)/dt^{k-1}$. After time discretisation, these processes can be presented as processes with a reduced range of finite differences of order k . This could help to reduce the calculation time. The case where $k = 0$ corresponds to martingale duality was studied in [17, 52, 2, 39, 4, 9, 6, 7].

5. The proofs.

Proof of Lemma 2.1. For the case where $G \neq 0$, the proof follows from the standard properties of the quadratic forms. The proof for $G = 0$ repeats the proof from [6]. \square

Proof of Lemma 2.2. To prove statement (i), it suffices to observe that $|u(t)^\top X(t)| \leq L|X(t)|$ if $G = 0$ and that optimal u cannot be too large for $G \neq 0$. Statement (ii) follows from the definition of J and from the fact that $U(t, \hat{y}) \subset U(t, y)$. Let us prove statement (iii). Let $y_j < \hat{y}_j$, $y = (y_1, \dots, y_n)^\top$, $\hat{y} = (y_1, \dots, y_{j-1}, \hat{y}_j, y_{j+1}, \dots, y_n)^\top$. Clearly, $J(t, \hat{y}) \leq J(t, y)$ a.s. for any t . In addition,

$$\begin{aligned} J(t, y) &\leq J(t, \hat{y}) + \operatorname{ess\,sup}_{v \in U(t, y): \int_t^T v(s) ds - (\hat{y} - y) \in \Gamma_0} \mathbf{E}_t \int_t^T |X(s)^\top v(s)| ds \\ &\leq J(t, \hat{y}) + \operatorname{const} \cdot \mathbf{E}_t \operatorname{ess\,sup}_{s \in [t, T]} |X(s)| (\hat{y}_j - y_j) \quad \text{a.s.} \end{aligned}$$

This completes the proof of Lemma 2.2. \square

Proof of Lemma 2.3. It suffices to show that $\frac{1}{2}[J(y_1, t) + J(y_2, t)] \leq J((y_1 + y_2)/2, t)$ a.s. We have

$$\begin{aligned} \frac{1}{2}[J(y_2, t) + J(y_1, t)] &= \sup_{u \in U(t, y_1)} \frac{1}{2} F(X, u, t) + \sup_{u \in U(t, y_2)} \frac{1}{2} F(X, u, t) \\ &= \sup_{u_1 \in U(t, y_1), u_2 \in U(t, y_2)} \frac{1}{2} [F(X, u_1, t) + F(X, u_2, t)] \\ &\leq \sup_{u_1 \in U(t, y_1), u_2 \in U(t, y_2)} F(X, (u_1 + u_2)/2, t) \leq \sup_{u \in U(t, (y_1 + y_2)/2)} F(X, u, t) \\ &= J((y_1 + y_2)/2, t) \quad \text{a.s.} \end{aligned}$$

This completes the proof of Lemma 2.3. \square

The remaining part of this section is devoted to the proof of Theorem 3.1.

5.1. Some preliminary results. For $y \in \Gamma$, $t \in [0, T]$, $\theta \in (t, T]$, let $U(t, \theta, y)$ be the set of processes $u(s) = (u_1(s), \dots, u_n(s)) : [t, \theta] \times \Omega \rightarrow K$ being adapted to \mathcal{F}_s and such that $y + \int_t^\theta u(s) ds \in \Gamma$ a.s.

LEMMA 5.1 (dynamic programming principle). *For any $y \in [0, 1]$ and any time $\theta \in (t, T)$,*

$$J(t, y) = \sup_{u \in U(t, \theta, y)} \mathbf{E}_t \left[\int_t^\theta f(X(s), u(s), t) ds + J(y + \nu_u, \theta) \right].$$

Proof. It suffices to prove that

$$J(t, y) \leq \sup_{u \in U(t, \theta, y)} \mathbf{E}_t \left(\int_t^\theta f(X(s), u(s), s) ds + J(y + \nu_u, \theta) \right),$$

where $\nu_u \triangleq \int_t^\theta u(s) ds$. We have

$$\begin{aligned} J(t, y) &\leq \sup_{u \in U(t, y)} \mathbf{E}_t \left(\int_t^\theta f(X(s), u(s), s) ds + \int_\theta^T f(X(s), u(s), s) ds \right) \\ &= \sup_{u \in U(t, \theta, y)} \sup_{v \in U(\theta, T, y + \nu_u)} \mathbf{E}_t \left(\int_t^\theta f(X(s), u(s), s) ds + \mathbf{E}_\theta \int_\theta^T f(X(s), v(s), s) ds \right) \\ &= \sup_{u \in U(t, y)} \mathbf{E}_t \left(\int_t^\theta f(X(s), u(s), s) ds + \sup_{v \in U(\theta, T, y + \nu_u)} \mathbf{E}_\theta \int_\theta^T f(X(s), v(s), s) ds \right). \end{aligned}$$

This completes the proof. □

Let $t_k = kT/N$, $N \in \{1, 2, \dots\}$, $k = 0, 1, 2, \dots, N$.

Let $U_N(s, y)$ be the set of all admissible $u \in U(s, N)$ such that $u(t) \equiv u(t_k)$, $t \in [t_k, t_{k+1})$, $t_k = kT/(N - 1)$, $k = 0, 1, \dots, N - 1$.

Let

$$\begin{aligned} \mathcal{V}(t, y) = \left\{ v \in L_1(\Omega, \mathcal{F}_{t_k}, \mathbf{P}, \mathbf{R}^m), \quad \text{where } t_k = \max\{t_m : t_m \leq t\}, \right. \\ \left. \text{such that } v \in K \text{ a.e., } y + (t_{k+1} - t)v \in \Gamma \text{ a.s.} \right\}. \end{aligned}$$

LEMMA 5.2. *Assume that $X(t) = X(t_k)$ for $t \in [t_k, t_{k+1})$, $k = 0, 1, \dots, N - 1$. Then, for any (s, y) , there exists optimal in $U(t, y)$ process $u \in U_N(s, y)$. In addition,*

$$(5.1) \quad J(t, y) = \text{ess sup}_{v \in \mathcal{V}(t, y)} [(t_{k+1} - t)[X(t_k)^\top v - v^\top Gv] + \mathbf{E}_t J(t_{k+1}, y + (t_{k+1} - t)v)]$$

for $t \in [t_k, t_{k+1})$. The corresponding $v \in \mathcal{V}(t, y)$ exists for all (t, y) .

Proof of Lemma 5.2. Let k be given, $t \in [t_k, t_{k+1})$. For $u \in U(t, t_k, y)$, let $\nu_u = \int_t^{t_{k+1}} u(s) ds$, $\bar{\nu}_u = \mathbf{E}_t \nu_u$, and $\tilde{\nu}_u = \bar{\nu}_u / (t_{k+1} - t)$. We have that

$$\begin{aligned} (5.2) \quad J(t, y) &= \sup_{u \in U(t, t_k, y)} \mathbf{E}_t \left[\int_t^{t_{k+1}} f(X(s), u(s), s) ds + J(t_k, y + \nu_u) \right] \\ &= \sup_{u \in U(t, t_k, y)} \left[X(t_k)^\top \bar{\nu}_u - \mathbf{E}_t \int_t^{t_{k+1}} u(s)^\top G u(s) ds + \mathbf{E}_t J(t_{k+1}, y + \nu_u) \right]. \end{aligned}$$

By the concavity of $J(t, y)$ in y , it follows that $\mathbf{E}_t J(t_{k+1}, y + \nu_u) \leq \mathbf{E}_t J(t_{k+1}, y + \bar{\nu}_u)$. By the concavity of $f(x, u, s)$ in u , it follows that

$$J(t, y) = \sup_{u \in U(t, t_k, y)} \left[X(t_k)^\top \bar{\nu} - (t_{k+1} - t) \tilde{\nu}_u^\top G \tilde{\nu}_u + \mathbf{E}_t J(t_{k+1}, y + \nu_u) \right].$$

Clearly, any \mathcal{F}_{t_k} -measurable $u(s)|_{s \in [t, t_{k+1}]}$ is optimal for (5.2) if $(t_{k+1} - t)v = \nu_u = \bar{\nu}_u$ for v being optimal for (5.1). This completes the proof of Lemma 5.2. □

5.2. The case of piecewise constant X .

PROPOSITION 5.1. *Assume that $X(t) = X(t_k)$ for $t \in [t_k, t_{k+1})$ (i.e., the assumptions of Lemma 5.2 hold). Then Theorem 3.1 holds.*

Proof. For $k = N - 1, N - 2, \dots, 1, 0$, let

$$V_k = \{v \in L_\infty([t_k, t_{k+1}], \mathbf{R}^n) : v(t) \in K\},$$

and let functions $J_k : [t_k, t_{k+1}] \times \Gamma \times \Omega \rightarrow \mathbf{R}$ be defined consequently as the value functions for deterministic (on the conditional probability spaces given \mathcal{F}_{t_k}) control problems

$$(5.3) \quad \begin{aligned} \text{Maximize} \quad & \widehat{J}_{k+1}(\tau_k^{x,s}, y^{x,s}(\tau_k^{x,s})) + \int_s^{\tau_k^{x,s}} f(X(t), v(t), t) dt \quad \text{over} \quad v \in V_k, \\ \text{subject to} \quad & \frac{dy^{x,s}}{dt}(t) = v(t), \quad y^{x,s}(s) = x, \\ & \tau_k^{x,s} = t_{k+1} \wedge \inf\{t > s : y^{x,s}(t) \notin \Gamma\}. \end{aligned}$$

Here

$$\widehat{J}_k(t, y) \triangleq \mathbf{E}_t J_k(t, y).$$

We assume that $\widehat{J}_N(T, y) \equiv 0$.

In other words,

$$J_k(y, s) = \sup_{v \in V_k} \left(\widehat{J}_{k+1}(\tau_k^{x,s}, y^{x,s}(\tau_k^{x,s})) + \int_s^{\tau_k^{x,s}} f(X(t), v(t), t) dt \right).$$

By the assumptions that X is piecewise constant, we have that

$$(5.4) \quad \mathcal{F}_t = \mathcal{F}_{t_k}, \quad t \in [t_k, t_{k+1}).$$

It follows that $J_k(t, \cdot)$ are \mathcal{F}_{t_k} -measurable for $t \in [t_k, t_{k+1}]$. On the conditional probability space given \mathcal{F}_{t_k} , the values $J_k(t, x)$ can be deemed to be deterministic for $t \in [t_k, t_{k+1}]$.

In addition, we have that each J_k is the value function for the problem

$$\begin{aligned} \text{Maximize} \quad & \widehat{J}_{k+1}(t_{k+1}, y^{x,s}(t_{k+1})) + \int_s^{t_{k+1}} f(X(t), v(t), t) dt \quad \text{over} \quad v \in \bar{U}_k(s, x), \\ \text{subject to} \quad & \frac{dy^{x,s}}{dt}(t) = v(t), \quad y^{x,s}(s) = x. \end{aligned}$$

Here $x \in \Gamma$, $s \in [t_k, t_{k+1})$, $\widehat{J}_{k+1}(t, y) = \mathbf{E}_{t_k} J_{k+1}(t, y)$,

$$\bar{U}_k(s, x) = \left\{ v \in L_\infty([t_k, t_{k+1}], \mathbf{R}^n) : v(t) \in K, \quad x + \int_s^{t_{k+1}} v(r) dr \in \Gamma \right\}.$$

By Lemma 5.1, we obtain, consequently for $k = N - 1, N - 2, \dots, 1, 0$, that

$$(5.5) \quad J(t, y) = J_k(t, y), \quad t \in [t_k, t_{k+1}).$$

The proof below is for the special case where $n = m$, $X_k(t) \geq 0$, and either $G \neq 0$ or $K = \mathbf{R}^n$.

Let us show that $J = J_k$ are unique viscosity solutions of the boundary value problems

$$(5.6) \quad \frac{\partial J}{\partial t}(t, y) + \sup_{v \in K} (f(y, v, t) + v^\top D_y^+ J(t, y)), \quad t \in [t_k, t_{k+1}], \quad y \in \Gamma,$$

$$J(t_{k+1}, y) = \widehat{J}_{k+1}(t_{k+1}, y), \quad J(t_{k+1}, y)|_{y \in \partial\Gamma} = 0.$$

Possibly, this statement follows from the known theory for deterministic controlled equations with first order HJB equations. For this, we would need the existence of a solution for HJB equations (5.6) plus a verification theorem connecting them with the control problem. So far, we were unable to find a result covering our case; the closest one is Theorem 5.2 from [35]; unfortunately, condition (iv) in this theorem does not hold in our case. Other existing literature covers either domains without boundaries or bounded domains; see, e.g., [43]. The existing results for solvability of first order HJB equations require additional conditions such as C^1 -smooth boundary functions; see, e.g., [43, 8, 11, 12]. To overcome these difficulties, we derived below equation (5.6) for $J = J_k$ directly using the features imposed by our special setting.

Let $\widehat{y}(t, \theta) = y + v(\theta - t)$, $t \in [t_k, t_{k+1}]$, $\theta \in [t, t_{k+1}]$, for an optimal v in (5.1). Clearly, the strategy $u(s) = v$ is the part on $[t_k, t_{k+1})$ of the optimal solution for the problem (2.3) with $t = t_k$. Respectively, the process $(u(s), y(s)) = (v, \widehat{y}(t, s))$ is the part on $[t_k, t_{k+1})$ of the optimal process for problem (2.3). Therefore, $v \in \mathcal{V}(t, y)$ is an optimal point for (5.1) if and only if, for any $\theta \in [t, t_{k+1}]$, it is a maximum point for the problem

$$(5.7) \quad J(t, y) = \operatorname{ess\,sup}_{v \in \mathcal{V}(t, y)} [(\theta - t)[v^\top X(t_k) - v^\top Gv] + \mathbf{E}_t J(\theta, \widehat{y}(t, \theta))].$$

LEMMA 5.3. *Assume that $X(t) = X(t_k)$ for $t \in [t_k, t_{k+1}]$, $k = 0, 1, \dots, N - 1$. Assume that $G \neq 0$ and $K = \mathbf{R}^n$. Then the following holds.*

- (i) *The only optimal point of (5.7) is $v = \frac{1}{2}G^{-1}[X(t_k) + D_y^+ J(\theta, \widehat{y}(t, \theta))]$.*
- (ii) *For any $t \in [t_k, t_{k+1}]$, $\theta \in [t, t_{k+1}]$,*

$$(5.8) \quad \begin{aligned} J(t, y) &= (\theta - t)[v^\top X(t_k) - v^\top Gv] + J(\theta, y + v(\theta - t)) \\ &= (\theta - t)[v^\top X(t_k) - v^\top Gv] + J(\theta, \widehat{y}(t, \theta)). \end{aligned}$$

- (iii) *For $t \in [t_k, t_{k+1}]$,*

$$(5.9) \quad \begin{aligned} J(t, y) &= \frac{1}{4} \int_t^{t_{k+1}} (X(s) + D_y^+ J(s, y))^\top G^{-1} (X(s) + D_y^+ J(s, y)) \, ds \\ &\quad + \widehat{J}_{k+1}(t_{k+1}, y). \end{aligned}$$

Proof of Lemma 5.3. Statements (i)–(ii) follow immediately from (5.8) and from the optimality of v . Let us show that

$$(5.10) \quad J(t, y) = v^\top \int_t^{t_{k+1}} (X(s) - Gv + D_y^+ J(s, y)) \, ds + \widehat{J}_{k+1}(t_{k+1}, y).$$

By (5.8), we have that

$$\begin{aligned} D_t J(t, y) &= -X(t_k)^\top v + v^\top Gv - D_y^+ J(\theta, y + v(\theta - t))^\top v \\ &= -X(t_k)^\top v + v^\top Gv - D_y^+ J(\theta, \widehat{y}(t, \theta))^\top v. \end{aligned}$$

It follows that $D_y^+ J(\theta_1, \hat{y}(t, \theta_1))^\top v = D_y^+ J(\theta_2, \hat{y}(t, \theta_2))^\top v$ for any $\theta_1, \theta_2 \in [t_k, t_{k+1}]$. If we assume some additional regularity of $D_y^+ J$, we have that $D_y^+ J(t, y)^\top v = D_y^+ J(\theta, \hat{y}(t, \theta))^\top v$ and, therefore,

$$\begin{aligned} D_t J(t, y) &= -v^\top (X(t) + D_y^+ J(t, y)) + v^\top Gv \\ &= -\frac{1}{4} (X(t) + D_y^+ J(t, y))^\top G^{-1} (X(t) + D_y^+ J(t, y)). \end{aligned}$$

This gives (5.9). Without these additional regularity assumptions, we prove (5.9) as the following. We observe that

$$\begin{aligned} J(t, y) &= v^\top \int_t^{t_{k+1}} (X(s) - Gv + D_y^+ J(t_{k+1}, \hat{y}(s, t_{k+1}))) ds + \hat{J}_{k+1}(t_{k+1}, y) \\ &= v^\top \int_t^{t_{k+1}} (X(s) - Gv + D_y^+ J(s + \varepsilon, \hat{y}(s, s + \varepsilon))) ds + \hat{J}_{k+1}(t_{k+1}, y) \end{aligned}$$

for any $\varepsilon \in (0, t_{k+1} - t_k)$. Hence

$$\begin{aligned} J_k(t, y) &= \frac{1}{4} \int_t^{t_{k+1}} (X(s) + D_y^+ J(s, y))^\top G^{-1} (X(s) + D_y^+ J(s, y)) ds + \hat{J}_{k+1}(t_{k+1}, y) \\ &\quad + \xi_\varepsilon(t, y), \end{aligned}$$

where

$$\begin{aligned} \xi_\varepsilon(t, y) &= \frac{1}{4} \int_t^{t_{k+1}} \left[(X(s) + D_y^+ J(s + \varepsilon, \hat{y}(s, s + \varepsilon)))^\top \right. \\ &\quad \left. G^{-1} (X(s) + D_y^+ J(s + \varepsilon, \hat{y}(s, s + \varepsilon))) \right. \\ &\quad \left. - (X(s) + D_y^+ J(s, y))^\top G^{-1} (X(s) + D_y^+ J(s, y)) \right] ds. \end{aligned}$$

For an arbitrarily selected $\eta \in L_\infty(0, 1)$, let $\gamma_\varepsilon(t) = \int_0^1 \xi_\varepsilon(t, y) \eta(y) dy$. By Luzin's theorem, we have $\gamma_\varepsilon(t) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence (5.10) holds. Substitution of optimal v gives the desired formula. This completes the proof of Lemma 5.3. \square

Let

$$\begin{aligned} \hat{D}_{y_i} J(t, y) &= h \in [D_{y_i}^+ J(t, y), D_{y_i}^- J(t, y)] \\ \text{such that } |h + X_i(t)| &= \min_{g \in [D_{y_i}^+ J(t, y), D_{y_i}^- J(t, y)]} |g + X_i(t)|. \end{aligned}$$

Clearly, h in this definition is unique.

LEMMA 5.4. Assume that $X(t) = X(t_k)$ for $t \in [t_k, t_{k+1})$, $k = 0, 1, \dots, N - 1$. Assume that $K = [0, L]^n$ and $G = 0$. Let $t \in [t_k, t_{k+1})$ and $y \in \Gamma$ be given, and let $v \in \mathcal{V}(t, y)$ be an optimal point in (5.1) and hence optimal in (5.7) for all $\theta \in [t, t_{k+1})$, where $\hat{y}(t, \theta) = y + v(\theta - t)$, $t \in [t_k, t_{k+1})$, $\theta \in [t, t_{k+1})$.

Then, under the assumptions of Lemma 5.2, the following holds.

- (i) $J(t, y) = v^\top X(t_k)(\theta - t) + J(\theta, \hat{y}(t, \theta))$ for $t \in [t_k, t_{k+1})$ and $\theta \in [t, t_{k+1})$.
- (ii) If $v_i = 0$, then $X_i(t_k) + D_{y_i}^+ J(\theta, \hat{y}(t, \theta)) \leq 0$.
If $v_i = L$, then $X_i(t_k) + D_{y_i}^+ J(\theta, \hat{y}(t, \theta)) \geq 0$.
- (iii) If $v_i \in (0, L)$, then $X_i(t_k) + \hat{D}_{y_i} J(\theta, y(t, \theta)) = 0$.

(iv) For $t \in [t_k, t_{k+1})$,

$$(5.11) \quad J(t, y) = \sum_{i=1}^n L \int_t^{t_{k+1}} \left(X_i(s) + \widehat{D}_{y_i} J(s, y) \right)_+ ds + \widehat{J}_{k+1}(t_{k+1}, y).$$

Proof of Lemma 5.4. Statement (i) follows immediately from (5.7) applied with the optimal v . Statement (ii) follows from (i). Consider optimization on the conditional probability space given \mathcal{F}_{t_k} . Let us prove statement (iii). If $v_i \in (0, L)$ is optimal for (5.7), then it is optimal for t_{k+1} replaced by any $\theta \in (t_k, t_{k+1}]$, then

$$0 \in [X_i(t_k) + D_{y_i}^+ J(t, \widehat{y}(t, \theta)), X_i(t_k) + D_{y_i}^- J(t, \widehat{y}(t, \theta))].$$

Hence

$$(5.12) \quad X_i(t_k) + \widehat{D}_{y_i} J(t, \widehat{y}(t, \theta)) = 0.$$

□

Let us prove statement (iv). By the definition of \widehat{D} and by concavity of J in y_i , we obtain the following.

- If $X_i(t) + D_{y_i}^+ J(t, \widehat{y}(t, \theta)) \geq 0$, then $X_i(t) + \widehat{D}_{y_i} J(t, \widehat{y}(t, \theta)) \geq 0$.
- If $X_i(t) + D_{y_i}^+ J(t, \widehat{y}(t, \theta)) < 0$, then

$$(X_i(t) + D_{y_i}^+ J(t, \widehat{y}(t, \theta)))_+ = (X_i(t) + \widehat{D}_{y_i} J(t, \widehat{y}(t, \theta)))_+ = 0.$$

For all cases listed in statements (ii)–(iii), we have

$$v_i(X_i(t) + D_{y_i}^+ J(t, \widehat{y}(t, \theta))) = L(X_i(t) + D_{y_i}^+ J(t, \widehat{y}(t, \theta)))_+.$$

Therefore, for all cases listed in statements (ii)–(iii), we have

$$\begin{aligned} v_i(X_i(t) + D_{y_i}^+ J(t, \widehat{y}(t, \theta))) &= L(X_i(t) + D_{y_i}^+ J(t, \widehat{y}(t, \theta)))_+ \\ &= v_i(X_i(t) + \widehat{D}_{y_i} J(t, \widehat{y}(t, \theta))) \\ &= L(X_i(t) + \widehat{D}_{y_i} J(t, \widehat{y}(t, \theta)))_+. \end{aligned}$$

By statement (i), we have that

$$D_t J(t, y) = -X(t_k)^\top v - D_{y_i}^+ J(\theta, \widehat{y}(t, \theta))^\top v.$$

It follows that $D_y^+ J(\theta_1, \widehat{y}(t, \theta_1))^\top v = D_y^+ J(\theta_2, \widehat{y}(t, \theta_2))^\top v$ for any $\theta_1, \theta_2 \in [t_k, t_{k+1}]$. Some additional regularity of $D_y^+ J$ gives that $D_y^+ J(t, y)^\top v = D_y^+ J(\theta, \widehat{y}(t, \theta))^\top v$ and, therefore,

$$D_t J(t, y) = -v^\top (X(t) + D_y^+ J(t, y)).$$

Without these additional regularity assumptions, we prove (5.11) as the following. We observe that

$$\begin{aligned} J(t, y) &= v^\top \int_t^{t_{k+1}} (X(s) + D_y^+ J(t_{k+1}, \widehat{y}(s, t_{k+1}))) ds + \widehat{J}_{k+1}(t_{k+1}, y) \\ &= \sum_{i=1}^n L \int_t^{t_{k+1}} (X_i(s) + D_{y_i}^+ J(s + \varepsilon, \widehat{y}(s, s + \varepsilon)))_+ ds + \widehat{J}_{k+1}(t_{k+1}, y) \end{aligned}$$

for any $\varepsilon \in (0, t_{k+1} - t_k)$. Hence

$$J_k(t, y) = \sum_{i=1}^n L \int_t^{t_{k+1}} (X_i(s) + D_{y_i}^+ J(s, y))_+ ds + \widehat{J}_{k+1}(t_{k+1}, y) + \xi_\varepsilon(t, y),$$

where

$$\begin{aligned} \xi_\varepsilon(t, y) &= \sum_{i=1}^n L \int_t^{t_{k+1}} \left[(X_i(s) + D_{y_i}^+ J(s + \varepsilon, \widehat{y}(s, s + \varepsilon)))_+ - (X_i(s) + D_{y_i}^+ J(s, y))_+ \right] ds. \end{aligned}$$

For an arbitrarily selected $\eta \in L_\infty(0, 1)$, let $\gamma_\varepsilon(t) = \int_0^1 \xi_\varepsilon(t, y) \eta(y) dy$. By Luzin's theorem, we have $\gamma_\varepsilon(t) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence (5.11) holds. This completes the proof of Lemma 5.4.

We now in the position to prove Proposition 5.1. By (5.6), we obtain that

$$\begin{aligned} (5.13) \quad J_k(t, y) &= \widehat{J}_{k+1}(t_{k+1}, y) + \int_t^{t_{k+1}} \sup_{v \in K} [f(X(s), v, s) + v^\top D_y^+ J_k(s, y)] ds, \\ &\quad t \in [t_k, t_{k+1}], \quad y \in \Gamma, \\ J_k(t_{k+1}, y)|_{y \in \Gamma} &= 0. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} (5.14) \quad J_k(t, y) &= \mathbf{E} \left\{ \widehat{J}_{k+1}(t_{k+1}, y) + \int_t^{t_{k+1}} \sup_{v \in K} [f(X(s), v, s) + v^\top D_y^+ J_k(s, y)] ds \middle| \mathcal{F}_{t_k} \right\}, \\ &\quad t \in [t_k, t_{k+1}], \quad y \in \Gamma, \\ J_k(t_{k+1}, y)|_{y \in \Gamma} &= 0. \end{aligned}$$

It follows from (5.5) that (3.2) holds for J . This completes the proof of Proposition 5.1. \square

PROPOSITION 5.2. *For $N = 1, 2, \dots$, consider piecewise constant processes defined as*

$$\begin{aligned} X_N(t) &= \frac{1}{t_{k+1} - t_k} \mathbf{E}_{t_k} \int_{t_k}^{t_{k+1}} X(s) ds, \quad t \in [t_k, t_{k+1}), \\ t_k &= T(k-1)/(N-1), \quad k = 0, 1, 2, \dots, N. \end{aligned}$$

Let J_N be the corresponding functions J obtained for X replaced by X_N . Then

$$(5.15) \quad \mathbf{E} \int_0^T |X_N(t) - X(t)|^2 dt \rightarrow 0 \quad \text{as } N \rightarrow +\infty,$$

$$(5.16) \quad J_N(y, t) \leq J(y, t) \quad \text{a.s. for all } t, \quad J_N \rightarrow J \quad \text{a.e. as } N \rightarrow +\infty,$$

$$(5.17) \quad D_y^+ J_N \rightarrow D_y^+ J \quad \text{a.e. as } N \rightarrow +\infty.$$

Proof. The limit in (5.15) follows from the properties of conditional expectations applied to the averaging on $[0, T] \times \Omega$; see, e.g., [33].

Let us prove (5.16). If $G = 0$, then it follow immediately from the definitions that

$$(5.18) \quad \begin{aligned} \operatorname{ess\,sup}_{u \in U(t,y)} \mathbf{E}_t F(X_N, u, t) &= \operatorname{ess\,sup}_{u \in U_N(t,y)} \mathbf{E}_t F(X_N, u, t) = \operatorname{ess\,sup}_{u \in U_N(t,y)} \mathbf{E}_t F(X, u, t) \\ &\rightarrow \operatorname{ess\,sup}_{u \in U(t,y)} \mathbf{E}_t F(X, u, t) \\ &\text{as } N \rightarrow +\infty \text{ a.s for all } t. \end{aligned}$$

If $G \neq 0$, then it suffices to observe that

$$\mathbf{E}_t \int_t^T u(s)^\top X_N(s) ds = \mathbf{E}_t \int_t^T \bar{u}(s)^\top X_N(s) ds$$

and

$$- \int_t^T u(s)^\top G u(s) ds \leq - \int_t^T \bar{u}_N(s)^\top G \bar{u}_N(s) ds \quad \text{a.s for all } t,$$

where

$$\bar{u}_N(t) = \frac{1}{t_{k+1} - t_k} \mathbf{E}_{t_k} \int_{t_k}^{t_{k+1}} u(s) ds, \quad s \in [t_k, t_{k+1}).$$

Let us prove (5.17). We will be using the following lemma.

LEMMA 5.5. *For $\alpha \in \mathbf{R}$ and $\beta \in (\alpha, +\infty)$, let $f : [\alpha, \beta] \rightarrow \mathbf{R}$ and $f_N : [\alpha, \beta] \rightarrow \mathbf{R}$ be bounded, concave, and uniformly Lipschitz functions such that $f_N(s) \rightarrow f(s)$ as $N \rightarrow +\infty$, $f_N(s) \leq f(s)$ for all $s \in [\alpha, \beta]$. Then $D^+ f_N(s) \rightarrow D^+ f(s)$ for a.e. s as $N \rightarrow +\infty$.*

Proof of Lemma 5.5. For $s \in (\alpha, \beta)$ and $h > 0$ such that $s + h \in (\alpha, \beta]$, let $\widehat{\Delta}_h^{(2)} f \triangleq f(s + h) - f(s) - D^+ f(x)h$. By Kachurovskii's theorem, we have that $\widehat{\Delta}_h^{(2)} f \leq 0$ and $\widehat{\Delta}_h^{(2)} f_N \leq 0$ for all h and N . Let $g_N \triangleq f - f_N$. We have that $f = f_N + g_N$ and $\widehat{\Delta}_h^{(2)} f = \widehat{\Delta}_h^{(2)} f_N + \widehat{\Delta}_h^{(2)} g_N$. Hence

$$(5.19) \quad |(\widehat{\Delta}_h^{(2)} g_N)_-| \leq |\widehat{\Delta}_h^{(2)} f| \quad \text{for all } s \in (\alpha, \beta), h, N,$$

where $(x)_- = \min(x, 0)$.

Suppose that the statement of the lemma is incorrect. In this case, there exists an interval $(\bar{\alpha}, \bar{\beta}) \subset [\alpha, \beta]$, $\bar{\alpha} < \bar{\beta}$, and a mapping $r : (\bar{\alpha}, \bar{\beta}) \rightarrow (0, +\infty)$ such that $r(s) > 0$ and $|D^+ g_N(s)| \geq r(s)$ for all $s \in (\bar{\alpha}, \bar{\beta})$. Clearly, $D^+ g_N \rightarrow 0$ weakly in $L_2(\bar{\alpha}, \bar{\beta})$ as $N \rightarrow +\infty$. It follows that $\mu_N \rightarrow 0$ as $k \rightarrow +\infty$, where

$$\mu_N \triangleq \max\{|\beta_1 - \alpha_1|; \bar{\alpha} \leq \alpha_1 < \beta_1 \leq \bar{\beta}, \text{ sign } D^+ g_N(s) = \text{const}, s \in (\alpha_1, \beta_1)\}.$$

This increasing oscillations of g_N implies that (5.19) does not hold. This implies (5.17) and completes the proof of Lemma 5.5. \square

We have that, for any given $y = (y_1, \dots, y_n)$ and any $j \in \{1, \dots, n\}$, the paths $f(y_j) = J(y, t)$ and $f_N(y_j) = J_N(y, t)$ satisfy the assumptions of Lemma 5.5, for a given $(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n)$, for $y_j \in [\alpha, \beta]$ and a interval $[\alpha, \beta]$ such that $y = (y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_n) \in \Gamma$. This completes the proof of Proposition 5.2. \square

We are now in the position to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $X_N = (X_{N1}, \dots, X_{Nn})^\top$ and J_N be such as described in Proposition 5.2. Starting from now, we will consider a subsequence $N = N_k$, $k = 1, 2, \dots$, such that $X_N \rightarrow X$ a.e.. It can be noted that, for any y , there is an integrable process $\xi(t, \omega)$ such that $|X_N(t)| + |D_y J_N(t, y)| \leq \xi(t, \omega)$.

Assume that $K = [0, L]^n$, and $f(t, x, u) = u^\top X$. By (5.17), we have for $t < T$ and a.e. $y \in \Gamma$ that

$$\mathbf{E}_t \int_t^T \left(\sum_{i=1}^n (X_{Ni}(s) + D_{y_i}^+ J_N(s, y))_+ \right) ds \rightarrow \mathbf{E}_t \int_t^T \left(\sum_{i=1}^n (X_i(s) + D_{y_i}^+ J(s, y))_+ \right) ds$$

a.e. as $N \rightarrow +\infty$.

By (5.16), we obtain the theorem statement for this case. The proof for the case where $G \neq 0$ is similar. \square

Proof of Lemma 4.1. It is straightforward and will be omitted here. It is based on the fact that, by the martingale representation theorem, $\mu(t) = \int_t^T \hat{\mu}(s) dw(s)$ for some $\hat{\mu} \in \mathcal{V}$. \square

Proof of Theorem 4.1. The first equality follows from Lemma 4.1. Furthermore, we have that $\mathcal{L}(u, v)$ is concave in $u \in U$ and affine in $v \in V$. In addition, $\mathcal{L}(u, v)$ is continuous in $u \in L_2([0, T] \times \Omega)$ given $v \in V$, and $\mathcal{L}(u, v)$ is continuous in $v \in V$ given $u \in U$. The statement of the theorem follows Proposition 2.3 from [32, Chapter VI], Statement (ii) follows from (i) and Proposition 1.2 from [32, Chapter VI]. \square

6. Discussion and future development. The paper considers stochastic control problems in a setting without specifying the evolution law on underlying input processes. The first order BSPDEs representing analogues of HJB equations are derived; they represent further development of the concept originally suggested in [6, 7]. The paper extends this approach on the multidimensional state space, with different proofs, and establishes existence of solutions for these equations and the fact the value functions for underlying control problems satisfy these equations. This opens ways to extend this theory on quite general models. For example, a possible extension is for the case where the constraints on $\int_t^T u(s) ds$ are replaced by constraints on solutions of more general equations with input u . We leave it for the future research.

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