Department of Mathematics and Statistics

Study of Various Stochastic Differential Equation Models for Finance

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Declaration

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made. This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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Abstract

Stochastic Calculus has been widely used for modeling financial variables such as asset prices, derivative price as well as portfolio allocation since it was introduced by Robert Merton into the study of finance in 1990. Unlike deterministic models such as ordinary differential equation models, we begin with study in the form of continuous-time stochastic processes instead of a unique solution for each appropriate initial condition.

This study consists of two parts. The first part of the research focuses on option pricing, including European option pricing and American option pricing. Option pricing is one of the predominant concerns in the financial market. Developed from the classical Black-Scholes option pricing formula, we start exploring Jump-diffusion models with stochastic volatility in order to explain numerous empirical phenomena such as large and sudden movements in prices, heavy tails, volatility clustering, the incompleteness of markets, the concentration of losses in a few large downward moves. Different from the Black-Scholes framework, we use jump-diffusion to describe the price dynamics for European options. The market of the model is allowed to be incomplete; that is, it is not possible to replicate the payoff of every contingent claim by a portfolio, and there are several equivalent martingale measures. The general equilibrium framework method is used as a unique martingale measure to deal with the option pricing in an incomplete market. As for the American option, we use the minimal martingale measure and consider the switch of asset prices with jump diffusion and stochastic volatility, and obtain the Radon-Nikodym derivative and a linear complementarity problem

for the pricing of American option. Unlike European options, American options, which can be exercised any time before the expiry date, have to be priced numerically. Comparing to European options which lead to a partial integro-differential equation, American options lead to inequalities in the form of a linear complementarity problem (LCP).

The second major part of this thesis is on Portfolio Optimization. As an important topic in financial market, Portfolio Optimization has been studied by a vast of researchers since the first publication by Markowitz[1]. The key in portfolio optimization is, for a given selected utility function and stochastic models, optimizing the value of control parameters to maximize the final value of the utility. Various utilities have been used in portfolio optimization including Mean-variance utility, the endogenous habit formation, hyperbolic discounting, and the classic CRRA. Among these utilities, optimal asset allocation under mean-variance is one of the most interesting and thought-provoking topic in the classic results of financial economics. Under this framework, we find that the optimal asset allocation problem with multiple-periods is time-inconsistent which prohibits the application of the classical Bellman Optimality principle. To solve this time-inconsistent problem, we formulate the problem in the game theoretic framework. The control process under multi-period time is dealt with like a game where the periods are treated as multiple players. Each player presents the accordingly present favor and the incarnations of future tastes. The game ends when the sub-game perfect Nash equilibrium point is found. Under this framework, a series of extended HJB equations have been developed and solved numerically.

List of Publications

1. Shuang Li, Yanli Zhou, Xinfeng Ruan and Yonghong Wu. "Pricing of American put option under a jump diffusion process with stochastic volatility in an incomplete market." Abstract and Applied Analysis. Vol. 2014. Hindawi Publishing Corporation, 2014.

2. Xinfeng Ruan, Wenli Zhu, Shuang Li, Jiexiang Huang. "Equilibrium asset and option pricing under jump-diffusion model with stochastic volatility." Abstract and Applied Analysis. Vol. 2013. Hindawi Publishing Corporation, 2013.

3. Xinfeng Ruan, Wenli Zhu, Jiexiang Huang and Shuang Li. "Continuous-time portfolio selection and option pricing under risk-minimization criterion in an incomplete market." Journal of Applied Mathematics 2013 (2013).

4. Xinfeng Ruan, Wenli Zhu, Shuang Li and Jiexiang Huang. "Option pricing under risk-minimization criterion in an incomplete market with the finite difference method." Mathematical Problems in Engineering 2013 (2013).

5. Shuang Li, Yanli Zhou, Yonghong Wu, Xiangyu Ge. "Equilibrium approach of asset and option pricing under Lvy process and stochastic volatility." Australian Journal of Management (2016): 0312896215619966.

6. Shuang Li, Chuong Luong, Francisca Angkola and Yonghong Wu. "Optimal asset portfolio with stochastic volatility under the mean-variance utility with state-dependent risk aversion." (2016).

Contents

CHAPTER 1

Introduction

1.1 Background

The previous fifty years has witnessed the emergence and rapid expansion of a new scientific discipline, the "theory of finance". Along the work of Harry Markowitz[1], William Sharpe[2], and Merton Miller[3], who lay the groundwork of the mathematical theory of finance, we have a basic understanding of how financial markets work, how to make them more efficient, and how they should be regulated. The theory of finance not only interprets and plays an important role in capital allocation and risk reduction, but also becomes more and more mathematical, so that nowadays problems in finance are driving research in mathematics to some point.

Harry Markowitz's Ph.D. thesis Portfolio Selection is the first significant work to quantify the diversification of financial market through the use of a concept of mean return and covariances for common stocks. The thesis concerns how to compute the mean return and variance and that for a given set of assets, what portfolio investors should hold in order to minimize the variance for a given mean return. As time goes by, the language of finance keeps developing. The underlying theme of the modern theory and practice of quantitative finance today involves stochastic calculus, management of risk.

Stochastic calculus was firstly introduced to the study of finance by Robert Merton in 1969. With the desire to explore how prices are set in financial markets which has been regarded as the classical economics question of "equilibrium", Merton solved this issue by using the machinery of stochastic calculus. With Merton's help, Fischer and Black and Myron Scholes developed their famous classical Black-Scholes option pricing formula at the same time. What does the Black-Scholes option pricing formula provide? It provides the theoretical method for fairly pricing a risk-hedging security at the first time. Under the formula, the investment bankers can offer the derivative securities at their fair prices which have been determined in advance to maintain market efficiency. If the price of a derivative security is higher than the fair price, it may not be bid. On the other hand, if it is lower than the fair price, it may cause substantial loss. Thus the practical problem of how to find a fair price for a European call option, a right to buy one share of a given stock at a specified price and time, was solved with a satisfactory solution within Fischer and Black and Myron Scholes's work. After selling the derivative security at the fair price, another question arises to the investment bank. That is how to hedge the derivative security and how to manage the new risk. Based on the Black-Scholes option formula, mathematical theories have been developed to deals with both the pricing and hedging problems. From 1979 to 1983, Harrison, Kreps, and Pliska[4-6] not only further laid the solid foundation for the Black-Scholes option-pricing formula but also extended the method to other financial derivatives through the general theory of continuous-time stochastic processes.

Although the theory and its practice of Black-Scholes option pricing formula have been widely used among researchers, there are some phenomena which the theory cannot explain, such as large and sudden movements in prices, heavy tails, volatility clustering, the incompleteness of markets, the concentration of losses in a few large downward moves. In order to overcome the drawbacks of the Black-

Scholes model, this research explores the jump-diffusion models with stochastic volatility.

Portfolio optimization is a challenging problem in investment and risk management. The objective is to optimize the proportions of the assets to be held in a portfolio according to certain criterion on the expected rate of investment return and/or the level of financial risk. By quantifying the criterion by certain utility function, the problem can be formulated as maximizing the expected value of the utility function at certain pre-determined time in future such as at the end of certain investment period. In a standard continuous time stochastic optimization problem, dynamic programing is a standard way to deal with the problem. We have the Bellman optimality principle which roughly says that the optimal control is independent of the initial point. More precisely, if the control law is optimal on the full time interval, then it is also optimal for any subinterval. Given the Bellman principle, it is easy to informally derive the Hamilton-Jacobi-Bellman (henceforth HJB) equation. However, there are some circumstances such as hyperbolic discounting, mean variance utility and endogenous habit formation where the Bellman optimality principle does not hold. The above three aspects belong to a time inconsistent family of problems. Mean Variance analysis for optimal asset allocation is one of the classical results of financial economics. Since Markowitz[1] first introduced the asset optimization analysis for mean-variance, a large volume of literature has been published concerning this topic, and the one period case is often taken into consideration. However, in practice, this is not reasonable because the optimal portfolio optimization problems are multi-period optimizing problems, and should be modeled within the multi-period framework. On the other hand, the mean-variance based portfolio optimization in the multi-period framework is time inconsistent as the Bellman Optimization principle does not hold in this case [7]. Thus, it is an interesting and challenging topic to solve the portfolio optimization problem under mean-variance.

1.2   Objectives

The main purpose of this thesis is to connect the knowledge from stochastic calculus with the real financial market by exploring and constructing various stochastic differential equation models for finance. Specifically, the aims of this work are as follows.

1) Develop option pricing models under various conditions, then look for analytical solutions, if possible or use numerical methods to get the approximate solutions, then analyze the risk premia under each issue established above from the fundamental model, and finally validate the model with the S&P 500 index from 1985-2005 to analyse the jump size and volatility smile.

2) Study the equity premium and option pricing under the jump diffusion model with stochastic volatility, and obtain the pricing kernel which acts like the physical and risk-neutral densities and the moments in the economy, and finally discuss the relationship of central moments between the physical measure and the risk-neutral measure.

3) Study the pricing of American options in an incomplete market in which the dynamics of the underlying risky asset is driven by a jump diffusion process with stochastic volatility, then obtain the Radon-Nikodym derivative for the minimal martingale measure and consequently a linear complementarity problem (LCP) for American option prices by employing a risk-minimization criterion, and then establish the iterative method to solve the LCP problem for American put option prices.

4) Explore the portfolio optimization of mean-variance utility with states-dependent risk aversion with the assets driven by stochastic processes, then use sub-game perfect Nash equilibrium strategies and the extended Hamilton-Jacobi-Bellman equation to derive the system of non-linear partial differential equations, and then analyze a special case where the risk aversion proportional to the wealth

has been studied with numerical method.

5) Study the portfolio optimization of mean-variance utility with multiscale stochastic volatility, then use the asymptotics approximations to give the expansion of the value function which is the solution of the Hamilton-Jacobi-Bellman PDE for the Merton problem with the constant parameters given explicitly in terms of the derivatives of the leading order value function, and then solve the portfolio optimization problem under the Mean-variance utility by using an analytical and/or numerical method.

1.3 Outlines of the Thesis

This thesis consists of seven chapters.

Chapter 1 gives a brief introduction of the study and highlights the objectives of the research.

Chapter 2 gives a general review of the background, and previous work relevant to the research to be carried out.

Chapter 3 is on pricing American put option under a jump-diffusion process with stochastic volatility in an incomplete market. By employing a risk-minimization criterion, we obtain the Radon-Nikodym derivative for the minimal martingale measure and consequently a linear complementarity problem (LCP) for American option price. An iterative method is then established to solve the LCP problem for American put option price.

Chapter 4 concerns equilibrium asset and option pricing under the jump diffusion model with stochastic volatility, where we present the model and solutions as well as some explanation of the significance of this topic. We study the equity premium and option pricing under the jump diffusion model with stochastic volatility based on the model in[38]. We obtain the pricing kernel which acts like the physical and risk-neutral densities and the moments in the economy.

Chapter 5 begins with the research about option asset portfolio under the mean-variance utility. This chapter focuses on the topic: option asset portfolio with stochastic volatility and interest rate under the mean-variance utility with multi-states dependent risk aversion. We study the portfolio optimization of mean-variance utility with states-dependent risk aversion with the assets driven by stochastic processes. Sub-game perfect Nash equilibrium strategies and the extended Hamilton-Jacobi-Bellman equation have been used to derive the system of non-linear partial differential equations. From the economic point of view, a special case where the risk aversion proportional to the wealth has been studied with numerical method. The parameters in the model are determined through calibration of real financial market. Results show that the asset driven by a stochastic volatility and stochastic interest rate is more general and reasonable than that driven by constant ones.

Chapter 6 deals with mean-variance portfolio optimization with stochastic volatility asymptotics. Sub-game perfect Nash equilibrium strategies and the extended Hamilton-Jacobi-Bellman equation have been used to get the system of non-linear partial differential equations. By using the asympototics approximations, the expansion of the value function which is the solution of the Hamilton-Jacobi-Bellman PDE for the Merton problem with constant parameters is given explicitly in terms of the derivatives of the leading order value function, and the optimal strategy is given explicitly in terms of the derivatives of the leading order value function. Thus, we solve the portfolio optimization problem under the Mean-variance utility by using an analytical method.

Chapter 7 briefly summarizes the research findings and discusses further research work.

CHAPTER 2

Literature Review

2.1   General

The research consists of two major parts. The first part is on option pricing and determination of risk premium, and the second part is on portfolio optimization. Under the option pricing and risk premium, there are three topics: European option, American option and risk premium under equilibrium framework. For portfolio optimization, different utilities may be used including CRRA, Habit Formation and Mean-variance Utility. We will reviews research progress and results highly relevant to the proposed research, including the Black-Scholes optimal pricing model, the Equilibrium asset, option pricing under jump diffusion and portfolio optimization under mean-variance utility.

2.2   The Black-Scholes Option Pricing Models

Harry Markowitz's Ph.D. thesis, Portfolio Selection[1], firstly quantifies the diversification of financial market by introducing a concept of mean return and covariances for common stocks. In 1990, Robert Merton[3] introduced stochastic calculus into the study of finance. Fischer and Black and Myron Scholes[42] started developing their famous and classical option pricing formula from 1979. In 1990, Harry Markowitz, William Sharpe and Merton Miller obtained the No-

bel Prize in economics which brings the world attention to this new scientific discipline , the "theory of finance". The Black-Scholes option pricing formula provides the theoretical method of fairly pricing a risk-hedging security at the first time. Under the formula, the investment bankers can offer the derivative securities at their fair prices which have been determined in advance to maintain market efficiency. If the price of a derivative security is higher than the fair price, it may not be bid. On the other hand, if it is lower than the fair price, it may cause substantial loss.

Here, we briefly describe the process of deriving the Black-Scholes-Merton partial differential equation for the price of an option on an asset modeled as geometric Brownian motion by Shreve[8]. The idea behind this derivation is to determine the initial capital required to perfectly hedge a short position in the option.

Consider an agent who at each time t has a portfolio value $X(t)$. This portfolio invests in a money market account paying a constant rate of interest r and in a stock modeled by the geometric Brownian motion $dS(t) = \alpha S(t) + \sigma S(t)dW(t)$. Suppose at each time t, the investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t), t \geq 0$. The reminder of the portfolio value, $X(t) - \Delta(t)S(t)$, is invested in the money market account. The differential $dX(t)$ for the investor's portfolio value at each time t is due to two factors, the capital gain $\Delta(t)dS(t)$ on the stock position and the interest earnings $r(X(t) - \Delta(t)S(t))dt$ on the cash position. In other words

$$
\begin{aligned}
dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\
&= \Delta(\alpha S(t)dt + \sigma S(t)dW(t)) + r(X(t) - \Delta(t)S(t))dt \quad (2.1) \\
&= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t).
\end{aligned}
$$

The three terms appearing in the last line of the above equation can be un-

derstood as follows: (i) an average underlying rate of return r on the portfolio, which is reflected by the term $rX(t)dt$, (ii) a risk premium $\alpha - r$ for investment in the stock, which is reflected by the term $\Delta(t)(\alpha - r)S(t)dt$, and (iii) a volatility term proportional to the size of the stock investment, which is the term $\Delta(t)\sigma S(t)dW(t)$.

The discrete-time analogue of equation (2.1) appears in Shreve[8] volume 1 as $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$, we may arrange terms in this equation to obtain $X_{n+1} - X_n = \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n)$, which is analogous to the first line of (2.1), except in the above equation, time steps forward one unit at a time, whereas in (2.1) time moves forward continuously.

We shall often consider the discounted stock price $e^{-rt}S(t)$ and the discounted portfolio value of an agent, $e^{-rt}X(t)$. According to the Ito-Doeblin formula with $f(t, x) = e^{-rt}x$, the differential of the discounted stock price is

$$
\begin{aligned}
d(e^{-rt}S(t)) &= df(t, S(t)) \\
&= f_t(t, S(t))dt + f_x(t, S(t))dS(t) + \frac{1}{2}f_{xx}(t, S(t))dS(t)dS(t) \\
&= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\
&= (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t),
\end{aligned}
\tag{2.2}
$$

and the differential of the discounted portfolio value is

$$
\begin{aligned}
d(e^{-rt}X(t)) &= df(t, X(t)) \\
&= f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))dX(t)dX(t) \\
&= -re^{-rt}X(t)dt + e^{-rt}dX(t) \\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \\
&= \Delta(t)d(e^{-rt}S(t)).
\end{aligned}
$$

$$(2.3)$$

Discounting the stock price reduces the mean rate of return from α, the term

multiplying $S(t)dt$ in $dS(t) = \alpha S(t) + \sigma S(t)dW(t)$, to $\alpha - r$, the term multiplying $e^{-rt}S(t)dt$ in (2.3). Discounting the portfolio value removes the underlying rate of return r, compare the last line of (2.1) to the next-to-last line of (2.3). The last line of (2.3) shows that change in the discounted portfolio value is solely due to change in the discounted stock price.

Consider a European call option that pays $(S(T) - K)^+$ at time T. The strike price K is some nonnegative constant. Black, Scholes, and Merton argued that the value of this call at any time should depend on the time (more precisely, on the time of expiration) and on the value of the stock price at that time, and of course it should also depend on the model parameters r and *sigma* and the contractual strike price K. Only two of these quantities, time and stock price, are available. Following this reasoning, we let $c(t, x)$ denote the value of the call at time t if the stock price at that time is $S(t) = x$. There is nothing random about the function $c(t, x)$. However, the value of the option is random; it is the stochastic process $c(t, S(t))$ obtained by replacing the dummy variable x by the random stock price $S(t)$ in this function. At the initial time, we do not know the future stock prices $S(t)$ and hence do not know the future option values $c(t, S(t))$. Our goal is to determine the function $c(t, x)$ so we at least have a formula for the future option values in terms of the future stock prices.

We begin by computing the differential of $c(t, S(t))$. According to the Ito-Doeblin formula, we have

$$
\begin{aligned}
dc(t, S(t)) &= c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}c_{xx}(t, S(t))dS(t)dS(t) \\
&= c_t(t, S(t))dt + c_x(t, S(t))(\alpha S(t) + \sigma S(t)dW(t)) \\
&\quad + \frac{1}{2}c_{xx}(t, S(t))\sigma^2 S^2(t)dt \\
&= [c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))]dt \\
&\quad + \sigma S(t)c_x(t, S(t))dW(t).
\end{aligned}
\tag{2.4}
$$

We next compute the differential of the discounted option price $e^{-rt}c(t, S(t))$. Let $f(t, x) = e^{-rt}x$. According to the Ito-Doeblin formula, we have

$$
\begin{aligned}
d(e^{-rt}c(t, S(t))) &= df(t, c(t, S(t))) \\
&= f_t(t, c(t, S(t)))dt + f_x(t, c(t, S(t)))dc(t, S(t)) \\
&\quad + \frac{1}{2}f_{xx}(t, c(t, S(t)))dc(t, S(t))dc(t, S(t)) \\
&= -re^{-rt}c(t, S(t))dt + e^{-rt}dc(t, S(t)) \\
&= e^{-rt}[-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) \\
&\quad + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))]dt + e^{-rt}\sigma S(t)c_x(t, S(t))dW(t).
\end{aligned}
\tag{2.5}
$$

A (short option) hedging portfolio starts with some initial capital $X(0)$ and invests in the stock and money market account so that the portfolio value $X(t)$ at each time $t \in [0, T]$ agrees with $c(t, S(t))$. This happens if and only if $e^{-rt}X(t) = e^{-rt}c(t, S(t))$ for all t. One way to ensure this equality is to make sure that

$$
d(e^{-rt}X(t)) = d(e^{-rt}c(t, S(t))) \ for \ all \ t \in [0, T)
\tag{2.6}
$$

and $X(0) = c(0, S(0))$. Integration of (2.6) from 0 to t yields

$$
e^{-rt}X(t) - X(0) = e^{-rt}c(t, S(t)) - c(0, S(0)) \ for \ all \ t \in [0, T).
\tag{2.7}
$$

If $X(0) = c(0, S(0))$, then we can cancel this term in (2.7) and get the desired equality. Comparing (2.3) and (2.5), we see that (2.6) holds if and only if

$$
\begin{aligned}
&\Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \\
&= [-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) \\
&\quad + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))]dt + e^{-rt}\sigma S(t)c_x(t, S(t))dW(t).
\end{aligned}
\tag{2.8}
$$

We examine what is required in order for (2.8) to hold. we first equate the $dW(t)$

terms in (2.8), which gives

$$\Delta(t) = c_x(t, S(t)) \ for \ all \ t \in [0, T]. \tag{2.9}$$

This is called the delta-hedging rule. At each time t prior to expiration, the number of shares held by the hedge of the short option position is the partial derivative with respect to the stock price of the option value at that time. This quantity, $c_x(t, S(t))$, is called the delta of the option. We next equate the dt terms in (2.8), using (2.9), to obtain

$$(\alpha - r)S(t)c_x(t, S(t))$$
$$= [-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) \tag{2.10}$$
$$+ \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t)) \ for \ all \ t \in [0, T].$$

The term $\alpha S(t)c_x(t, S(t))$ appears on both sides of (2.10), and after canceling it, we obtain

$$rc(t, S(t)) = c_t(t, S(t)) + rS(t)c_x(t, S(t))$$
$$+ \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t)) \ for \ all \ t \in [0, T]. \tag{2.11}$$

In conclusion, we should seek a continuous function $c(t, x)$ that is a solution to the Black-Scholes-Merton partial differential equation

$$c_t(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t, x) = rc(t, x) \ for \ all \ t \in [0, T), x \geq 0, \tag{2.12}$$

and that satisfies the terminal condition

$$c(T, x) = (x - K)^+. \tag{2.13}$$

Suppose we have found this function. If an investor starts with initial capital

$X(0) = c(0, S(0))$ and uses the hedge $\Delta(t) = c_x(t, S(t))$, then (2.8) will hold for all $t \in [0, T]$. Indeed, the $dW(t)$ terms on the left and right sides of (2.8) agree because $\Delta(t) = c_x(t, S(t))$, and the dt term agrees because (2.12) guarantee (2.11). Equality in (2.8) gives us (2.7). Canceling $X(0) = c(0, S(0))$ and e^{-rt} in this equation, we see that $X(t) = c(t, S(t))$ for all $t \in [0, T]$. Taking the limit as $t \uparrow T$ and using the fact that both $X(t)$ and $c(t, S(t)$ are continuous, we conclude that $X(T) = c(T, S(T)) = (S(T) - K)^+$. This means that the short position has been successfully hedged. No matter which of its possible paths the stock price follows, when the option expires, the agent hedging the short position has a portfolio whose value agrees with the option payoff. The Black-Scholes-Merton equation (2.12) does not involve probability. It is a partial differential equation, and the argument t and x are dummy variables, not random variables. One can solve it by partial equation methods.

The Black-Scholes-Merton formula is based on the geometric Brownian motion model for stock prices. However, no-arbitrage pricing theory has now moved far beyond this assumption. As studies been conducted by researchers, this theory and the accompanying risk-neutral pricing formula can be applied in the presence of a time-varying random volatility, a time-varying random mean rate of return, and a time-varying random interest rate. Many finance books, including Hull[9], Dothan[10], and Duffie[11], include sections on Ito's integrals and the Ito-Doeblin formula. Some other books on dynamic models in finance are Cox and Rubinstein[12], Huang and Litzenberger [13], Ingersoll [14], and Jarrow [15]. A comprehensive text is Wilmott [16]. Some good references for practitioners are Baxter and Rennie [17] (reviewed in [18]), Bjork [19] (reviewed in [20]), and Musiela and Rutkowski [21]. More mathematical texts on stochastic calculus with applications to finance are Lamberton and Lapeyre [22] and Steele [23]. Other texts on stochastic calculus are Chung and Williams [24], Karatzas and Shreve [25], 0ksendal [26], and Protter [27]. Karatzas and Shreve [28] is a sequel to [29]

that focuses on finance. Protter [27] is the easiest place to learn about stochastic calculus for processes with jumps, and this is not at all easy. No-arbitrage pricing theory and the accompanying risk-neutral pricing formula is predicated on the assumption that there is no arbitrage in the market. An arbitrage is defined to be a trading strategy which begins with zero capital and at a later time has positive capital with positive probability without having any risk of loss. An empirical study supporting the efficient market hypothesis is Fama [30], which also discusses distributions that fit stock prices better than geometric Brownian motion. A criticism of the efficient market hypothesis is provided by LeRoy [31], and a recent paper that finds long-range dependence (but not much) in stock price data is Willinger, Taqqu, and Teverovsky [32]. A provocative article on the source of stock price movements is due to Black [33]. Geometric fractional Brownian motion has been proposed as an alternative model for stock prices because it has fatter tails than the geometric Brownian motion. One can assume such a model and compute the prices of derivative securities as their expected discounted payoffs, but the model is inconsistent with the usual delta-hedging formula. Indeed, geometric fractional Brownian motion violates the efficient market hypothesis so strongly that it admits arbitrage (not just "statistical arbitrage" but actual arbitrage). An example of this is provided by Rogers [34]. Further examples of arbitrage and a market trading restriction that prevents arbitrage in such markets are provided by Cheridito [35]. Our research relaxes the risk-neutral pricing formula by taking consideration of a time-varying random volatility.

2.3 Expansion of Equilibrium Asset and Option Pricing under Jump Diffusion

A great deal of research has been carried out to model rare events as jumps. There is a large amount of literature on jump-diffusion methods in finance, including

the books by Cont and Tankov[36] and Kijima[37] and many papers such as the paper on equilibrium asset and option pricing under jump diffusion[38] and the paper An equilibrium model of rare-event premia and its implication for Option Smirks[39]. Studies on jump-diffusion including various aspects:

(i) In finance literature, equilibrium models are often built to explain the expected returns of the underlying assets, including index and individual stock returns. Generally, two ways are exploited to establish an equilibrium model. One of them is consumption-based asset pricing that is built on an exchange economy. With a stochastic aggregate endowment (dividend),a representative investor maximizes the expected utility by choosing an optimal level of consumption at each period .The other is production-based asset pricing that is built on a production economy. Having a production technology to grow its commodity stochastically and with a fixed amount of initial endowment, a representative investor choose an optimal level of consumption at each period and leaves the rest in the production to grow for the future consumption. Above are the two basic economical background in which jump diffusion problems are addressing.

(ii) Option pricing is another angle of jump diffusion related problems, e.g. Dynamic Derivate Strategies[40], Jump Diffusion Models for Asset Pricing in Financial Engineering[41] and so on. Branches are made under the main angle, including aspects of premiums, price kernel, stochastic variables, multivariate jump diffusion models and volatility smirks, etc.

(iii) Failing to conjecture the analytical formulas of the option pricing, numerical as well as other skills, involving Laplace transforms, physical, risk-neutral and martingale measure are also used in the process of finding the final expressions.

(IV) Dynamic derivatives strategies for modifying market incompleteness. Derivatives trading is now the world's biggest business, with an estimated daily turnover of over US 2.5 trillion and an annual growth rate of around 14 percentage. Despite increasing usage and growing interest; little is known about the optimal

trading strategies with derivatives as part of an investment portfolio. Indeed, academic studies on dynamic asset allocation typically exclude derivatives from the investment portfolios. In a complete market setting, such an exclusion can very well be justified by the fact that derivative securities are redundant.(e.g., Black and Scholes[42] and Cox and Ross[43]). When the completeness of the market breaks down either because of infrequent trading or by the presence of additional sources of uncertainty, it then becomes sub-optimal to exclude derivatives. Otherwise, further questions are arising: What are the optimal dynamic strategies for an investor who can control not just the holding in the aggregate stock market and a riskless bond, but also derivatives? How much can be benefit by including derivatives? Therefore, several new problems and difficulties expose in our model.

2.4 Optimal Asset portfolio under The Mean-variance Utility

Portfolio Optimization, an important topic in financial market, has been studied by a vast of researchers after the first publication by Markowitz[1]. Briefly speaking, portfolio optimization can be described as follows: with the given utility function and parameters, choose the ideal value of control parameters to maximize the final utility. Previous research works in this field can be cataloged by different utilities such as Mean-variance utility[44], endogenous habit formation[45], hyperbolic discounting[46], and the classic CRRA[38]. Among those utilities, optimal asset allocation under mean-variance is one of the very interesting and thought-provoking topic in the classic results of financial economics. Under this framework, we find that optimal asset allocation problems within multiple-period are time-inconsistent which prohibits the application of the classical Bellman optimality principal. Over the last couple of decades, various types

of utility functions have been used in financial modeling and portfolio optimization. In the work of[38], in developing an equilibrium asset and option pricing model, it is assumed that the representative investor seeks to maximize the expected value of the utility function over a period of time and a constant relative risk aversion (CRRA) utility function is used. An instantaneous utility function with habit forming performance is proposed in formulating the preference of the representative investor in [38]; while in[47], a mean-variance utility function is used for a dynamic asset allocation problem. In this paper, we focus on the mean-variance (MV) utility.

Commonly, two different approaches have been used to deal with this kind of time-inconsistent optimal control problems. One of the approaches is called pre-committed, namely the initial point $(0, x_0)$ is fixed and the control law $\mathbf{u}^*$ is chosen to maximize $Q(0, x_0, \mathbf{u})$. Here we simply regard that the optimal law we choose at initial time will be the best choice in the later time point (s, X_s) for the functional $Q(s, X_0, \mathbf{u})$. Kydland and Prescott[48] discuss the economical meaning of this pre-committed strategy. While Richardon[49], followed by Bajeux-Besnainou[50], is the first one to explore the portfolio optimization under mean-variance in continuous time even though the author just sets one single stock with a constant risk-free rate. By Li and Ng[51], the original MV problem can be transformed into a stochastic linear-quadratic control problem. Further research mainly focuses on model extensions and improvements[52-55], such as the inclusion of cost transaction by Dai, Xu, Zhou[52].

An alternative approach is to formulate the time inconsistent problem in the framework of game theory. Game theory approaches have a long history since Markowitz[1] first introduces the mean-variance portfolio optimization analysis. Along this road, Goldman[56], Krusell and Smith[57], Peleg and Yaari[58] and Pollak[59] have also formulated their problems within the game theory framework. Particularly, in the MV analysis, Basak and Chabakauri[44] firstly studied

the game theoretic approach towards portfolio optimization in continuous time. Later, Bjork and Murgoci[7] extend Basak's model by relaxing the risk aversion rate in mean-variance utility from a constant to a state dependent one. This is a very provoking development both from modeling and economic significance. More and more researchers are attracted to game theoretic approach because it is more rigorous.

With stochastic volatility asymptotics, for the multiscale stochastic volatility, as Chacko and Viceira[60] mentioned, from the estimation of reversion parameter Kappa, there seems to exist two volatility factors, with one being fast and the other one being slow, and both factors should be taken into consideration at the same time. Asymptotic analysis, which already has long history in dealing with the option pricing problem, has also been carried out in Fouque[61]. In the work of Fouque[61] and Sircar and Zariphopoulou [62], the authors use the multiscale stochastic volatility and asymptotic analysis within the power utilities framework. Based on previous research, in our work, we continue to explore the option assets portfolio under the Mean-variance utility by replacing the constant volatility in the Bjork's work by the multiscale stochastic volatility. For solving the problem, we use the asymptotic analysis to deal with the value function in the extended HJB equations which is also our contribution to this specific problem.

New and more rigorous way of dealing with the time-inconsistent problem is to put it into the game theoretic framework. Briefly speaking, the control process under multi-period time is like a game where the periods are treated as multiple players. Each player presents the accordingly present favor and the incarnations of future tastes. The game ends when we find the sub-game perfect Nash equilibrium point. Following the work by Markowitz[1] who first introduced the game theory approach into the MV portfolio optimization analysis, Goldman[63], Krusell and Smith[64], Peleg and Yaari[65], Pollak[66] and Vieille and Wweibull[67] solved the continuous and discrete time problems along the same line. Particularly,

for the continuous portfolio optimization problem, Basak and Chabakauri[44] firstly studied the game theoretic approach. And the development, both from the modeling and the actual economic meaning, is made by Bjork and Murgoci[7] through extending the results to other objective functions than the mean variance criterion. Along the work of Bjork and Murgoci[7], Bjork, Murgoci and Zhou[47] studied the mean-variance portfolio optimization by relaxing the constant risk aversion into a dynamic one which depends on the current wealth which is more economically reasonable.

2.5 Concluding Remarks

For the option pricing, in this thesis, we focus on studying the jump-diffusion model with stochastic volatility. Different from the Black Scholes framework, we use jump-diffusion to describe the price dynamics of the underlying asset. The market of our model is incomplete; that is, it is not possible to replicate the payoff of every contingent claim by a portfolio, and there are several equivalent martingale measures. A key issue is to choose a proper consistent pricing measure from the set of equivalent martingale measures. Duffie, Pan and Singleton[68], Chacko and Das[69] presented a transform analysis to price the valuation of options for affine jump-diffusions with stochastic volatility. Lorig and Lozano-Carbasse[70] studied option pricing in exponential Levy-type models with stochastic volatility and stochastic jump-intensity. Lewis[71] used Fourier transformation methods to obtain the transform-based solution of option price. We will thus extend the results of Zhang, Zhao and Chang[38] to the pricing kernel with stochastic volatility.

For the portfolio optimization, some significant work has been done and many results have been achieved[7][47]. However on previous work, some key parameters in the models such as interest rate and volatility are taken as con-

stants[47]. Hence, in the thesis, we relax both constant volatility and interest rate into stochastic ones. The parameters in the stochastic processes are determined through calibration of real financial market data which is more reasonable. Consequently, we will establish the general extended Hamilton-Jacobi-Bellman equation using the theory of time-inconsistent control by Bjork and Murgoci[7]. Some special cases where the risk aversion is proportioned to the wealth, will also be investigated.

CHAPTER 3

Pricing of American Put Option under A Jump-Diffusion Process with Stochastic Volatility in An Incomplete Market

3.1 General

In this Chapter, we propose an iterative method for pricing American options in an incomplete market when the dynamics of the risky underlying asset is driven by a jump diffusion process with stochastic volatility. By employing the risk-minimization criterion, we obtain the Radon-Nikodym derivative for the minimal martingale measure and inequalities for the American option in the form of a linear complementarity problem(LCP). Then, an iterative method is adopted to solve the LCP for American put options.

3.2 Formulation of the Model under Minimal Martingale Measure

Taking into account the jumps in the evolution process of the underlying asset, we introduce the following stochastic volatility model based on the work in

reference[73],

$$
\begin{cases}
\frac{dX_t}{X_{t^-}} = \mu dt + \sqrt{V_t}dW_t^1 + \int_{R_0}(y-1)\bar{N}(du,dy) \\
dV_t = \alpha(\varphi - V_t)dt + \sigma\sqrt{V_t}dW_t^2
\end{cases}
\tag{3}
$$

where $W_t = (W_t^1, W_t^2)$ defined on the filtered complete space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a $2-$dimensional standard Brownian motion with the correlation coefficient $dW_t^1 dW_t^2 = \rho dt, \rho \in [-1,1]$. The asset price has drift μ and $\bar{N}(du,dy) = N(du,dy) - v(dy)dt$ is the compensated jump measure, where $N(du,dy)$ is the jump measure and $v(dy)$ is the Lévy measure, σ governs the volatility of the variance process and α, φ are the mean reversion coefficients to ensure that the variance will drift back to φ at the rate of α.

According to H. Föllmer and M. Schweizer[74], the optimal hedging strategy can be computed with the minimal martingale measure and it has been proved that the minimal martingale measure could be uniquely determined. Using Theorem 3.1 in reference[73], the Radon-Nikodym derivative under the minimal martingale measure q is

$$
\begin{aligned}
Z_t =& exp\Big(-\int_0^t \theta_u\sqrt{V_u}dW_u^1 - \frac{1}{2}\int_0^t \theta_u^2 V_u du \\
&+ \int_0^t \int_{R_0} ln(1 - \theta_u(y-1))N(du,dy) + \int_0^t \int_{R_0} \theta_u(y-1)v(dy)du \Big)
\end{aligned}
\tag{3.1}
$$

where

$$
\theta_u = \frac{\mu_u - r_u}{V_u + \int_{R_0}(y-1)^2 v(dy)}
$$

and r_t is the risk-free rate of interest in the discounted risky asset price process $\hat{S}_t = e^{-\int_0^t r_u du} S_t$. Correspondingly, the Brownian motion and compensated Poisson process under the minimal martingale measure are derived respectively as follows,

$$
\tilde{W}_t^1 = W_t^1 + \int_0^t \theta_u \sqrt{V_u} du
\tag{3.2}
$$

$$\tilde{W}_t^2 = W_t^2 + \rho \int_0^t \theta_u \sqrt{V_u} du \tag{3.3}$$

$$\tilde{v}(dy)du = (1 - \theta_u(y-1))v(dy)du \tag{3.4}$$

$$\tilde{N}(du, dy) = N(du, dy) - \tilde{v}(dy)du \tag{3.5}$$

In our model, there are at least two dimensions of risks from the Brownian motion and the Lévy process. Thus the financial market is not complete and such a claim has an intrinsic risk. In this context, we can construct the pricing model under the risk minimization criterion by employing the minimal martingale measure method, and therefore equation 3 can be written as follows:

$$\begin{cases} \frac{dX_t}{X_{t-}} = \mu dt + \sqrt{V_t} d\tilde{W}_t^1 + \int_{R_0} (y-1)\tilde{N}(du, dy) - \theta_t V_t dt - \int_{R_0} \theta_t(y-1)^2 v(dy) dt \\ dV_t = \alpha(\varphi - V_t) dt + \sigma \sqrt{V_t} d\tilde{W}_t^2 - \rho\sigma\theta_t V_t dt \end{cases}$$

The value of the American put option $P(t, X_t, V_t)$ at time t with strike price K and maturity date T is

$$P(t, X_t, V_t) = \max_\tau \boldsymbol{E}^q [e^{-\int_t^\tau r_u du} (K - X_\tau)^+ | F_t] \tag{3.6}$$

Under the minimal martingale measure, we can obtain the following partial integro-differiential equation

$$\begin{aligned} \frac{\partial P}{\partial t} = & (r + K_0 - \theta_t K_1) P(t, X, V) - \frac{\partial P}{\partial X} (r - K_1 + \theta_t K_2) X \\ & - \frac{\partial P}{\partial V} (\alpha(\varphi - V) - \rho\sigma\theta_t V) - \frac{1}{2} \frac{\partial^2 P}{\partial X^2} V X^2 - \frac{\partial^2 P}{\partial X \partial V} X V \sigma\rho - \frac{1}{2} \frac{\partial^2 P}{\partial V^2} V \sigma^2 \\ & - \int_R P(t, Xy, V) \nu(dy) + \theta_t \int_R P(t, Xy, V)(y-1) \nu(dy) \end{aligned}$$

$$\tag{3.7}$$

where

$$K_0 = \int_{R_0} \nu(dy)$$

$$K_1 = \int_{R_0} (y-1)\nu(dy)$$

$$K_2 = \int_{R_0} (y-1)^2 \nu(dy)$$

Under Merton's jump diffusion model, the stochastic volatility V is from the log-normal distribution with the density

$$f(y) = \frac{1}{y\delta\sqrt{2\pi}} exp(-\frac{(\log y - \gamma)^2}{2\delta^2}) \tag{3.8}$$

Thus, the notations under the log-normal density functions above are $K_0 = 1, K_1 = e^{\delta/2+\gamma} - 1$ and $K_2 = \frac{1}{\sqrt{1-2\delta^2}} e^{\gamma^2/(1-2\delta^2)} - 2e^{\delta/2+\gamma} + 1$.

Now we take the time transform $\tau = T - t$ and denote the operator by

$$
\begin{aligned}
L\tilde{P} =\ & -(r + K_0 - \theta K_1)\tilde{P}(\tau, X, V) + \frac{\partial \tilde{P}}{\partial X}(r - K_1 + \theta K_2)X \\
& + \frac{\partial \tilde{P}}{\partial V}(\alpha(\varphi - V) + \rho\sigma\theta V) + \frac{1}{2}\frac{\partial^2 \tilde{P}}{\partial X^2}VX^2 + \frac{\partial^2 \tilde{P}}{\partial X\partial V}XV\sigma\rho + \frac{1}{2}\frac{\partial^2 \tilde{P}}{\partial V^2}V\sigma^2 \\
& + \int_R \tilde{P}(\tau, Xy, V)\nu(dy) - \theta \int_R \tilde{P}(\tau, Xy, V)(y-1)\nu(dy)
\end{aligned}
\tag{3.9}
$$

Then, the value of an American put option satisfies the following linear complementarity problem(LCP)

$$
\begin{cases}
\frac{\partial \tilde{P}}{\partial \tau} - L\tilde{P}_\tau \geq 0, \quad \tilde{P}_\tau \geq (K - X_\tau)^+ \\
(\frac{\partial \tilde{P}}{\partial \tau} - L\tilde{P}_\tau)[\tilde{P}_\tau - (K - X_\tau)^+] = 0
\end{cases}
$$

with the boundary conditions

$$\tilde{P}(\tau, 0, V) = K \tag{3.10}$$

$$\frac{\partial \tilde{P}(\tau, \bar{X}, V)}{\partial X} = 0 \tag{3.11}$$

$$\tilde{P}(\tau, X, 0) = (K - X)^+ \tag{3.12}$$

$$\frac{\partial \tilde{P}(\tau, X, \bar{V})}{\partial V} = 0 \tag{3.13}$$

and the initial condition

$$\tilde{P}(0, X, V) = (K - X)^+. \tag{3.14}$$

3.3 Numerical Solution

We use the uniform grid $\Delta\tau = \Gamma/n$, $\Delta x = \bar{X}/m$, $\Delta v = \bar{V}/l$ on the domain $[0, \Gamma] \times [0, \bar{X}] \times [0, \bar{V}]$. Denoting $\tilde{P}_{i,j} = \tilde{P}(\tau, i\Delta x, j\Delta v)$ and using the central difference scheme, we have

$$\frac{\partial \tilde{P}}{\partial x}(\tau, x_i, v_j) = \frac{\tilde{P}_{i+1,j}(\tau) - \tilde{P}_{i-1,j}(\tau)}{2\Delta x} \tag{3.15}$$

$$\frac{\partial^2 \tilde{P}}{\partial x^2}(\tau, x_i, v_j) = \frac{\tilde{P}_{i+1,j}(\tau) - 2\tilde{P}_{i,j}(\tau) + \tilde{P}_{i-1,j}(\tau)}{(\Delta x)^2} \tag{3.16}$$

$$\frac{\partial \tilde{P}}{\partial v}(\tau, x_i, v_j) = \frac{\tilde{P}_{i,j+1}(\tau) - \tilde{P}_{i,j-1}(\tau)}{2\Delta v} \tag{3.17}$$

$$\frac{\partial^2 \tilde{P}}{\partial v^2}(\tau, x_i, v_j) = \frac{\tilde{P}_{i,j+1}(\tau) - 2\tilde{P}_{i,j}(\tau) + \tilde{P}_{i,j-1}(\tau)}{(\Delta v)^2} \tag{3.18}$$

$$\frac{\partial^2 \tilde{P}}{\partial x \partial v}(\tau, x_i, v_j) = \frac{\tilde{P}_{i+1,j+1}(\tau) + \tilde{P}_{i-1,j-1}(\tau) - \tilde{P}_{i+1,j-1}(\tau) - \tilde{P}_{i-1,j+1}(\tau)}{4\Delta x \Delta v} \tag{3.19}$$

$$D_{i,j} = -\frac{ij\sigma\rho}{4}\tilde{P}_{i-1,j-1} + \frac{\alpha(\varphi - j\Delta v) - \rho\sigma\theta_t j\Delta v - j\sigma^2}{2\Delta v}\tilde{P}_{i,j-1} + \frac{ij\sigma\rho}{4}\tilde{P}_{i+1,j-1}$$

$$+ \frac{i(r - k_1 + \theta_t k_2) - j\Delta vi^2}{2}\tilde{P}_{i-1,j} + (r + k_0 - \theta_t k_1 + j\Delta vi^2 + \frac{j\sigma^2}{\Delta v})\tilde{P}_{i,j}$$

$$- \frac{i(r - k_1 + \theta_t k_2) - j\Delta vi^2}{2}\tilde{P}_{i+1,j}$$

$$- \frac{ij\sigma\rho}{4}\tilde{P}_{i-1,j+1} - \frac{\alpha(\varphi - j\Delta v) - \rho\sigma\theta_t j\Delta v - j\sigma^2}{2\Delta v}\tilde{P}_{i,j+1} - \frac{ij\sigma\rho}{4}\tilde{P}_{i+1,j+1}$$

$$(3.20)$$

For simplicity in presentation, we write $a_{ij} = -\frac{ij\sigma\rho}{4}$, $b_{ij} = \frac{\alpha(\varphi - j\Delta v) - \rho\sigma\theta_t j\Delta v - j\sigma^2}{2\Delta v}$,

$c_{ij} = \frac{i(r - k_1 + \theta_t k_2) - j\Delta vi^2}{2}$,

$d_{ij} = r + k_0 - \theta_t k_1 + j\Delta vi^2 + \frac{j\sigma^2}{\Delta v}$

$$\boldsymbol{D}_{1,j-1} = \begin{pmatrix} b_{0,j-1} & -a_{1,j-1} & 0 & 0 & \dots & 0 & 0 & 0 \\ a_{0,j-1} & b_{1,j-1} & -a_{2,j-1} & 0 & \dots & 0 & 0 & 0 \\ 0 & a_{1,j-1} & b_{2,j-1} & -a_{3,j-1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{m-2,j-1} & b_{m-1,j-1} & -a_{m,j-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_{m-1,j-1} & b_{m,j-1} \end{pmatrix}$$

$$\boldsymbol{D}_{2,j} = \begin{pmatrix} d_{0,j} & -c_{1,j} & 0 & 0 & \dots & 0 & 0 & 0 \\ c_{0,j} & d_{1,j} & -c_{2,j} & 0 & \dots & 0 & 0 & 0 \\ 0 & c_{1,j} & d_{2,j} & -c_{3,j} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & c_{m-2,j} & d_{m-1,j} & -c_{m,j} \\ 0 & 0 & 0 & 0 & \dots & 0 & c_{m-1,j} & d_{m,j} \end{pmatrix}$$

$$\boldsymbol{D}_{3,j+1} = \begin{pmatrix} b_{0,j+1} & a_{1,j+1} & 0 & 0 & \ldots & 0 & 0 & 0 \\ a_{0,j+1} & b_{1,j+1} & a_{2,j+1} & 0 & \ldots & 0 & 0 & 0 \\ 0 & a_{1,j+1} & b_{2,j+1} & a_{3,j+1} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_{m-2,j+1} & b_{m-1,j+1} & a_{m,j+1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & a_{m-1,j+1} & b_{m,j+1} \end{pmatrix}$$

Therefore, the coefficient matrix of the operator LP without integrals is given by

$$\boldsymbol{D} = \begin{pmatrix} D_{2,0} & D_{3,1} & 0 & 0 & \ldots & 0 & 0 & 0 \\ D_{1,0} & D_{2,1} & D_{3,2} & 0 & \ldots & 0 & 0 & 0 \\ 0 & D_{1,1} & D_{2,2} & D_{3,3} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & D_{1,l-2} & D_{2,l-1} & D_{3,l} \\ 0 & 0 & 0 & 0 & \ldots & 0 & D_{1,l-1} & D_{2,l} \end{pmatrix}$$

Let $z = Sy$, then

$$I_1(\tau, Xy, V) = \int_0^\infty \tilde{P}(\tau, Xy, V) f(y) dy = \int_0^\infty \tilde{P}(\tau, z, V) f(z/X)/X dz \quad (3.21)$$

$$I_2(\tau, Xy, V) = \int_0^\infty \tilde{P}(\tau, Xy, V)(y-1) f(y) dy = \int_0^\infty \tilde{P}(\tau, z, V)(z/X-1) f(z/X)/X dz \quad (3.22)$$

Using integral discretization by applying linear interpolation, we get

$$I_{1(i,j)}^\tau = \int_0^\infty \tilde{P}(\tau, z, j\Delta v) f(z/i\Delta x)/(i\Delta x) dz \approx \sum_{k=0}^{m-1} A_{i,j}^{\tau,k} \quad (3.23)$$

where

$$A_{i,j}^{\tau,k} = \int_{k\Delta x}^{(k+1)\Delta x} \tilde{P}(\tau, z, j\Delta v) f(z/i\Delta x)/(i\Delta x) dz$$

$$= \int_{k\Delta x}^{(k+1)\Delta x} [\frac{(k+1)\Delta x - z}{\Delta x} \tilde{P}(\tau, k\Delta x, j\Delta v)$$

$$+ \frac{z - k\Delta x}{\Delta x} \tilde{P}(\tau, (k+1)\Delta x, j\Delta v)] f(z/i\Delta x)/(i\Delta x) dz$$

$$= \frac{1}{2}[k\tilde{P}(\tau, (k+1)\Delta x, j\Delta v) - (k+1)\tilde{P}(\tau, k\Delta x, j\Delta v)] \qquad (3.24)$$

$$\times [erf(\frac{\mu - \ln((k+1)/i)}{\sqrt{2}\sigma}) - erf(\frac{\mu - \ln(k/i)}{\sqrt{2}\sigma})]$$

$$+ \frac{1}{2}i\exp(\sigma^2/2 + \mu)[\tilde{P}(\tau, k\Delta x, j\Delta v) - \tilde{P}(\tau, (k+1)\Delta x, j\Delta v)]$$

$$\times [erf(\frac{\mu - \ln((k+1)/i) + \sigma^2}{\sqrt{2}\sigma}) - erf(\frac{\mu - \ln(k/i) + \sigma^2}{\sqrt{2}\sigma})]$$

Let $e_{1i}^k = \frac{1}{2}(erf(\frac{\mu - \ln((k+1)/i)}{\sqrt{2}\sigma}) - erf(\frac{\mu - \ln(k/i)}{\sqrt{2}\sigma}))$,

and $e_{2i}^k = \frac{1}{2}i\exp(\sigma^2/2 + \mu)(erf(\frac{\mu - \ln((k+1)/i) + \sigma^2}{\sqrt{2}\sigma}) - erf(\frac{\mu - \ln(k/i) + \sigma^2}{\sqrt{2}\sigma}))$.

We rewrite (3.24) as follows

$$A_{i,j}^{\tau,k} = (e_{2i}^k - (k+1)e_{1i}^k)\tilde{P}(\tau, k\Delta x, j\Delta v) + (ke_{1i}^k - e_{2i}^k)\tilde{P}(\tau, (k+1)\Delta x, j\Delta v) \quad (3.25)$$

$$I_{1(i,j)}^\tau \approx \sum_{k=0}^{m-1} A_{i,j}^{\tau,k} = e_{2,i}^0 - 2e_{1,i}^0 + \sum_{k=1}^{m-1}[(k-1)e_{1,i}^{k-1} - e_{2,i}^{k-1} + e_{2,i}^k - (k+1)e_{1,i}^k] + (m-1)e_{1,i}^{m-1} - e_{2,i}^{m-1}$$

$$(3.26)$$

$$\boldsymbol{A}_0 = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \dots & a_{0,m-1} & a_{0,m} \\ a_{1,0} & a_{1,1} & a_{1,2} & \dots & a_{1,m-1} & a_{1,m} \\ a_{2,0} & a_{2,1} & a_{2,2} & \dots & a_{2,m-1} & a_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m,0} & a_{m,1} & a_{m,2} & \dots & a_{m,m-1} & a_{m,m} \end{pmatrix}$$

where $a_{i,0} = e_{2,i}^0 - e_{1,i}^0$, $a_{i,k} = (k-1)e_{1,i}^{k-1} - e_{2,i}^{k-1} + e_{2,i}^k - (k+1)e_{1,i}^k$, $k = 1, 2, \dots, m-1$ and $a_{i,m} = (m-1)e_{1,i}^{m-1} - e_{2,i}^{m-1}$, $i = 0, 1, \dots, m$.

$$\boldsymbol{A} = \begin{pmatrix} A_0 & 0 & \dots & 0 \\ 0 & A_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{pmatrix}$$

In a similar way, we can obtain

$$I^\tau_{2(i,j)} = \int_0^\infty \tilde{P}(\tau, z, j\Delta v)(z/(i\Delta x) - 1)f(z/(i\Delta x))/(i\Delta x)dz \approx \sum_{k=0}^{m-1} B^{\tau,k}_{i,j} \quad (3.27)$$

where

$$\begin{aligned}
B^{\tau,k}_{i,j} &= \int_{k\Delta x}^{(k+1)\Delta x} \tilde{P}(\tau, z, j\Delta v)(z/(i\Delta x) - 1)f(z/i\Delta x)/(i\Delta x)dz \\
&= \int_{k\Delta x}^{(k+1)\Delta x} [\frac{(k+1)\Delta x - z}{\Delta x}\tilde{P}(\tau, k\Delta x, j\Delta v) \\
&\quad + \frac{z - k\Delta x}{\Delta x}\tilde{P}(\tau, (k+1)\Delta x, j\Delta v)](z/(i\Delta x) - 1)f(z/i\Delta x)/(i\Delta x)dz \\
&= \frac{1}{2}\exp(\sigma^2/2 + \mu)[k\tilde{P}(\tau, (k+1)\Delta x, j\Delta v) - (k+1)\tilde{P}(\tau, k\Delta x, j\Delta v)] \\
&\quad \times [erf(\frac{\mu - \ln((k+1)/i) + \sigma^2}{\sqrt{2}\sigma}) - erf(\frac{\mu - \ln(k/i) + \sigma^2}{\sqrt{2}\sigma})] \\
&\quad + \frac{1}{2i\sqrt{1 - 2\sigma^2}}\exp\frac{\mu^2}{1 - 2\sigma^2}[\tilde{P}(\tau, k\Delta x, j\Delta v) - \tilde{P}(\tau, (k+1)\Delta x, j\Delta v)] \\
&\quad \times [erf(\frac{\sqrt{1 - 2\sigma^2}(\mu - \ln((k+1)/i))}{\sqrt{2}\sigma} - \frac{\sqrt{2}\mu\sigma}{\sqrt{1 - 2\sigma^2}}) \\
&\quad - erf(\frac{\sqrt{1 - 2\sigma^2}(\mu - \ln(k/i))}{\sqrt{2}\sigma} - \frac{\sqrt{2}\mu\sigma}{\sqrt{1 - 2\sigma^2}})] - A^{\tau,k}_{i,j}
\end{aligned}$$

$$(3.28)$$

Let $e^k_{3i} = \frac{1}{2}\exp(\sigma^2/2 + \mu) \times [erf(\frac{\mu - \ln((k+1)/i) + \sigma^2}{\sqrt{2}\sigma}) - erf(\frac{\mu - \ln(k/i) + \sigma^2}{\sqrt{2}\sigma})]$ and $e^k_{4i} =$

$\frac{1}{2i\sqrt{1 - 2\sigma^2}}\exp\frac{\mu^2}{1 - 2\sigma^2} \times [erf(\frac{\sqrt{1 - 2\sigma^2}(\mu - \ln((k+1)/i))}{\sqrt{2}\sigma} - \frac{\sqrt{2}\mu\sigma}{\sqrt{1 - 2\sigma^2}}) -$

$erf(\frac{\sqrt{1 - 2\sigma^2}(\mu - \ln(k/i))}{\sqrt{2}\sigma} - \frac{\sqrt{2}\mu\sigma}{\sqrt{1 - 2\sigma^2}})]$

We use the notation above and then rewrite (3.28) as follows

$$B_{i,j}^{\tau,k} = (e_{4i}^k - (k+1)e_{3i}^k)\tilde{P}(\tau, k\Delta x, j\Delta v) + (ke_{3i}^k - e_{4i}^k)\tilde{P}(\tau, (k+1)\Delta x, j\Delta v) - A_{i,j}^{\tau,k}$$

$$(3.29)$$

$$I_{2(i,j)}^\tau \approx \sum_{k=0}^{m-1} B_{i,j}^{\tau,k} = e_{4i}^0 - 2e_{3i}^0 - e_{2i}^0 - 2e_{1i}^0$$

$$+ \sum_{k=1}^{m-1} [(k-1)e_{3,i}^{k-1} - e_{4,i}^{k-1} + e_{4,i}^k - (k+1)e_{3,i}^k - (k-1)e_{1,i}^{k-1} + e_{2,i}^{k-1}$$

$$- e_{2,i}^k + (k+1)e_{1,i}^k] + (m-1)e_{3,i}^{m-1} - e_{4,i}^{m-1} - (m-1)e_{1,i}^{m-1} + e_{2,i}^{m-1}$$

$$(3.30)$$

$$\boldsymbol{B_0} = \begin{pmatrix} b_{00} & b_{01} & b_{02} & \ldots & b_{0,m-1} & b_{1,m} \\ b_{10} & b_{11} & b_{12} & \ldots & b_{1,m-1} & b_{1,m} \\ b_{20} & b_{21} & b_{22} & \ldots & b_{2,m-1} & b_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m0} & b_{m1} & b_{m2} & \ldots & b_{m,m-1} & b_{m,m} \end{pmatrix}$$

where $b_{i0} = e_{4i}^0 - 2e_{3i}^0 - e_{2i}^0 - 2e_{1i}^0$,

$b_{i,k} = (k-1)e_{3,i}^{k-1} - e_{4,i}^{k-1} + e_{4,i}^k - (k+1)e_{3,i}^k - (k-1)e_{1,i}^{k-1} + e_{2,i}^{k-1} - e_{2,i}^k$

$+ (k+1)e_{1,i}^k, k = 1, 2, \ldots, m-1$,

and $b_{im} = (m-1)e_{3,i}^{m-1} - e_{4,i}^{m-1} - (m-1)e_{1,i}^{m-1} + e_{2,i}^{m-1}, i = 0, 1, \ldots, m$.

$$\boldsymbol{B} = \begin{pmatrix} B_0 & 0 & \ldots & 0 \\ 0 & B_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_0 \end{pmatrix}$$

Thus, the integral part of the operator $L\tilde{P}$ is denoted by $\boldsymbol{R} = \boldsymbol{A} - \theta_\tau \boldsymbol{B}$.

The finite difference space discretization leads to a semi-discrete LCP

$$\begin{cases} \frac{\partial \tilde{\boldsymbol{P}}}{\partial \tau} - \boldsymbol{C}\tilde{\boldsymbol{P}} \geq 0, & \tilde{\boldsymbol{P}} \geq \boldsymbol{g} \\ (\frac{\partial \tilde{\boldsymbol{P}}}{\partial \tau} - \boldsymbol{C}\tilde{\boldsymbol{P}})^T (\tilde{\boldsymbol{P}} - \boldsymbol{g}) = 0 \end{cases}$$

where $\boldsymbol{C}$ is an $(m+1)(l+1) \times (m+1)(l+1)$ matrix and $\boldsymbol{P}$ is a vector of length $(m+1)(l+1)$.

$$\boldsymbol{C} = \boldsymbol{D} - \boldsymbol{R}$$

$$\boldsymbol{P} = (P_{00} \quad P_{10} \quad P_{20}\ldots P_{m0} \quad P_{01} \quad P_{11} \quad P_{21}\ldots P_{m1}\ldots P_{0l} \quad P_{1l} \quad P_{2l}\ldots P_{ml})^T.$$

Now for American options we obtain a semi-discrete LCP. We use Crank-Nicolson method for time discretization. Let $\Delta\tau = \Gamma/n$, we then obtain

$$(\boldsymbol{I} + \frac{1}{2}\Delta\tau \boldsymbol{C}^{(n)})\tilde{\boldsymbol{P}}^{(n)} = (\boldsymbol{I} - \frac{1}{2}\Delta\tau \boldsymbol{C}^{(n)})\tilde{\boldsymbol{P}}^{(n+1)} \qquad (3.31)$$

$$\begin{cases} (\boldsymbol{I} + \frac{1}{2}\Delta\tau \boldsymbol{C}^{(n)})\tilde{\boldsymbol{P}}^{(n)} \geq (\boldsymbol{I} - \frac{1}{2}\Delta\tau \boldsymbol{C}^{(n)})\tilde{\boldsymbol{P}}^{(n+1)}, & \tilde{\boldsymbol{P}}^{(k)} \geq \boldsymbol{g} \\ ((\boldsymbol{I} + \frac{1}{2}\Delta\tau \boldsymbol{C}^{(n)})\tilde{\boldsymbol{P}}^{(n)} - (\boldsymbol{I} - \frac{1}{2}\Delta\tau \boldsymbol{C}^{(n)})\tilde{\boldsymbol{P}}^{(n+1)})^T (\tilde{\boldsymbol{P}}^{(n)} - \boldsymbol{g}) = 0 \end{cases}$$

To simplify notation, we denote the problem above by

$$LCP(\boldsymbol{M}, \tilde{\boldsymbol{P}}^{(n+1)}, \boldsymbol{N}\tilde{\boldsymbol{P}}^{(n)}, \boldsymbol{g}) \qquad (3.32)$$

where $\boldsymbol{M} = \boldsymbol{I} + \frac{1}{2}\Delta\tau \boldsymbol{C}^{(n)}$ and $\boldsymbol{N} = \boldsymbol{I} - \frac{1}{2}\Delta\tau \boldsymbol{C}^{(n)}$.

The projected SOR algorithm PSOR($\boldsymbol{M}, \tilde{\boldsymbol{P}}^{(n+1)}, \boldsymbol{N}\tilde{\boldsymbol{P}}^{(n)}, \boldsymbol{g}$) can then be used to solve the LCP problem. The algorithm is as follows

Do $n = 1, \dim(\boldsymbol{M})$

$$\boldsymbol{r}_n = \boldsymbol{N}\tilde{\boldsymbol{P}}_n - \sum_j \boldsymbol{M}_{i,j}\tilde{\boldsymbol{P}}_j$$

$$\tilde{\boldsymbol{P}}_n = \max\{\tilde{\boldsymbol{P}}_j + \omega\boldsymbol{r}_n/\boldsymbol{M}_{i,j}, \boldsymbol{g}_n\}$$

End do.

where the relaxation parameter $\omega = \frac{2}{1+\sqrt{1-\varphi^2}}$, wherein $\varphi = \max_i\{\frac{1}{\boldsymbol{M}_{i,i}} \sum_{j \neq i} |\boldsymbol{M}_{i,j}|\}$.

3.4 Results and Discussion

In this section, we report computational results pertaining to the finite difference schemes and numerical solutions of the discretized LCPs by adopting the basic set of appropriate parameter values of the model shown in the following table 3.1.

Table 3.1: Basic Set of Appropriate Parameter Values of The Model

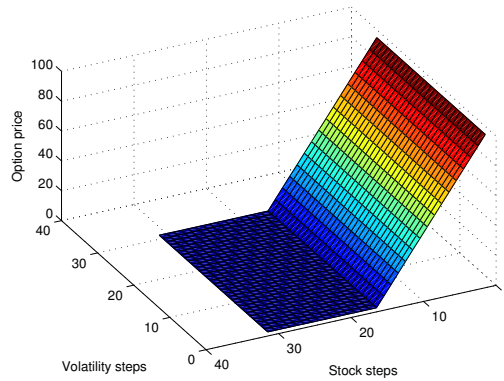| μ	0.23	r	0.039	ρ	-0.82	K	100
κ	3.46	σ	0.14	T	1	N	3000
φ	0.0894^2	λ	0.77	S_{max}	200	I	30
δ	0.0001	γ	-0.086	v_{max}	1	J	30

Figure 3.1: The payoff with the price and the volatility

Figure 1 shows the payoff with the price and the volatility.And the solution of equation (3.32). All numerical experiments have been implemented by the MATLAB2012b software on a 2.0-GHz Intel Core PC.

We also compute the value of the American option price under various values of volatility. Other parameters are as given in Table 1. The results are plotted in Figure 3.3.
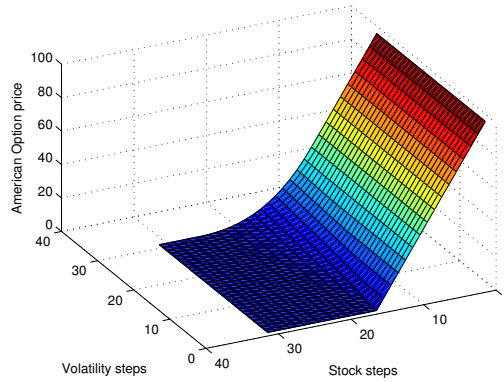
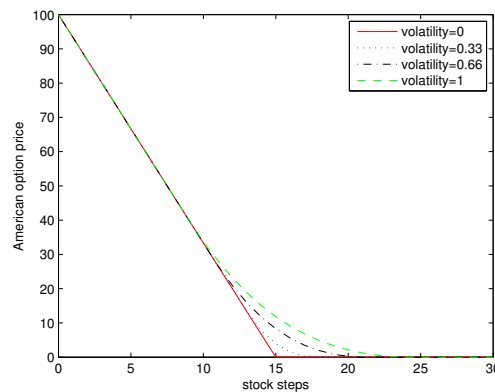Figure 3.2:   The payoff with the price and the volatility

Figure 3.3:   American option price under different volatilities

3.5   Concluding Remarks

In this chapter, an iterative method has been proposed for pricing American options in an incomplete market when the dynamics of the risky underlying asset driven by a jump diffusion with stochastic volatility. By employing the risk-minimization criterion, we obtain the Radon-Nikodym derivative for the minimal martingale measure and the inequalities of American option in the form of a linear complementarity problem(LCP). Then, an iterative method is adopted to solve the LCP for American put option. Our numerical results show that the model and numerical scheme are robust in capturing the feature of incomplete finance market, particularly the influence of market volatility on the price of American

options.

CHAPTER 4

Equilibrium Asset and Option Pricing under a Jump Dissusion Model with Stochastic Volatility

4.1 General

In this chapter, we study the equity premium and option pricing under a jump diffusion model with stochastic volatility based on the model in [38]. We obtain the pricing kernel which acts like the physical and risk-neutral densities and the moments in the economy. Moreover, the exact expression of option valuation is derived by the Fourier transformation method. We also discuss the relationship of central moments between the physical measure and the risk-neutral measure. Our numerical results show that our model is more realistic than the previous model.

4.2 The Wealth Evolution Model and Utility Function

In this paper, we consider the financial market with the following two basic assets,

(1) A Bond whose price R_t at time t is given by

$$dR_t = rR_t dt, R_0 = 1;$$

(2) A Stock whose price S_t at time t is given by

$$
\begin{cases}
\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t}dW_t^S + (e^x - 1)dN_t - \lambda E(e^x - 1)dt, S_0 > 0 \\
dV_t = \kappa(\theta - V_t)dt + \epsilon\sqrt{V_t}dW_t^V, V_0 > 0
\end{cases}
\quad (4-1)
$$

where $t \in [0, T]$ and $T > 0$; on the filtered complete space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, there are W_t^S, W_t^V which both are 1-dimensional Brownian motions with $dW_t^S dW_t^V = \rho dt$, and dN_t is a Poisson process with the constant jump intensity λ, $E[\cdot]$ is the expectation under the physical measure. The jump size x is stochastic and r, μ, κ, q are constants. The integration form of stock process in (4) is given by

$$In\frac{S_T}{S_t} = \int_t^T (\mu - \frac{1}{2}V_\tau)d\tau - [\lambda E(e^x - 1)]\tau + \int_t^T \sqrt{V_\tau}dW_\tau^S + \sum_{i=1}^{N_\tau} x_i \quad (4.1)$$

We suppose that the portfolio is $(\omega, 1 - \omega)$ which respectively means the fraction of wealth invested in the stock and the money market, then the wealth process W(t) and the initial wealth $X(0) = X_0 > 0$ satisfy the equations as follows,

$$
\begin{cases}
\frac{dX_t}{X_t} = [r + \omega\phi - \omega\lambda E(e^x - 1)]dt + \omega\sqrt{V_t}dW_t^S + \omega(e^x - 1)dN_t \\
dV_t = \kappa(\theta - V_t)dt + \epsilon\sqrt{V_t}dW_t^V
\end{cases}
\quad (4-2)
$$

where $\phi = \mu - r$ is the equity premium.

The representative investor maximizes his/her expected utility,

$$J(X(t), V(t), t) = max E_t[U(X(T))], \tag{4.2}$$

where $E_t[\cdot]$ is the conditional expectation and equals $E[\cdot | \mathcal{F}_t]$.

For tractability, we concentrate our attention on the case of constant relative risk aversion (CRRA) utility function:

$$U(X_T, T) = \frac{X_T^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 1, \tag{4.3}$$

where the constant γ is the relative risk aversion coefficient.

4.3 Equity Premium

The equity premium is very important for option pricing in the general equilibrium framework. Following the idea of Santa-Clara and Yan[75] and Zhang, Zhao and Chang[38], we obtain the equilibrium equity premium by modeling the general equilibrium economy in the following proposition.

Proposition 4.2 In general equilibrium framework, the equilibrium equity premium is given by

$$\phi = V_t[\gamma - \epsilon\rho A(t)] + \lambda E[(1 - e^{-\gamma x})(e^x - 1)] \tag{4.4}$$

where

$$A(t) = \gamma(1 - \gamma)\frac{1 - e^{-\alpha(t-T)}}{\beta_- - \beta_+ e^{-\alpha(t-T)}}, \beta_\pm = \kappa \pm \alpha, \alpha = \sqrt{\kappa^2 - \epsilon^2\gamma(1-\gamma)}$$

Proof From the optimal control problem (4.2), we get the Bellman equation as follows.

$$0 = J_t + \mathcal{L}(J) \tag{4.5}$$

where

$$\mathcal{L}(J) = [r + \omega\phi - \omega\lambda E(e^x - 1)]XJ_X + \frac{1}{2}\omega^2 X^2 V J_{XX} + \kappa(\theta - V)J_V +$$
$$\frac{1}{2}\epsilon^2 V J_{VV} + \epsilon\omega XV\rho J_{XV} + \lambda E[J(X(1 + \omega(e^x - 1)), V, t) - J]$$

Equating the derivatives of the Bellman equation (4.5) with respect to X to zero, we have the following equation.

$$0 = [\phi - \lambda E(e^x - 1)]XJ_X + \omega^2 X^2 V J_{XX} + \epsilon\omega XV\rho J_{XV} +$$
$$\lambda E[J(X(1 + \omega(e^x - 1)), V, t)X(e^x - 1)] \tag{4.6}$$

In equilibrium, the money market is in zero net supply. Therefore, the representative investor holds all the wealth in the stock market, that is $\omega = 1$. Then we can get the expression of ϕ from equation (4.6).

$$\phi = -V\frac{XJ_{XX}}{J_X} + \lambda E(e^x - 1) - \epsilon V\rho\frac{J_{XV}}{J_X} - \frac{\lambda}{J_X}E[J_X(Xe^x, V, t)(e^x - 1) \tag{4.7}$$

Then Bellman equation (4.5) can be written as follows.

$$0 = J_t + rXJ_X - \frac{1}{2}X^2 V J_{XX} + \kappa(\theta - V)J_V + \frac{1}{2}\epsilon^2 V J_{VV} -$$
$$\lambda X E[J_X(Xe^x, V, t)(e^x - 1)] + \lambda E[J(Xe^x, V, t)] - \lambda J \tag{4.8}$$

From (4.3), we conjecture that

$$J(X_t, V, t) = exp(A(t)V_t + B(t))\frac{X_t^{1-\gamma}}{1 - \gamma} \tag{4.9}$$

Then, substituting (4.6) into (4.9), we obtain

$$0 = \dot{A}(t)V + \dot{B}(t) + r(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)V + \kappa(\theta-V)A(t) + \frac{1}{2}\epsilon^2 V A^2(t) +$$
$$\lambda\gamma E[e^{(1-\gamma)x}] + \lambda(1-\gamma)E[e^{-\gamma x}] - \lambda$$

This leads to a system of two ordinary differential equations (ODEs),

$$\begin{cases} \dot{A}(t) + \frac{1}{2}\epsilon^2 A^2(t) - \kappa A(t) + \frac{1}{2}\gamma(1-\gamma) = 0 \\ A(T) = 0 \end{cases} \qquad (4-3)$$

and

$$\begin{cases} \dot{B}(t) + \kappa\theta A(t) + a = 0 \\ B(T) = 0 \end{cases} \qquad (4-4)$$

where

$$a = r(1-\gamma) + \lambda\gamma E[e^{(1-\gamma)x}] + \lambda(1-\gamma)E[e^{-\gamma x}] - \lambda$$

The systems (4-3) and (4-4) can be solved explicitly. First, we solve the first ODE (4**), which is the Riccati differential equation. Making the substitution

$$A(t) = \frac{2}{\epsilon^2}\frac{C'(t)}{C(t)},$$

we obtain the second order differential equation

$$\begin{cases} C''(t) + \kappa C'(t) + \frac{\epsilon^2}{4}\gamma(1-\gamma)C(t) = 0 \\ C'(T) = 0 \end{cases} \qquad (4-5)$$

A general solution has the form

$$C(t) = d_+ e^{\frac{1}{2}\beta_+ t} + d_- e^{\frac{1}{2}\beta_- t}$$

where

$$\beta_\pm = \kappa + \alpha, \alpha = \sqrt{\kappa^2 - \epsilon^2 \gamma(1 - \gamma)}$$

$$d_+ = \frac{\beta_- C(T)}{(\beta_- - \beta_+)e^{\frac{1}{2}\beta_+ T}}, d_- = \frac{\beta_+ C(T)}{(\beta_+ - \beta_-)e^{\frac{1}{2}\beta_- T}}$$

thus,

$$A(t) = \gamma(1 - \gamma)\frac{1 - e^{-\alpha(t-T)}}{\beta_- - \beta_+ e^{-\alpha(t-T)}}$$

Then, the solution of the second ODE (4-4) is

$$B(t) = \frac{2\kappa\theta}{\epsilon^2}[\frac{1}{2}\beta_+(t - T) + ln\frac{\beta_+ e^{-\alpha(t-T)} - \beta_-}{2\alpha}] + a(t - T)$$

Substituting (4.9) into (4.7), we get Proposition 4.1.

Remark 4.1 In the special case where there is no stochastic volatility and jumps, $V_t = \sigma^2$, $\epsilon = \kappa = 0$, and consequently $\phi = \sigma^2\gamma$ which is constant in Merton[40]. In the special case where there is no stochastic volatility, $\epsilon = 0$, and consequently $\phi = \sigma^2\gamma + \lambda E[(1 - e^{-\gamma x})(e^x - 1)]$ which is constant in Zhang, Zhao and Chang[38].

4.4   Option Pricing

In this section, we will study the pricing kernel and the option pricing in the general equilibrium framework. We first derive the pricing kernel which acts like the physical and risk-neutral densities in the economy and is the key to obtain the PDE of option price as follows.

Proposition 4.1 In the general equilibrium framework, the pricing kernel is given in differential form by

$$\frac{d\pi_t}{\pi_t} = -rdt - (\gamma - \epsilon\rho A)\sqrt{V_t}dW_t^S + (e^y - 1)dN_t - \lambda E(e^y - 1)dt$$

and the integration is given by

$$\frac{\pi_T}{\pi_t} = exp\{[-r-\lambda E(e^y-1)\tau-\frac{1}{2}\int_t^T(\gamma-\epsilon\rho A_\tau)^2 V_\tau d\tau-\int_t^T(\gamma-\epsilon\rho A_\tau)\sqrt{V_\tau}dW_\tau^S+\sum_{i=1}^{N_\tau}y_i\}$$

(4.10)

where $\tau = T - t$.

The martingale condition, $\pi_t S_t = E_t[\pi_T S_T]$, requires that the jump size y satisfies the following restriction

$$E[(e^y - e^{-\gamma x})(e^x - 1)] = 0$$

(4.11)

Proof To satisfy the martingale condition, $\pi_t S_t = E_t[\pi_T S_T]$, from (4.1), (4.10), we have

$$Eexp\{[\int_t^T[\phi_\tau - \frac{1}{2}V_\tau - \frac{1}{2}(\gamma - \epsilon\rho A_\tau)^2 V_\tau]d\tau - \lambda[E(e^y - 1)\tau + E(e^x - 1)]\tau +$$

$$\int_t^T[1 - (\gamma - \epsilon\rho A_\tau)]\sqrt{V_\tau}dW_\tau^S + \sum_{i=1}^{N_\tau}x_i + \sum_{i=1}^{N_\tau}y_i\} = 1$$

(4.12)

Substituting (4.4) into (4.12) , we have

$$Eexp\{\int_t^T \lambda E[(1 - e^{-\gamma x})(e^x - 1)]\tau - \lambda[E(e^y - 1)\tau + E(e^x - 1)]\tau + \sum_{i=1}^{N_\tau}x_i +$$

$$\sum_{i=1}^{N_\tau}y_i\} = 1$$

(4.13)

thus it is easy to obtain

$$E(\prod_{i=1}^{N_\tau}e^{x_i+y_i})exp[\lambda E[(1 - e^{-\gamma x})(e^x - 1)]\tau - \lambda[E(e^y - 1)\tau - E(e^x - 1)]\tau) = 1$$

Using the property of Poisson process, $E(\prod_{i=1}^{N_\tau}x_i + y_i) = exp[\lambda E(e^{x+y} - 1)\tau]$, we

have

$$exp[\lambda E(e^{x+y} - 1)\tau]exp[\lambda E[(1 - e^{-\gamma x})(e^x - 1)]\tau - \lambda[E(e^y - 1)\tau - E(e^x - 1)]\tau) = 1,$$

$$exp[\lambda E(e^{x+y} - 1 + (1 - e^{-\gamma x})(e^x - 1) - e^y + 1 - (e^x - 1))\tau] = 1,$$

$$E(e^{x+y} - e^{(1-\gamma)x} + e^{-\gamma x} - e^y) = 0,$$

$$E[(e^y - e^{-\gamma x})(e^x - 1)] = 0,$$

Remark 4.2 In this market, there is only one tradable asset, a stock with price S_t , but there are at least two dimensions of risk, diffusive risk and jump risk. Therefore, the market is incomplete and the pricing kernel is not unique. The nonuniqueness of the pricing kernel can be justified by the fact that the distribution of jump size y in the pricing kernel can be arbitrary as long as it satisfies the martingale restriction(4.11). In a special case, we can choose $y = -gx$, as in Liu, Pan, and Wang[39].

Remark 4.3 With Proposition 4.1, we define a new probability measure H ,

$$\frac{dH}{dP} = H_T' = e^{rT}\pi_T = exp\{-\lambda E(e^y - 1)T - \frac{1}{2}\int_0^T (\gamma - \epsilon\rho A_\tau)^2 V_\tau d\tau -$$
$$\int_0^T (\gamma - \epsilon\rho A_\tau)\sqrt{V_\tau}dW_\tau^S + \sum_{i=1}^{N_T} y_i\}$$

Since, for any assets P_t at time t , we have

$$E_t^H[P_T] = \frac{1}{H_t'}E_t[H_T'P_T] = e^{r(T-t)}E_t[\frac{\pi_T}{\pi_t}P_T] = e^{r(T-t)}P_t,$$

which means H is a risk-neutral probability measure.

Lemma 4.1 Define a new probability measure, H* by the following Radon-

Nikodym derivative

$$\frac{dH}{dP} = exp\{[-\lambda E(e^y - 1)]T + \sum_{i=1}^{N_T} y_i\}$$

then the following relation

$$E[e^y f(x)] = E(e^y)E^{H^*}[f(x)]$$

is true.

Proof The change of probability measure formula gives

$$E^{H^*}[f(x)]$$
$$= E[\frac{dH^*}{dP}f(x)]$$
$$= exp\{[-\lambda E(e^y - 1)]T\}E[f(x)\prod_{i=1}^{N_T} e^{y_i}]$$
$$= exp\{[-\lambda E(e^y - 1)]T\}\sum_{n=1}^{+\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}E[f(x)\prod_{i=1}^{N_T} e^{y_i}]$$

Since $y_i, i = 1, 2, ...i, ..., n$ is i.i.d. and y and x are correlated. This means that only one of the y_i is correlated with x. Without a loss of generality, we assume that y_n is correlated with x and other y_is are independent of x. Then we have

$$E^{H^*}[f(x)] = E[\frac{dH^*}{dP}f(x)] = exp\{[-\lambda E(e^y - 1)]T\}E[f(x)\prod_{i=1}^{N_T} e^{y_i}]$$
$$= exp\{[-\lambda E(e^y - 1)]T\}\sum_{n=1}^{+\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}E[f(x)\prod_{i=1}^{N_T} e^{y_i}]$$
$$= E[f(x)e^y]exp\{[-\lambda E(e^y - 1)]T\}\sum_{n=1}^{+\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}E[f(x)\prod_{i=1}^{N_T} e^{y_i}]$$
$$= \frac{E[e^y f(x)]}{E(e^y)}$$

Remark 4.4 These results are also true in H measure, because the difference

between H and H^* is the Brownian motion that is independent of the jumps. Now, we consider a European call written on the stock price S_t at time t. The option has a payoff function $(S_T - K)^+$ at time T . Its price is denoted as $c(St, Vt, t)$ at time t. We derive a PDE in which $c(St, Vt, t)$ has to satisfy in following proposition.

Proposition 4.2 In the general equilibrium framework, the price of European call option satisfies the following PDE,

$$
\begin{cases}
0 = c_t + \frac{1}{2}VS^2c_{SS} + \frac{1}{2}\epsilon^2Vc_{VV} + \epsilon\rho VSc_{SV} + [r - \lambda^H E^H(e^x - 1)]Sc_S - \\
rc + [\kappa^H(\theta^H - V)]c_V + \lambda^H\{E^H[c(Se^x, V, t)] - c(S, V, t)\} \\
c(S, V, T) = (S - K)^+
\end{cases}
$$

$$(4 - 6)$$

where

$$
E^{H^*}[f(x)] = \frac{E[e^y f(x)]}{E(e^y)}, \lambda^H = \lambda E(e^y),
$$

$$
\kappa^H = \kappa + (\gamma - \epsilon\rho A)\epsilon\rho, \theta^H = \frac{\kappa\theta}{\kappa + (\gamma - \epsilon\rho A)\epsilon\rho}.
$$

Proof First, we rewrite the stock with continue part and jump part,

$$
dS = d^C S + S(e^x - 1)dN_t
$$

where

$$
d^C S = [r + \phi - \lambda E(e^x - 1)]Sdt + \sqrt{V}SdW^S
$$

$$
dV_t = \kappa(\theta - V_t)dt + \epsilon\sqrt{V_t}dW_t^V
$$

Similarly,

$$
d\pi = d^C\pi + (e^y - 1)\pi dN_t
$$

in which

$$
d^C\pi = [-r - \lambda E(e^y - 1)]\pi dt - (\gamma - \epsilon\rho A)\sqrt{V}\pi dW_t^S
$$

and

$$dc = d^C c + [c(Se^x, V, t) - c(S, V, t)]dN_t$$

where

$$d^C c = \{c_t + (r + \phi - \lambda E(e^x - 1))Sc_S + \frac{1}{2}VS^2 c_{SS} + [\kappa(\theta - V)]c_V +$$
$$\frac{1}{2}\epsilon^2 V c_{VV} + \epsilon\rho V Sc_{SV}\}dt + \sqrt{V}Sc_S dW^S + \epsilon\sqrt{V}c_V dW_t^V$$

From (4.7)and (4.8), we get

$$d(\pi c) = d^C(\pi c) + [\pi e^y c(Se^x, V, t) - \pi c(S, V, t)]dN_t$$

where

$$d^C(\pi c) = c[-r - \lambda E(e^y - 1)\pi dt - c(\gamma - \epsilon\rho A)\sqrt{V}\pi dW_t^S +$$
$$\pi\{c_t + (r + \phi - \lambda E(e^x - 1))Sc_S + \frac{1}{2}VS^2 c_{SS} +$$
$$[\kappa(\theta - V)]c_V + \frac{1}{2}\epsilon^2 V c_{VV} + \epsilon\rho V Sc_{SV}\}dt +$$
$$\pi\sqrt{V}Sc_S dW^S + \pi\epsilon\sqrt{V}c_V dW_t^V -$$
$$V\pi(\gamma - \epsilon\rho A)(Sc_s dt + \epsilon\rho c_V dt)$$

The martingale condition $E[d(\pi c)] = 0$ requires

$$0 = c_t + \frac{1}{2}VS^2 c_{SS} + \frac{1}{2}\epsilon^2 V c_{VV} + [(r + \phi - V(\gamma - \epsilon\rho A) - \lambda E(e^x - 1))]Sc_S -$$
$$[r + \lambda E(e^y - 1)]c + [\kappa(\theta - V)] - V(\gamma - \epsilon\rho A)\epsilon\rho]c_V + \epsilon\rho V Sc_{SV} +$$
$$\lambda E[e^y c(Se^x, V, t)] - \lambda c(S, V, t)$$

Using Lemma 4.1, we have

$$E^H[f(x)] = \frac{E[e^y f(x)]}{E(e^y)},$$

Denoting $\lambda^H = \lambda E(e^y)$, and using the restriction condition $E[(e^y - e^{-\gamma x})(e^x - 1)] = 0$, $\phi = V_t[\gamma - \epsilon\rho A(t)] + \lambda E[(1 - e^{-\gamma x})$ in Proposition 3.1 and the terminal payoff

function $c(S, V, T) = (S - K)^+$, we obtain

$$
\begin{cases}
0 = c_t + \frac{1}{2}VS^2c_{SS} + \frac{1}{2}\epsilon^2 V c_{VV} + \epsilon\rho V S c_{SV} + [r - \lambda^H E^H(e^x - 1)]Sc_S - \\
rc + [\kappa^H(\theta - V) - V(\gamma - \epsilon\rho A)\epsilon\rho]c_V + \lambda^H\{E^H[c(Se^x, V, t)] - c(S, V, t)\} \\
c(S, V, T) = (S - K)^+
\end{cases}
$$

$$(4 - 7)$$

If we denote $\kappa^H = \kappa + (\gamma - \epsilon\rho A)\epsilon\rho$ and $\theta^H = \frac{\kappa\theta}{\kappa + (\gamma - \epsilon\rho A)\epsilon\rho}$, then (4-7) can be written

as

$$
\begin{cases}
0 = c_t + \frac{1}{2}VS^2c_{SS} + \frac{1}{2}\epsilon^2 V c_{VV} + \epsilon\rho V S c_{SV} + [r - \lambda^H E^H(e^x - 1)]Sc_S - \\
rc + [\kappa(\theta - V)]c_V + \lambda^H\{E^H[c(Se^x, V, t)] - c(S, V, t)\} \\
c(S, V, T) = (S - K)^+
\end{cases}
$$

$$(4 - 8)$$

Remark 4.5 The stock process (4-1) in a risk-neutral measure H can be written

as

$$
\begin{cases}
\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t}d\widetilde{W}_t^S + (e^x - 1)dN_t - \lambda E(e^x - 1)dt, S_0 > 0 \\
dV_t = \kappa^H(\theta^H - V_t)dt + \epsilon\sqrt{V_t}d\widetilde{W}_t^V, V_0 > 0
\end{cases}
$$

$$(4 - 9)$$

where

$$
d\widetilde{W}_t^S = dW_t^S + (\gamma - \epsilon\rho A)\sqrt{V_t}dt
$$

$$
d\widetilde{W}_t^V = dW_t^V + \rho(\gamma - \epsilon\rho A)\sqrt{V_t}dt
$$

$$
\lambda^H = \lambda E(e^y), \kappa^H = \kappa + (\gamma - \epsilon\rho A)\epsilon\rho
$$

$$
E^H[f(x)] = \frac{E[e^y f(x)]}{E(e^y)}, \theta^H = \frac{\kappa\theta}{\kappa + (\gamma - \epsilon\rho A)\epsilon\rho}
$$

The proof is straightforward. Substituting all equations in Remark 4.4 to (4-9), and using the equilibrium equity premium (4.4) and the restriction (4.11), we will get (4-1).

4.5 Numerical Results

In this section, we use our model to solve the equity premium puzzle. First, we get the long equilibrium risk premium $\hat{\phi}$ for $t \to T$ where T is sufficiently large. In this case, $V_t \to \theta$, $A_t \to 0$, then

$$\hat{\phi} = \mu - r = \theta\gamma + \lambda(1 - e^{-\gamma x})(e^x - 1) \tag{4.14}$$

To compare with the results from the model in [38], we take nonrandom constant jump size $x = -0.08$ and $r = 0.05, \lambda = 0.48, \mu = 0.11, \theta = 0.162$. Then,we obtain the relative risk aversion coefficient $\gamma = 2.13$ which is the same as the numerical value in [38]. However, the risk premium ϕ is stochastic. We get the expectation of *phi* as follows,

$$E[\phi] = [e^{-\kappa t}V_0 + \theta(1 - e^{-\kappa t})][\gamma - \epsilon\rho A(t)] + \lambda(1 - e^{-\gamma x})(e^x - 1)$$

If we assume other parameter values in our model are $\kappa = 0.03, V_0 = 0.1, \epsilon = 0.1, \rho = -0.25$, we will find the expectation of ϕ tend to 0.6 from Figure 5.1, which is consistent with above analysis. As we know that the risk premium ϕ
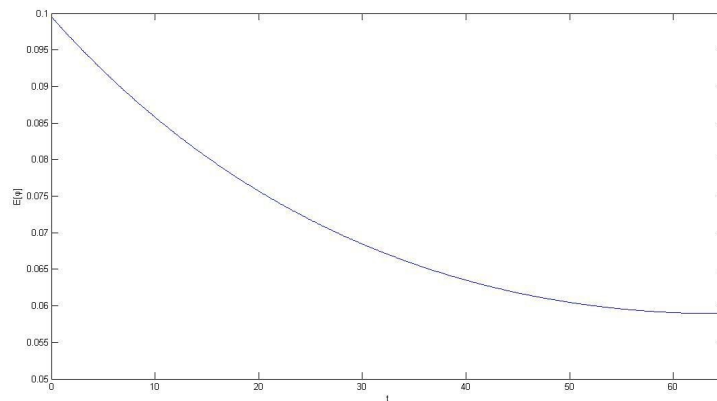

Figure 4.1: The expectation of ϕ

is stochastic and it has same stochastic characteristic with volatility V_t, the risk

premium ϕ can be generated by a discrete scheme of system as follows,

$$V(t + \Delta t) = V(t) + \kappa(\theta - V(t))\Delta t + \epsilon\sqrt{V(t)}\rho\omega\sqrt{\Delta t}, V_0 > 0$$

where Δt is the time interval, ω is a sample from the standard normal distribution. One path of ϕ is given by
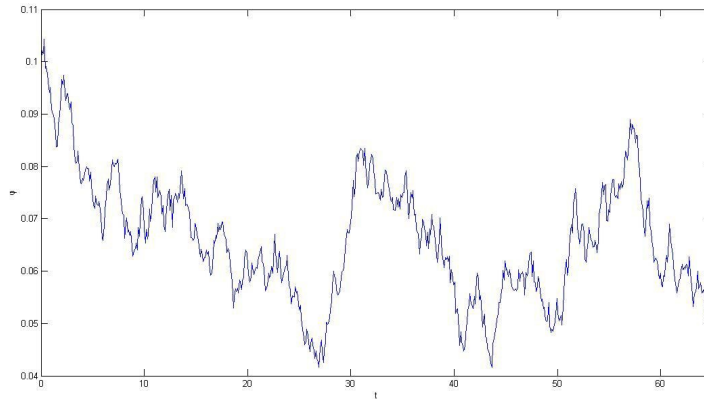
Figure 4.2: One path of ϕ

4.6 Concluding Remarks

In this chapter, the equity premium and option pricing under jump diffusion model with stochastic volatility based on the model in [38] have been studied. The pricing kernel which acts like the physical and risk-neutral densities and the moments in the economy has been constructed. Our numerical results show that our model is more realistic than the previous model in explaining equity premium puzzle.

CHAPTER 5

Optimal Asset Portfolio with a Stochastic Volatility and Interest Rate under the Mean-Variance Utility with Multistates-dependent Risk Aversion

5.1 General

This chapter studies the portfolio optimization of mean-variance utility with the states-dependent risk aversion with the assets driven by stochastic processes. Sub-game perfect Nash equilibrium strategies and extended Hamilton-Jacobi-Bellman equation have been used to derive the system of non- linear partial differential equations. From the economic point of view, a special case where the risk aversion proportional to the wealth has been studied with numerical methods. The parameters in the model are determined though calibration of real financial market. Our results show that the asset driven by a stochastic volatility and stochastic interest rate is more general and reasonable than that driven by constant ones.

5.2 Formulation of the Asset Portfolio Optimization Problem

We still consider two assets in our model, the stock and the bond with stochastic interest rate. The price of the bond is assumed to evaluate according to the following stochastic process

$$dR_t = \kappa_r(\theta_r - R_t)dt + \varepsilon_r\sqrt{R_t}dW_t^R, R_0 = 1, \tag{5.1}$$

in which the stock prize is modeled by the following stochastic process,

$$dS_t = \alpha S_t dt + \sqrt{V_t}S_t dW_t^S, \tag{5.2}$$

$$dV_t = \kappa_v(\theta_v - V_t)dt + \varepsilon_v\sqrt{V_t}dW_t^V, \tag{5.3}$$

where the dynamic stochastic volatility is similar to the square root process introduced by Cox Ingerscoll and Ross[43]. κ_r, κ_v, θ_r, θ_v, ε_r, ε_v are constants. For κ_r, κ_v, θ_r, $\theta_v > 0$, (5.1) and (5.2) are the continuous time first-order auto regressive processes where the randomly moving interest and volatility are elastically pulled toward a central location or a long-term value, θ_r and θ_v. The parameters κ_r and κ_v indicate speeds of adjustment. W_t^R, W_t^S and W_t^V are dependent Winner Processes with the correlation coefficients ρ_{sr}, ρ_{sv}, ρ_{rv}.

Here, we denote the total wealth as X_t with the initial wealth X_0, and we invest the wealth on the above two assets. The money we invest on the stock is denoted by u_t. Thus, we have the stochastic process of the total wealth.

$$dX_t = [\kappa_r \frac{\theta_r - R_t}{R_t} X_t + (\alpha - \kappa_r \frac{\theta_r - R_t}{R_t}) u_t] dt + u_t \sqrt{V_t} dW_t^S + (X_t - u_t) \varepsilon_r R_t^{-\frac{1}{2}} dW_t^R$$

(5.4)

Then, we choose the Mean-variance utility with risk aversion depending on wealth, volatility and interest rate $\gamma(x, v, r)$ as follows

$$J(t, x, v, r, u) = sup_{u \in U} \{E[X_T] - \frac{\gamma(x, v, r)}{2} Var[X_T]\}$$

(5.5)

Our purpose is to find a best control law $\widehat{u}$ to maximize the expected return with a penalty term for the risk. Thus

$$Q(t, x, v, r) = max_{u \in U} \{E_{t,x,v,r}[X_T^U] - \frac{\gamma(x, v, r)}{2} Var_{t,x,v,r}[X_T^U]\}.$$

(5.6)

5.3 The Game Theoretic Framework

Followed the idea of Bjork, Murgoci and Zhou[47], we use the subgame perfect Nash equilibrium point for this game. Here, we give a brief explanation of the basic idea. Suppose that we have several players or one player at several time points, and each player will have a control u. If at the later time point t, the player have chosen the optimal control $u(\cdot, t)$, we simply regard that the player at time s, $s \leq t$, will still use this control $u(\cdot, s)$ as the optimal one. Based on this concept, we give the formal expression of the optimal equilibrium control

$$u_{\Delta t}(s, \cdot) = \begin{cases} u, \ for \ t \leq s < t + \Delta t, \\ \widehat{u}(s, \cdot), \ for \ t + \Delta t \leq s \leq T, \end{cases}$$

where $u \in R^k$, $\Delta t > 0$, and $(t, x, v) \in [0, T] \times R^n \times R^n \times R^n$ are arbitrarily choosen. If

$$lim \ inf_{\Delta t \to 0} \frac{J(t, x, v, r, \widehat{u}) - J(t, x, v, r, u_{\Delta t})}{\Delta t} \geq 0,$$

for all $u \in R^k$, and $(t, x, v, r) \in [0, T] \times R^n \times R^n \times R^n$, we define the $\widehat{u}$ as the equilibrium control, and then we have the equilibrium value function

$$Q(t, x, v, r) = J(t, x, v, r, \widehat{u}).$$

As a extension of the work of Bjork and Murgoci[72], we have developed the derivation into the one with stochastic interest rate and volatility. Steps are as follows,

Definition 5.1. *We denote the infinitesimal generator as A. Thus, for any fixed $u \in U$, the corresponding infinitesimal generator is A^u defined as follows*

$$
\begin{aligned}
A^u = {} & \frac{\partial}{\partial t} + a\frac{\partial}{\partial X} + [\kappa_v(\theta_v - V_t)]\frac{\partial}{\partial V} + \kappa_r(\theta_r - R_t)\frac{\partial}{\partial R} + b\frac{\partial^2}{\partial X^2} \\
& + \frac{1}{2}\varepsilon_v^2 V_t\frac{\partial^2}{\partial V^2} + \frac{1}{2}\varepsilon_r^2 R_t\frac{\partial^2}{\partial R^2} + c\frac{\partial^2}{\partial X\partial V} + d\frac{\partial^2}{\partial X\partial R} + e\frac{\partial^2}{\partial R\partial V}
\end{aligned}
\tag{5.7}
$$

where a,b,c,d,e have the following expressions,

$$a = \kappa_r\frac{\theta_r - R_t}{R_t}X + (\alpha - \kappa_r\frac{\theta r - R_t}{R_t})u_t, \tag{5.8}$$

$$b = \frac{1}{2}[u_t^2 V_t + 2u_t\sqrt{V_t}(X - u_t)\varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} + (X - u_t)^2\varepsilon_r^2 R_t^{-1}], \tag{5.9}$$

$$c = u_t\varepsilon_v V_t\rho_{sv} + (X - u_t)\varepsilon_r\varepsilon_v(\frac{V_t}{R_t})^{\frac{1}{2}}\rho_{rv}, \tag{5.10}$$

$$d = u_t\varepsilon_r\sqrt{V_t}\sqrt{R_t}\rho_{sr} + (X - u_t)\varepsilon_r^2, \tag{5.11}$$

$$e = \varepsilon_r\varepsilon_v\sqrt{R_t}\sqrt{V_t}\rho_{rv}. \tag{5.12}$$

According to the study of Bjork and Murgoci[72], we have the following extended HJB equations for the Nash equilibrium problem with stochastic volatility.

Theorem 5.1.

$$Q_t + \sup_{u \in U} \{ a Q_x + [\kappa_v(\theta_v - v_t)] Q_v + [\kappa_r(\theta_r - r_t)] Q_r + b Q_{xx} + \frac{1}{2} \varepsilon_v^2 v_t Q_{vv}$$

$$+ \frac{1}{2} \varepsilon_r^2 r_t Q_{rr} + c Q_{xv} + d Q_{xr} + e Q_{rv} - a f_{y_1} - \kappa_v(\theta_v - v_t) f_{y_2} - \kappa_r(\theta_r - r_t) f_{y_3}$$

$$- b(2 f_{xy_1} + f_{y_1 y_1}) - \frac{1}{2} \varepsilon_v^2 v_t f_{y_2 y_2} - \frac{1}{2} \varepsilon_r^2 r_t f_{y_3 y_3} - c(f_{xy_2} + f_{y_1 y_2}) - d(f_{xy_3} + f_{y_1 y_3})$$

$$- e f_{y_3 y_2} - a G_x - [\kappa_v(\theta_v - v_t)] G_v - [\kappa_r(\theta_r - r_t)] G_r - b[G_{xx} + 2 G_{xg} g_x + G_{gg} g_x^2]$$

$$- \frac{1}{2} \varepsilon_v^2 v_t G_{vv} - \frac{1}{2} \varepsilon_r^2 r_t G_{rr} - c(G_{xv} + G_{gv} g_x) - d(G_{xr} + G_{gr} g_x) - e G_{vr} \} = 0,$$

$$\text{(5.13)}$$

with the boundary conditions,

$$A^{\widehat{u}} f^{y_1, y_2, y_3}(t, x) = 0, 0 \leq t \leq T, \tag{5.14}$$

$$A^{\widehat{u}} g(t, x) = 0, 0 \leq t \leq T, \tag{5.15}$$

$$Q(T, x, v, r) = F(x, v, r, x) + G(x, v, r, x), \tag{5.16}$$

$$f^{y_1, y_2, y_3}(T, x) = F(y_1, y_2, y_3, x), \tag{5.17}$$

$$g(T, x) = x. \tag{5.18}$$

In the above theorem, $f.$ represents $f(t, x, y_1, y_2, y_3)$ and $f_.^{y_1, y_2, y_3}$ represents $f_.^{y_1, y_2, y_3}(t, x)$. $f.$ is different from $f_.^{y_1, y_2, y_3}$. The former is the function of variables t, x, y_1, y_2, y_3, the later is a function of t, x. When the parameters y_1, y_2, y_3 are fixed, $f(t, x, y_1, y_2, y_3)$ is equal to $f^{y_1, y_2, y_3}(t, x)$.

5.4 Analysis for the General Case of Mean-Variance Utility with State-dependent Risk Aversion

In this part, we first give the extended HJB equation with the stochastic volatility under the mean-variance utility with constant risk aversion.

For this special case, because of the assumption of constant risk aversion, $f(\cdot)$ is the same as $f^y(\cdot)$. Thus, in Theorem 1, F, G, f and g take the following forms

$$F(y_1, y_2, y_3, X_T) = X_T - \frac{\gamma(y_1, y_2, y_3)}{2}(X_T)^2, \tag{5.19}$$

$$G(y_1, y_2, y_3, g) = \frac{\gamma(y_1, y_2, y_3)}{2}g^2. \tag{5.20}$$

$$f(t, x, y_1, y_2, y_3) = E_{t,x,v,r}[F(y_1, y_2, y_3, X_T^{\widehat{u}})], \tag{5.21}$$

$$g(t, x) = E_{t,x,v,r}[X_T^{\widehat{u}}], \tag{5.22}$$

Here we regard the risk aversion as a deterministic function of states x, v and r denoted by $\gamma(x, v, r)$. We have:

$$
\begin{aligned}
F(x, v, r, X_t) &= X_t - \frac{\gamma(x, v, r)}{2}(X_t)^2, & (5.23)\\
G(x, v, r, g) &= \frac{\gamma(x, v, r)}{2}g^2, & (5.24)\\
G_x &= \frac{\gamma_x(x, v, r)}{2}g^2, & (5.25)\\
G_v &= \frac{\gamma_v(x, v, r)}{2}g^2, & (5.26)\\
G_r &= \frac{\gamma_r(x, v, r)}{2}g^2, & (5.27)\\
G_{xx} &= \frac{\gamma_{xx}(x, v, r)}{2}g^2, & (5.28)\\
G_{vv} &= \frac{\gamma_{vv}(x, v, r)}{2}g^2, & (5.29)\\
G_{rr} &= \frac{\gamma_{rr}(x, v, r)}{2}g^2, & (5.30)\\
G_{xv} &= \frac{\gamma_{xv}(x, v, r)}{2}g^2, & (5.31)\\
G_{xr} &= \frac{\gamma_{xr}(x, v, r)}{2}g^2, & (5.32)\\
G_{vr} &= \frac{\gamma_{vr}(x, v, r)}{2}g^2, & (5.33)\\
G_g &= \gamma(x, v, r)g, & (5.34)\\
G_{gg} &= \gamma(x, v, r), & (5.35)
\end{aligned}
$$

$$G_{xg} = \gamma_x g, \tag{5.36}$$

$$G_{rg} = \gamma_r g, \tag{5.37}$$

$$G_{vg} = \gamma_v g. \tag{5.38}$$

Taking these equations and apply them to Theorem 5.1, we obtain the following

HJB equation,

Theorem 5.2.

$$
\begin{aligned}
Q_t + \sup_{u \in U} &\Big\{ a\Big(Q_x - f_{y_1} - \frac{\gamma_x(x,v,r)}{2}g^2\Big) + [\kappa_v(\theta_v - v_t)]\Big(Q_v - f_{y_2} - \frac{\gamma_v(x,v,r)}{2}g^2\Big) \\
&+ [\kappa_r(\theta_r - r_t)]\Big(Q_r - f_{y_3} - \frac{\gamma_r(x,v,r)}{2}g^2\Big) + b\Big(Q_{xx} - 2f_{xy_1} - f_{y_1 y_1} - \frac{\gamma_{xx}(x,v,r)}{2}g^2 \\
&- 2\gamma_x(x,v,r)g_x g - \gamma(x,v,r)g_x^2\Big) + \frac{1}{2}\varepsilon_v^2 v_t\Big(Q_{vv} - f_{y_2 y_2} - \frac{\gamma_{vv}(x,v,r)}{2}g^2\Big) \\
&+ \frac{1}{2}\varepsilon_r^2 r_t\Big(Q_{rr} - f_{y_3 y_3} - \frac{\gamma_{rr}(x,v,r)}{2}g^2\Big) \\
&+ c\Big(Q_{xv} - f_{xy_2} - f_{y_1 v} - f_{y_1 y_2} - \frac{\gamma_{xv}(x,v,r)}{2}g^2 - \gamma_v(x,v,r)gg_x\Big) \\
&+ d\Big(Q_{xr} - f_{xy_3} - f_{y_1 y_3} - \frac{\gamma_{xr}(x,v,r)}{2}g^2 - \gamma_r(x,v,r)gg_x\Big) \\
&+ e\Big(Q_{rv} - f_{y_3 y_2} - \frac{\gamma_{vr}(x,v,r)}{2}g^2\Big)\Big\} = 0,
\end{aligned}
\tag{5.39}
$$

with the boundary condition,

$$A^{\widehat{u}} f = 0, \tag{5.40}$$

$$A^{\widehat{u}} g = 0, \tag{5.41}$$

$$f(T, y_1, y_2, y_3, x) = x - \frac{\gamma(y_1, y_2, y_3)}{2}x^2, \tag{5.42}$$

$$g(T, x) = x. \tag{5.43}$$

Let X_T^u be the terminal wealth corresponding to the optimal equilibrium con-

trol law $\widehat{u}$ from (5.6), (5.21) and (5.22), we have

$$Q(t, x, y_1, y_2, y_3) = E_{t,x,v,r}[X_T^{\widehat{U}}] - \frac{\gamma(y_1, y_2, y_3)}{2} Var_{t,x,v}[X_T^{\widehat{U}}], \tag{5.44}$$

$$f(t, y_1, y_2, y_3, x) = E_{t,x,v,r}[X_T^{\widehat{U}}] - \frac{\gamma(y_1, y_2, y_3)}{2} E_{t,x,v,r}[X_T^{\widehat{U}}]^2, \tag{5.45}$$

$$g(t, x) = E_{t,x,v,r}[X_T^{\widehat{U}}] \tag{5.46}$$

$$Q(t, x, y_1, y_2, y_3) = f(t, y_1, y_2, y_3, x) + \frac{\gamma(y_1, y_2, y_3)}{2} g^2(t, x). \tag{5.47}$$

where $y_1 = x, y_2 = v, y_3 = r$ in our case.

From (5.47), we have

$$
\begin{aligned}
Q_t &= f_t + \gamma g g_t, \\
Q_x &= f_x + f_{y_1} + \frac{\gamma_x}{2} g^2 + \gamma g g_x, \\
Q_v &= f_{y_2} + \frac{\gamma_v}{2} g^2, \\
Q_r &= f_{y_3} + \frac{\gamma_r}{2} g^2, \\
Q_{xx} &= f_{xx} + 2f_{xy_1} + f_{y_1 y_1} + \frac{\gamma_{xx}}{2} g^2 + 2\gamma_x g g_x + \gamma g_x^2 + \gamma g g_{xx}, \\
Q_{vv} &= f_{y_2 y_2} + \frac{\gamma_{vv}}{2} g^2, \\
Q_{rr} &= f_{y_3 y_3} + \frac{\gamma_{rr}}{2} g^2, \\
Q_{xv} &= f_{xy_2} + f_{y_1 y_2} + \frac{\gamma_{xv}}{2} g^2 + \gamma v g g_x, \\
Q_{xr} &= f_{xy_3} + f_{y_1 y_3} + \frac{\gamma_{xr}}{2} g^2 + \gamma r g g_x, \\
Q_{vr} &= f_{y_2 y_3} + \frac{\gamma_{vr}}{2} g^2.
\end{aligned}
\tag{5.48}
$$

Substituting the above into Theorem 5.2, we obtain:

$$f_t + \gamma g g_t + sup_{u \in U}\{a[f_x + \gamma(x, v, r)g g_x] + b[f_{xx} + \gamma g g_{xx}]\} = 0, \tag{5.49}$$

where

$$a = \kappa_r \frac{\theta_r - R_t}{R_t} X_t + (\alpha - \kappa_r \frac{\theta r - R_t}{R_t}) u_t, \tag{5.50}$$

$$b = \frac{1}{2}[u_t^2 V_t + 2u_t \sqrt{V_t}(X_t - u_t)\varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} + (X_t - u_t)^2 \varepsilon_r^2 R_t^{-1}], \tag{5.51}$$

or

$$f_t + \gamma g g_t + sup_u Z = 0. \tag{5.52}$$

The optimization problem in (5.52) requires $\frac{dZ}{du}|_{\hat{u}} = 0$. From the definitions of a, b, c, d and e, we have

$$\frac{dZ}{du}|_{\hat{u}} = (\alpha - \kappa_r \frac{\theta r - R_t}{R_t})(f_x + \gamma g g_x) + (\hat{U}_t V_t + \sqrt{V_t} X_t \varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr}$$
$$- 2\hat{U}_t \sqrt{V_t}\varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} - X_t \varepsilon_r^2 R_t^{-1} + \hat{U}_t \varepsilon_r^2 R_t^{-1})(f_{xx} + \gamma g g_{xx}) = 0, \tag{5.53}$$

Therefore,

$$\hat{U}_t = -\frac{(f_x + \gamma g g_x)(\alpha - \kappa_r \frac{\theta_r - R_t}{R_t}) + (f_{xx} + \gamma g g_{xx})(\sqrt{V_t} X_t \varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} - X_t \varepsilon_r^2 R_t^{-1})}{(f_{xx} + \gamma g g_{xx})(V_t - 2\sqrt{V_t}\varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} + \varepsilon_r^2 R_t^{-1})} \tag{5.54}$$

Remark 5.1 If $\rho_{sr} = \rho_{vr} = 0, \theta_r = \varepsilon_r = 0$. Then the Bond reduces to the risk free bond. Thus, (5.54) reduces to,

$$\hat{U}_t = -\frac{(\alpha + \kappa_r)(f_x + \gamma(x,v)g g_x)}{(f_{xx} + \gamma(x,v)g g_{xx})V_t} \tag{5.55}$$

which is the same as S. LI[72] if $\kappa_r = -r$.

Remark 5.2 If $\rho_{sr} = \rho_{sv} = \rho_{vr} = 0, \theta_r = \varepsilon_r = 0, \kappa_v = \theta_v = \varepsilon_v = 0$. Then the stochastic volatility reduces to a constant volatility and the stochastic bond

process reduces to a risk-free bond. Thus, our result reduces to

$$\hat{U}_t = -\frac{(f_x + \gamma(x)gg_x)\beta}{(f_{xx} + \gamma(x)gg_{xx})\sigma^2} \qquad (5.56)$$

which is the same as Zhou[47] if $\alpha + \kappa_r = \beta$, $V_t = \sigma^2$.

5.5 Analysis for Special Case of Mean-Variance Utility with 'Natural' Risk Aversion

Now, in order to simplify but without the generality, we choose the 'natural' risk aversion with the following expression

$$\gamma(x, v, r) = \frac{\gamma}{xvr}. \qquad (5.57)$$

Thus, we have

$$\gamma_x = -\frac{\gamma}{x^2 vr}, \qquad (5.58)$$

$$\gamma_v = -\frac{\gamma}{xv^2 r}. \qquad (5.59)$$

$$\gamma_r = -\frac{\gamma}{xvr^2}. \qquad (5.60)$$

We have

$$f_t + \gamma gg_t + sup_{u \in U}\{a[f_x + \frac{\gamma}{xvr}gg_x] + b[f_{xx} + \gamma gg_{xx}]\} = 0, \qquad (5.61)$$

and

$$\hat{U}_t = -\frac{(f_x + \frac{\gamma}{xvr}gg_x)(\alpha - \kappa_r\frac{\theta_r - R_t}{R_t}) + (f_{xx} + \frac{\gamma}{xvr}gg_{xx})(\sqrt{V_t}X_t\varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} - X_t\varepsilon_r^2 R_t^{-1})}{(f_{xx} + \frac{\gamma}{xvr}gg_{xx})(V_t - 2\sqrt{V_t}\varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} + \varepsilon_r^2 R_t^{-1})} \tag{5.62}$$

According to the previous research, here we conjecture that $\hat{u}$ is linear in X, so for this case, $\hat{u}$ can be expressed by $\hat{u}(t) = c(t)X$, and we have

$$E_{t,x,v,r}(X_T^{\hat{u}}) = p(t)x, \tag{5.63}$$

$$E_{t,x,v,r}[(X_T^{\hat{u}})^2] = q(t)x^2. \tag{5.64}$$

This leads to the Ansatz,

$$f(t, y_1, y_2, y_3, x) = p(t)x - \frac{\gamma}{2y_1y_2y_3}q(t)x^2, \tag{5.65}$$

$$g(t, x) = p(t)x. \tag{5.66}$$

$$f_t(t, y_1, y_2, y_3, x) = p'(t)x - \frac{\gamma}{2y_1y_2y_3}q'(t)x^2, \tag{5.67}$$

$$f_x(t, y_1, y_2, y_3, x) = p(t) - \frac{\gamma}{y_1y_2y_3}q(t)x, \tag{5.68}$$

$$f_{xx}(t, y_1, y_2, y_3, x) = -\frac{\gamma}{y_1y_2y_3}q(t), \tag{5.69}$$

$$g_t(t, x) = p'(t)x, \tag{5.70}$$

$$g_x(t, x) = p(t), \tag{5.71}$$

$$g_{xx}(t, x) = 0. \tag{5.72}$$

By substituting (5.65)-(5.72) into (5.62), we get

$$\hat{u}_t = c(t)x = \frac{[(pvr + \gamma(p^2 - q))(\alpha - \kappa_r\frac{\theta_r - R_t}{R_t}) - \gamma q(\sqrt{V_t}\varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} - \varepsilon_r^2 R_t^{-1})]}{\gamma q(V_t - 2\sqrt{V_t}\varepsilon_r R_t^{-\frac{1}{2}}\rho_{sr} + \varepsilon_r^2 R_t^{-1})}x \tag{5.73}$$

By substituting (5.65)-(5.72) into (5.40) and (5.41), we get

$$p'x - \frac{\gamma}{2y_1 y_2 y_3} q'x^2 + (\kappa_r \frac{\theta_r - R_t}{R_t} x + (\alpha - \kappa_r \frac{\theta_r - R_t}{R_t})c(t)x)(p - \frac{\gamma}{y_1 y_2 y_3} qx)$$
$$+ (\frac{1}{2}c^2 x^2 v + cx^2 \sqrt{v}(1-c)\varepsilon_r R_t^{-\frac{1}{2}} \rho_{sr} + \frac{1}{2}(1-c)^2 x^2 \varepsilon_r^2 R_t^{-1})(-\frac{\gamma}{y_1 y_2 y_3} q) = 0,$$

$$\tag{5.74}$$

$$p'x + p(\kappa_r \frac{\theta_r - R_t}{R_t} x + (\alpha - \kappa_r \frac{\theta_r - R_t}{R_t})c(t)x) = 0. \tag{5.75}$$

By splitting the above two equations, we have the following two ordinary differential equations for the determination of p and q,

$$p' + p(\kappa_r \frac{\theta_r - R_t}{R_t} + (\alpha - \kappa_r \frac{\theta_r - R_t}{R_t})c(t)) = 0. \tag{5.76}$$

$$q' + 2q\{(\kappa_r \frac{\theta_r - R_t}{R_t} + (\alpha - \kappa_r \frac{\theta_r - R_t}{R_t})c(t)$$
$$+ (\frac{1}{2}c^2 x^2 v + cx^2 \sqrt{v}(1-c)\varepsilon_r R_t^{-\frac{1}{2}} \rho_{sr} + \frac{1}{2}(1-c)^2 x^2 \varepsilon_r^2 R_t^{-1})\} = 0, \tag{5.77}$$

$$p(T) = 1, \tag{5.78}$$

$$q(T) = 1. \tag{5.79}$$

Instead of solving (5.76)-(5.79) for p and q, we can express p and q in terms of the $c(t)$ using integral equations:

$$p = e^{\int_t^T \{(\kappa_r \frac{\theta_r - r}{r} + (\alpha - \kappa_r \frac{\theta_r - r}{r})c(s)\}ds},$$

$$q = e^{2\int_t^T \{(\kappa_r \frac{\theta_r - R_t}{R_t} + (\alpha - \kappa_r \frac{\theta_r - R_t}{R_t})c(s)) + (\frac{1}{2}c^2 x^2 v + cx^2 \sqrt{v}(1-c(s))\varepsilon_r R_t^{-\frac{1}{2}} \rho_{sr} + \frac{1}{2}(1-c)^2 x^2 \varepsilon_r^2 R_t^{-1})\}ds}.$$

5.6 Numerical Investigation

We demonstrate the numerical results for the suggested model by discretizing the wealth process using the Euler discretization scheme, with various choices of γ. We obtained the financial data from SIRCA and we calibrated the stock price process, volatility and interest rate to derive the other model parameters. For the simulation purpose, we use these calibrated parameters to present the wealth process and the amount of money invested into stock, under the stochastic process of interest rate and volatility.

Figure 5.1 plots a simulated stock price process together with its volatility and the interest rate. Figure 5.2 shows how the proportion of wealth invested in the simulated stock price would change for various choice of γ. Finally, Figure 5.3 presents the expected values of wealth with respect to these investment strategies.

5.7 Concluding Remarks

This chapter studies the portfolio optimization of mean-variance utility with states-dependent risk aversion with the assets driven by stochastic processes. Subgame perfect Nash equilibrium strategies and extended Hamilton-Jacobi-Bellman equation have been used to derive the system of non- linear partial differential equations. From the economic point of view, a special case where the risk aversion proportional to the wealth has been studied with numerical methods. The parameters in the model are determined though calibration of real financial market. Our results show that the asset driven by a stochastic volatility and stochastic interest rate is more general and reasonable than that driven by constant ones.
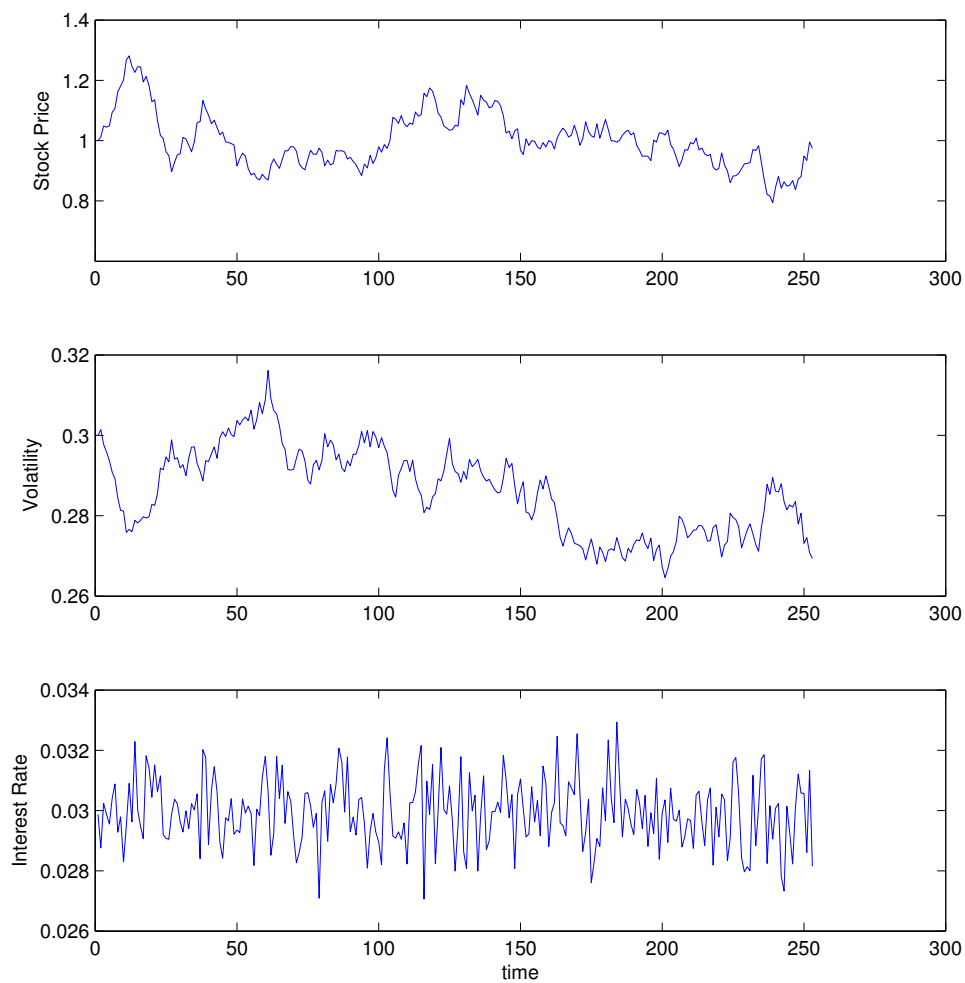
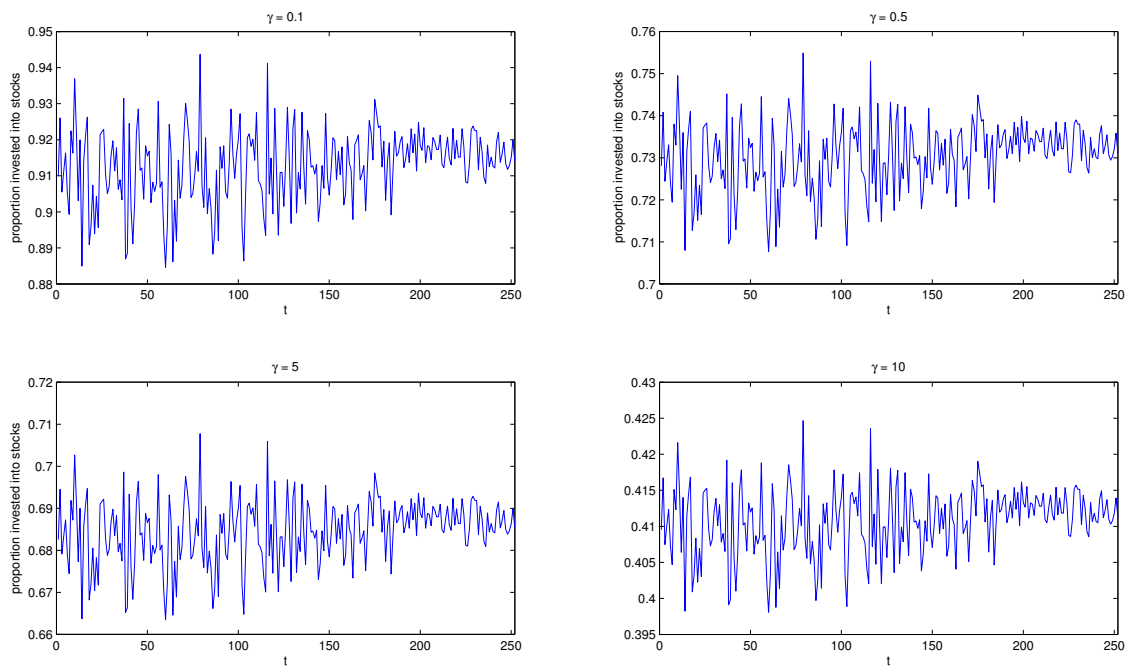Figure 5.1: Simulated Stock Price, Volatility and Interest Rate

Figure 5.2: c(t) for various choices of γ using the iterative scheme (4.14)-(4.15) as discussed in [47]
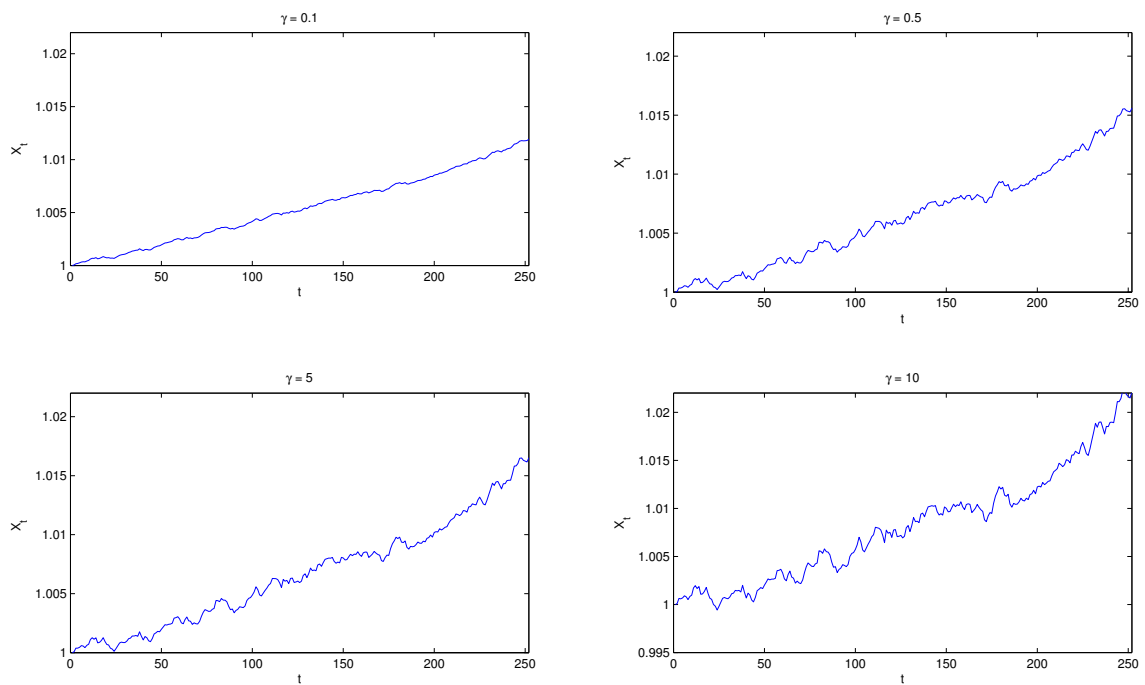

Figure 5.3: Expected values of wealth for various choices of γ

CHAPTER 6

Mean-Variance Portfolio Optimization with Stochastic Volatility Asymptotics

6.1   General

This chapter studies the portfolio optimization of mean-variance utility with multiscale stochastic volatility. The sub-game perfect Nash equilibrium strategies and the extended Hamilton-Jacobi-Bellman equation are used to derive the system of non-linear partial differential equations for the optimization problem. By using asympototics approximations, the expansion of the value function which is the solution of the Hamilton-Jacobi-Bellman equation for the Merton problem with constant parameters is given explicitly in terms of the derivatives of the leading order value function. The optimal strategy is also established explicitly in terms of the derivatives of the leading order value function. Thus, we solve the portfolio optimization problem under the Mean-variance utility by using an analytical method.

6.2 The Underlying Governing Equations

We consider two assets in our model, with one being the stock and the other being the risk-free bond. Let the price of the risk-free bond be as follows

$$dR_t = rR_t dt, R_0 = 1 \tag{6.1}$$

For the stock, following the work of Fougue[61], the price of stock is assumed to follow the stochastic equation

$$dS_t = \mu(Y_t, Z_t)S_t dt + \sigma(Y_t, Z_t)S_t dW_t^S \tag{6.2}$$

where the growth rate and the volatility of the stock are functions of the fast and slow factors which are governed by

$$dY_t = \frac{1}{\varepsilon}b(Y_t)dt + \frac{1}{\sqrt{\varepsilon}}a(Y_t)dW_t^Y \tag{6.3}$$

$$dZ_t = \delta c(Z_t)dt + \sqrt{\delta}g(Z_t)dW_t^Z \tag{6.4}$$

where (W_t^S, W_t^Y, W_t^Z) are the standard Brownian motions with the correlations as follows, $d < W^S, W^Y >_t = \rho_{sy}$, $d < W^S, W^Z >_t = \rho_{sz}$, $d < W^Y, W^Z >_t = \rho_{yz}$.

In order to ensure the positive definiteness of the covariance matrix among the above three Brownian motions, $|\rho_{sy} < 1|, |\rho_{sz}| < 1, |\rho_{yz}| < 1$, and $1 + 2\rho_{sy}\rho_{sz}\rho_{yz} - \rho_{sy}^2 - \rho_{sz}^2 - \rho_{yz}^2 > 0$. When the parameters ε and δ are small, the stochastic processes Y_t and Z_t represent the slow and fast volatility processes respectively.

Let $X_t, t \in [0, T]$ be the wealth process and u_t donate the money we invest in the risky assets. The remaining, that is $X_t - u_t$, is invested in the risk-free money market. Thus the dynamic wealth process should follows the following stochastic

differential equation:

$$dX_t = [rX_t + (\mu(Y_t, Z_t) - r)u]dt + u\sigma(Y_t, Z_t)dW_t^S \qquad (6.5)$$

The investor has a terminal utility which in our work is the Mean-variance utility with state dependent risk version

$$J(t, x, y, z, u) = E[X_T^u] - \frac{\gamma(x, y, z)}{2}Var[X_T^u] \qquad (6.6)$$

Thus our aim is to maximize the terminal utility by choosing the optimal control strategy $\widehat{u}_t$. Mathematically the problem can be formulated as follows:

$$V(t, X_t, Y_t, Z_t) = max_{u \in U}\{E_{t,x,y,z}[X_T^U] - \frac{\gamma(x, y, z)}{2}Var_{t,x,y,z}[X_T^U]\}. \qquad (6.7)$$

where F, G, f and g are as follows

$$F(y_1, y_2, y_3, X_T) = X_T - \frac{\gamma(y_1, y_2, y_3)}{2}(X_T)^2, \qquad (6.8)$$

$$G(y_1, y_2, y_3, g) = \frac{\gamma(y_1, y_2, y_3)}{2}g^2. \qquad (6.9)$$

$$f(t, x, y_1, y_2, y_3) = E_{t,x,y,z}[F(y_1, y_2, y_3, X_T^{\widehat{u}})], \qquad (6.10)$$

$$g(t, x) = E_{t,x,y,z}[X_T^{\widehat{u}}], \qquad (6.11)$$

$$V(t, x, y_1, y_2, y_3) = f(t, x, y_1, y_2, y_3) + \frac{\gamma(y_1, y_2, y_3)}{2}g^2(t, x). \qquad (6.12)$$

6.3 The Extended HJB Equations for the Problem

Following the idea of Bjork, Murgoci and Zhou[47], we use the sub-game perfect Nash equilibrium point for this problem. Here, we give a brief explanation of the

basic idea. Suppose that we have several players or one player at several time points, and each player has a control u. If at the later time point t, the player has chosen the optimal control $u(\cdot, t)$, we simply assume that the player at time s, $s \leq t$, will still use this control $u(\cdot, s)$ as the optimal one. Accordingly we give the formal expression of the optimal equilibrium control by

$$
u_{\Delta t}(s, \cdot) =
\begin{cases}
u, \ for \ t \leq s < t + \Delta t, \\
\widehat{u}(s, \cdot), \ for \ t + \Delta t \leq s \leq T,
\end{cases}
$$

where $u \in R^k$, $\Delta t > 0$, and $(t, x, y, z) \in [0, T] \times R^n \times R^n \times R^n$ is arbitrarily chosen. If

$$
limit f_{\Delta t \to 0} \frac{J(t, x, y, z, \widehat{u}) - J(t, x, y, z, u_{\Delta t})}{\Delta t} \geq 0,
$$

for all $u \in R^k$, and $(t, x, y, z) \in [0, T] \times R^n \times R^n \times R^n$, we define the $\widehat{u}$ as the equilibrium control, and then we have the equilibrium value function

$$
V(t, x, y, z) = J(t, x, y, z, \widehat{u}).
$$

As a extension of the work of Bjork and Murgoci[7], we have developed an extended HJB equation for the problem with multiscale stochastic volatility.

Definition 6.1. *We denote the infinitesimal generator as A. Then, for any fix $u \in U$, the corresponding infinitesimal generator is A^u, which is as follows*

$$
A^u = \frac{\partial}{\partial t} + \alpha_1 \frac{\partial}{\partial X} + \alpha_2 \frac{\partial}{\partial Y} + \alpha_3 \frac{\partial}{\partial Z} + \alpha_4 \frac{\partial^2}{\partial X^2} + \alpha_5 \frac{\partial^2}{\partial Y^2} + \alpha_6 \frac{\partial^2}{\partial Z^2} + \alpha_7 \frac{\partial^2}{\partial X \partial Y}
$$

$$\tag{6.13}$$

$$
+ \alpha_8 \frac{\partial^2}{\partial X \partial Z} + \alpha_9 \frac{\partial^2}{\partial Y \partial Z}
$$

$$\tag{6.14}$$

where

$$
\alpha_1 = [rX + (\mu(Y_t, Z_t) - r)u],
$$

$$\alpha_2 = \frac{b(Y_t)}{\varepsilon},$$

$$\alpha_3 = \delta c(Z_t),$$

$$\alpha_4 = \frac{u^2 \sigma^2(Y_t, Z_t)}{2},$$

$$\alpha_5 = \frac{a^2(Y_t)}{2\varepsilon},$$

$$\alpha_6 = \frac{\delta g^2(Z_t)}{2},$$

$$\alpha_7 = \frac{u a(Y_t) \sigma(Y_t, Z_t) \rho_{sy}}{\sqrt{\varepsilon}},$$

$$\alpha_8 = u\sqrt{\delta} g(Z_t) \sigma(Y_t, Z_t) \rho_{sz},$$

$$\alpha_9 = \frac{\sqrt{\delta} a(Y_t) g(Z_t) \rho_{yz}}{\sqrt{\varepsilon}}.$$

Based on the work of Bjork and Murgoci[7], we establish the following extended HJB equation for the Nash equilibrium problem with stochastic volatility.

Theorem 6.1. *The value function as defied in (6.7) satisfies the following extended HJB equation,*

$$
\begin{aligned}
&V_t + sup_{u \in U}\{\alpha_1(V_x - f_{y_1} - \frac{\gamma_x(x,y,z)}{2}g^2) + \alpha_2(V_y - f_{y_2} - \frac{\gamma_y(x,y,z)}{2}g^2) \\
&+ \alpha_3(V_z - f_{y_3} - \frac{\gamma_z(x,y,z)}{2}g^2) + \alpha_4(V_{xx} - 2f_{xy_1} - f_{y_1y_1} - \frac{\gamma_{xx}(x,y,z)}{2}g^2 \\
&- 2\gamma_x(x,y,z)g_x g - \gamma(x,y,z)g_x^2) + \alpha_5(V_{yy} - f_{y_2y_2} - \frac{\gamma_{yy}(x,y,z)}{2}g^2) \\
&+ \alpha_6(V_{zz} - f_{y_3y_3} - \frac{\gamma_{zz}(x,y,z)}{2}g^2) \\
&+ \alpha_7(V_{xy} - f_{xy_2} - f_{y_1y_2} - \frac{\gamma_{xy}(x,y,z)}{2}g^2 - \gamma_y(x,y,z)gg_x) \\
&+ \alpha_8(V_{xz} - f_{xy_3} - f_{y_1y_3} - \frac{\gamma_{xz}(x,y,z)}{2}g^2 - \gamma_z(x,y,z)gg_x) \\
&+ \alpha_9(V_{yz} - f_{y_3y_2} - \frac{\gamma_{yz}(x,y,z)}{2}g^2)\} = 0,
\end{aligned}
$$

(6.15)

subject to the boundary conditions,

$$A^{\widehat{u}} f = 0, \tag{6.16}$$

$$A^{\widehat{u}} g = 0, \tag{6.17}$$

$$f(T, x, y_1, y_2, y_3) = x - \frac{\gamma(y_1, y_2, y_3)}{2} x^2, \tag{6.18}$$

$$g(T, x) = x. \tag{6.19}$$

Remark 6.1 In comparison to the HJB equation in [47] for the constant volatility, the extended HJB equation has several new terms including $\alpha_2 = \dfrac{b(Y_t)}{\varepsilon}, \alpha_3 = \delta c(Z_t), \alpha_5 = \dfrac{a^2(Y_t)}{2\varepsilon}, \alpha_6 = \dfrac{\delta g^2(Z_t)}{2}, \alpha_7 = \dfrac{ua(Y_t)\sigma(Y_t, Z_t)\rho_{sy}}{\sqrt{\varepsilon}},$
$\alpha_8 = u\sqrt{\delta} g(Z_t)\sigma(Y_t, Z_t)\rho_{sz}, \alpha_9 = \dfrac{\sqrt{\delta} a(Y_t) g(Z_t)\rho_{yz}}{\sqrt{\varepsilon}}.$

Proof   (Derivation of the extended HJB equations). As for the HJB equation in [7], by (6.7), we have

$$V(t, X_t, Y_t, Z_t) = sup_{u \in U} J(t, X_t, Y_t, Z_t, U), \tag{6.20}$$

where

$$J(t, X_t, Y_t, Z_t, U) = E_{t, X_t, Y_t, Z_t}[F(y, X_T^U)] + G(y, E_{t, X_t, Y_t, Z_t}[X_T^U]), \tag{6.21}$$

in which F and G are as given in (6.8) and (6.9). For $s > t$,

$$J(s, X_s, Y_s, Z_s, U) = E_{s, X_s, Y_s, Z_s}[F(y_s, X_T^U)] + G(y_s, E_{s, X_s, Y_s, Z_s}[X_T^U]), \tag{6.22}$$

From the Markovian structure and the definitions of (6.8) and (6.9), we get

$$E_{s, X_s, Y_s, Z_s}[F(y_s, X_T^U)] = f^U(s, X_s, Y_s, Z_s, y_s), \tag{6.23}$$

$$E_{s,X_s,Y_s,Z_s}[X_T^U] = g^U(s, X_s, Y_s, Z_s). \tag{6.24}$$

Then, (6.21) can be written as follows,

$$J(s, X_s, Y_s, Z_s, U) = f^U(s, X_s, Y_s, Z_s, y_s) + G^U(y_s, g^U(s, X_s, Y_s, Z_s)). \tag{6.25}$$

Taking expectations on both sides yields,

$$\begin{aligned} E_{t,X_t,Y_t,Z_t}[J(s, X_s, Y_s, Z_s, U)] &= E_{t,X_t,Y_t,Z_t}[f^U(s, X_s, Y_s, Z_s, y_s)] \\ &+ E_{t,X_t,Y_t,Z_t}[G(y_s, g^U(s, X_s, Y_s, Z_s))], \end{aligned} \tag{6.26}$$

Further from the definition of (6.20), we have

$$\begin{aligned} E_{t,X_t,Y_t,Z_t}[J(s, X_s, Y_s, Z_s, U)] &= J(t, X_t, Y_t, Z_t, U) + E_{t,X_t,Y_t,Z_t}[f^U(s, X_s, Y_s, Z_s, y_s)] \\ - E_{t,X_t,Y_t,Z_t}[F(y, X_T^U)] &+ E_{t,X_t,Y_t,Z_t}[G(y_s, g^U(s, X_s, Y_s, Z_s))] - G(y, E_{t,X_t,Y_t,Z_t}[X_T^U]). \end{aligned} \tag{6.27}$$

From the iterated conditioning, we obtain

$$\begin{aligned} E_{t,X_t,Y_t,Z_t}[F(y, X_T^U)] &= E_{t,X_t,Y_t,Z_t}[E_{s,X_s,Y_s,Z_s}[F(y, X_T^U)]] \\ &= E_{t,X_t,Y_t,Z_t}[f^U(s, X_s, Y_s, Z_s, y_s)], \end{aligned} \tag{6.28}$$

and that

$$\begin{aligned} E_{t,X_t,Y_t,Z_t}[X_T^U] &= E_{t,X_t,Y_t,Z_t}[E_{s,X_s,Y_s,Z_s}X_T^U]] \\ &= E_{t,X_t,Y_t,Z_t}[g^U(s, X_s, Y_s, Z_s)], \end{aligned} \tag{6.29}$$

Substituting (6.27) and (6.28) into (6.26), we obtain

$$\begin{aligned} &E_{t,X_t,Y_t,Z_t}[J(s, X_s, Y_s, Z_s, U)] - J(t, X_t, Y_t, Z_t, U) - E_{X_t,Y_t,Z_t}[f^U(s, X_s, Y_s, Z_s, y_s)] \\ &+ E_{t,X_t,Y_t,Z_t}[f^U(s, X_s, Y_s, Z_s, y_s)] - E_{t,X_t,Y_t,Z_t}[G(y_s, g^U(s, X_s, Y_s, Z_s))] \\ &+ G(y, E_{t,X_t,Y_t,Z_t}[g^U(s, X_s, Y_s, Z_s)]) = 0. \end{aligned} \tag{6.30}$$

Then

$$sup_{u \in U}\{E_{t,X_t,Y_t,Z_t}[J(s, X_s, Y_s, Z_s, U)] - J(t, X_t, Y_t, Z_t, U)$$

$$- E_{t,X_t,Y_t,Z_t}[f^U(s, X_s, Y_s, Z_s, y_s)] + E_{t,X_t,Y_t,Z_t}[f^U(s, X_s, Y_s, Z_s, y_s)]$$

$$- E_{t,X_t,Y_t,Z_t}[G(y_s, g^U(s, X_s, Y_s, Z_s))] + G(y_t, E_{t,X_t,Y_t,Z_t}[g^U(s, X_s, Y_s, Z_s)])\} = 0.$$

$$(6.31)$$

From (6.19) and the definition of the control law, we get that U coincides with the equilibrium law $\widehat{u}$ in $[s, T]$, and thus we have the following formula,

$$J(s, X_s, Y_s, Z_s, \widehat{u}) = Q(s, X_s, Y_s, Z_s), \qquad (6.32)$$

$$f^U(s, X_s, Y_s, Z_s, y) = f(s, X_s, Y_s, Z_s, y), \qquad (6.33)$$

$$g^U(s, X_s, Y_s, Z_s) = g(s, X_s, Y_s, Z_s). \qquad (6.34)$$

Hence, (6.30) can be written as

$$sup_{u \in U}\{E_{t,X_t,Y_t,Z_t}[Q(s, X_s, Y_s, Z_s)] - Q(t, X_t, Y_t, Z_t) - E_{t,X_t,Y_t,Z_t}[f(s, X_s, Y_s, Z_s, y_s)]$$

$$+ E_{t,X_t,Y_t,Z_t}[f(s, X_s, Y_s, Z_s, y_s)] - E_{t,X_t,Y_t,Z_t}[G(y_s, g(s, X_s, Y_s, Z_s))]$$

$$+ G(y_t, E_{t,X_t,Y_t,Z_t}[g^U(s, X_s, Y_s, Z_s)])\} = 0.$$

$$(6.35)$$

Denote

$$E_{t,X_t,Y_t,Z_t}[Q(s, X_s, Y_s, Z_s)] - Q(t, X_t, Y_t, Z_t) = A^u Q, \qquad (6.36)$$

$$E_{t,X_t,Y_t,Z_t}[f(s, X_s, Y_s, Z_s)] = A^u f, \qquad (6.37)$$

$$E_{t,X_t,Y_t,Z_t}[f(s, X_s, Y_s, Z_s, y_s) = A^u f^y, \qquad (6.38)$$

$$E_{t,X_t,Y_t,Z_t}[G(y_s, g(s, X_s, Y_s, Z_s))] = A^u G, \qquad (6.39)$$

$$G(y_t, E_{t,X_t,Y_t,Z_t}[g^U(s, X_s, Y_s, Z_s)]) = H^u g. \qquad (6.40)$$

Then from the extended HJB equation in [7], we get for the case with stochas-

tic volatility,

$$V_t + sup_u\{\alpha_1 V_Y + \alpha_2 V_Y + \alpha_3 V_Z + \alpha_4 V_{XX} + \alpha_5 V_{YY} + \alpha_6 V_{ZZ} + \alpha_7 V_{XY} + \alpha_8 V_{XZ}$$

$$+ \alpha_9 V_{YZ} - A^u f + A^u f^y - A^u G(X, Y, Z, g) + H^u g\} = 0,$$

$$(6.41)$$

We then obtain

$$- A^u f + A^u f^y = \alpha_1(-f_{y_1}) + \alpha_2(-f_{y_2})$$

$$+ \alpha_3(-f_{y_3}) + \alpha_4(-2f_{xy_1} - f_{y_1y_1}) + \alpha_5(-f_{y_2y_2})$$

$$+ \alpha_6(-f_{y_3y_3})$$

$$+ \alpha_7(-f_{xy_2} - f_{y_1y_2})$$

$$+ \alpha_8(-f_{xy_3} - f_{y_1y_3})$$

$$+ \alpha_9(-f_{y_3y_2})$$

$$(6.42)$$

$$- A^u G + H^u g = \alpha_1(-\frac{\gamma_x(x,y,z)}{2}g^2) + \alpha_2(-\frac{\gamma_y(x,y,z)}{2}g^2)$$

$$+ \alpha_3(-\frac{\gamma_z(x,y,z)}{2}g^2) + \alpha_4(-\frac{\gamma_{xx}(x,y,z)}{2}g^2$$

$$- 2\gamma_x(x,y,z)g_x g - \gamma(x,y,z)g_x^2) + \alpha_5(-\frac{\gamma_{yy}(x,y,z)}{2}g^2)$$

$$+ \alpha_6(-\frac{\gamma_{zz}(x,y,z)}{2}g^2)$$

$$+ \alpha_7(-\frac{\gamma_{xy}(x,y,z)}{2}g^2 - \gamma_y(x,y,z)gg_x)$$

$$+ \alpha_8(-\frac{\gamma_{xz}(x,y,z)}{2}g^2 - \gamma_z(x,y,z)gg_x)$$

$$+ \alpha_9(-\frac{\gamma_{yz}(x,y,z)}{2}g^2)$$

$$(6.43)$$

Substituting (6.41) and (6.42) into (6.40), we obtain Theorem 6.1.

Let X_T^u be the terminal wealth corresponding to the optimal equilibrium control law $\widehat{u}$. From (6.7), (6.10) and (6.11), we have

$$V(t, x, y_1, y_2, y_3) = E_{t,x,y,z}[X_T^{\widehat{U}}] - \frac{\gamma(y_1, y_2, y_3)}{2} Var_{t,x,y,z}[X_T^{\widehat{U}}],$$

$$(6.44)$$

$$f(t, x, y_1, y_2, y_3) = E_{t,x,y,z}[X_T^{\widehat{U}}] - \frac{\gamma(y_1, y_2, y_3)}{2} E_{t,x,y,z}[X_T^{\widehat{U}}]^2, \qquad (6.45)$$

$$g(t, x) = E_{t,x,y,z}[X_T^{\widehat{U}}] \qquad (6.46)$$

$$V(t, x, y_1, y_2, y_3) = f(t, x, y_1, y_2, y_3) + \frac{\gamma(y_1, y_2, y_3)}{2} g^2(t, x), \qquad (6.47)$$

where $y_1 = x, y_2 = y, y_3 = z$ in our case. From (6.22), we have

$$
\begin{aligned}
V_t &= f_t + \gamma g g_t, \\
V_x &= f_x + f_{y_1} + \frac{\gamma_x}{2} g^2 + \gamma g g_x, \\
V_y &= f_{y_2} + \frac{\gamma_y}{2} g^2, \\
V_z &= f_{y_3} + \frac{\gamma_z}{2} g^2, \\
V_{xx} &= f_{xx} + 2 f_{xy_1} + f_{y_1 y_1} + \frac{\gamma_{xx}}{2} g^2 + 2 \gamma_x g g_x + \gamma g_x^2 + \gamma g g_{xx}, \\
V_{yy} &= f_{y_2 y_2} + \frac{\gamma_{yy}}{2} g^2, \\
V_{zz} &= f_{y_3 y_3} + \frac{\gamma_{zz}}{2} g^2, \\
V_{xy} &= f_{xy_2} + f_{y_1 y_2} + \frac{\gamma_{xy}}{2} g^2 + \gamma y g g_x, \\
V_{xz} &= f_{xy_3} + f_{y_1 y_3} + \frac{\gamma_{xz}}{2} g^2 + \gamma_z g g_x, \\
V_{yz} &= f_{y_2 y_3} + \frac{\gamma_{yz}}{2} g^2.
\end{aligned}
\qquad (6.48)
$$

Substituting the above into Theorem 2, we obtain:

$$f_t + \gamma g g_t + \sup_{u \in U}\{\alpha_1[f_x + \gamma(x, y, z) g g_x] + \alpha_4[f_{xx} + \gamma(x, y, z) g g_{xx}]\} = 0, \quad (6.49)$$

where

$$\alpha_1 = [rX + (\mu(Y_t, Z_t) - r)u],$$

$$\alpha_4 = \frac{u^2 \sigma^2(Y_t, Z_t)}{2}.$$

or

$$f_t + \gamma g g_t + sup_u Z = 0. \tag{6.50}$$

The optimization problem in (6.25) requires $\frac{dZ}{du}|_{\hat{u}} = 0$. Consequently we have

$$\hat{u}_t = -\frac{(\mu(Y_t, Z_t) - r)(f_x + \gamma(x, y, z)g g_x)}{\sigma^2(Y_t, Z_t)(f_{xx} + \gamma(x, y, z)g g_{xx})}. \tag{6.51}$$

Remark 6.1. If μ, σ are constants, and the risk aversion depends only on the wealth, then the multiscale stochastic volatility reduces to a constant volatility and our result reduces to

$$\hat{u}_t = -\frac{(f_x + \gamma(x)g g_x)\beta}{(f_{xx} + \gamma(x)g g_{xx})\sigma^2} \tag{6.52}$$

which is the same as Zhou[47] if $\mu - r = \beta$, $\sigma(Y_t, Z_t) = \sigma$, $r(x, y, z) = r(x)$.

6.4 Analytical Results for a Special Case

In order to simplify the work but without loss of the generality, we choose the 'natural' risk aversion with the following expression $\gamma(x, y, z) = \dfrac{\gamma}{xyz}$. Hence (6.26) becomes

$$\hat{u}_t = -\frac{(\mu(Y_t, Z_t) - r)(f_x + \dfrac{\gamma}{xyz}g g_x)}{\sigma^2(Y_t, Z_t)(f_{xx} + \dfrac{\gamma}{xvr}g g_{xx})} \tag{6.53}$$

From the previous research[47], we conjecture that $\hat{u}$ is linear in X, namely $\hat{u}(t) = c(t)X$, and thus we have

$$E_{t,x,y,z}(X_T^{\hat{u}}) = p(t)x, \tag{6.54}$$

$$E_{t,x,y,z}[(X_T^{\hat{u}})^2] = q(t)x^2. \tag{6.55}$$

This leads to the Ansatz,

$$f(t, x, y_1, y_2, y_3) = p(t)x - \frac{\gamma}{2y_1y_2y_3}q(t)x^2, \tag{6.56}$$

$$g(t, x) = p(t)x. \tag{6.57}$$

$$f_t(t, x, y_1, y_2, y_3) = p'(t)x - \frac{\gamma}{2y_1y_2y_3}q'(t)x^2, \tag{6.58}$$

$$f_x(t, x, y_1, y_2, y_3) = p(t) - \frac{\gamma}{y_1y_2y_3}q(t)x, \tag{6.59}$$

$$f_{xx}(t, x, y_1, y_2, y_3) = -\frac{\gamma}{y_1y_2y_3}q(t), \tag{6.60}$$

$$g_t(t, x) = p'(t)x, \tag{6.61}$$

$$g_x(t, x) = p(t), \tag{6.62}$$

$$g_{xx}(t, x) = 0. \tag{6.63}$$

By substituting (6.31)-(6.38) into (6.26), we get

$$\hat{u}_t = c(t)x = \frac{(\mu(Y_t, Z_t) - r)[pyz + \gamma(p^2 - q)]}{\gamma q \sigma^2(Y_t, Z_t)}x \tag{6.64}$$

By substituting (6.31)-(6.38) into (6.15) and (6.16), we obtain

$$p'x - \frac{\gamma}{2y_1y_2y_3}q'x^2 + [rx + (\mu(Y_t, Z_t) - r)c(t)x](p - \frac{\gamma}{y_1y_2y_3}qx)$$
$$+ \frac{1}{2}c^2(t)x^2\sigma^2(Y_t, Z_t)(-\frac{\gamma}{y_1y_2y_3}q) = 0, \tag{6.65}$$

$$p'x + p[rx + (\mu(Y_t, Z_t) - r)c(t)x] = 0. \tag{6.66}$$

By splitting the above two equations, we have the following ordinary differential equations for the determination of p and q,

$$p' + p[r + (\mu(Y_t, Z_t) - r)c(t)] = 0. \tag{6.67}$$

$$q' + 2q\{([r + (\mu(Y_t, Z_t) - r)c(t)] + \frac{1}{2}c^2(t)\sigma^2(Y_t, Z_t)\} = 0, \tag{6.68}$$

$$p(T) = 1, \tag{6.69}$$

$$q(T) = 1. \tag{6.70}$$

Solving (6.42)-(6.45) we can express p and q in terms of $c(t)$ by

$$p = e^{\int_t^T \{r + (\mu(Y_t, Z_t) - r)c(s)\}ds},$$

$$q = e^{2\int_t^T \{r + (\mu(Y_t, Z_t) - r)c(s) + (\frac{1}{2}c^2(s)\sigma^2(Y_t, Z_t)\}ds}.$$

6.5   Concluding Remarks

This chapter studies the portfolio optimization of mean-variance utility with multiscale stochastic volatility. The sub-game perfect Nash equilibrium strategies and the extended Hamilton-Jacobi-Bellman equation have been used to establish the system of non-linear partial differential equations. By using asympototics approximations, the expansion of the value function which is the solution of the Hamilton-Jacobi-Bellman PDE for the Merton problem with constant parameters is derived explicitly in terms of the derivatives of the leading order value function. The optimal strategy is also given explicitly in terms of the derivatives of the leading order value function. Thus, we solve the portfolio optimization problem under the Mean-variance utility buy using an analytical method. It is found that multiscale volatility is more accurate in capturing the stock fluctuations. Moreover the model generates stochastic correlation between volatility and stock returns, provides more flexible modeling of the time variation in the smirk and flexible modeling of the volatility term structure.

CHAPTER 7

Summary and Future Research

7.1 Summary

In this thesis, we study various finance problems using stochastic calculus and optimal control theories. The research includes two major parts. The first part of the research, covered in chapters 3-4, focuses on equilibrium and option pricing, including European Option and American Option. The second part of research , covered in chapters 5-6, is Portfolio Optimization under mean variance utility. Through the research, many results and findings have been obtained which can be summarized in five aspects as detailed below

1) The developed model takes into account stochastic volatility and jump diffusion. A sophisticated option pricing model has been developed. In comparison with existing models, numerical methods have been established to get the approximate solutions. The risk premia under various conditions has also been established from the fundamental model. Moreover, the pricing kernel with stochastic volatility is established in this paper. The model has been validated with the S&P 500 index from 1985-2005 and it is found that the equilibrium equity premium in general equilibrium framework links not only the jump risk but also the stochastic volatility risk.

2) The price formula for American options in an incomplete market has been

established in which the dynamics of the underlying risky asset is driven by a jump diffusion process with stochastic volatility. The Radon-Nikodym derivative for the minimal martingale measure and consequently a linear complementarity problem (LCP) for American option price have been derived by employing a risk-minimization criterion. An iterative method has also been developed to solve the LCP problem for American put option prices. The result also shows the relationship between the payoff of the option price and the volatility.

3) The portfolio optimization problem under the mean-variance utility with state dependent risk aversion has been solved. A system of non-linear partial differential equations has been derived for the problem by using the sub-game perfect Nash equilibrium strategies and the extended Hamilton-Jacobi-Bellman equation. Some analytical results have been obtained for a special case where the risk aversion is proportional to the wealth. Comparison made between our model and previous ones shows that our model is more realistic.

4) The portfolio optimization of mean-variance utility with multiscale stochastic volatility has been studied by using the asymptotics approximations. The expansion of the value function has been established to approximate the solution of the Hamilton-Jacobi-Bellman PDE for the Merton problem with the constant parameters given explicitly in terms of the derivatives of the leading order value function. The portfolio optimization problem under the Mean-variance utility has been solved by using analytical method.

7.2 Further Research

In this thesis, our main work is the development of stochastic models for option pricing and portfolio optimization. It is observed that these models are very effective for solving all the problems under consideration. To make significant advancement, new and more accurate models could be derived for solving existing

stochastic problems and new unconventional problems arising in the study of real practical financial problems. Further possible improvements and advancements may be made in the following directions:

1) Different from the BlackScholes framework, we can use jump-diffusion to describe the price dynamics of the underlying asset. The market of our model is incomplete; that is, it is not possible to replicate the payoff of every contingent claim by a portfolio, and there are several equivalent martingale measures. Different measures may lead to different results. Hence the problem of how to choose a consistent pricing measure from the set of equivalent martingale measures need further exploration.

2) For time inconsistent problems, new methods could be explored in addition to the game theory. Other utilities might be adopted in order to construct more accurate and convincing financial models.

3) In order to give better explanation to existing and new financial problems, including, but not limited to, large and sudden movements in prices, heavy tails, volatility clustering, the incompleteness of markets, the concentration of losses in a few large downward moves, new forms of stochastic volatility should be investigated.

4) In this thesis, numerical methods have been used for solving stochastic differential equations, and also in estimating parameters in the stochastic models. To make significant advancement, new and more efficient computational algorithms could be derived for solving existing stochastic problems and new unconventional problems arising in the study of real world practical financial problems.

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