INERTIAL ACCELERATED ALGORITHMS FOR SOLVING A SPLIT FEASIBILITY PROBLEM

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Abstract. Inspired by the inertial proximal algorithms for finding a zero of a maximal monotone operator, in this paper, we propose two inertial accelerated algorithms to solve the split feasibility problem. One is an inertial relaxed-CQ algorithm constructed by applying inertial technique to a relaxed-CQ algorithm, the other is a modified inertial relaxed-CQ algorithm which combines the KM method with the inertial relaxed-CQ algorithm. We prove their asymptotical convergence under some suitable conditions. Numerical results are reported to show the effectiveness of the proposed algorithms.

1. Introduction. We are concerned with the following split feasibility problem (SFP): Find a point $x$ satisfying

$$x \in C, \ Ax \in Q,$$

(1.1)

where $C$ and $Q$ are nonempty closed convex subsets of real Hilbert spaces $H^1$ and $H^2$, respectively, and $A : H^1 \to H^2$ is a bounded linear operator. The SFP was introduced in [9], which has broad applications in many fields such as image reconstruction problem [8, 18], approximation theory [13], control [17], and so on. Many projection methods have been developed for solving the SFP, see [6, 10, 23, 25]. Denote by $P_C$ the orthogonal projection onto $C$; that is, $P_C(x) = \arg \min_{y \in C} \|x - y\|$, over all $x \in C$. In [5], Byrne introduced the so-called CQ algorithm, taking an initial point $x^0$ arbitrarily, and defining the iterative step as:

$$x^{k+1} = P_C[(I - \gamma A^T(I - P_Q)A)(x^k)],$$

where $0 < \gamma < 2/\rho(A^TA)$ and $\rho(A^TA)$ is the spectral radius of $A^TA$. Another approach is the so-called Krasnoselski-Mann (KM) algorithm, which was proposed...

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to a CQ algorithm for solving the SFP. Subsequently, Zhao [26] applied KM iteration
to a perturbed CQ algorithm, Dang and Gao [14] combined the KM iterative method
with a modified CQ algorithm to construct a KM-CQ-Like algorithm for solving the
SFP. The implementation of the algorithms mentioned above is under the condition
that the orthogonal projections onto \( C \) and \( Q \) are easily calculated. However, in
most cases, it is impossible or needs too much work to compute the exact orthogonal
projection. Similar to the inexact techniques used in optimization (e.g., [15, 16]),
an extension of the CQ algorithm that incorporate an inexact technique has been
proposed by Qu and Xiu [22]. In [24], by using the inexact projection technique,
Yang developed a relaxed CQ algorithm for solving the SFP, where he used two
halfspaces \( C_k \) and \( Q_k \) in place of \( C \) and \( Q \), respectively, at the \( k \)–th iteration such
that the orthogonal projections onto \( C_k \) and \( Q_k \) are easily done.

In nonlinear analysis, as a powerful unified framework for solving problems such
as minimization problem, complementarity problem and variational inequality prob-
lem, the problem of finding a zero of a maximal monotone operator \( G \) on a real
Hilbert space \( H \) is formulated as

\[
\text{finding } x \in H \text{ such that } 0 \in G(x). \tag{1.2}
\]

A classical method for solving this problem is the proximal method, in which the
next iteration \( x^{k+1} \) is generated by solving the subproblem

\[
0 \in \lambda_k G(x) + (x - x^k), \tag{1.3}
\]

where \( x^k \) is the current iteration and \( \lambda_k \) is a regularization parameter. Attouch
and Alvarez [2] applied an inertial technique to (1.3) to develop an inertial prox-
imal method for solving (1.2). It works as follows. Given \( x^{k-1}, x^k \in H \) and two
parameters \( \theta_k \in [0, 1), \lambda_k > 0 \), find \( x^{k+1} \in H \) such that

\[
0 \in \lambda_k G(x^{k+1}) + x^{k+1} - x^k - \theta_k(x^k - x^{k-1}). \tag{1.4}
\]

Here, the inertia is induced by the term \( \theta_k(x^k - x^{k-1}) \).

It is well known that the proximal iteration (1.3) may be interpreted as an implicit
one-step discretization method for the evolution differential inclusion

\[
0 \in \frac{dx}{dt}(t) + G(x(t)) \text{ a.e. } t \geq 0. \tag{1.5}
\]

Similarly, (1.4) may be thought of as coming from the implicit discretization of the
second-order differential system

\[
0 \in \frac{d^2x}{dt^2}(t) + \rho \frac{dx}{dt}(t) + G(x(t)) \text{ a.e. } t \geq 0. \tag{1.6}
\]

where \( \rho > 0 \) is a damping or a friction parameter. This point of view inspired various
numerical methods related to the inertial terminology, all those methods had nice
convergence properties [1-3, 19, 20] by incorporating second order information.

In this paper, we apply the inertial technique to the relaxed CQ method [24] to
propose inertial relaxed CQ algorithms for the SFP. Under certain suitable condi-
tions, the asymptotical convergence are proved. Preliminary numerical results are
reported to show the effectiveness of the algorithms.

The paper is organized as follows. In Section 2, we state some basic definitions
and lemmas. In Section 3, we present an inertial relaxed CQ algorithm and show
its convergence. In Section 4, a modified inertial relaxed CQ algorithm is presented.
and its convergence is also proved. In Section 5, numerical experiment results are given.

2. Preliminaries. Throughout the rest of the paper, \( I \) denotes the identity operator, \( \text{Fix}(T) \) denotes the set of the fixed points of an operator \( T \) i.e., \( \text{Fix}(T) := \{ x \mid x = T(x) \} \).

Recall that an operator \( T \) is called nonexpansive if
\[
\| T(x) - T(y) \| \leq \| x - y \| ,
\]
firmly nonexpansive if
\[
\| T(x) - T(y) \|^2 \leq \langle x - y, T(x) - T(y) \rangle .
\]
It is well known that the orthogonal projection operator \( P_C \) for any \( x, y \), is characterized by the inequalities
\[
\langle x - P_C(x), c - P_C(x) \rangle \leq 0, \ c \in C
\]
and
\[
\langle P_C(y) - P_C(x), y - x \rangle \geq \| P_C(y) - P_C(x) \|^2 .
\]
Therefore, the operator \( P_C \) is firmly nonexpansive. From Cauchy inequality we conclude that
\[
\| P_C(x) - P_C(y) \| \leq \| x - y \| ,
\]
that is, the operator \( P_C \) is nonexpansive.

Recall the notion of the subdifferential for an appropriate convex function.

**Definition 2.1 [12].** Let \( f : H \rightarrow \mathbb{R} \) be appropriate convex. The subdifferential of \( f \) at \( x \) is defined as
\[
\partial f(x) = \{ \xi \in H \mid f(y) \geq f(x) + \langle \xi, y - x \rangle , \ \forall y \in H \} .
\]

The lemmas below are necessary for the convergence analysis in the next section.

**Lemma 2.1 [20].** Assume \( \varphi_k \in [0, \infty) \) and \( \delta_k \in [0, \infty) \) satisfy:

1. \( \varphi_{k+1} - \varphi_k \leq \theta_k (\varphi_k - \varphi_{k-1}) + \delta_k , \)
2. \( \sum_{k=1}^{+\infty} \delta_k < \infty , \)
3. \( \{ \theta_k \} \subset [0, \theta] , \) where \( \theta \in (0, 1) . \)

Then, the sequence \( \{ \varphi_k \} \) is convergent with \( \sum_{k=1}^{+\infty} \sum_{k=1}^{+\infty} \| \varphi_{k+1} - \varphi_k \| < \infty \), where \( [t]_+ := \max\{t, 0\} \) ( for any \( t \in \mathbb{R} \)).

**Lemma 2.2 (Opial [21]).** Let \( H \) be a Hilbert space and \( \{ x^k \} \) a sequence such that there exists a nonempty set \( S \subset H \) verifying:

1. For every \( z \in S, \lim_{k \rightarrow \infty} \| x^k - z \| \) exists.
2. If \( x^k \rightharpoonup x^* \) weakly in \( H \) for a sequence \( k_j \rightarrow \infty \) then \( x^* \in S . \)

Then, there exists \( x \in S \) such that \( x^k \rightharpoonup x \) weakly in \( H \) as \( k \rightarrow \infty . \)

3. The inertial relaxed CQ algorithm and its asymptotic convergence.

3.1. The inertial relaxed CQ algorithm. As in [24], we make the following blanket assumptions.

1. The solution set of the SFP is nonempty.
2. The set \( C \) is denoted as
\[
C = \{ x \in H^1 \mid c(x) \leq 0 \} ,
\]
where \( c : H^1 \rightarrow \mathbb{R} \) is appropriate convex and \( C \) is nonempty.
The set $Q$ is denoted by

$$Q = \{ h \in H^2 \mid q(h) \leq 0 \},$$

where $q : H^2 \to R$ is appropriate convex and $Q$ is nonempty.

(3) For any $x \in H^1$, at least one subgradient $\xi \in \partial c(x)$ can be calculated; and for any $h \in H^2$, at least one subgradient $\eta \in \partial q(h)$ can be computed.

Now, we define two sets at point $x^k$,

$$C_k = \{ x \in H^1 \mid c(x^k) + \langle \xi^k, x - x^k \rangle \leq 0 \},$$

where $\xi^k$ is an element in $\partial c(x^k)$, and

$$Q_k = \{ h \in H^2 \mid q(Ax^k) + \langle \eta^k, h - Ax^k \rangle \leq 0 \},$$

where $\eta^k$ is an element in $\partial q(Ax^k)$.

By the definition of subgradient, it is clear that the halfspaces $C_k$ and $Q_k$ contain $C$ and $Q$, respectively. Due to the specific form of $C_k$ and $Q_k$, the orthogonal projections onto $C_k$ and $Q_k$ may be computed directly, see [4].

The following lemma provides an important boundedness property for the subdifferential.

Lemma 3.1 [12]. Suppose that $f : H \to R$ is convex. Then its subdifferential are uniformly bounded on any bounded subsets of finite dimensional space $H$.

Next, we state our inertial relaxed CQ algorithm.

Algorithm 3.1

Initialization: Take $x^0, x^1$ in $H^1$.

Iterative step: For $k \geq 0$, given the points $x^k, x^{k-1}$, the next iterative point $x^{k+1}$ is generated by

$$x^{k+1} = P_{C_k}[U_k(x^k + \theta_k(x^k - x^{k-1}))],$$

where $U_k = I - \gamma F_k, F_k = A^T(I - P_{Q_k})A, \gamma \in (0, 2/L)$, $L$ denotes the spectral radius of $A^TA$, $\theta_k \in [0, 1)$, $C_k$ and $Q_k$ are given by (3.3) and (3.4), respectively.

Evidently, when $\theta_k \equiv 0$, (3.5) happens to be the standard relaxed CQ method.

3.2. Asymptotic convergence of the inertial relaxed CQ algorithm. In this subsection, we establish the asymptotic convergence of Algorithm 3.1.

Define $f(x) = \frac{1}{2}\|Ax - P_{Q}Ax\|^2, x \in H^1$. It is not hard to see that $x$ solves the SFP (1.1) if and only if $x$ solves the minimization $f_{\text{min}} := \min_{x \in C} f(x)$ with $f_{\text{min}} = 0$. It is well known that the gradient of $f : F = \nabla f = A^T(I - P_{Q})A$ is L-Lipschitz continuous with $L = \rho(A^TA)$, and thus it is $\frac{1}{L}$-cocoercive (see [6]), that is $(F(x) - F(y), x - y) \geq \frac{1}{L}\|F(x) - F(y)\|^2$. The same is true for the operators $F_k = A^T(I - P_{Q_k})A$ for $k = 0, 1, \ldots$.

Theorem 3.1. Choose parameter $\theta_k \in [0, \bar{\theta}_k]$ with $\bar{\theta}_k := \min\{\theta, (\max\{k^2\|x^k - x^{k-1}\|^2, k^2\|x^k - x^{k-1}\|\})^{-1}\}, \theta \in (0, 1)$, then the sequence $\{x^k\}$ generated by (3.5) converges weakly to a point $x^*$ as $k \to \infty$, where $x^*$ is a solution of (1.1).

Proof. The case for $\theta_k \equiv 0$, we can see the detailed proof in [23].

Now we see the case for $\theta_k > 0$ for some $k \in N$. Let $z$ be a solution of the SFP. Since $C \subset C_k, Q \subset Q_k$, then $z = P_C(z) = P_{C_k}(z) \equiv F_k(z) = F_k(z) = 0$. Define the auxiliary real sequence $\varphi_k := \frac{1}{2}\|x^k - z\|^2$. Let $y^k = x^k + \theta_k(x^k - x^{k-1})$. 

Therefore, from the nonexpansive of the operator \( P_C \) and (3.5), we have
\[
\varphi_{k+1} = \frac{1}{2} \| x^{k+1} - z \|^2 = \frac{1}{2} \| P_{C_\theta}[U_k(x^k + \theta_k(x^k - x^{k-1}))] - z \|^2 \\
\leq \frac{1}{2} \| y^k - z \|^2 - \gamma F_k(y^k)^2.
\]
\[
\varphi_{k+1} = \frac{1}{2} \| P_{C_\theta}[y^k - \gamma F_k(y^k)] - P_{C_\theta}(z) \|^2 \\
\leq \frac{1}{2} \| y^k - z \|^2 - \gamma F_k(y^k)^2.
\]
Furthermore, by the cocoercivity of the operator \( F_k \), we have
\[
\varphi_{k+1} \leq \frac{1}{2} \| y^k - z \|^2 - \left( \frac{\gamma}{L} - \frac{\gamma^2}{2} \right) \| A^T(P_{Q_k} - I)A y^k \|^2.
\]
Note that we have
\[
\frac{1}{2} \| y^k - z \|^2 = \frac{1}{2} \| x^k + \theta_k(x^k - x^{k-1}) - z \|^2 \\
= \frac{1}{2} \| x^k - z \|^2 + \theta_k \langle x^k - z, x^k - x^{k-1} \rangle + \frac{\theta_k^2}{2} \| x^k - x^{k-1} \|^2 \\
= \varphi_k + \theta_k \langle x^k - z, x^k - x^{k-1} \rangle + \frac{\theta_k^2}{2} \| x^k - x^{k-1} \|^2.
\]
It is easy to check that \( \varphi_k = \varphi_{k-1} + \langle x^k - z, x^k - x^{k-1} \rangle - \frac{1}{2} \| x^k - x^{k-1} \|^2 \). Hence
\[
\frac{1}{2} \| y^k - z \|^2 = \varphi_k + \theta_k(\varphi_{k} - \varphi_{k-1}) + \frac{\theta_k + \theta_k^2}{2} \| x^k - x^{k-1} \|^2.
\]
Substituting (3.7) into (3.6), we get
\[
\varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_{k} - \varphi_{k-1}) + \frac{\theta_k + \theta_k^2}{2} \| x^k - x^{k-1} \|^2 - \left( \frac{\gamma}{L} - \frac{\gamma^2}{2} \right) \| A^T(P_{Q_k} - I)A y^k \|^2.
\]
Since \( 0 < \gamma < 2/L \), we have \( (\gamma/L - \frac{\gamma^2}{2}) > 0 \). According to \( \theta_k^2 \leq \theta_k \) and (3.8), we derive
\[
\varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_{k} - \varphi_{k-1}) + \theta_k \| x^k - x^{k-1} \|^2.
\]
Evidently,
\[
\sum_{k=1}^{+\infty} \theta_k \| x^k - x^{k-1} \|^2 < \infty,
\]
due to \( \theta_k \| x^k - x^{k-1} \|^2 \leq \frac{1}{L^2} \). Let \( \delta_k := \theta_k \| x^k - x^{k-1} \|^2 \) in Lemma 2.1. We deduce that the sequence \( \{ \| x^k - z \| \} \) is convergent (hence \( \{ x^k \} \) is bounded). By (3.9) and Lemma 2.1, we obtain \( \sum_{k=1}^{+\infty} \| x^k - z \|^2 - \| x^{k-1} - z \|^2 \) is bounded. In view of (3.6), we have
\[
\left( \frac{\gamma}{L} - \frac{\gamma^2}{2} \right) \| A^T(P_{Q_k} - I)A y^k \|^2 \leq \varphi_k - \varphi_{k+1} + \theta_k(\varphi_{k} - \varphi_{k-1}) + \theta_k \| x^k - x^{k-1} \|^2.
\]
Therefore
\[
\sum_{k=1}^{+\infty} \left( \frac{\gamma}{L} - \frac{\gamma^2}{2} \right) \|A^T(PQ_k - I)A y^k\|^2 < \infty.
\]

By \(0 < \gamma < 2/L\), we get
\[
\|A^T(PQ_k - I)A y^k\|^2 \to 0 \tag{3.11}
\]
and
\[
\|(PQ_k - I)A y^k\|^2 \to 0. \tag{3.12}
\]
Obviously, \(F_k(y^k) \to 0\). We next show that
\[
\|x^{k+1} - x^k\| \to 0. \tag{3.13}
\]
To do this, we proceed as follows.
\[
\|x^{k+1} - y^k\|^2 = \|x^{k+1} - y^k + y^k - z\|^2
= \|x^{k+1} - y^k\|^2 + \|y^k - z\|^2 + 2\langle x^{k+1} - y^k, y^k - z \rangle
= \|x^{k+1} - y^k\|^2 + \|y^k - z\|^2
+ 2\langle x^{k+1} - y^k, y^k - x^{k+1} \rangle + 2\langle x^{k+1} - y^k, x^{k+1} - z \rangle.
\]

Hence
\[
\|x^{k+1} - y^k\|^2 = \|y^k - z\|^2 - \|x^{k+1} - z\|^2 + 2\langle x^{k+1} - y^k, x^{k+1} - z \rangle. \tag{3.14}
\]

Substituting (3.7) into (3.14), we have
\[
\|x^{k+1} - y^k\|^2 = 2\varphi_k + 2\theta_k(\varphi_k - \varphi_{k-1}) - \|x^{k+1} - z\|^2 + (\theta_k + \theta_k^2)\|x^k - x^{k-1}\|^2
+ 2\langle x^{k+1} - y^k, x^{k+1} - z \rangle
= \|x^k - z\|^2 - \|x^{k+1} - z\|^2 + \theta_k(\|x^k - z\|^2 - \|x^{k+1} - z\|^2)
+ (\theta_k + \theta_k^2)\|x^{k+1} - x^{k-1}\|^2 + 2\langle x^{k+1} - y^k, x^{k+1} - z \rangle.
\]

By \(\theta_k \in [0, \bar{\theta}_k]\), \(\bar{\theta}_k := \min\{\theta, \max\{t_+\|x^k - x^{k-1}\|^2, t_+\|x^k - x^{k-1}\|^2\}\}, \theta \in [0, 1)\), we get
\[
\|x^{k+1} - y^k\|^2 \leq (\|x^k - z\|^2 - \|x^{k+1} - z\|^2)_+ + \theta(\|x^k - z\|^2 - \|x^{k-1} - z\|^2)_+
+ 2\theta_k\|x^k - x^{k-1}\|^2 + 2\langle x^{k+1} - y^k, x^{k+1} - z \rangle, \tag{3.15}
\]
where \(t_+ = \max\{t, 0\}\). On the other hand, \(x^{k+1} = PQ_k(x - \gamma F_k(y^k))\) implies
\[
\langle (y^k - \gamma F_k(y^k)) - x^{k+1}, z - x^{k+1} \rangle \leq 0.
\]

Then,
\[
\langle x^{k+1} - y^k, x^{k+1} - z \rangle \leq \gamma\langle F_k(y^k), z - x^{k+1} \rangle \to 0. \tag{3.16}
\]

From (3.10), we obtain
\[
\lim_{k \to +\infty} \theta_k\|x^k - x^{k-1}\|^2 = 0 \tag{3.17}
\]
Combining (3.15) with (3.16) and (3.17) leads to
\[
\|x^{k+1} - y^k\| \to 0. \tag{3.18}
\]
By the triangle inequality, we get
\[
\|x^{k+1} - x^k\| \leq \|x^{k+1} - y^k\| + \|y^k - x^k\| = \|x^{k+1} - y^k\| + \theta_k\|x^k - x^{k-1}\|. \tag{3.19}
\]
Since \(\theta_k \in [0, \bar{\theta}_k]\) with \(\bar{\theta}_k := \min\{\theta, \max\{t_+\|x^k - x^{k-1}\|^2, t_+\|x^k - x^{k-1}\|^2\}\}, \theta \in [0, 1)\), we have
\[
\sum_{k=1}^{+\infty} \theta_k\|x^k - x^{k-1}\| < \infty, \tag{3.20}
\]
which implies
\[ \theta_k \| x^k - x^{k-1} \| \to 0. \quad (3.21) \]

Using (3.18) and (3.21), from (3.19), we derive that
\[ \| x^{k+1} - x^k \| \to 0. \]

We have known that \( \{ x^k \} \) is bounded, which implies that \( \xi_k \) is bounded. Assume that \( x^* \) is an accumulation point of \( \{ x^k \} \) and \( x^{k_i} \to x^* \), where \( \{ x^{k_i} \} \) is a subsequence of \( \{ x^k \} \). Since \( \theta_k \| x^k - x^{k-1} \| \to 0 \), we have \( y^{k_i} \to x^* \). Then, from (3.11), it follows
\[ P_{Q_{k_i}}(Ax^{k_i}) \to Ax^*, \quad k_i \to +\infty. \quad (3.22) \]

Finally, we show that \( x^* \) is a solution of the SFP. Since \( x^{k+1} \in C_{k_i} \), We obtain
\[ c(x^{k_i}) + \langle \xi^{k_i}, x^{k+1} - x^{k_i} \rangle \leq 0. \]

Thus
\[ c(x^{k_i}) \leq -\langle \xi^{k_i}, x^{k+1} - x^{k_i} \rangle \leq \xi \| x^{k+1} - x^{k_i} \|. \]

where \( \xi \) satisfies \( \| \xi \| \leq \xi \) for all \( k \). By virtue of the continuity of function \( c \) and \( \| x^{k+1} - x^k \| \to 0 \), we get that
\[ c(x^*) = \lim_{l \to \infty} c(x^{k_i}) \leq 0. \]

Therefore, \( x^* \in C \).

Now we show that \( Ax^* \in Q \). To do this, let \( z^k = Ay^k - P_{Q_{k_i}}(Ay^k) \to 0 \) and let \( \eta \) be such that \( \| \eta_k \| \leq \eta \) for all \( k \). Since \( Ay^{k_i} - z^{k_i} = P_{Q_{k_i}}(Ay^{k_i}) \in Q_{k_i} \), we have
\[ q(Ax^{k_i}) + \langle \eta^{k_i}, (Ay^{k_i} - z^{k_i}) - Ax^{k_i} \rangle \leq 0. \]

Hence,
\[ q(Ax^{k_i}) \leq \langle \eta^{k_i}, Ax^{k_i} - Ay^{k_i} \rangle + \langle \eta^{k_i}, z^{k_i} \rangle \leq \eta \| A \| \theta_k \| x^k - x^{k-1} \| + \eta \| z^{k_i} \| \to 0. \]

By the continuity of \( q \) and \( A(x^{k_i}) \to Ax^* \), we arrive at the conclusion
\[ q(Ax^*) = \lim_{l \to \infty} q(Ax^{k_i}) \leq 0, \]

namely \( Ax^* \in Q \).

\[ \square \]

**Remark 3.1.** Since the current value of \( \| x^k - x^{k-1} \| \) is known before choosing the parameter \( \theta_k \), then \( \theta_k \) is well-defined in Theorem 3.1. In fact, from the process of proof Theorem 3.1, we can claim that the convergence result of Theorem 3.1 always holds provided that we take \( \theta_k \in [0, \theta], \theta \in (0, 1), \forall k \geq 0 \), with
\[ \sum_{k=1}^{+\infty} \theta_k \| x^k - x^{k-1} \|^2 < \infty \]

and
\[ \sum_{k=1}^{+\infty} \theta_k \| x^k - x^{k-1} \| < \infty. \]
4. A modified inertial relaxed CQ algorithm and its asymptotic convergence. In this section, a modified inertial relaxed CQ algorithm is presented, the asymptotic convergence is shown under some conditions.

Algorithm 4.1. Take \( x^0, x^1 \in H^1 \), the sequence \( \{x^k\}_{k \geq 0} \) is generated by the iterative process

\[
x^{k+1} = (1 - \alpha_k)y^k + \alpha_kPC_k[U_k(y^k)],
\]

where \( U_k = I - \gamma F_k, F_k = A^T(I-P_{Q_k})A, y^k = x^k + \theta_k(x^k - x^{k-1}), \alpha_k \in (0, 1), \theta_k \in [0, 1), \gamma, C_k, Q_k \) are given as in Algorithm 3.1.

Now, we establish the asymptotic convergence of the algorithm 4.1.

Theorem 4.1. Suppose (1.1) is consistent, \( \theta_k \in [0, \theta], \theta \in [0, 1), \forall k \geq 0 \). If (3.12), (3.21) and the following condition holds

\[
1 > R_1 = \inf_{k \geq 0} \alpha_k > 0,
\]

then the sequence \( \{x^k\} \) generated by (4.1) converges weakly to a point \( x^* \) contained in the set of solution of (1.1).

Proof. Let \( z \) be a solution of the SFP. \( C \subset C_k \) implies \( z = PC(z) = PC_k(z) \). Define the auxiliary real sequence \( \varphi_k := \frac{1}{2}\|x^k - z\|^2 \). From (4.1), we have

\[
\varphi_{k+1} = \frac{1}{2}\|x^{k+1} - z\|^2 \\
= \frac{1}{2}\| (1 - \alpha_k)y^k + \alpha_kPC_k[U_k(x^k + \theta_k(x^k - x^{k-1})) - z] \|^2 \\
= \frac{1}{2}\| (1 - \alpha_k)y^k + \alpha_kPC_k[y^k - \gamma F_k(y^k)] - z \|^2 \\
\leq \frac{1}{2}(1 - \alpha_k)\|y^k - z\|^2 + \frac{1}{2}\alpha_k\|PC_k[y^k - \gamma F_k(y^k)] - PC_kz\|^2 \\
\leq \frac{1}{2}(1 - \alpha_k)\|y^k - z\|^2 + \frac{1}{2}\alpha_k\|y^k - \gamma F_k(y^k) - z\|^2 \\
= \frac{1}{2}(1 - \alpha_k)\|y^k - z\|^2 + \frac{1}{2}\alpha_k\|y^k - z\|^2 + \alpha_k\gamma^2 \|F_k(y^k)\|^2 \\
- \alpha_k\gamma\langle y^k - z, F_k(y^k) - F_k(z) \rangle.
\]

So we get

\[
\varphi_{k+1} \leq \frac{1}{2}\|y^k - z\|^2 + \alpha_k\frac{\gamma^2}{2}\|F_k(y^k)\|^2 - \alpha_k\gamma \frac{1}{L}\|F_k(y^k)\|^2,
\]

therefore

\[
\varphi_{k+1} \leq \frac{1}{2}\|y^k - z\|^2 - \alpha_k(\frac{\gamma}{L} - \frac{\gamma^2}{2})\|F_k(y^k)\|^2.
\]

Substituting (3.10) into (4.4), we get

\[
\varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2 - \alpha_k(\frac{\gamma}{L} - \frac{\gamma^2}{2})\|A^T(P_{Q_k} - I)Ay^k\|^2.
\]

Since \( 0 < \gamma < 2/L \) and \( 0 < \inf \alpha_k < 1 \), we have

\[
\varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2 - R_1(\frac{\gamma}{L} - \frac{\gamma^2}{2})\|A^T(P_{Q_k} - I)Ay^k\|^2,
\]

moreover

\[
\varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2.
\]
Suppose $\sum_{k=1}^{+\infty} \theta_k \|x^k - x^{k-1}\|^2 < \infty$, choose $\delta_k := \theta_k \|x^k - x^{k-1}\|^2$ in Lemma 2.1, we deduce that the sequence $\{\|x^k - z\|\}$ is convergent (hence \{x^k\} is bounded). From (4.7) and Lemma 2.1 we obtain $\sum_{k=1}^{+\infty} \|x^k - z\|^2 - \|x^{k-1} - z\|^2 < \infty$, while from (4.6) we have

$$R_1(\frac{\gamma}{L} - \frac{\gamma^2}{2}) A^T (P_{Q_k} - I) A y_k \leq \varphi_k - \varphi_{k+1} + \theta_k (\varphi_k - \varphi_{k-1}) + \theta_k \|x^k - x^{k-1}\|^2.$$ 

Obviously,

$$\sum_{k=1}^{+\infty} R_1(\frac{\gamma}{L} - \frac{\gamma^2}{2}) A^T (P_{Q_k} - I) A y_k < \infty.$$ 

Since $0 < \gamma < 2/L$ and $0 < R_1 < 1$, we have

$$\|A^T (P_{Q_k} - I) A y_k\|^2 \to 0. \quad (4.8)$$

The rest part of the proof is similar to that of Theorem 3.1, and hence it is omitted.

5. Numerical results. In this section, we will test three numerical examples.

Throughout the computational experiments, we set $\varepsilon = 10^{-4}$. In the algorithms, we take $\gamma = 1/L$, $L$ denotes the spectral radius of $A^T A$, $\theta = 0.8$. If $\theta \leq \left[ \max \{ k^2 \|x^k - x^{k-1}\|^2, k^2 \|x^k - x^{k-1}\| \} \right]^{-1}$, we take $\theta_k = \frac{\theta}{2} = 0.4$; Otherwise, we take $\theta_k = \left[ \max \{ (k+1)^2 \|x^k - x^{k-1}\|^2, (k+1)^2 \|x^k - x^{k-1}\| \} \right]^{-1}$, $k = 1, 2 \cdots$.

Example 5.1. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - 9 \leq 0 \}; Q = \{ x \in \mathbb{R}^3 \mid x_1 + x_2^3 - 3 \leq 0 \}$. Find $x \in C$ with $Ax \in Q$.

Example 5.2. Let

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 & 3 \\ 1 & 2 & 5 & 2 & 1 \\ 2 & 0 & 2 & 1 & -2 \\ 2 & -1 & 0 & -3 & 5 \end{bmatrix}$$

$C = \{ x \in \mathbb{R}^5 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 0.25 \leq 0 \}; Q = \{ x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 + 0.6 \leq 0 \}$. Find $x \in C$ with $Ax \in Q$.

Example 5.3. Let $A = (a_{ij})_{M \times N}, a_{ij} \in (0, 1)$ be a random matrix, $M, N$ are two positive integers. $C = \{ x \in \mathbb{R}^N \mid \sum_{i=1}^{N} x_i^2 \leq r^2 \}; Q = \{ x \in \mathbb{R}^M \mid x \leq b \}$. To ensure the existence of the solution of the problem, the vector $b$ is generated by using the following way: Given a random $N$-dimensional vector $x^* \in C, r = \|x^*\|$, taking $b = Ax^*$. Find $x \in C$ with $Ax \in Q$.

The numerical results of Examples 5.1-5.3 can be seen from Tables 1-5. In Tables 1-5, “R-Iter”, “Iner-R-Iter” and “Iner-KM-R-Iter” denote the relaxed CQ algorithm, the inertial relaxed CQ algorithm and the modified inertial relaxed CQ algorithm, respectively. “$k^*$”, “$s^*$” and “$x^*$” denote the number of iterations, cpu time in seconds and the solution, respectively. To compare conveniently, we take the initial point $x^1$ in the latter two algorithms as in the R-iter, that is, the point $x^1$ is generated by the R-iter.
Table 1. The numerical results of example 5.1

<table>
<thead>
<tr>
<th>Initiative point</th>
<th>R-Iter</th>
<th>Iner-R-Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^0 = (3.2, 4.2, 5.2)$</td>
<td>$k = 74; s = 0.068$</td>
<td>$k = 5; s = 0.016$</td>
</tr>
<tr>
<td>$x^1 = (-0.5843, 2.3078, 3.3435)$</td>
<td>$x^* = (-0.6260, 1.6180, 1.6216)$</td>
<td>$x^* = (-1.1281, 1.0720, 1.9694)$</td>
</tr>
<tr>
<td>$x^0 = (10, 0, 10)$</td>
<td>$k = 93; s = 0.090$</td>
<td>$k = 84; s = 0.085$</td>
</tr>
<tr>
<td>$x^1 = (2.0825, -5.2575, 6.4589)$</td>
<td>$x^* = (0.9000, -1.7152, 1.7074)$</td>
<td>$x^* = (-0.1061, -1.4514, 2.1596)$</td>
</tr>
<tr>
<td>$x^0 = (2, -5, 2)$</td>
<td>$k = 73; s = 0.075$</td>
<td>$k = 35; s = 0.035$</td>
</tr>
<tr>
<td>$x^1 = (1.3327, -3.2657, 1.9328)$</td>
<td>$x^* = (1.5112, -2.7679, 1.8616)$</td>
<td>$x^* = (0.9010, -2.1029, 1.8169)$</td>
</tr>
</tbody>
</table>

Table 2. The numerical results of example 5.1

<table>
<thead>
<tr>
<th>Initiative point</th>
<th>$\alpha_k$</th>
<th>Iner-KM-R-Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^0 = (3.2, 4.2, 5.2)$</td>
<td>0.4</td>
<td>$k = 3; s = 0.016$</td>
</tr>
<tr>
<td>$x^1 = (-0.5843, 2.3078, 3.3435)$</td>
<td>0.8</td>
<td>$k = 3; s = 0.013$</td>
</tr>
<tr>
<td>$x^0 = (10, 0, 10)$</td>
<td>0.4</td>
<td>$k = 76; s = 0.086$</td>
</tr>
<tr>
<td>$x^1 = (2.0825, -5.2575, 6.4589)$</td>
<td>0.8</td>
<td>$k = 74; s = 0.085$</td>
</tr>
<tr>
<td>$x^0 = (2, -5, 2)$</td>
<td>0.6</td>
<td>$k = 62; s = 0.056$</td>
</tr>
<tr>
<td>$x^1 = (1.3327, -3.2657, 1.9328)$</td>
<td>0.8</td>
<td>$k = 45; s = 0.040$</td>
</tr>
</tbody>
</table>

Table 3. The numerical results of example 5.2

<table>
<thead>
<tr>
<th>Initiative point</th>
<th>R-Iter</th>
<th>Iner-R-Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^0 = (0, 0, 0, 0, 0)$</td>
<td>$k = 15; s = 0.675$</td>
<td>$k = 5; s = 0.018$</td>
</tr>
<tr>
<td>$x^1 = (-0.0092, 0, -0.0136, -0.026, -0.0092)$</td>
<td>$x^* = (-0.0208, 0)$</td>
<td>$x^* = (0.0015, 0)$</td>
</tr>
<tr>
<td>$x^0 = (1.1, 1.1, 1, 1, 1)$</td>
<td>$k = 20; s = 0.083$</td>
<td>$k = 3; s = 0.0272$</td>
</tr>
<tr>
<td>$x^1 = (0.3237, 0.5471, 0.2280, 0.4833, 0.3237)$</td>
<td>$x^* = (0.0171, 0.3622, -0.1394, 0.2779, 0.0171)$</td>
<td>$x^* = (-0.0784, 0.2935, -0.2378, 0.1873, -0.0784)$</td>
</tr>
<tr>
<td>$x^0 = (20, 10, 20, 10, 20)$</td>
<td>$k = 22; s = 0.090$</td>
<td>$k = 6; s = 0.067$</td>
</tr>
<tr>
<td>$x^1 = (6.1605, 5.0023, 4.5130, 3.9040, 6.1605)$</td>
<td>$x^* = (0.0837, 0.3910, -0.2155, 0.1915, 0.0837)$</td>
<td>$x^* = (-0.2490, -0.2177, -0.1742, -0.1619, -0.2490)$</td>
</tr>
</tbody>
</table>

Table 4. The numerical results of example 5.2

<table>
<thead>
<tr>
<th>Initiative point</th>
<th>$\alpha_k$</th>
<th>Iner-KM-R-Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^0 = (0, 0, 0, 0, 0)$</td>
<td>0.6</td>
<td>$k = 6; s = 0.020$</td>
</tr>
<tr>
<td>$x^1 = (-0.0092, 0, -0.0136, -0.0092)$</td>
<td>0.8</td>
<td>$k = 5; s = 0.018$</td>
</tr>
<tr>
<td>$x^0 = (1.1, 1.1, 1, 1, 1)$</td>
<td>0.4</td>
<td>$k = 3; s = 0.034$</td>
</tr>
<tr>
<td>$x^1 = (0.3237, 0.5471, 0.2280, 0.4833, 0.3237)$</td>
<td>0.6</td>
<td>$k = 3; s = 0.034$</td>
</tr>
<tr>
<td>$x^0 = (20, 10, 20, 10, 20)$</td>
<td>0.6</td>
<td>$k = 9; s = 0.072$</td>
</tr>
<tr>
<td>$x^1 = (6.1605, 5.0023, 4.5130, 3.9040, 6.1605)$</td>
<td>0.8</td>
<td>$k = 7; s = 0.071$</td>
</tr>
</tbody>
</table>
Table 5. The numerical results of example 5.3

<table>
<thead>
<tr>
<th>$M, N$</th>
<th>R-Iter</th>
<th>Iner-R-Iter</th>
<th>Iner-KM-R-Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 20, N = 10$</td>
<td>$k = 436, s = 0.970$</td>
<td>$k = 174, s = 0.500$</td>
<td>$k = 210, s = 0.270$</td>
</tr>
<tr>
<td>$M = 100, N = 90$</td>
<td>$k = 3788, s = 0.130$</td>
<td>$k = 602, s = 0.680$</td>
<td>$k = 534, s = 0.690$</td>
</tr>
</tbody>
</table>

Table 1 and Table 2 give the numerical results of Example 5.1 with the R-Iter, the Iner-R-Iter, and the Iner-KM-R-Iter for different $\alpha_k$, respectively. Table 3 and Table 4 show the numerical results of Example 5.2 with the R-Iter, the Iner-R-Iter, and the Iner-KM-R-Iter for different $\alpha_k$, respectively. Table 5 gives the numerical results of Example 5.3 with $\alpha_k = 0.6, k = 1, 2, \ldots$.

From Tables 1-5, we can see that our algorithms are effective. It appears that they converge more quickly than the relaxed CQ algorithm in [23]. It also appears to suggest greater values of $\alpha_k$ because the convergence of the Iner-KM-R-Iter took less number of iterations in the experiment.

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