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Nonovershooting state feedback and dynamic output feedback tracking controllers for descriptor systems

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Abstract

The use of linear multivariable feedback control to achieve a nonovershooting step response is considered for multi-input multi-output (MIMO), linear, time-invariant (LTI) descriptor systems. The use of dynamic output feedback control to improve the transient response to a step input is also considered for MIMO and LTI descriptor systems. We design a state feedback controller and a dynamic output feedback for MIMO and LTI descriptor systems to asymptotically stabilize and track a step reference with zero overshoot and arbitrarily small rise time, under some mild assumptions.

Keywords: Tracking control, step response, nonovershooting linear controllers and observers, MIMO systems.

1 Introduction

In the past few decades, there has been an increasing interest in the study of descriptor systems, also known as singular or differential-algebraic systems. Descriptor systems have many applications in circuit theory, large-scale systems, constrained mechanical systems, robotics, aircraft modeling, biological systems, see e.g. [6], [13], [17], [25]. Many classical control problems of standard linear time-invariant (LTI) systems have been extended to descriptor systems, such as the pole and eigenstructure assignment, observer design, optimal control, disturbance decoupling, the solution of the generalized Sylvester matrix equation, robust stability and stabilization, see e.g. [1], [4], [5], [7], [10], [14], [18], [20], [27]-[30]. The difficulties of extending results from standard LTI systems to descriptor systems is due to their richer mathematical structure, see for example [26] or [6].

The problem of designing control laws to improve the transient response is important in several applications such as manufacturing processes, where overshoot may compromise tolerances and damage the product. The problem of improving transient response for standard LTI multi-input

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multi-output (MIMO) systems was investigated in the papers [22]-[24]. In particular, nonovershooting state feedback tracking controllers were designed in [23], nonovershooting and nonundershooting state feedback tracking controllers were designed in [24], and nonovershooting dynamic output feedback tracking controllers were designed in [22]. However, to date there have been very few results on the problem of improving transient response for descriptor systems. In [16], an output feedback gain for descriptor systems was designed, which ensures that the closed loop has satisfactory transient response with significantly reduced instantaneous jumps with random initial states. In [15] and [8], the transient response of linear continuous-time and discrete-time descriptor systems, respectively, with input saturation was improved via composite nonlinear feedback (CNF) control.

In this paper, we investigate the problem of designing nonovershooting state feedback tracking controllers and dynamic output feedback tracking controllers of full order for linear MIMO descriptor systems, generalizing the results of [23] and [22]. The design methods proposed here make use of the eigenvalue and eigenvector assignment method given by Moore in [19] generalized for descriptor systems. We give conditions under which a linear state feedback controller and a linear dynamic output feedback can be obtained to asymptotically stabilize the descriptor system and track a step reference with zero overshoot, with arbitrarily small rise time, from any initial condition. The results of this paper are presented for continuous-time descriptor systems but are also applicable for discrete-time descriptor systems, with only minor modifications.

The paper is structured as follows. In Section 2, we introduce the nonovershooting control problem for continuous-time descriptor systems and provide some preliminary results on descriptor systems. In Section 3, we design nonovershooting feedback controllers for square descriptor systems. In Section 4, we consider the nonovershooting problem for square descriptor systems via the use of dynamic output feedback control based on a Luenberger observer. The methods are illustrated by an example in Section 5 and conclusions are given in Section 6.

Notation. The origin of a vector space is denoted by $\{0\}$. The image and the kernel of a matrix A are represented by $\operatorname{im} A$ and $\ker A$, respectively. The Moore-Penrose pseudo-inverse of A is denoted by $A^\dagger$. For convenience, a linear mapping between finite-dimensional spaces and a matrix representation with respect to a particular basis are not distinguished notationally. The symbol $\oplus$ will stand for the direct sum of subspaces. Finally, the symbol $\mathfrak{i}$ represents the imaginary unit, i.e., $\mathfrak{i} = \sqrt{-1}$, while the symbol $\overline{\alpha}$ represents the complex conjugate of $\alpha \in \mathbb{C}$.

2 Problem formulation

Consider an LTI continuous-time descriptor system Σ governed by

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t), \qquad E\,x(0) = E\,x_0 \in \mathbb{R}^n, \tag{1a}$$

$$y(t) = C\,x(t). \tag{1b}$$

For all $t \in \mathbb{R}^+$, the symbol $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output, and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. We assume that B has full column rank and C has full row rank. Let ℓ denote the rank of E, and we consider the general case where $\ell \leq n$. In the case where $\ell = n$, we say that the system is *explicit*.

The matrix pencil $\lambda E - A$ is said to be *regular* if $\det(\lambda E - A)$ is not identically zero. The pair (E, A) is said to be *asymptotically stable* if all the *finite generalized eigenvalues* of a regular matrix pencil $\lambda E - A$, i.e., the roots of $\det(\lambda E - A)$, are in the open left-half complex plane $\mathbb{C}^-$. The descriptor system Σ is called *stabilizable* if $\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\{\lambda\} \geq 0\}$; and *detectable* if $\operatorname{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n$ for all $\{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\{\lambda\} \geq 0\}$, see e.g. [6].

In this paper, we are concerned with the problem of designing a state-feedback controller and a dynamic output feedback for (1) to asymptotically stabilize the descriptor system and track a given constant reference $r \in \mathbb{R}^p$ without overshoot for any initial condition $E x_0$. The descriptor system (1) is said to have a *nonovershooting response* for r if the output $y(t)$ arising from the initial condition $E x_0 \in \mathbb{R}^n$ yields a tracking error $\epsilon(t) \stackrel{\text{def}}{=} r - y(t)$ that converges to 0 as t goes to infinity without changing sign in any component, i.e., for all $i \in \{1, \ldots, p\}$, the sign of $\epsilon_i(t)$ is constant for all $t \in \mathbb{R}^+$. The descriptor system (1) is said to have a *globally nonovershooting response* for r if the output $y(t)$ is nonovershooting for all initial conditions $E x_0$.

2.1 Preliminary results

We now present some results on descriptor systems that will be used in our analysis and design methods, see e.g. [6]. First, using a singular value decomposition we may obtain nonsingular matrices P and Q to write (1) in the *dynamics decomposition form*

$$Q E P = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}, \quad Q A P = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad Q B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C P = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{2}$$

with $P^{-1} x(t) = \begin{bmatrix} \tilde{x}(t) \\ \hat{x}(t) \end{bmatrix}$. Without loss of generality, we assume that Σ is already in the dynamics decomposition form (2), i.e., we assume that $Q = P = I_n$.

Descriptor systems may exhibit impulsive behavior, which is typically not desired as it may cause instantaneous jumps and damage or destroy an engineering system. A descriptor system is called *impulse-free* if

$$\deg\big(\det(\lambda E - A)\big) = \operatorname{rank} E = \ell, \tag{3}$$

i.e., it has ℓ finite generalized eigenvalues, or, equivalently, A_{22} is nonsingular. Clearly, an impulse-free system is regular, because (3) implies that $\det(\lambda E - A) \neq 0$.

If there exists a matrix L of suitable size such that $\deg\big(\det(\lambda E - (A + L C))\big) = \ell$, then the descriptor system is called *impulse observable*. The descriptor system Σ is impulse observable if and only if $\big(A^{-1} \operatorname{im} E\big) \cap \ker E \cap \ker C = \{0\}$, or, equivalently, if and only if $\operatorname{rank} \begin{bmatrix} A_{22} \\ C_2 \end{bmatrix} = n - \ell$, see e.g. [11], [6, Ch.4]. If Σ is impulse observable, partitioning L conformably with (2) as $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, then L can be constructed in such a way that $\det(A_{22} + L_2 C_2) \neq 0$; since L_1 is arbitrary, it can be taken to be equal to the zero matrix.

Descriptor systems are said to be *impulse controllable* if impulsive modes can be removed by means of a state feedback, i.e., there exists a matrix H such that $\deg\big(\det(\lambda E - (A + B H))\big) = \ell$. The descriptor system Σ is impulse controllable if and only if rank $\begin{bmatrix} E & A E_\infty & B \end{bmatrix} = n$, where E_∞ is a basis matrix for $\ker E$, or, equivalently, if and only if rank $\begin{bmatrix} A_{22} & B_2 \end{bmatrix} = n - \ell$. Indeed, if Σ is impulse controllable, we can apply any state feedback $u(t) = H_1 \tilde{x}(t) + H_2 \hat{x}(t) + v(t)$ such that $\det(A_{22} + B_2 H_2) \neq 0$, i.e., in such a way that the closed-loop system is impulse-free. It is clear that H_1 is arbitrary and can be taken to be equal to the zero matrix. The closed-loop system $\hat{\Sigma} = \big(E, \hat{A}, B, C\big)$, where $\hat{A} \stackrel{\text{def}}{=} A + B H$, $H \stackrel{\text{def}}{=} \begin{bmatrix} 0 & H_2 \end{bmatrix}$, is governed by

$$E \dot{x}(t) = \hat{A} x(t) + B v(t), \tag{4a}$$

$$y(t) = C x(t), \tag{4b}$$

where $\hat{A} = \begin{bmatrix} A_{11} & \hat{A}_{12} \\ A_{21} & \hat{A}_{22} \end{bmatrix}$, $\hat{A}_{12} \stackrel{\text{def}}{=} A_{12} + B_1 H_2$, $\hat{A}_{22} \stackrel{\text{def}}{=} A_{22} + B_2 H_2$. Using a further change of coordinates, we can bring the system into an equivalent impulse-free form as

$$\tilde{Q} E \tilde{P} = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}, \tilde{Q} \hat{A} \tilde{P} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{n-\ell} \end{bmatrix}, \tilde{Q} B = \begin{bmatrix} \tilde{B} \\ B_2 \end{bmatrix}, C \tilde{P} = \begin{bmatrix} \tilde{C} & \tilde{C}_2 \end{bmatrix}, \begin{bmatrix} \tilde{x}(t) \\ \check{x}(t) \end{bmatrix} = \tilde{P}^{-1} \begin{bmatrix} \tilde{x}(t) \\ \hat{x}(t) \end{bmatrix}$$

where $\tilde{Q} \stackrel{\text{def}}{=} \begin{bmatrix} I_\ell & -\hat{A}_{12} \hat{A}_{22}^{-1} \\ 0 & I_{n-\ell} \end{bmatrix}$, $\tilde{P} \stackrel{\text{def}}{=} \begin{bmatrix} I_\ell & 0 \\ -\hat{A}_{22}^{-1} A_{21} & \hat{A}_{22}^{-1} \end{bmatrix}$, $\tilde{P}^{-1} = \begin{bmatrix} I_\ell & 0 \\ A_{21} & \hat{A}_{22} \end{bmatrix}$ and $\tilde{A} \stackrel{\text{def}}{=} A_{11} - \hat{A}_{12} \hat{A}_{22}^{-1} A_{21}$, $\tilde{B} \stackrel{\text{def}}{=} B_1 - \hat{A}_{12} \hat{A}_{22}^{-1} B_2$, $\tilde{C} \stackrel{\text{def}}{=} C_1 - C_2 \hat{A}_{22}^{-1} A_{21}$, $\tilde{C}_2 \stackrel{\text{def}}{=} C_2 \hat{A}_{22}^{-1}$, $\check{x}(t) \stackrel{\text{def}}{=} A_{21} \tilde{x}(t) + \hat{A}_{22} \hat{x}(t)$, see e.g. [27], so that the descriptor system can be written as

$$\dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{B} v(t), \tag{5a}$$

$$0 = \check{x}(t) + B_2 v(t), \tag{5b}$$

$$y(t) = \tilde{C} \tilde{x}(t) + \tilde{C}_2 \check{x}(t). \tag{5c}$$

Introducing $\widehat{x}(t) \stackrel{\text{def}}{=} \check{x}(t) + B_2 v(t)$, we can rewrite the system (5) as

$$\dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{B} v(t), \tag{6a}$$

$$0 = \widehat{x}(t), \tag{6b}$$

$$y(t) = \tilde{C} \tilde{x}(t) + \tilde{D} v(t). \tag{6c}$$

where $\tilde{D} \stackrel{\text{def}}{=} -\tilde{C}_2 B_2 \in \mathbb{R}^{p \times m}$. Equations (6a) and (6c) form an explicit system, that we denote by $\tilde{\Sigma} \stackrel{\text{def}}{=} \big(I_\ell, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\big)$.

The *Rosenbrock system matrix pencil* of the descriptor system Σ is defined as $P_\Sigma(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} A - \lambda E & B \\ C & 0 \end{bmatrix}$. The *invariant zeros* of Σ are the $\lambda \in \mathbb{C}$ for which rank $P_\Sigma(\lambda) <$ normrank $P_\Sigma(\lambda)$. For the regular system $\hat{\Sigma}$, the invariant zeros are the $\lambda \in \mathbb{C}$ for which rank $P_{\hat{\Sigma}}(\lambda) < n +$ normrank $G_{\hat{\Sigma}}(\lambda)$, where $G_{\hat{\Sigma}}(\lambda) \stackrel{\text{def}}{=} C \left(\lambda E - \hat{A}\right)^{-1} B$, see e.g. [3, Ch.6]. The invariant zeros of Σ coincide with the invariant zeros of $\hat{\Sigma}$, because it is easy to see that $P_{\hat{\Sigma}}(\lambda) = P_\Sigma(\lambda) \begin{bmatrix} I_n & 0 \\ H & I_m \end{bmatrix}$, and with the invariant zeros of the associated explicit system $\tilde{\Sigma}$, because $C \left(\lambda E - \hat{A}\right)^{-1} B = \tilde{C} \left(\lambda I_\ell - \tilde{A}\right)^{-1} \tilde{B} + \tilde{D} \stackrel{\text{def}}{=} G_{\tilde{\Sigma}}(\lambda)$, which is the transfer function matrix of $\tilde{\Sigma}$.

The next proposition shows that $P_\Sigma(\lambda)$ is right invertible as a rational matrix if and only if $P_{\hat{\Sigma}}(\lambda)$ and $P_{\tilde{\Sigma}}(\lambda) \stackrel{\text{def}}{=} \left[\begin{smallmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \\ \tilde{C} & \tilde{D} \end{smallmatrix}\right]$ are right invertible as rational matrices.

Proposition 2.1 *The Rosenbrock system matrix pencil $P_\Sigma(\lambda)$ has full row rank if and only if $P_{\hat{\Sigma}}(\lambda)$ and $P_{\tilde{\Sigma}}(\lambda)$ have full row rank.*

Proof: This is a consequence of the decomposition

$$P_\Sigma(\lambda) = P_{\hat{\Sigma}}(\lambda) \begin{bmatrix} I_n & 0 \\ -H & I_m \end{bmatrix} = P_1 \begin{bmatrix} P_{\tilde{\Sigma}}(\lambda) & 0 \\ 0 & I_{n-\ell} \end{bmatrix} P_2 \begin{bmatrix} I_n & 0 \\ -H & I_m \end{bmatrix},$$

where P_1, P_2 are unimodular matrices, see Lemma 5.1 and Remark 5.1 in [12]. ■

2.2 Solvability conditions for the tracking control problem

The following set of assumptions is essential to ensure that any given reference r can be tracked from any given initial condition, [15].

Assumption 2.1 *We assume that Σ is*

(i) impulse controllable;

(ii) stabilizable;

(iii) right invertible and has no invariant zeros at $\lambda = 0$.[1]

Since we assume impulse controllability, the assumption of regularity of the pencil $\lambda E - A$ is not required, because impulse controllability implies regularizability, see [12].

The method for designing a state-feedback tracking controller for a step reference signal is the following. Assumption 2.1 (i) and (ii) imply the existence of a feedback gain matrix F such that $(E, A+B F)$ is asymptotically stable and the matrix pencil $\lambda E - (A+B F)$ is regular, while Assumption 2.1 (iii) ensures that for any $r \in \mathbb{R}^p$ there exist two vectors $x_{\text{ss}} \in \mathbb{R}^n$ and $u_{\text{ss}} \in \mathbb{R}^m$ that satisfy

$$\begin{aligned} 0 &= A\, x_{\text{ss}} + B\, u_{\text{ss}}, \\ r &= C\, x_{\text{ss}}. \end{aligned} \tag{7}$$

Applying the state feedback control law

$$u(t) = F\,(x(t) - x_{\text{ss}}) + u_{\text{ss}} \tag{8}$$

to Σ and using the change of variable $\xi(t) \stackrel{\text{def}}{=} x(t) - x_{\text{ss}}$, we obtain the closed-loop homogeneous system

$$\begin{aligned} E\,\dot{\xi}(t) &= (A+B\,F)\,\xi(t), \qquad E\,\xi(0) = E\,(x_0 - x_{\text{ss}}), \\ y(t) &= C\,\xi(t) + r. \end{aligned} \tag{9}$$

Since $(E, A+B\,F)$ is asymptotically stable, x converges to x_{ss}, y converges to r and ϵ converges to zero as t goes to infinity.

[1] Following the terminology of [9], we recall that Σ is right invertible in the strong sense if and only if $P_\Sigma(\lambda)$ is right invertible as a rational matrix. Since Σ is assumed to be impulse controllable, we have $\operatorname{rank} [\, A_{22} \;\; B_2 \,] = n - \ell$, so that $\operatorname{rank} [\, A - \lambda E \;\; B \,] = n$, i.e., $[\, A - \lambda E \;\; B \,]$ is right invertible as a rational matrix. In that case, weak and strong right invertibility are equivalent, see Corollary 4.13 in [9].

3 Design of nonovershooting feedback controllers

In [23], several methods were given to design a state-feedback matrix that yields a nonovershooting step response for explicit systems. In this section, we consider how to adapt these methods to descriptor systems.

3.1 Eigenstructure assignment of descriptor systems

The methods of [23] adapted the classic eigenstructure assignment result of [19] for the purposes of the tracking controller design. We begin by extending the result of [19] to descriptor systems.

Proposition 3.1 *Let (E, A, B) be an impulse controllable descriptor system described by (2). For a given $\lambda \in \mathbb{C}$, let the columns of $\begin{bmatrix} S_\lambda \\ T_\lambda \end{bmatrix}$ span* $\ker \begin{bmatrix} A - \lambda E & B \end{bmatrix}$. *Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_\ell\}$ be a self-conjugate set of distinct complex numbers. Let $\{v_1, \ldots, v_\ell\}$ be a set of vectors in $\mathbb{C}^n$. There exists a real F such that $(A + B\,F)\,v_i = \lambda_i\,E\,v_i$ and the closed-loop system $(E, A + B\,F)$ is impulse-free, so that the matrix pencil $\lambda\,E - (A + B\,F)$ is regular, if and only if for all $i \in \{1, \ldots, \ell\}$ the following conditions are satisfied:*
(i) The vectors $E\,v_i$ are linearly independent in $\mathbb{C}^n$;
(ii) $v_j = \overline{v}_i$ whenever $\lambda_j = \overline{\lambda}_i$;
(iii) $v_i \in \operatorname{im} S_{\lambda_i}$.

The above result is a special case of Theorem 2.1 in [20] for assigning $\ell = \operatorname{rank} E$ finite generalized eigenvalues, and a proof is given in Appendix A. The proof provides an algorithm for the construction of a real matrix F with the desired properties. Since regularity is not assumed, the construction of F will be carried out in two steps: first, find a preliminary state feedback H such that $\big(E, \hat{A}, B\big)$ is impulse-free, where $\hat{A} = A + BH$. Then, construct a state-feedback matrix $\hat{F}$ for the closed-loop system $\big(E, \hat{A}, B\big)$, so that $F = H + \hat{F}$ and $(E, A + B\,F)$ is impulse-free.[2] Although we have proved in Proposition 3.1 that we can construct a matrix $\hat{F}$ in two ways, in the sequel we choose to use the second method for numerical efficiency. The following corollary summarizes this procedure.

Corollary 3.1 *Let $\big(E, \hat{A}, B\big)$ be an impulse-free descriptor system as in (4). Let $\begin{bmatrix} S_\lambda \\ \hat{T}_\lambda \end{bmatrix}$ be a basis for* $\ker \begin{bmatrix} \hat{A} - \lambda\,E & B \end{bmatrix}$. *Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_\ell\}$ be a self-conjugate set of distinct complex numbers. Let $v_i \stackrel{def}{=} S_{\lambda_i}\,k_i,\ \hat{w}_i \stackrel{def}{=} \hat{T}_{\lambda_i}\,k_i,\ i \in \{1, \ldots, \ell\}$ and k_i be parameter vectors of suitable dimension chosen such that $E\,v_i \stackrel{def}{=} \begin{bmatrix} \tilde{v}_i \\ 0 \end{bmatrix}$ are linearly independent and $k_j = \overline{k}_i$ whenever $\lambda_j = \overline{\lambda}_i$. A matrix $\hat{F}$ such that $\big(\hat{A} + B\hat{F}\big)\,v_i = \lambda_i\,E\,v_i$ and the closed-loop system $(E, A{+}B\,F)$ is impulse-free, so that $\lambda\,E - \big(\hat{A}{+}B\,\hat{F}\big)$ is regular, is computed by $\hat{F} = \hat{W} \begin{bmatrix} \tilde{V} \\ 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} \hat{W}\,\tilde{V}^{-1} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{F} & 0 \end{bmatrix}$, where $\hat{W} = \begin{bmatrix} \hat{w}_1 & \ldots & \hat{w}_\ell \end{bmatrix}$, $\tilde{V} = \begin{bmatrix} \tilde{v}_1 & \ldots & \tilde{v}_\ell \end{bmatrix}$, $\tilde{F} = \hat{W}\,\tilde{V}^{-1}$.*

The next lemma will be an important tool in the development of an eigenstructure to obtain a desirable tracking response.

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[2] If Σ is not in the dynamics decomposition form (2), then $F = \big(H + \hat{F}\big)\,P^{-1}$.

Lemma 3.1 *Let Σ be an impulse controllable descriptor system as in (2) and $H = [\ 0 \ \ H_2\ ]$ be a real matrix such that the closed-loop system $\hat{\Sigma}$ is impulse-free and described by (4). Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_\ell\}$ be a self-conjugate set of ℓ distinct complex numbers and $\mathcal{S} = \{s_1, \ldots, s_\ell\}$ be a set of ℓ (not necessarily distinct) vectors in $\mathbb{R}^p$. Assume that the matrix equation*

$$\begin{bmatrix} \hat{A} - \lambda_i E & B \\ C & 0 \end{bmatrix} \begin{bmatrix} v_i \\ \hat{w}_i \end{bmatrix} = \begin{bmatrix} 0 \\ s_i \end{bmatrix} \tag{10}$$

has solutions sets $\mathcal{V} = \{v_1, \ldots, v_\ell\} \subset \mathbb{C}^n$ and $\hat{\mathcal{W}} = \{\hat{w}_1, \ldots, \hat{w}_\ell\} \subset \mathbb{C}^m$ for each $i \in \{1, \ldots, \ell\}$. If $E\,\mathcal{V}$ is linearly independent, then a real feedback matrix F exists such that $(E, A + B\,F)$ is impulse-free, $\lambda\,E - (A + B\,F)$ is regular and for all $i \in \{1, \ldots, \ell\}$

$$\begin{aligned} (A + B\,F)\,v_i &= \lambda_i\,E\,v_i, \\ C\,v_i &= s_i. \end{aligned} \tag{11}$$

Proof: The sets $\mathcal{L}$ and $\mathcal{V}$ satisfy all the assumptions of Proposition 3.1 and therefore we can use Corollary 3.1 to construct a real $\hat{F}$ satisfying $\hat{F}\,v_i = \hat{w}_i$ for all $i \in \{1, \ldots, \ell\}$, so that

$$\begin{aligned} \big(\hat{A} + B\,\hat{F}\big)\,v_i &= \lambda_i\,E\,v_i, \\ C\,v_i &= s_i \end{aligned}$$

and $\big(E, \hat{A} + B\,\hat{F}\big)$ is impulse-free and $\lambda\,E - \big(\hat{A} + B\,\hat{F}\big)$ is regular. Since $\hat{A} = A + B\,H$, we have

$$\begin{aligned} \big(A + B\,\big(H + \hat{F}\big)\big)\,v_i &= \lambda_i\,E\,v_i, \\ C\,v_i &= s_i, \end{aligned}$$

so that $F = H + \hat{F}$ satisfies (11), $(E, A + B\,F)$ is impulse-free and $\lambda\,E - (A + B\,F)$ is regular.

■

3.2 Descriptor systems with $\ell - p$ stable invariant zeros

We now present the main result of this paper on the design of state feedback control laws to yield a nonovershooting response for descriptor systems.

Assumption 3.1 *The descriptor system Σ*

(i) is square, i.e., $p = m$;

(ii) has at least $\ell - p$ distinct invariant zeros in the open left-half complex plane.

The assumption $p = m$ does not cause any significant loss of generality, see Remark 3.4 in the sequel. Under Assumptions 2.1 and 3.1*(i)*, the descriptor system Σ is invertible, which implies that $\operatorname{rank} P_\Sigma(\lambda) = n + p = n + m$ if and only if $\lambda \in \mathbb{C}$ is not an invariant zero of Σ.

Let us choose the self-conjugate set $\mathcal{L} = \{\lambda_1, \ldots, \lambda_\ell\} \subset \mathbb{C}^-$ of distinct stable finite generalized eigenvalues of $(E, A + B\,F)$. We choose $\lambda_i = z_i$, $i \in \{1, \ldots, \ell - p\}$, where $\{z_1, \ldots, z_{\ell-p}\} \subset \mathbb{C}^-$ is freely chosen from the distinct stable invariant zeros of Σ. Since Σ is invertible, any uncontrollable modes of Σ are also invariant zeros of Σ. Indeed, rank $\begin{bmatrix} \lambda E - A & B \end{bmatrix} < n$ implies that rank $P_{\Sigma}(\lambda) =$ rank $P_{\hat{\Sigma}}(\lambda) < n+p$. Consequently, under the assumption of stabilizability, all the uncontrollable finite generalized eigenvalues are included among the λ_i for $i \in \{1, \dots, \ell - p\}$, so that we can freely choose λ_i, $i \in \{\ell - p + 1, \dots, \ell\}$ to be any real distinct stable modes that are different from the invariant zeros of Σ.

With this choice of $\mathcal{L}$, we can solve (10) for all $i \in \{1, \dots, \ell\}$. Indeed, since λ_i, $i \in \{1, \dots, \ell - p\}$ are chosen to be equal to distinct invariant zeros, the null-spaces of $P_{\hat{\Sigma}}(\lambda_i)$ are 1-dimensional subspaces of $\mathbb{R}^{n+p}$ and $\left[\begin{smallmatrix} v_i \\ \hat{w}_i \end{smallmatrix}\right] \in \ker P_{\hat{\Sigma}}(\lambda_i)$ satisfy (10) for $s_i = 0$. For $\lambda_j = \overline{\lambda}_i$ we need to ensure that v_i and v_j are chosen to satisfy $v_j = \overline{v}_i$. For $i \in \{\ell - p + 1, \dots, \ell\}$, $\ker P_{\hat{\Sigma}}(\lambda_i) = \{0\}$, because these λ_i are not chosen from the set of invariant zeros. This fact and the right invertibility of Σ imply that rank $P_{\hat{\Sigma}}(\lambda_i) =$ rank $\begin{bmatrix} \hat{A} - \lambda_i E & B \end{bmatrix} + p$, which guarantees that (10) can be solved for all $i \in \{\ell - p + 1, \dots, \ell\}$ for any $s_i \in \mathbb{R}^p$.

Let now $\{\mathbf{e}_1, \dots, \mathbf{e}_p\}$ be the canonical basis of $\mathbb{R}^p$ and let $\mathcal{S} = \{s_1, \dots, s_{\ell-p}, s_{\ell-p+1}, \dots, s_\ell\} \subset \mathbb{R}^p$ be such that

$$s_i = \begin{cases} 0 & i \in \{1, \dots, \ell - p\}, \\ \mathbf{e}_1 & i = \ell - p + 1, \\ \vdots & \\ \mathbf{e}_p & i = \ell. \end{cases}$$

Then the solution of (10) is $\left[\begin{smallmatrix} v_i \\ \hat{w}_i \end{smallmatrix}\right] = P_{\hat{\Sigma}}^{-1}(\lambda_i) \left[\begin{smallmatrix} 0 \\ \mathbf{e}_{i-(\ell-p)} \end{smallmatrix}\right]$. If we solve (10) for all the vectors in $\mathcal{S}$, we obtain $\mathcal{V} = \{v_1, \dots, v_\ell\} \subset \mathbb{C}^n$ and $\hat{\mathcal{W}} = \{\hat{w}_1, \dots, \hat{w}_\ell\} \subset \mathbb{C}^p$. If $E\,\mathcal{V} = \left\{\left[\begin{smallmatrix} \tilde{v}_1 \\ 0 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} \tilde{v}_\ell \\ 0 \end{smallmatrix}\right]\right\}$ or, equivalently, $\{\tilde{v}_1, \dots, \tilde{v}_\ell\}$ is linearly independent, then, from Lemma 3.1, we can use $\mathcal{V}$ and $\hat{\mathcal{W}}$ to construct a feedback matrix F such that the finite eigenstructure of $(E, A + B\,F)$ is given by $\mathcal{L}$ and $\mathcal{V}$ and there hold

$$\begin{aligned} (A + B\,F)\,v_i &= \lambda_i\,E\,v_i, \quad i \in \{1, \dots, \ell\}, \\ C\,v_i &= \begin{cases} 0, & i \in \{1, \dots, \ell - p\}, \\ \mathbf{e}_{i-(\ell-p)}, & i \in \{\ell - p + 1, \dots, \ell\}. \end{cases} \end{aligned} \tag{12}$$

The following theorem shows that we can use the matrix F constructed above to obtain a state feedback control law which gives rise to a closed-loop system response that converges to any given step reference $r \in \mathbb{R}^p$ without overshoot, from *all* initial conditions $E\,x_0$.

Theorem 3.1 *Consider the descriptor system Σ in (2) satisfying Assumptions 2.1 and 3.1. Let $\mathcal{L}$ be a set of desired closed-loop poles, and assume that the set $E\,\mathcal{V}$ is linearly independent, where $\mathcal{V}$ is obtained from the solution of (10). Assume that F satisfies (12) with respect to $\mathcal{L}$ and $\mathcal{V}$. Let $r \in \mathbb{R}^p$ be any step reference and let $E\,x_0 = \left[\begin{smallmatrix} \tilde{x}_0 \\ 0 \end{smallmatrix}\right] \in \mathbb{R}^n$ be any initial condition. Then, the output $y(t)$ obtained from applying $u(t) = F\,(x(t) - x_{\rm ss}) + u_{\rm ss}$ to Σ tracks r asymptotically without overshoot.*

Proof: Applying $u(t)$ to Σ and employing the change of variable $\xi(t) \stackrel{\text{def}}{=} x(t) - x_{\rm ss}$, which is partitioned as $\begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{x}(t)-\tilde{x}_{\text{ss}} \\ \hat{x}(t)-\hat{x}_{\text{ss}} \end{bmatrix}$, we obtain (9), or, equivalently,

$$\begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\tilde{\xi}}(t) \\ \dot{\hat{\xi}}(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}\begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \end{bmatrix} = \begin{bmatrix} A_{11}+B_1\,\tilde{F} & A_{12}+B_1\,H_2 \\ A_{21}+B_2\,\tilde{F} & A_{22}+B_2\,H_2 \end{bmatrix}\begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \end{bmatrix},$$

$$y(t) = [\; C_1 \;\; C_2 \;] \begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \end{bmatrix} + r.$$

Using a further change of coordinates, we can bring (9) into an equivalent form as follows

$$\tilde{Q}\,E\,\check{P} = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}, \; \tilde{Q}\,(A+B\,F)\,\check{P} = \begin{bmatrix} \tilde{A}_F & 0 \\ 0 & I_{n-\ell} \end{bmatrix}, \; C\,\check{P} = [\; \tilde{C}_F \;\; \tilde{C}_2 \;], \; \begin{bmatrix} \tilde{\xi}(t) \\ \check{\xi}(t) \end{bmatrix} = \check{P}^{-1} \begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \end{bmatrix}$$

where $\check{P} \stackrel{\text{def}}{=} \begin{bmatrix} I_\ell & 0 \\ -\hat{A}_{22}^{-1}\,\hat{A}_{21} & \hat{A}_{22}^{-1} \end{bmatrix}$, $\check{P}^{-1} = \begin{bmatrix} I_\ell & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$, $\tilde{A}_F \stackrel{\text{def}}{=} \hat{A}_{11} - \hat{A}_{12}\,\hat{A}_{22}^{-1}\,\hat{A}_{21}$, $\tilde{C}_F \stackrel{\text{def}}{=} C_1 - C_2\,\hat{A}_{22}^{-1}\,\hat{A}_{21}$ and $\check{\xi}(t) \stackrel{\text{def}}{=} \hat{A}_{21}\,\tilde{\xi}(t) + \hat{A}_{22}\,\hat{\xi}(t)$, so that

$$\begin{aligned} \dot{\tilde{\xi}}(t) &= \tilde{A}_F\,\tilde{\xi}(t), \\ 0 &= \check{\xi}(t), \\ y(t) &= \tilde{C}_F\,\tilde{\xi}(t) + \tilde{C}_2\,\check{\xi}(t) + r. \end{aligned}$$

The state response of the above descriptor system is given by $\begin{bmatrix} \tilde{\xi}(t) \\ \check{\xi}(t) \end{bmatrix} = \begin{bmatrix} e^{\tilde{A}_F\,t}\,\tilde{\xi}_0 \\ 0 \end{bmatrix}$, see e.g. [20] and [6, Ch.3], so that the state response of (9) is $\xi(t) = \begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \end{bmatrix} = \check{P} \begin{bmatrix} e^{\tilde{A}_F\,t}\,\tilde{\xi}_0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_\ell \\ -\hat{A}_{22}^{-1}\,\hat{A}_{21} \end{bmatrix} e^{\tilde{A}_F\,t}\,\tilde{\xi}_0$.

The tracking error $\epsilon(t) = r - y(t)$ is

$$\begin{aligned} \epsilon(t) &= -[\; C_1 \;\; C_2 \;] \begin{bmatrix} I_\ell \\ -\hat{A}_{22}^{-1}\,\hat{A}_{21} \end{bmatrix} e^{\tilde{A}_F\,t}\,\tilde{\xi}_0 \\ &= -\big(C_1 - C_2\,\hat{A}_{22}^{-1}\,\hat{A}_{21}\big)\,e^{\tilde{A}_F\,t}\,\tilde{\xi}_0 \\ &= -\big(\tilde{C} + \tilde{D}\,\tilde{F}\big)\,e^{\left(\tilde{A}+\tilde{B}\,\tilde{F}\right)t}\,\tilde{\xi}_0, \end{aligned}$$

which coincides with the tracking error for the associated explicit system $\tilde{\Sigma}$ using the feedback control law $\tilde{F}$, see [23]. Since $E\,\mathcal{V}$ is linearly independent, the matrix $\tilde{V} = [\; \tilde{v}_1 \;\; \ldots \;\; \tilde{v}_\ell \;]$ is invertible. Introducing $\alpha \stackrel{\text{def}}{=} [\; \alpha_1 \;\; \ldots \;\; \alpha_\ell \;]^\top = \tilde{V}^{-1}\,\tilde{\xi}_0$, from (12) and since

$$\begin{aligned} s_i &= C\,v_i = [\; C_1 \;\; C_2 \;] \begin{bmatrix} \tilde{v}_i \\ \hat{v}_i \end{bmatrix} \\ &= [\; C_1 \;\; C_2 \;] \begin{bmatrix} \tilde{v}_i \\ -\hat{A}_{22}^{-1}\,\big(A_{21} + B_2\,\tilde{F}\big)\tilde{v}_i \end{bmatrix} \\ &= \big(C_1 - C_2\,\hat{A}_{22}^{-1}\,\hat{A}_{21}\big)\,\tilde{v}_i = \big(\tilde{C} + \tilde{D}\,\tilde{F}\big)\,\tilde{v}_i, \end{aligned}$$

it follows that the tracking error can be expressed as

$$\epsilon(t) = -\sum_{i=1}^{\ell} \big(\tilde{C}+\tilde{D}\tilde{F}\big)\,\tilde{v}_i\,\alpha_i\,e^{\lambda_i t} = -\sum_{i=\ell-p+1}^{\ell} \mathbf{e}_{i-(\ell-p)}\,\alpha_i\,e^{\lambda_i t} = -\begin{bmatrix} \alpha_{\ell-p+1}\,e^{\lambda_{\ell-p+1} t} \\ \vdots \\ \alpha_\ell\,e^{\lambda_\ell t} \end{bmatrix}.$$

Thus, every component of $\epsilon(t)$ contains exactly one mode, i.e., $\epsilon_i(t) = -\alpha_{\ell-p+i}\, e^{\lambda_{\ell-p+i}\, t}$, $i \in \{1, \ldots, p\}$. Since all the finite generalized eigenvalues are in the open left-half complex plane, the descriptor system (9) is asymptotically stable and $\epsilon(t)$ converges to 0 as t goes to infinity. The λ_i for $i \in \{\ell - p + 1, \ldots, \ell\}$ have been chosen so that $e^{\lambda_i t}$ do not change sign. Thus, $\epsilon(t)$ does not change sign in any component and $y(t)$ converges to r without overshoot. ■

Remark 3.1 The transient response of the closed-loop system depends on the closed-loop finite eigenvectors for a specified set of closed-loop finite generalized eigenvalues, given an initial condition. The preliminary state feedback will not affect the transient response because it does not affect the closed-loop finite eigenvectors. Indeed, from the proof of Proposition 3.1 in Appendix A, the closed-loop finite eigenvectors are computed by $v_i = S_{\lambda_i}\, k_i$ and $\mathrm{im}\, \hat{S}_{\lambda_i} = \mathrm{im}\, S_{\lambda_i}$, $i \in \{1, \ldots, \ell\}$. If we choose another H', then we can write

$$
\begin{aligned}
0 &= \left[\, A + B\,H - \lambda_i\,E \;\; B \,\right] \begin{bmatrix} v_i \\ \hat{w}_i \end{bmatrix} \\
&= \left[\, A + B\,H' - \lambda_i\,E \;\; B \,\right] \begin{bmatrix} I_n & 0 \\ H - H' & I_m \end{bmatrix} \begin{bmatrix} v_i \\ \hat{w}_i \end{bmatrix} \\
&= \left[\, A + B\,H' - \lambda_i\,E \;\; B \,\right] \begin{bmatrix} v_i \\ (H - H')\; v_i + \hat{w}_i \end{bmatrix}
\end{aligned}
$$

for $i \in \{1, \ldots, \ell\}$, which shows that the closed-loop finite eigenvectors are the same for distinct closed-loop finite generalized eigenvalues.

3.3 Descriptor systems with $\ell - 2\,p$ stable invariant zeros

We now weaken the Assumption 3.1(ii) that the descriptor system Σ has at least $\ell - p$ invariant zeros in the open left-half complex plane with the following:

Assumption 3.2 *The descriptor system Σ is square and has at least $\ell - 2\,p$ distinct invariant zeros in the open left-half complex plane.*

Let $\mathcal{L} = \{\lambda_1, \;\ldots,\; \lambda_\ell\}$ denote the distinct stable finite generalized eigenvalues of $(E, A + B\,F)$ to be chosen. We assume that the descriptor system Σ has exactly $\ell - 2\,p$ distinct stable invariant zeros, denoted by $z_1, \ldots, z_{\ell-2\,p}$. Mimicking the procedure in Section 3.2, we choose $\lambda_i = z_i$ for $i \in \{1, \ldots, \ell - 2\,p\}$, and λ_i for $i \in \{\ell - 2\,p + 1, \ldots, \ell\}$ may be freely chosen to be any distinct real stable modes. Let $\mathcal{S} = \{s_1, \ldots, s_{\ell-2\,p}, s_{\ell-2\,p+1}, s_{\ell-2\,p+2} \ldots, s_{\ell-1}, s_\ell\} \subset \mathbb{R}^p$ be such that

$$
s_i = \begin{cases} 0 & \text{for } i \in \{1, \ldots, \ell - 2\,p\}, \\ \mathbf{e}_1 & i \in \{\ell - 2\,p + 1, \ell - 2\,p + 2\}, \\ \mathbf{e}_2 & i \in \{\ell - 2\,p + 3, \ell - 2\,p + 4\}, \\ \vdots & \\ \mathbf{e}_p & i \in \{\ell - 1, \ell\}. \end{cases} \tag{13}
$$

Solving (10) for all the vectors in $\mathcal{S}$, we obtain $\mathcal{V} = \{v_1, \ldots, v_\ell\} \subset \mathbb{C}^n$ and $\hat{\mathcal{W}} = \{\hat{w}_1, \ldots, \hat{w}_\ell\} \subset \mathbb{C}^p$. If $E\,\mathcal{V} = \left\{ \left[\begin{smallmatrix} \tilde{v}_1 \\ 0 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} \tilde{v}_\ell \\ 0 \end{smallmatrix}\right] \right\}$ or, equivalently, $\tilde{\mathcal{V}} \stackrel{\text{def}}{=} \{\tilde{v}_1, \ldots, \tilde{v}_\ell\} \subset \mathbb{C}^\ell$ is linearly independent, then, from

Lemma 3.1, we can use $\mathcal{V}$ and $\hat{\mathcal{W}}$ to construct a feedback matrix F such that the finite eigenstructure of $(E, A+B\,F)$ is given by $\mathcal{L}$ and $\mathcal{V}$ and there hold

$$\begin{aligned}(A+B\,F)\,v_i &= \lambda_i\,E\,v_i, \quad i\in\{1,\ldots,\ell\},\\ C\,v_i &= \begin{cases} 0 & i\in\{1,\ldots,\ell-2\,p\},\\ \mathbf{e}_1 & i\in\{\ell-2\,p+1,\ell-2\,p+2\},\\ \mathbf{e}_2 & i\in\{\ell-2\,p+3,\ell-2\,p+4\},\\ \vdots & \\ \mathbf{e}_p & i\in\{\ell-1,\ell\}.\end{cases}\end{aligned} \tag{14}$$

The following notation allows us to succinctly state Theorem 3.1 for descriptor systems satisfying Assumption 3.2.

Notation 3.1 For each $k\in\{1,\ldots,p\}$, let

(i) $v_{k,1}$ and $v_{k,2}$ denote the finite eigenvectors in $\mathcal{V}$ associated with $\mathbf{e}_k$ in (13), partitioned as $v_{k,1}\stackrel{\text{def}}{=}\begin{bmatrix}\tilde{v}_{k,1}\\ \hat{v}_{k,1}\end{bmatrix}$ and $v_{k,2}\stackrel{\text{def}}{=}\begin{bmatrix}\tilde{v}_{k,2}\\ \hat{v}_{k,2}\end{bmatrix}$. Let $\tilde{V}$ be the matrix formed by the columns of $\tilde{\mathcal{V}}$. Then

$$\tilde{V}\stackrel{\text{def}}{=}[\,\tilde{v}_1\quad\ldots\quad\tilde{v}_{\ell-2\,p}\quad\tilde{v}_{1,1}\quad\tilde{v}_{1,2}\quad\ldots\quad\tilde{v}_{p,1}\quad\tilde{v}_{p,2}\,]; \tag{15}$$

(ii) $\lambda_{k,1}$ and $\lambda_{k,2}$ be the finite generalized eigenvalues corresponding to $v_{k,1}$ and $v_{k,2}$, ordered such that $\lambda_{k,1}<\lambda_{k,2}$;

(iii) $\xi(t)=x(t)-x_{\text{ss}}$ and $E\xi(0)=E\,\xi_0=E\,(x_0-x_{\text{ss}})$, which are partitioned respectively as $\begin{bmatrix}\tilde{\xi}(t)\\ \hat{\xi}(t)\end{bmatrix}\stackrel{\text{def}}{=}\begin{bmatrix}\tilde{x}(t)-\tilde{x}_{\text{ss}}\\ \hat{x}(t)-\hat{x}_{\text{ss}}\end{bmatrix}$ and $\begin{bmatrix}\tilde{\xi}_0\\ 0\end{bmatrix}\stackrel{\text{def}}{=}\begin{bmatrix}\tilde{x}_0-\tilde{x}_{\text{ss}}\\ 0\end{bmatrix}$;

(iv) $\alpha\stackrel{\text{def}}{=}\tilde{V}^{-1}\,\tilde{\xi}_0$. Then we may write

$$\alpha\stackrel{\text{def}}{=}[\,\alpha_1\ \ \ldots\ \ \alpha_{\ell-2\,p}\ \ \alpha_{1,1}\ \ \alpha_{1,2}\ \ \ldots\ \ \alpha_{p,1}\ \ \alpha_{p,2}\,]^\top; \tag{16}$$

(v) $\mathcal{H}_k\stackrel{\text{def}}{=}\text{span}\{\tilde{v}_{k,1},\tilde{v}_{k,2}\}$;

(vi) $\mathcal{J}_k\subseteq\mathcal{H}_k$ be the region $\mathcal{J}_k\stackrel{\text{def}}{=}\{\gamma_{k,1}\,\tilde{v}_{k,1}+\gamma_{k,2}\,(\tilde{v}_{k,1}-\tilde{v}_{k,2})\mid\gamma_{k,1}\,\gamma_{k,2}\le 0\}$;

(vii) Let $\tilde{x}_k$ denote the orthogonal projection of $\tilde{x}$ onto $\mathcal{H}_k$ and let $\mathcal{J}\subseteq\mathbb{R}^\ell$ consist of those points in $\tilde{x}\in\mathbb{R}^\ell$ for which $\tilde{x}_k\in\mathcal{J}_k$ for all $k\in\{1,\ldots,p\}$.

In Theorem 3.2 a set of initial conditions $E\,x_0=\begin{bmatrix}\tilde{x}_0\\ 0\end{bmatrix}\in\mathbb{R}^n$ is given from which, for a given $r\in\mathbb{R}^p$, the closed-loop system in (9) yields a nonovershooting response. The following lemma from [23] is needed for the proof of the theorem and is included for completeness.

Lemma 3.2 ([23]) *Let $\lambda_1<\lambda_2<0$ and define $f(t)\stackrel{\text{def}}{=}\alpha_1\,e^{\lambda_1\,t}+\alpha_2\,e^{\lambda_2\,t}$. Then, $f(t)$ changes sign for some $t\in\mathbb{R}^+$ if and only if $(\alpha_1,\alpha_2)=(\gamma_1+\gamma_2,-\gamma_2)$ for some real numbers γ_1 and γ_2 such that $\gamma_1\,\gamma_2>0$.*

Theorem 3.2 *Consider the descriptor system Σ in (2) satisfying Assumptions 2.1 and 3.2. Let $\mathcal{L}$ be a set of desired closed-loop poles, and assume that the set $E\,\mathcal{V}$ is linearly independent, where $\mathcal{V}$ is obtained from the solution of (10) with s_i in (13). Assume that F satisfies (14) with respect to $\mathcal{L}$ and $\mathcal{V}$. Let $r\in\mathbb{R}^p$ be a given step reference and let $E\,x_0=\begin{bmatrix}\tilde{x}_0\\ 0\end{bmatrix}\in\mathbb{R}^n$ be an initial condition. Then, the output $y(t)$ obtained from applying $u(t)=F\,(x(t)-x_{\text{ss}})+u_{\text{ss}}=F\begin{bmatrix}\tilde{x}(t)-\tilde{x}_{\text{ss}}\\ \hat{x}(t)-\hat{x}_{\text{ss}}\end{bmatrix}+u_{\text{ss}}$ to Σ is nonovershooting for $E\,x_0$ if and only if $\tilde{x}_0-\tilde{x}_{\text{ss}}\in\mathcal{J}$.*

Proof: Defining

$$\epsilon_k(t) \stackrel{\text{def}}{=} \alpha_{k,1}\, e^{\lambda_{k,1}\, t} + \alpha_{k,2}\, e^{\lambda_{k,2}\, t}, \tag{17}$$

the tracking error can be expressed as

$$\epsilon(t) = -\sum_{k=1}^{p} \mathbf{e}_k\, \epsilon_k(t) = -\begin{bmatrix} \alpha_{1,1}\, e^{\lambda_{1,1}\, t} + \alpha_{1,2}\, e^{\lambda_{1,2}\, t} \\ \vdots \\ \alpha_{p,1}\, e^{\lambda_{p,1}\, t} + \alpha_{p,2}\, e^{\lambda_{p,2}\, t} \end{bmatrix}.$$

Since all the finite generalized eigenvalues are in the open left-half complex plane, the descriptor system (9) is asymptotically stable and $\epsilon(t)$ converges to 0 as t goes to infinity. The signs of the tracking errors remain unchanged if and only if $\epsilon_k(t)$ does not change sign for every $k \in \{1,\ldots,p\}$ and for all $t \in \mathbb{R}^+$. From (15) and (16), we may represent $\tilde{\xi}_0 = \tilde{x}_0 - \tilde{x}_{\text{ss}}$ as

$$\tilde{\xi}_0 = \sum_{i=1}^{\ell-2\,p} \alpha_i\, \tilde{v}_i + \sum_{k=1}^{p} \Big(\alpha_{k,1}\, \tilde{v}_{k,1} + \alpha_{k,2}\, \tilde{v}_{k,2}\Big),$$

and thus the projection of $\tilde{\xi}_0$ onto $\mathcal{H}_k$, is

$$\tilde{\xi}_{0_k} = \alpha_{k,1}\, \tilde{v}_{k,1} + \alpha_{k,2}\, \tilde{v}_{k,2}. \tag{18}$$

(Sufficiency). Let $\tilde{\xi}_0 \in \mathcal{J}$. Since $\tilde{\xi}_{0_k} \in \mathcal{J}_k$ for every $k \in \{1,\ldots,p\}$, we have $\gamma_{k,1}$ and $\gamma_{k,2}$ such that

$$\tilde{\xi}_{0_k} = \gamma_{k,1}\, \tilde{v}_{k,1} + \gamma_{k,2}\, (\tilde{v}_{k,1} - \tilde{v}_{k,2}) \tag{19}$$

and $\gamma_{k,1}\, \gamma_{k,2} \le 0$. Comparing (18) and (19), we find $(\alpha_{k,1}, \alpha_{k,2}) = (\gamma_{k,1}+\gamma_{k,2}, -\gamma_{k,2})$. From Lemma 3.2 and the assumption that $\lambda_{k,1} < \lambda_{k,2} < 0$, each $\epsilon_k(t)$ does not change sign for $t \in \mathbb{R}^+$. Consequently, $y(t)$ converges to r without overshoot.

(Necessity). If $\tilde{\xi}_0 \notin \mathcal{J}$, for some $\kappa \in \{1,\ldots,p\}$ we have $\tilde{\xi}_{0_\kappa} \notin \mathcal{J}_\kappa$, and hence there exist $\gamma_{\kappa,1}$ and $\gamma_{\kappa,2}$ such that $\tilde{\xi}_{0_\kappa} = \gamma_{\kappa,1}\, \tilde{v}_{\kappa,1} + \gamma_{\kappa,2}\, (\tilde{v}_{\kappa,1} - \tilde{v}_{\kappa,2})$ and $\gamma_{\kappa,1}\, \gamma_{\kappa,2} > 0$. Applying Lemma 3.2 again, we conclude that $\epsilon_\kappa(t)$ changes sign for some $t \in \mathbb{R}^+$, so that $\epsilon(t)$ changes sign in the κ-th component. ■

Remark 3.2 (i) In applying Theorem 3.2 to see if the descriptor system in (9) has nonovershooting response for a given $E\, x_0 \in \mathbb{R}^n$, we may construct the matrix

$$P = [\, \tilde{v}_1 \quad \ldots \quad \tilde{v}_{\ell-2\,p} \quad \tilde{v}_{1,1} \quad \tilde{v}_{1,1} - \tilde{v}_{1,2} \quad \ldots \quad \tilde{v}_{p,1} \quad \tilde{v}_{p,1} - \tilde{v}_{p,2}\,],$$

then calculate $\gamma = [\, \gamma_1 \quad \ldots \quad \gamma_{\ell-2\,p} \quad \gamma_{1,1} \quad \gamma_{1,2} \quad \ldots \quad \gamma_{p,1} \quad \gamma_{p,2}\,]^\top$, where $\gamma = P^{-1}\, (\tilde{x}_0 - \tilde{x}_{\text{ss}})$ and check if $\gamma_{k,1}\, \gamma_{k,2} \le 0$ for all $k \in \{1,\ldots,p\}$.

Remark 3.3 If the descriptor system Σ has $\ell - 2p + q$ stable invariant zeros, where $1 \le q < p$, then we may modify (13) so that the $\mathbf{e}_k$ correspond to a unique s_i for $i \in \{\ell - 2p + 1, \ldots, \ell - 2p + q\}$ and then pairs of s_i for $i \in \{\ell - 2p + q + 1, \ldots, \ell - 2p\}$. Then (17) will only contain a single exponential term for $k \in \{1,\ldots,q\}$ and those output components will be nonovershooting for any initial condition $E\, x_0 = \left[\begin{smallmatrix} \tilde{x}_0 \\ 0 \end{smallmatrix}\right]$.

Remark 3.4 When a descriptor system has more inputs than outputs, the design method can be generalized as discussed in Section 4 of [23] by defining $m-p$ fictitious outputs and placing as many additional modes as required. Moreover, if the system has fewer than $\ell - 2\,p$ invariant zeros in the open left-half complex plane, then a similar method as in Section 3.4 of [23] can be used.

4 Nonovershooting dynamic output feedback tracking controllers

In this section, we deal with the problem of designing a dynamic output feedback control law for the descriptor system Σ described by (2), such that the output $y(t)$ tracks a step reference r with zero steady-state error with no overshoot.

Assumption 4.1 *The descriptor system Σ is:*

(i) impulse controllable and impulse observable;

(ii) stabilizable and detectable;

(iii) right invertible and has no invariant zeros at $\lambda = 0$.

The *Luenberger observer* system Σ_o for Σ is described by

$$E\,\dot{z}(t) = A\,z(t) + B\,u(t) - G\,(y(t) - y_o(t)), \tag{20a}$$

$$y_o(t) = C\,z(t), \tag{20b}$$

with $E\,z(0) = 0$, where F is designed according to the scheme of the previous section, and G will be defined subsequently. There exist vectors $x_{\rm ss} \in \mathbb{R}^n$ and $u_{\rm ss} \in \mathbb{R}^m$ that satisfy (7) for any $r \in \mathbb{R}^p$ and we apply the output feedback control law

$$u(t) = F\,(z(t) - x_{\rm ss}) + u_{\rm ss} \tag{21}$$

to the descriptor system (20), so that

$$\begin{aligned} E\,\dot{z}(t) &= (A + B\,F + G\,C)\,z(t) - (A + B\,F)\,x_{\rm ss} - G\,y(t), \\ y_o(t) &= C\,z(t). \end{aligned} \tag{22}$$

Our aim in this section is to choose suitable G such that the dynamic output feedback control law (21) yields a nonovershooting response for the observer system (20) in all output components.

Substituting (21) in (1), we obtain

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,F\,z(t) - B\,F\,x_{\rm ss} + B\,u_{\rm ss}, \\ y(t) &= C\,x(t), \end{aligned}$$

which, in view of (7), yields

$$\begin{aligned} E\,\dot{x}(t) &= A\,(x(t) - x_{\rm ss}) + B\,F\,z(t) - B\,F\,x_{\rm ss}, \\ y(t) &= C\,(x(t) - x_{\rm ss}) + r \end{aligned}$$

and changing coordinates as $\xi(t) = x(t) - x_{\rm ss}$, we get

$$\begin{aligned} E\,\dot{\xi}(t) &= (A + B\,F)\,\xi(t) + B\,F\,(z(t) - x(t)),\\ y(t) - r &= C\,\xi(t). \end{aligned}$$

Moreover, subtracting (20a) from (1a), we obtain

$$E(\dot{x}(t) - \dot{z}(t)) = (A + G\,C)(x(t) - z(t)).$$

Therefore we have the homogeneous closed-loop system

$$\begin{aligned} \begin{bmatrix} E\,\dot{\xi}(t) \\ E\,\dot{e}(t) \end{bmatrix} &= \begin{bmatrix} A + B\,F & -B\,F \\ 0 & A + G\,C \end{bmatrix} \begin{bmatrix} \xi(t) \\ e(t) \end{bmatrix},\\ \epsilon(t) &= -[\,C \;\; 0\,] \begin{bmatrix} \xi(t) \\ e(t) \end{bmatrix}, \end{aligned} \tag{23}$$

where $e(t) \stackrel{\rm def}{=} x(t) - z(t)$ is the *estimation error*, and $\epsilon(t) = r - y(t)$ is the tracking error.

Again using the decomposition (2), and the impulse observability of Σ, there exists an $L = \left[\begin{smallmatrix} 0 \\ L_2 \end{smallmatrix}\right]$, such that $\det\,(A_{22}{+}L_2\,C_2) \neq 0$. To ensure regularity of the matrix pencil $\lambda E{-}(A+B\,F{+}G\,C)$ of (22), the matrix L_2 must be constructed to satisfy also $\det\,(A_{22} + B_2\,H_2 + L_2\,C_2) = \det\,\big(\hat{A}_{22} + L_2\,C_2\big) \neq 0$, or, equivalently, $\det\,\big(I_p + C_2\,\hat{A}_{22}^{-1}\,L_2\big) \neq 0$.[3] Let us choose the set $\mathcal{L}_o = \{\lambda_{\ell+1}, \ldots, \lambda_{2\ell}\}$ of negative real numbers such that $\mu_0 \stackrel{\rm def}{=} \max\{\lambda_{\ell+1}, \ldots, \lambda_{2\ell}\}$ satisfies

$$\mu_0 < \min\{\lambda_{\ell-p+1}, \ldots, \lambda_\ell\}. \tag{24}$$

We shall refer to $\mathcal{L}_o$ as the *observer poles*. From the detectability of Σ and in view of Corollary 3.1, we can obtain a real $\hat{G}^\top = \big[\,\tilde{G}^\top \;\; 0\,\big]$ for the dual impulse-free descriptor system $\big(E^\top, A^\top + C^\top\,L^\top, C^\top\big)$, such that the closed-loop system $\big(E^\top, A^\top + C^\top\,L^\top + C^\top\,\hat{G}^\top, C^\top\big)$ has finite generalized eigenvalues given by $\mathcal{L}_o$. We then construct G in (20) using $G = L + \hat{G} = \left[\begin{smallmatrix} \tilde{G} \\ L_2 \end{smallmatrix}\right]$.[4] It follows that $(E, A + G\,C)$ is asymptotically stable. Thus, this choice of F and G makes the estimation and tracking errors vanish, leading to asymptotic tracking.

[3] There holds $\det\,\big(\hat{A}_{22} + L_2\,C_2\big) = \det \hat{A}_{22}\,\det\,\big(I_p + C_2\,\hat{A}_{22}^{-1}\,L_2\big)$, see [2].

[4] If Σ is not in the dynamics decomposition form (2), then $G = Q^{-1}\,\big(L + \hat{G}\big)$.

Next, we partition conformably the estimation error $e(t) \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{e}(t) \\ \hat{e}(t) \end{bmatrix}$ and rewrite (23) as

$$
\begin{aligned}
\begin{bmatrix} \dot{\tilde{\xi}}(t) \\ 0 \\ \dot{\tilde{e}}(t) \\ 0 \end{bmatrix} &= \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & B_{11} & B_{12} \\ \hat{A}_{21} & \hat{A}_{22} & B_{21} & B_{22} \\ 0 & 0 & \check{A}_{11} & \check{A}_{12} \\ 0 & 0 & \check{A}_{21} & \check{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \\ \tilde{e}(t) \\ \hat{e}(t) \end{bmatrix} \\
&= \begin{bmatrix} A_{11} + B_1 \tilde{F} & A_{12} + B_1 H_2 & -B_1 \tilde{F} & -B_1 H_2 \\ A_{21} + B_2 \tilde{F} & A_{22} + B_2 H_2 & -B_2 \tilde{F} & -B_2 H_2 \\ 0 & 0 & A_{11} + \tilde{G} C_1 & A_{12} + \tilde{G} C_2 \\ 0 & 0 & A_{21} + L_2 C_1 & A_{22} + L_2 C_2 \end{bmatrix} \begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \\ \tilde{e}(t) \\ \hat{e}(t) \end{bmatrix}, \\
\epsilon(t) &= -\begin{bmatrix} C_1 & C_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}(t) \\ \hat{\xi}(t) \\ \tilde{e}(t) \\ \hat{e}(t) \end{bmatrix},
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
\left[\begin{array}{c} \dot{\tilde{\xi}}(t) \\ \dot{\tilde{e}}(t) \\ \hline 0 \\ 0 \end{array}\right] &= \left[\begin{array}{cc|cc} \hat{A}_{11} & B_{11} & \hat{A}_{12} & B_{12} \\ 0 & \check{A}_{11} & 0 & \check{A}_{12} \\ \hline \hat{A}_{21} & B_{21} & \hat{A}_{22} & B_{22} \\ 0 & \check{A}_{21} & 0 & \check{A}_{22} \end{array}\right] \left[\begin{array}{c} \tilde{\xi}(t) \\ \tilde{e}(t) \\ \hline \hat{\xi}(t) \\ \hat{e}(t) \end{array}\right], \\
\epsilon(t) &= -\left[\begin{array}{cc|cc} C_1 & 0 & C_2 & 0 \end{array}\right] \left[\begin{array}{c} \tilde{\xi}(t) \\ \tilde{e}(t) \\ \hline \hat{\xi}(t) \\ \hat{e}(t) \end{array}\right].
\end{aligned} \tag{25}
$$

The above descriptor system is impulse-free, because $\hat{A}_{22}$ and $\check{A}_{22}$ are nonsingular. In order to compute the state response, we rewrite (25) as

$$
\acute{E} \begin{bmatrix} \dot{\tilde{Z}}(t) \\ \dot{\hat{Z}}(t) \end{bmatrix} = \begin{bmatrix} \acute{A}_{11} & \acute{A}_{12} \\ \acute{A}_{21} & \acute{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{Z}(t) \\ \hat{Z}(t) \end{bmatrix},
$$

where $\acute{E} \stackrel{\text{def}}{=} \begin{bmatrix} I_{2\ell} & 0 \\ 0 & 0 \end{bmatrix}$, $\tilde{Z}(t) \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{\xi}(t) \\ \tilde{e}(t) \end{bmatrix}$, $\hat{Z}(t) \stackrel{\text{def}}{=} \begin{bmatrix} \hat{\xi}(t) \\ \hat{e}(t) \end{bmatrix}$, $\acute{A}_{11} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{A}_{11} & B_{11} \\ 0 & \check{A}_{11} \end{bmatrix}$, $\acute{A}_{12} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{A}_{12} & B_{12} \\ 0 & \check{A}_{12} \end{bmatrix}$, $\acute{A}_{21} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{A}_{21} & B_{21} \\ 0 & \check{A}_{21} \end{bmatrix}$, $\acute{A}_{22} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{A}_{22} & B_{22} \\ 0 & \check{A}_{22} \end{bmatrix}$, and we compute

$$
\acute{Q} \stackrel{\text{def}}{=} \begin{bmatrix} I_{2\ell} & -\acute{A}_{12} \acute{A}_{22}^{-1} \\ 0 & I_{2(n-\ell)} \end{bmatrix} = \left[\begin{array}{cc|cc} I_\ell & 0 & -\hat{A}_{12} \hat{A}_{22}^{-1} & -\big(B_{12} - \hat{A}_{12} \hat{A}_{22}^{-1} B_{22}\big) \check{A}_{22}^{-1} \\ 0 & I_\ell & 0 & -\check{A}_{12} \check{A}_{22}^{-1} \\ \hline 0 & 0 & I_{n-\ell} & 0 \\ 0 & 0 & 0 & I_{n-\ell} \end{array}\right],
$$

$$\acute{P} \stackrel{\text{def}}{=} \begin{bmatrix} I_{2\ell} & 0 \\ -\acute{A}_{22}^{-1}\acute{A}_{21} & \acute{A}_{22}^{-1} \end{bmatrix} = \left[\begin{array}{cc|cc} I_\ell & 0 & 0 & 0 \\ 0 & I_\ell & 0 & 0 \\ \hline -\hat{A}_{22}^{-1}\hat{A}_{21} & -\hat{A}_{22}^{-1}\left(B_{21} - B_{22}\check{A}_{22}^{-1}\check{A}_{21}\right) & \hat{A}_{22}^{-1} & -\hat{A}_{22}^{-1}B_{22}\check{A}_{22}^{-1} \\ 0 & -\check{A}_{22}^{-1}\check{A}_{21} & 0 & \check{A}_{22}^{-1} \end{array}\right],$$

so that $\acute{Q}\begin{bmatrix} \acute{A}_{11} & \acute{A}_{12} \\ \acute{A}_{21} & \acute{A}_{22} \end{bmatrix}\acute{P} = \begin{bmatrix} \Lambda & 0 \\ 0 & I_{2(n-\ell)} \end{bmatrix}$, where

$$\begin{aligned} \Lambda &\stackrel{\text{def}}{=} \begin{bmatrix} \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} & B_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}B_{21} - \left(B_{12} - \hat{A}_{12}\hat{A}_{22}^{-1}B_{22}\right)\check{A}_{22}^{-1}\check{A}_{21} \\ 0 & \check{A}_{11} - \check{A}_{12}\check{A}_{22}^{-1}\check{A}_{21} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{F} & -\tilde{B}\left(\tilde{F} - H_2\check{A}_{22}^{-1}\check{A}_{21}\right) \\ 0 & \tilde{A}_G + \tilde{G}\tilde{C}_G \end{bmatrix}, \end{aligned}$$

and $\tilde{A}_G \stackrel{\text{def}}{=} A_{11} - A_{12}\check{A}_{22}^{-1}\check{A}_{21},\ \tilde{C}_G \stackrel{\text{def}}{=} C_1 - C_2\check{A}_{22}^{-1}\check{A}_{21}$. The state response of (25) is

$$\begin{bmatrix} \tilde{Z}(t) \\ \hat{Z}(t) \end{bmatrix} = \acute{P}\begin{bmatrix} e^{\Lambda t}\tilde{Z}_0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_\ell & 0 \\ 0 & I_\ell \\ -\hat{A}_{22}^{-1}\hat{A}_{21} & -\hat{A}_{22}^{-1}\left(B_{21} - B_{22}\check{A}_{22}^{-1}\check{A}_{21}\right) \\ 0 & -\check{A}_{22}^{-1}\check{A}_{21} \end{bmatrix} e^{\Lambda t}\tilde{Z}_0,$$

where $\tilde{Z}_0 \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{\xi}_0 \\ \tilde{e}_0 \end{bmatrix}$, so that the tracking error for any initial condition $\begin{bmatrix} \tilde{Z}(0) \\ 0 \end{bmatrix} \in \mathbb{R}^{2n}$ is

$$\begin{aligned} \epsilon(t) &= -\begin{bmatrix} C_1 & 0 & C_2 & 0 \end{bmatrix}\begin{bmatrix} I_\ell & 0 \\ 0 & I_\ell \\ -\hat{A}_{22}^{-1}\hat{A}_{21} & -\hat{A}_{22}^{-1}\left(B_{21} - B_{22}\check{A}_{22}^{-1}\check{A}_{21}\right) \\ 0 & -\check{A}_{22}^{-1}\check{A}_{21} \end{bmatrix} e^{\Lambda t}\tilde{Z}_0 \\ &= -\begin{bmatrix} C_1 - C_2\hat{A}_{22}^{-1}\hat{A}_{21} & -C_2\hat{A}_{22}^{-1}\left(B_{21} - B_{22}\check{A}_{22}^{-1}\check{A}_{21}\right) \end{bmatrix} e^{\Lambda t}\tilde{Z}_0 \\ &= -\begin{bmatrix} \tilde{C} + \tilde{D}\tilde{F} & -\tilde{D}\left(\tilde{F} - H_2\check{A}_{22}^{-1}\check{A}_{21}\right) \end{bmatrix} e^{\Lambda t}\tilde{Z}_0, \end{aligned}$$

or $\epsilon(t) = \Gamma e^{\Lambda t}\tilde{Z}_0$, where $\Gamma \stackrel{\text{def}}{=} -\begin{bmatrix} \tilde{C} + \tilde{D}\tilde{F} & -\tilde{D}\left(\tilde{F} - H_2\check{A}_{22}^{-1}\check{A}_{21}\right) \end{bmatrix}$.[5] Then Λ has eigenvalues $\{\lambda_1, \ldots, \lambda_{2\ell}\}$, which are the finite generalized eigenvalues of $(E, A + BF)$ and $(E, A + GC)$, and corresponding finite eigenvectors $\check{\mathcal{V}} = \{\check{v}_1, \ldots, \check{v}_{2\ell}\} = \left\{\begin{bmatrix} \tilde{v}_1 \\ 0_\ell \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{v}_\ell \\ 0_\ell \end{bmatrix}, \check{v}_{\ell+1}, \ldots, \check{v}_{2\ell}\right\} \subset \mathbb{C}^{2\ell}$. The set $\check{\mathcal{V}}$ is linearly independent, so that the matrix $\check{V} \stackrel{\text{def}}{=} \begin{bmatrix} \check{v}_1 & \ldots & \check{v}_{2\ell} \end{bmatrix}$ is invertible. We introduce $\check{\alpha} \stackrel{\text{def}}{=} \begin{bmatrix} \check{\alpha}_1 & \ldots & \check{\alpha}_{2\ell} \end{bmatrix}^\top = \check{V}^{-1}\tilde{Z}_0$. It follows that the tracking error can be expressed by

$$\epsilon(t) = -\sum_{i=\ell-p+1}^{\ell} \mathbf{e}_{i-(\ell-p)}\check{\alpha}_i e^{\lambda_i t} + \sum_{i=\ell+1}^{2\ell} \Gamma\check{v}_i\check{\alpha}_i e^{\lambda_i t},$$

because $\Gamma\check{v}_i = -\begin{bmatrix} \tilde{C} + \tilde{D}\tilde{F} & -\tilde{D}\left(\tilde{F} - H_2\check{A}_{22}^{-1}\check{A}_{21}\right) \end{bmatrix}\begin{bmatrix} \tilde{v}_i \\ 0 \end{bmatrix} = -(\tilde{C} + \tilde{D}\tilde{F})\tilde{v}_i = -s_i$ for $i \in \{1, \ldots, \ell\}$. If we denote the j-th row of Γ by $\gamma_j^\top$, for $j \in \{1, \ldots, p\}$, then the j-th component of $\epsilon(t)$ is given by

$$\epsilon_j(t) = -\check{\alpha}_{\ell-p+j}e^{\lambda_{\ell-p+j}t} + \sum_{i=\ell+1}^{2\ell} \gamma_j^\top\check{v}_i\check{\alpha}_i e^{\lambda_i t}.$$

[5] If the descriptor system is impulse-free, then A_{22} is nonsingular and $H_2 = 0,\ L_2 = 0$. In such a case, we have $\hat{A}_{12} = A_{12},\ \hat{A}_{22} = \check{A}_{22} = A_{22},\ \check{A}_{21} = A_{21}$, so that $\Lambda = \begin{bmatrix} \tilde{A}+\tilde{B}\tilde{F} & -\tilde{B}\tilde{F} \\ 0 & \tilde{A}+\tilde{G}\tilde{C} \end{bmatrix}$, $\Gamma = -\begin{bmatrix} \tilde{C} + \tilde{D}\tilde{F} & -\tilde{D}\tilde{F} \end{bmatrix}$.

Since the observer poles satisfy (24), $\epsilon_j(t)$ will not change sign if

$$|\check{\alpha}_{\ell-p+j}| > \left|\sum_{i=\ell+1}^{2\ell} \gamma_j^\top \check{v}_i \check{\alpha}_i\right| e^{\mu_0-\lambda_j}. \tag{26}$$

It is clear that for any given initial state $E\,x_0$, (26) will be satisfied for all $j \in \{1,\dots,p\}$ if either the magnitude of the initial estimation error $|e(0)|$ is sufficiently small, or equivalently, $|\mu_0|$ is sufficiently large. We summarize the above as follows.

Theorem 4.1 *Assume that the descriptor system Σ satisfies Assumptions 4.1 and has at least $\ell - p$ stable invariant zeros. Let F and G be defined as above, let $r \in \mathbb{R}^p$ be any step reference and let $E\,x_0 \in \mathbb{R}^n$ be any initial condition. Then, applying the output feedback control law $u(t)$ in (21) to Σ, we obtain an output $y(t)$ asymptotically tracking r without overshoot, if the initial error $e(0)$ satisfies (26) for $j \in \{1,\dots,p\}$.*

Remark 4.1 Since the λ_i for $i \in \{\ell-p+1,\dots,\ell\}$ can be freely chosen to be any distinct real stable modes, provided they are distinct from the stable invariant zeros of Σ and that the resulting $E\,\mathcal{V}$ is linearly independent, the rate of convergence of the output to the target reference can be chosen to be arbitrarily fast. Note that F is independent of r and $E\,x_0$, so that the same F can be used to achieve nonovershooting convergence for any r and any $E\,x_0$. Moreover, the observer modes $i \in \{\ell+1,\dots,2\ell\}$ can also be freely chosen to be any distinct real stable modes that satisfy (24). Hence we can ensure that $e^{\mu_0-\lambda_j}$ is sufficiently small, and that (26) holds for any Ex_0.

Remark 4.2 The bound in (26) is rather conservative. A less conservative bound may be obtained as follows. For each $j \in \{1,\dots,p\}$, define k_j to be the largest integer such that $\gamma_j^\top \check{v}_i \check{\alpha}_i$, $i \in \{\ell+1,\dots,\ell+k_j\}$ have the same sign as $\check{\alpha}_{\ell-p+j}$. A sufficient condition to ensure that $\epsilon_j(t)$ does not change sign is

$$\left|-\check{\alpha}_{\ell-p+j} + \sum_{i=\ell+1}^{\ell+k_j} \gamma_j^\top \check{v}_i \check{\alpha}_i\right| > \left|\sum_{i=\ell+k_j+1}^{2\ell} \gamma_j^\top \check{v}_i \check{\alpha}_i\right| e^{\mu_0-\lambda_j}. \tag{27}$$

Observe that k_j may be equal to zero and then (26) and (27) coincide.

5 Numerical example

We consider the continuous-time descriptor system that appeared in [15]:

$$E=\left[\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline 0&0&0&0\\0&0&0&0\end{array}\right]=\begin{bmatrix}I_2&0\\0&0\end{bmatrix},\quad A=\left[\begin{array}{cc|cc}2&1&0&0\\0&1&1&0\\\hline 1&0&0&0\\0&0&1&1\end{array}\right]=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix},$$

$$B=\left[\begin{array}{c}0\\0\\\hline 1\\0\end{array}\right]=\begin{bmatrix}B_1\\B_2\end{bmatrix},\quad C=[\,1\;\;0\,|\,0\;\;0\,]=[\,C_1\;|\;C_2\,],\quad \ell=2,\;m=p=1.$$

In [15], a composite nonlinear feedback (CNF) controller for (E,A,B,C) was designed, given a tracking reference $r=1$ and an initial condition $E\,x_0=[\,0\;\;0\;\;0\;\;0\,]^\top$. Considering the same r and $E\,x_0$ as in [15], we design nonovershooting linear (NOSL) controllers and compare their performance with the performance of the CNF controller designed in [15].

The descriptor system is square, invertible and stabilizable with no invariant zeros ($\ell-2\,p=0$). The matrix pencil $\lambda\,E-A$ is regular, because $\det(\lambda\,E-A)=1\neq 0$, therefore we may design a state feedback matrix F, which would yield a nonovershooting response, in one step. Let us choose $\lambda_1=-6$, $\lambda_2=-3$ and solve (10) for $s_1=s_2=\mathbf{e}_1$. We obtain

$$V=[\,v_1\;\;v_2\,]=\begin{bmatrix}1&1\\-8&-5\\56&20\\-56&-20\end{bmatrix},\quad \hat{W}=[\,\hat{w}_1\;\;\hat{w}_2\,]=[\,-1\;\;-1\,]$$

and hence

$$F=\hat{W}\,V^\dagger=[\,-0.0828\;\;0.2752\;\;0.0115\;\;-0.0115\,].$$

To illustrate how we construct an F in two steps, we choose a state feedback matrix $H=[\,0\;\;H_2\,]$ such that $\det(A_{22}+B_2\,H_2)\neq 0$. Let us choose for example $H=[\,0\;\;0\;\;1\;\;-1\,]$ and compute:

$$\hat{A}=A+B\,H=\left[\begin{array}{cc|cc}2&1&0&0\\0&1&1&0\\\hline 1&0&1&-1\\0&0&1&1\end{array}\right]=\begin{bmatrix}A_{11}&\hat{A}_{12}\\A_{21}&\hat{A}_{22}\end{bmatrix}.$$

Since $\det\hat{A}_{22}\neq 0$, the system $\big(E,\hat{A}\big)$ is impulse-free. We solve (10) for $\lambda_1=-6$, $\lambda_2=-3$ and $s_1=s_2=\mathbf{e}_1$, and obtain

$$E\,V=E\,[\,v_1\;\;v_2\,]=E\begin{bmatrix}1&1\\-8&-5\\56&20\\-56&-20\end{bmatrix}=\left[\begin{array}{cc}1&1\\-8&-5\\\hline 0&0\\0&0\end{array}\right]=\left[\begin{array}{cc}\tilde{v}_1&\tilde{v}_2\\\hline 0&0\end{array}\right],\;\hat{W}=[\,\hat{w}_1\;\;\hat{w}_2\,]=[\,-113\;\;-41\,].$$

Thus,

$$\hat{F} = \hat{W}\,(EV)^{\dagger} = [\,79 \;\; 24 \;\; 0 \;\; 0\,], \quad F = H + \hat{F} = [\,79 \;\; 24 \;\; 1 \;\; -1\,].$$

Let us now choose another state feedback matrix $H = [\; H_1 \;\; H_2 \;]$ such that $(E, A + B\,H)$ is impulse-free, for example $H = [\,12 \;\; 5 \;\; 1 \;\; 0\,]$, which is the same as the linear state feedback matrix F designed in the example of [15]. Solving (10) for $\lambda_1 = -6$, $\lambda_2 = -3$ and $s_1 = s_2 = \mathbf{e}_1$, we obtain

$$E\,V = E\,[\,v_1 \;\; v_2\,] = E \left[\begin{array}{cc} 1 & 1 \\ -8 & -5 \\ 56 & 20 \\ -56 & -20 \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ -8 & -5 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] = \left[\begin{array}{cc} \tilde{v}_1 & \tilde{v}_2 \\ \hline 0 & 0 \end{array}\right], \; \hat{W} = [\,\hat{w}_1 \;\; \hat{w}_2\,] = [\,-29 \;\; -8\,],$$

and hence

$$\hat{F} = \hat{W}\,(EV)^{\dagger} = [\,27 \;\; 7 \;\; 0 \;\; 0\,], \quad F = H + \hat{F} = [\,39 \;\; 12 \;\; 1 \;\; 0\,].$$

Notice that v_1 and v_2 are the same in all the cases. To see if the initial condition $E\,x_0$ yields nonovershooting response for the particular choice of λ_1, λ_2, we compute

$$\gamma = \left[\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right] = [\; \tilde{v}_1 \;\; \tilde{v}_1 - \tilde{v}_2 \;]^{-1}\,(-\tilde{x}_{\rm ss}) = \left[\begin{array}{cc} 1 & 0 \\ -8 & -3 \end{array}\right]^{-1} \left[\begin{array}{c} -1 \\ 2 \end{array}\right] = \left[\begin{array}{c} -1 \\ 2 \end{array}\right].$$

Since $\gamma_1\,\gamma_2 = -2 < 0$, the output obtained under the control law in (8) using the state feedback controllers F designed above is nonovershooting.

The output response for the descriptor system under the control law in (8) using the three different F, and the amplitude of the control input are shown in Figure 1 and they coincide as expected.

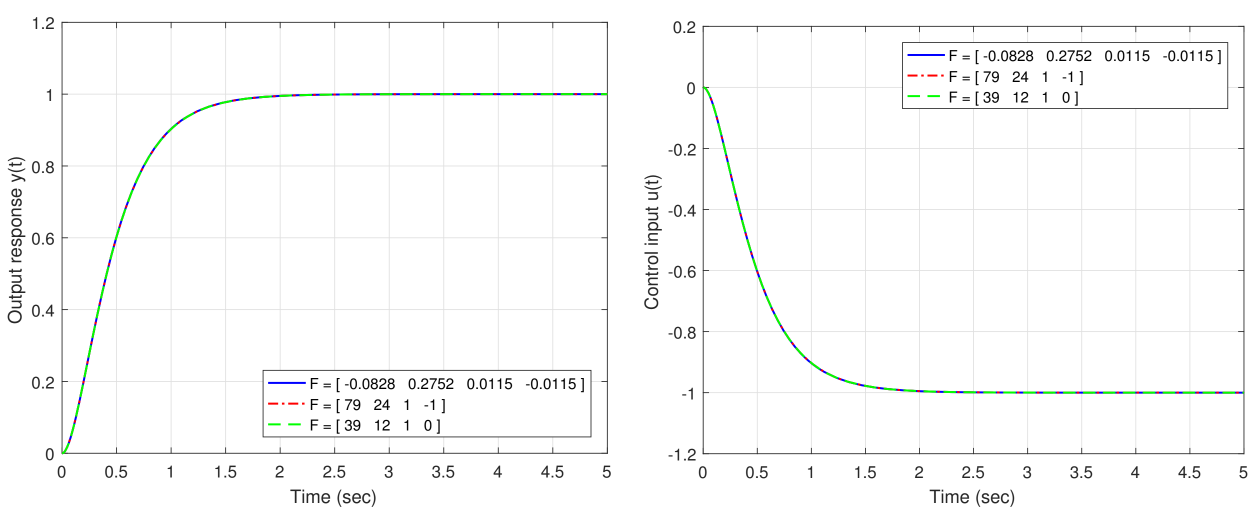

Figure 1: Output response and control amplitude using state feedback F.

The response of the NOSL controllers F is very similar to the CNF controller designed in [15], as they both avoid overshoot and have a 2% settling time of 1.5 seconds. The control amplitude curves are also very similar, with the control amplitude smoothly increasing from 0 to 1 over 1.5 seconds.

However, by contrast, the NOSL design method is considerably simpler as it only involves the design of up to two state feedback matrices.

To further compare the NOSL and CNF design methods, we follow the same procedure to obtain three feedback matrices F' by choosing $\lambda_1 = -9$, $\lambda_2 = -8$. The state feedback matrix F' designed in one step is

$$F' = [\ -0.032 \quad 0.176 \quad 0.0044 \quad -0.0044\ ].$$

Choosing $H = [\ 0 \quad 0 \quad 1 \quad -1\ ]$, we obtain

$$\hat{F}' = [\ 219 \quad 40 \quad 0 \quad 0\ ], \quad F' = H + \hat{F}' = [\ 219 \quad 40 \quad 1 \quad -1\ ].$$

Finally, choosing $H = [\ 12 \quad 5 \quad 1 \quad 0\ ]$, we obtain

$$\hat{F}' = [\ 97 \quad 15 \quad 0 \quad 0\ ], \quad F' = H + \hat{F}' = [\ 109 \quad 20 \quad 1 \quad 0\ ].$$

The initial condition $E\,x_0$ yields nonovershooting response for the particular choice of λ_1, λ_2, because

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = [\ \tilde{v}_1 \quad \tilde{v}_1 - \tilde{v}_2\ ]^{-1}(-\tilde{x}_{\rm ss}) = \begin{bmatrix} 1 & 0 \\ -11 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}, \quad \gamma_1\,\gamma_2 = -9 < 0.$$

The output response for the descriptor system under the control law in (8) using the three different F', and the amplitude of the control input are shown in Figure 2.

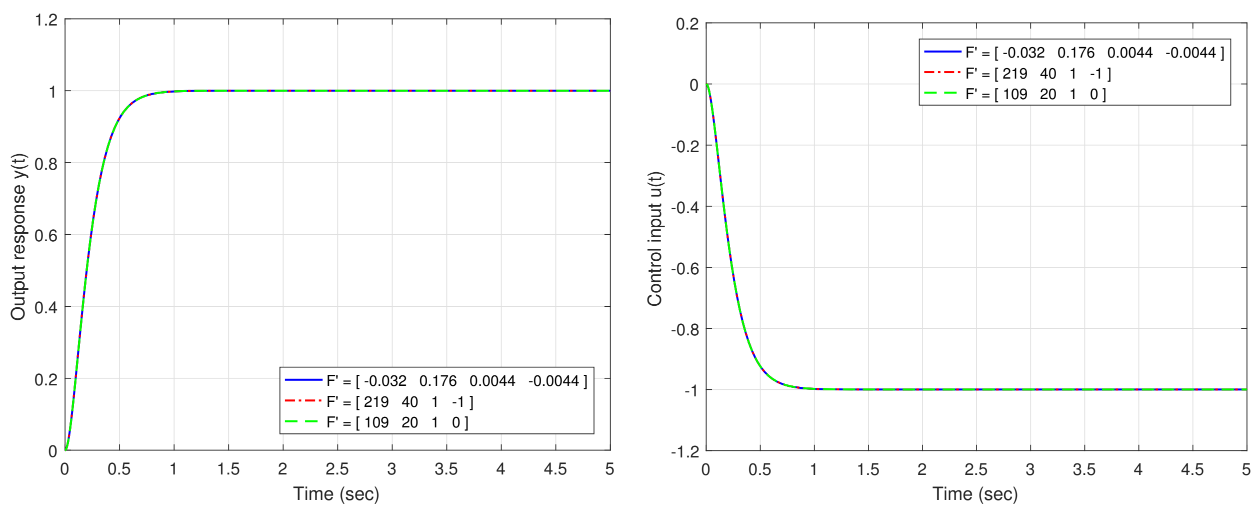

Figure 2: Output response and control amplitude using state feedback F'.

We observe that the NOSL controllers F' yield the same nonovershooting response and achieve a 2% settling time of 0.7 seconds. This faster response required the control amplitude to increase from 0 to 1 over approximately one second.

6 Concluding remarks and future work

In this paper, we investigated the problem of designing nonovershooting controllers for impulse controllable MIMO descriptor systems via state feedback and dynamic output feedback based on a Luenberger observer scheme. A unified method to improve transient response was given for a linear state feedback tracking controller and linear observer of full order, which may be applied to continuous-time and discrete-time descriptor systems.

The design of nonovershooting and nonundershooting multivariable tracking controllers and the extension of the NOUS toolbox of [21] to descriptor systems are natural topics for future work. We envisage that with the current work, we will be able to conduct further research directly on descriptor systems, in such a way that results will hold for explicit systems in a straightforward manner by simply setting $E = I_n$.

Appendix A

The following lemma shows the relation between the kernel of $\begin{bmatrix} \hat{A}-\lambda E & B \end{bmatrix}$ for the descriptor system as in (4a) with the kernel of $\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \end{bmatrix}$ for the associated explicit system $\tilde{\Sigma}$ as in (6a). The proof of the lemma, which we include for completeness, can be carried out along the same lines of the proofs of Lemmas 5.1 and 5.2 in [12].

Lemma 6.1 *Let* $(E, \hat{A}, B)$ *be as in* (4a)*, and let* $(I_\ell, \tilde{A}, \tilde{B})$ *be the associated explicit system as in* (6a)*. Then* $\begin{bmatrix} \hat{A}-\lambda E & B \end{bmatrix} = \tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Pi$*, where* $\tilde{Q}^{-1} = \begin{bmatrix} I_\ell & \hat{A}_{12}\hat{A}_{22}^{-1} \\ 0 & I_{n-\ell} \end{bmatrix}$, $\Pi = \begin{bmatrix} I_\ell & 0 & 0 \\ 0 & 0 & I_m \\ A_{21} & \hat{A}_{22} & B_2 \end{bmatrix}$ *and* $\ker\begin{bmatrix} \hat{A}-\lambda E & B \end{bmatrix} = \Pi^{-1}\big(\ker\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \end{bmatrix} \oplus \{0\}\big)$.

Proof: By direct computation, we have

$$\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Pi = \tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & 0 & \tilde{B} \\ A_{21} & \hat{A}_{22} & B_2 \end{bmatrix} = \begin{bmatrix} A_{11}-\lambda I_\ell & \hat{A}_{12} & B_1 \\ A_{21} & \hat{A}_{22} & B_2 \end{bmatrix}.$$

Notice that $\ker\Big(\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Big) = \ker\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \end{bmatrix} \oplus \{0\}$. Let $\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} \in \ker\Big(\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Big)$, then

$$\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} = \begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & \hat{A}_{12}\hat{A}_{22}^{-1} \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} = 0,$$

from which it follows $z' = 0$. Therefore $\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \end{bmatrix}\begin{bmatrix} v' \\ w' \end{bmatrix} = 0$ and $\ker\Big(\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Big) \subseteq \ker\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \end{bmatrix} \oplus \{0\}$. Now let $\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} \in \ker\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \end{bmatrix} \oplus \{0\}$. Then $\begin{bmatrix} v' \\ w' \end{bmatrix} \in \ker\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \end{bmatrix}$ and $z' = 0$, so that $\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} = 0$. Also $\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} = 0$ and thus $\begin{bmatrix} v' \\ w' \\ z' \end{bmatrix} \in \ker\Big(\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Big)$.

We show that $\ker\begin{bmatrix} \hat{A}-\lambda E & B \end{bmatrix} \subseteq \Pi^{-1}\big(\ker\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} \end{bmatrix} \oplus \{0\}\big)$. Let $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix} \in \ker\begin{bmatrix} \hat{A}-\lambda E & B \end{bmatrix}$, then we have $\begin{bmatrix} \hat{A}-\lambda E & B \end{bmatrix}\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix} = 0$ or, equivalently, $\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Pi\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix} = 0$, which is satisfied for $\Pi\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix} \in \ker\Big(\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Big)$ and implies that $\begin{bmatrix} \hat{v} \\ z \\ w \end{bmatrix} \in \Pi^{-1}\ker\Big(\tilde{Q}^{-1}\begin{bmatrix} \tilde{A}-\lambda I_\ell & \tilde{B} & 0 \\ 0 & 0 & I_{n-\ell} \end{bmatrix}\Big) =$

$\Pi^{-1}\big(\ker\left[\begin{smallmatrix}\tilde{A}-\lambda I_\ell & \tilde{B}\end{smallmatrix}\right]\oplus\{0\}\big)$. We show the opposite inclusion. Let $\left[\begin{smallmatrix}\hat{v}\\ z\\ w\end{smallmatrix}\right] \in \Pi^{-1}\big(\ker\left[\begin{smallmatrix}\tilde{A}-\lambda I_\ell & \tilde{B}\end{smallmatrix}\right]\oplus\{0\}\big)$. Then $\left[\begin{smallmatrix}\hat{v}\\ z\\ w\end{smallmatrix}\right] \in \Pi^{-1}\ker\left(\tilde{Q}^{-1}\left[\begin{smallmatrix}\tilde{A}-\lambda I_\ell & \tilde{B} & 0\\ 0 & 0 & I_{n-\ell}\end{smallmatrix}\right]\right)$ or, equivalently, $\Pi\left[\begin{smallmatrix}\hat{v}\\ z\\ w\end{smallmatrix}\right] \in \ker\left(\tilde{Q}^{-1}\left[\begin{smallmatrix}\tilde{A}-\lambda I_\ell & \tilde{B} & 0\\ 0 & 0 & I_{n-\ell}\end{smallmatrix}\right]\right)$. Thus, $\tilde{Q}^{-1}\left[\begin{smallmatrix}\tilde{A}-\lambda I_\ell & \tilde{B} & 0\\ 0 & 0 & I_{n-\ell}\end{smallmatrix}\right]\Pi\left[\begin{smallmatrix}\hat{v}\\ z\\ w\end{smallmatrix}\right] = 0$, or, equivalently, $\left[\begin{smallmatrix}\hat{A}-\lambda E & B\end{smallmatrix}\right]\left[\begin{smallmatrix}\hat{v}\\ z\\ w\end{smallmatrix}\right] = 0$, so that $\left[\begin{smallmatrix}\hat{v}\\ z\\ w\end{smallmatrix}\right] \in \ker\left[\begin{smallmatrix}\hat{A}-\lambda E & B\end{smallmatrix}\right]$. ■

Remark 6.1 Let $\left[\begin{smallmatrix}\hat{v}\\ z\\ w\end{smallmatrix}\right] \in \ker\left[\begin{smallmatrix}\hat{A}-\lambda E & B\end{smallmatrix}\right]$. In view of Lemma 6.1, we have $\Pi\left[\begin{smallmatrix}\hat{v}\\ z\\ w\end{smallmatrix}\right] = \left[\begin{smallmatrix}\hat{v}\\ w\\ A_{21}\hat{v}+\hat{A}_{22}z+B_2 w\end{smallmatrix}\right] \in \ker\left[\begin{smallmatrix}\tilde{A}-\lambda I_\ell & \tilde{B}\end{smallmatrix}\right]\oplus\{0\}$. Let now $\left[\begin{smallmatrix}\tilde{v}\\ \tilde{w}\end{smallmatrix}\right] \in \ker\left[\begin{smallmatrix}\tilde{A}-\lambda I_\ell & \tilde{B}\end{smallmatrix}\right]$. Comparing the above, it follows that $\left[\begin{smallmatrix}\hat{v}\\ w\end{smallmatrix}\right] = \left[\begin{smallmatrix}\tilde{v}\\ \tilde{w}\end{smallmatrix}\right]$ and $A_{21}\,\hat{v} + \hat{A}_{22}\,z + B_2\,w = 0$. Thus,

$$\ker\left[\begin{array}{cc}\hat{A}-\lambda E & B\end{array}\right] = \left\{\left[\begin{array}{c}\tilde{v}\\ -\hat{A}_{22}^{-1}\,(A_{21}\,\tilde{v}+B_2\,\tilde{w})\\ \tilde{w}\end{array}\right] : \left[\begin{array}{c}\tilde{v}\\ \tilde{w}\end{array}\right] \in \ker\left[\begin{array}{cc}\tilde{A}-\lambda I_\ell & \tilde{B}\end{array}\right]\right\}.$$

***Proof of Proposition* 3.1:**

(Sufficiency) The proof will be given for two cases. First, we consider regular descriptor systems, because in this case F can be computed in one step. Indeed, by impulse controllability, we can apply a state feedback, so that the closed-loop system is impulse-free with finite generalized eigenvalues $\lambda_1,\ldots,\lambda_\ell$. If the descriptor system is not regular, to ensure closed-loop system regularity, the state feedback F is computed in two steps: a preliminary state feedback is applied, so that the closed-loop system is impulse-free and thus regular (regardless of its finite generalized eigenvalues) and then another state feedback is applied to assign $\lambda_1,\ldots,\lambda_\ell$. Without loss of generality, assume that $\mathcal{L}=\{\lambda_1,\ldots,\lambda_{2\sigma},\lambda_{2\sigma+1},\ldots,\lambda_\ell\}$ are ordered in such a way that $\lambda_2=\overline{\lambda}_1,\ldots,\lambda_{2\sigma}=\overline{\lambda}_{2\sigma-1}$ and $\lambda_i\in\mathbb{R}$ for all $i\in\{2\sigma+1,\ldots,\ell\}$.

(I) Regular matrix pencil $\lambda\,E-A$

Suppose that $v_i \stackrel{\text{def}}{=} \left[\begin{smallmatrix}\acute{v}_i\\ \grave{v}_i\end{smallmatrix}\right]$, $i\in\{1,\ldots,\ell\}$ are chosen to satisfy conditions *(i)*-*(iii)*. By condition *(iii)*, we can write $v_i = S_{\lambda_i}\,k_i$ for some complex-valued vector k_i of suitable size with $k_{i+1}=\overline{k}_i$ for all odd $i<2\sigma$ and a real-valued parameter k_i, $i\in\{2\sigma+1,\ldots,\ell\}$ of suitable size. It follows that $(A-\lambda_i\,E)\,v_i + B\,T_{\lambda_i}\,k_i = 0$. Let $w_i \stackrel{\text{def}}{=} T_{\lambda_i}\,k_i$ and define $V \stackrel{\text{def}}{=} \left[\begin{array}{ccc}v_1 & \ldots & v_\ell\end{array}\right]$ and $W \stackrel{\text{def}}{=} \left[\begin{array}{ccc}w_1 & \ldots & w_\ell\end{array}\right]$. Since $E\,v_i = \left[\begin{smallmatrix}\acute{v}_i\\ 0\end{smallmatrix}\right]$ are linearly independent, the vectors v_i are also linearly independent. Thus, V is full column-rank and there exists a matrix F satisfying $F\,V=W$. In order to show that this matrix is real, define vectors

$$\breve{v}_i = \begin{cases}\frac{1}{2}\,(v_i+v_{i+1}), & \text{if } i<2\sigma \text{ is odd,}\\ \frac{1}{2}\,(v_i-v_{i-1})\,\mathrm{i}, & \text{if } i\le 2\sigma \text{ is even,}\\ v_i, & \text{if } i>2\sigma\end{cases}$$

and define $\breve{w}_i$ similarly. By condition *(ii)*, we conclude that $\breve{v}_i$ and $\breve{w}_i$ are real vectors. For all odd $i<2\sigma$, we have $\left[\begin{array}{cc}v_i & v_{i+1}\end{array}\right]R = \left[\begin{array}{cc}\breve{v}_i & \breve{v}_{i+1}\end{array}\right]$ and $\left[\begin{array}{cc}w_i & w_{i+1}\end{array}\right]R = \left[\begin{array}{cc}\breve{w}_i & \breve{w}_{i+1}\end{array}\right]$, where $R=\frac{1}{2}\left[\begin{smallmatrix}1 & -\mathrm{i}\\ 1 & \mathrm{i}\end{smallmatrix}\right]$.

Define the real matrices $\check{V} \stackrel{\text{def}}{=} \begin{bmatrix} \check{v}_1 & \dots & \check{v}_\ell \end{bmatrix}$ and $\check{W} \stackrel{\text{def}}{=} \begin{bmatrix} \check{w}_1 & \dots & \check{w}_\ell \end{bmatrix}$. There holds $\check{V} = V\,U$ and $\check{W} = W\,U$, where $U = \text{diag}\{R, \dots, R, I_{\ell-2\sigma}\}$. The vectors $\check{v}_i$ are linearly independent because the vectors v_i are linearly independent; thus $\check{V}$ is full column-rank and $F = W\,V^\dagger = W\,U\,U^{-1}\,V^\dagger = \check{W}\,\check{V}^\dagger$. Consequently, F is a real matrix and there holds $(A + B\,F - \lambda_i\,E)\,v_i = 0$. Since we assigned ℓ finite generalized eigenvalues to the regular open-loop system, so that the closed-loop system is impulse-free, the matrix pencil $\lambda\,E - (A + B\,F)$ is regular.

(II) Singular matrix pencil $\lambda\,E - A$

Since the descriptor system is impulse controllable, there exists a state feedback $u(t) = H\,x(t) + v(t)$, such that the closed-loop system $E\,\dot{x}(t) = \hat{A}\,x(t) + B\,v(t)$, is impulse-free and thus regular. Then we can construct a real $\hat{F}$, such that $\big(\hat{A} + B\,\hat{F}\big)\,v_i = \lambda_i\,E\,v_i$ and $\lambda\,E - \big(\hat{A} + B\,\hat{F}\big)$ is regular, in a similar way as in *(I)*. Indeed let the columns of $\begin{bmatrix} \hat{S}_\lambda \\ \hat{T}_\lambda \end{bmatrix}$ span $\ker \begin{bmatrix} \hat{A} - \lambda\,E & B \end{bmatrix}$, so that

$$0 = \begin{bmatrix} \hat{A} - \lambda\,E & B \end{bmatrix} \begin{bmatrix} \hat{S}_\lambda \\ \hat{T}_\lambda \end{bmatrix} = \begin{bmatrix} A - \lambda\,E & B \end{bmatrix} \begin{bmatrix} I_n & 0 \\ H & I_m \end{bmatrix} \begin{bmatrix} \hat{S}_\lambda \\ \hat{T}_\lambda \end{bmatrix} = \begin{bmatrix} A - \lambda\,E & B \end{bmatrix} \begin{bmatrix} \hat{S}_\lambda \\ H\,\hat{S}_\lambda + \hat{T}_\lambda \end{bmatrix}$$

and thus $\text{im}\,\hat{S}_\lambda = \text{im}\,S_\lambda$. It follows that $\big(\hat{A} - \lambda_i\,E\big)\,v_i + B\,\hat{T}_{\lambda_i}\,k_i = 0$. Define $\hat{w}_i \stackrel{\text{def}}{=} \hat{T}_{\lambda_i}\,k_i$ and $\hat{W} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{w}_1 & \dots & \hat{w}_\ell \end{bmatrix}$ and compute $\hat{F} = \hat{W}\,V^\dagger$, which satisfies $\hat{F}\,V = \hat{W}$. Moreover, since the descriptor system $\big(E, \hat{A}, B\big)$ is impulse-free, there is an alternative way to compute $\hat{F}$. Indeed, consider the associated explicit system described by $\big(I_\ell, \tilde{A}, \tilde{B}\big)$ as in (6a). Let the columns $\begin{bmatrix} \tilde{S}_\lambda \\ \tilde{T}_\lambda \end{bmatrix}$ span $\ker \begin{bmatrix} \tilde{A} - \lambda\,I_\ell & \tilde{B} \end{bmatrix}$ and define $\tilde{v}_i \stackrel{\text{def}}{=} \tilde{S}_{\lambda_i}\,k_i$, $\tilde{w}_i \stackrel{\text{def}}{=} \tilde{T}_{\lambda_i}\,k_i$, $i \in \{1, \dots, \ell\}$. In view of Remark 6.1, we have that $\tilde{v}_i = \acute{v}_i$, $\tilde{w}_i = \hat{w}_i$. Define $\tilde{V} \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{v}_1 & \dots & \tilde{v}_\ell \end{bmatrix}$. Since $E\,v_i = \begin{bmatrix} \tilde{v}_i \\ 0 \end{bmatrix}$ are linearly independent, the vectors $\tilde{v}_i$ are also linearly independent and $\tilde{V}$ is nonsingular. Applying Moore's algorithm for explicit systems, there exists a real $\tilde{F}$ such that $\tilde{F}\,\tilde{V} = \hat{W}$, which can be rewritten as $\begin{bmatrix} \tilde{F} & 0 \end{bmatrix} \begin{bmatrix} \tilde{V} \\ 0 \end{bmatrix} = \hat{W}$ or, equivalently, $\begin{bmatrix} \tilde{F} & 0 \end{bmatrix} E\,V = \hat{W}$. Thus, $\tilde{F} = \hat{W}\,\tilde{V}^{-1}$ and $\hat{F} = \hat{W}\,(E\,V)^\dagger = \hat{W} \begin{bmatrix} \tilde{V}^{-1} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{F} & 0 \end{bmatrix}$, which also satisfies $\hat{F}\,V = \hat{W}$. It follows that $\big(\hat{A} + B\,\hat{F} - \lambda_i\,E\big)\,v_i = 0$ or, equivalently, $\big(A + B\,\big(H + \hat{F}\big) - \lambda_i\,E\big)\,v_i = 0$. Consequently, $F = H + \hat{F}$. Since we assigned ℓ finite generalized eigenvalues to the regular closed-loop system $\hat{\Sigma}$, the matrix pencil $\lambda\,E - \big(\hat{A} + B\,\hat{F}\big) = \lambda\,E - (A + B\,F)$ is regular.

(Necessity) Let F be a linear map such that $\lambda\,E - (A + B\,F)$ is regular and $(A + B\,F)\,v_i = \lambda_i\,E\,v_i$, $i \in \{1, \dots, \ell\}$. Since E, A, B are real matrices, then *(ii)* holds. The vectors $E\,v_i$ are linearly independent, because v_i, $i \in \{1, \dots, \ell\}$ are the finite eigenvectors of $(E, A + BF)$, see e.g. [20], thus *(i)* holds. Finally, if $(A + B\,F)\,v_i = \lambda_i\,E\,v_i$, then $(A - \lambda_i\,E)\,v_i = -B\,F\,v_i$, which can be written as $\begin{bmatrix} A - \lambda_i\,E & B \end{bmatrix} \begin{bmatrix} v_i \\ F\,v_i \end{bmatrix} = 0$. The columns of $\begin{bmatrix} S_{\lambda_i} \\ T_{\lambda_i} \end{bmatrix}$ form a basis for $\ker \begin{bmatrix} A - \lambda_i\,E & B \end{bmatrix}$, which implies *(iii)*. ∎

Appendix B

In this appendix we highlight the minor changes of the approach in the discrete-time case. Let Σ be an LTI discrete-time descriptor system

$$
\begin{aligned}
E\,x(t+1) &= A\,x(t)+B\,u(t), \quad E\,x(0)=E\,x_0 \in \mathbb{R}^n,\\
y(t) &= C\,x(t).
\end{aligned}
$$

The pair (E,A) is asymptotically stable if all the finite generalized eigenvalues of a regular matrix pencil $\lambda\,E-A$ are in the unit disc. Assumption 2.1 becomes:

System Σ is:

(i) impulse controllable;

(ii) stabilizable, i.e., rank $\left[\begin{array}{cc}\lambda\,E-A & B\end{array}\right]=n$ for all $\{\lambda \in \mathbb{C}\,|\,|\lambda| \geq 1\}$;

(iii) right invertible and has no invariant zeros at $\lambda=1$.

Two vectors $x_{\rm ss} \in \mathbb{R}^n$ and $u_{\rm ss} \in \mathbb{R}^m$ exist that satisfy for any $r \in \mathbb{R}^p$

$$
\begin{aligned}
0 &= (A-I_n)\,x_{\rm ss}+B\,u_{\rm ss},\\
r &= C\,x_{\rm ss}.
\end{aligned}
$$

We apply the state feedback control law $u(t)=F\,(x(t)-x_{\rm ss})+u_{\rm ss}$, where $F=\left[\begin{array}{cc}\tilde{F} & H_2\end{array}\right]$, to Σ and employ the change of variable $\xi(t) \stackrel{\text{def}}{=} x(t)-x_{\rm ss}$ to obtain the closed-loop homogeneous system

$$
\begin{aligned}
E\,\xi(t+1) &= (A+B\,F)\,\xi(t), \qquad E\,\xi(0)=E\,(x_0-x_{\rm ss}),\\
y(t) &= C\,\xi(t)+r.
\end{aligned}
$$

Since $(E,A+B\,F)$ is asymptotically stable, x converges to $x_{\rm ss}$ and y converges to r as t goes to infinity. The state response of the above system is $\xi(t)=\left[\begin{smallmatrix} I_\ell \\ -\hat{A}_{22}^{-1}\,\hat{A}_{21}\end{smallmatrix}\right]\tilde{A}_F^t\,\tilde{\xi}_0$ and the tracking error is $\epsilon(t)=-\big(\tilde{C}+\tilde{D}\,\tilde{F}\big)\,\big(\tilde{A}+\tilde{B}\,\tilde{F}\big)^t\,\tilde{\xi}_0$. Finally, the tracking errors can be expressed as

$$
\epsilon(t)=-\sum_{i=1}^{\ell}\big(\tilde{C}+\tilde{D}\,\tilde{F}\big)\,\tilde{v}_i\,\alpha_i\,\lambda_i^t=-\sum_{i=\ell-p+1}^{\ell}\mathbf{e}_{i-(\ell-p)}\,\alpha_i\,{\lambda_i}^t=-\left[\begin{array}{c}\alpha_{\ell-p+1}\,\lambda_{\ell-p+1}^t\\ \vdots \\ \alpha_\ell\,\lambda_\ell^t\end{array}\right]
$$

and every component of $\epsilon(t)$ contains exactly one mode $\epsilon_i(t)=-\alpha_{\ell-p+i}\,\lambda_{\ell-p+i}^t$ for $i \in \{1,\ldots,p\}$.

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