A first-order statistical smoothing approximation for the coherent wave field in random porous media

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An important dissipation mechanism for waves in randomly inhomogeneous poroelastic media is the effect of wave-induced fluid flow. In the framework of Biot's theory of poroelasticity, this mechanism can be understood as scattering from fast into slow compressional waves. To describe this conversion scattering effect in poroelastic random media, the dynamic characteristics of the coherent wavefield using the theory of statistical wave propagation are analyzed. In particular, the method of statistical smoothing is applied to Biot's equations of poroelasticity. Within the accuracy of the first-order statistical smoothing an effective wave number of the coherent field, which accounts for the effect of wave-induced flow, is derived. This wave number is complex and involves an integral over the correlation function of the medium's fluctuations. It is shown that the known one-dimensional (1-D) result can be obtained as a special case of the present 3-D theory. The expression for the effective wave number allows to derive a model for elastic attenuation and dispersion due to wave-induced fluid flow. These wavefield attributes are analyzed in a companion paper. © *2005 Acoustical Society of America.* [DOI: 10.1121/1.1871754]

I. INTRODUCTION

Understanding elastic wave attenuation and dispersion in porous fluid-saturated materials is important in such diverse applications as architectural acoustics, soil and rock mechanics, and exploration seismology. One major cause of elastic wave attenuation is viscous dissipation due to the flow of the pore fluid induced by the passing wave. Wave-induced fluid flow occurs as a passing wave creates local pressure gradients within the fluid phase and the resulting fluid flow is accompanied with internal friction until the pore pressure is equilibrated. The fluid flow can take place on various length scales: for example, from compliant fractures into the equant pores (so-called squirt flow[1–3]), or between mesoscopic heterogeneities like fluid patches in partially saturated rocks.[4–7]

Theoretical studies of the elastic wave attenuation due to wave-induced flow go back to the 1970s. In such studies wave propagation in an inhomogeneous porous medium is usually analyzed using Biot's equations of poroelasticity with spatially varying coefficients.[8] The first models of attenuation due to wave-induced flow considered flow caused by a regular assemblage of inhomogeneities of ideal shape such as two concentric spheres or flat slabs.[9–11] A general theory of wave propagation in heterogeneous porous media using the double-porosity approach was recently developed by Pride and Berryman[12] and Pride *et al.*[13] In the case of patchy saturation, the results of this theory coincide with those obtained earlier by Johnson[14] using a slightly different method. Although mathematical formulations in the theories Pride and Berryman[12] and Johnson[14] allow for arbitrary geometry of the inhomogeneities, closed-form expressions for attenuation and phase velocity can only be obtained for a regular system of simple geometrical shapes such as concentric spheres.

In real rocks heterogeneities are more likely to be spatially distributed in a random fashion. Therefore, it is desirable, as suggested by Lopatnikov and Gurevich,[15] to model wave-induced flow using the theory of waves in random media.[16] For one-dimensional (1-D) porous media, Gurevich and Lopatnikov[17] showed that elastic wave attenuation in a randomly layered porous medium differs significantly from attenuation in periodically layered porous media.[10,18] This suggests that the effects of three-dimensional inhomogeneities on elastic wave wave attenuation and dispersion in porous media may also be different for random and periodic spatial configurations. One approach to this problem is to consider a homogeneous porous medium with randomly distributed discrete, regularly shaped 3-D inclusions. However, so far only the attenuation due to conversion scattering at spherical inclusions from fast *P* waves into Biot's slow *P* wave is known.[19]

Our purpose in this paper is to analyze the effect of wave-induced fluid flow on the dynamic characteristics of coherent elastic waves propagating in a porous medium whose properties are continuous random functions of position. Our approach is based on the statistical wave theory, as applied to Biot's equations of poroelasticity.[8] We restrict our analysis to the case of mesoscopic inhomogeneities, that is, inhomogeneities whose characteristic size a is much larger than the typical size of pores or grains a_{pore} but, at the same time, much smaller than the wavelength of the propagating elastic wave, λ:

$$a_{pore} \ll a \ll \lambda. \tag{1}$$

In other words, we ignore pore-scale heterogeneities, which

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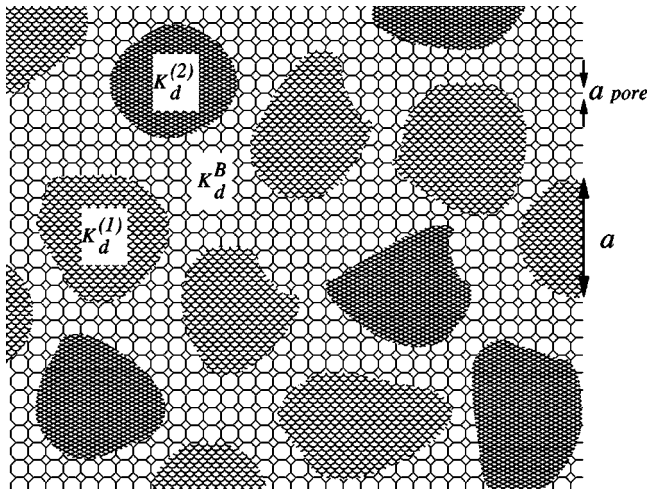[c)]URL: http://www.geophysics.curtin.edu.au

FIG. 1. Sketch of a heterogeneous porous medium where only the bulk modulus of the drained-framework-of-grains fluctuates. The homogeneous background medium is characterized by the bulk modulus K_d^B. There are inhomogeneities with higher bulk moduli, that is $K_d^{(1)} > K_d^B$, and other inhomogeneities with lower bulk modulus, $K_d^{(2)} < K_d^B$. The inhomogeneities have a length scale of a that is much larger than the typical pore scale a_{pore}. During the compression cycle of a wave there will be fluid flow from inhomogeneities with $K_d^{(2)}$ into the background and flow from the background into inhomogeneities with $K_d^{(1)}$. During the extension cycle of the wave the fluid flow becomes reversed.

allows the application of Biot's equations for a poroelastic continuum.[8] We model wave propagation in the inhomogeneous porous medium using Biot's equations of poroelasticity with randomly varying coefficients (see Fig. 1). By using a Green's function approach, these partial differential equations transform into a system of integral equations. This latter system is solved by means of the method of statistical smoothing which is widely used in problems of electromagnetic, acoustic, and elastic wave propagation.[20–22] More precisely, we employ a first-order statistical smoothing approximation that sometimes is referred to as a Bourret approximation.[16,23] For poroelastic media, this method was earlier employed to compute P-wave attenuation and dispersion due to conversion scattering from fast P waves into Biot's slow waves in a randomly layered porous medium.[17] We follow this strategy and analyze the conversion scattering from fast P waves into Biot's slow wave in 3-D randomly inhomogeneous porous media. Biot's slow wave is a highly dissipative wave mode. Therefore, the use of the first-order statistical smoothing approximation to the conversion scattering problem in Biot's equations of poroelasticity quantifies the dissipation of wave field energy due to energy transfer from the coherent component of the fast P wave into the dissipative slow P-wave mode. This is different from the usual application of the method of smoothing to energy conserving wave systems, where an apparent dissipation (so-called scattering attenuation) results from the energy transfer from the coherent component of the wave field into the incoherent component.

In light of previous results for randomly layered porous media, it is useful to introduce some simplifications from the outset. First of all, we restrict our analysis to low frequencies. Specifically, we assume that frequency ω is much smaller than the critical Biot frequency $\omega_B = \phi \eta / \kappa_0 \rho_f$:

$$\omega \ll \omega_B, \tag{2}$$

where ϕ and κ_0 denote the porosity and permeability of the composite material, while ρ_f and η are the density and viscosity of the pore fluid. Condition (2) implies that the standard Biot's viscoinertial attenuation and dispersion (the so-called Biot's global flow mechanism) is neglected and the P wave number k_p is real. Furthermore, at low frequencies the slow wave is much slower than the fast P wave and therefore the ratio of k_p to slow P wave number k_{ps} is a small parameter:

$$\frac{|k_p|}{|k_{ps}|} \ll 1. \tag{3}$$

Wherever applicable we make use of relation (3), being aware of the underlying low-frequency assumption.

The strategy of this paper is as follows. First, we apply the method of Green's functions to Biot's equations of poroelasticity in order to represent the wavefield due to point source excitation (Sec. II). Next, we derive an integral wavefield representation for the case when the coefficients in Biot's equations exhibit a randomly fluctuating component. This wavefield representation (also called the scattering equation) is then converted into an integral equation for Green's function for the inhomogeneous medium. In Sec. III we apply a perturbation method to compute an approximation for the mean of the Green's function. From this Green's function we derive and analyze an explicit expression for the effective, complex P wave number that accounts for the conversion scattering from P into slow P waves. The range of applicability of our theory is analyzed in Sec. IV. Finally, in Sec. V we show how the results of the aforementioned 1-D theory can be recovered as a limiting case of the more general 3-D theory. A detailed analysis of attenuation and dispersion as functions of frequency and the potential applicability of these wavefield signatures to interpret the effect of wave-induced flow in real rocks is presented in a companion paper.[24]

II. FORMULATION OF THE POROELASTIC SCATTERING PROBLEM

A. Green's function approach for Biot's equations of poroelasticity

In order to study dynamic effects of elastic wave propagation in porous media, we base our analysis on Biot's equation of poroelasticity.[8] Using index notation—summation over repeated indices is assumed and partial derivatives are denoted as $_{,i}$ or ∂_i—we can write the equations of motion in the frequency domain (the time-harmonic dependency $\exp(-i\omega t)$ is omitted),

$$\rho \omega^2 u_i + \rho_f \omega^2 w_i + \tau_{ij,j} = 0, \tag{4}$$

$$\rho_f \omega^2 u_i + q \omega^2 w_i - p_{,i} = 0, \tag{5}$$

where τ_{ij} is the total stress tensor, p the fluid pressure, while u_i and w_i are the components of the solid and relative fluid displacement vectors, respectively. The relative fluid displacement is defined as $w_i = \phi(U_i - u_i)$, where is U_i the

fluid displacement. The densities of the solid and fluid phase are denoted by ρ_g and ρ_f so that the bulk density is given by $\rho = \phi \rho_f + (1 - \phi) \rho_g$, where ϕ is porosity. The parameter q is defined as $q = i \eta / (\omega \kappa_0)$, where η is viscosity and κ_0 permeability. We note that this definition of q is a consequence of the low-frequency assumption (2).

In order to obtain a closed system of wave equations in the displacements u_i and w_i, we complement the equations of motion with the stress–strain relations for an isotropic poroelastic medium,[8]

$$\tau_{ij} = G[u_{i,j} + u_{j,i} - 2\delta_{ij}u_{j,j}] + \delta_{ij}[Hu_{j,j} + Cw_{j,j}], \quad (6)$$

$$p = -Cu_{j,j} - Mw_{j,j}. \quad (7)$$

Here G is the porous-material shear modulus, and H is the undrained, low-frequency P-wave modulus given by Gassmann's equation:

$$H = P_d + \alpha^2 M, \quad (8)$$

where

$$M = [(\alpha - \phi)/K_g + \phi/K_f]^{-1}. \quad (9)$$

In Eqs. (8)–(9), $P_d = K_d + 4/3G$ is the P-wave modulus of the drained frame, $\alpha = 1 - K_d/K_g$ is the Biot–Willis coefficient, $C = \alpha M$, and K_g, K_d, and K_f denote the bulk moduli of the solid phase, the drained frame, and the fluid phase, respectively. Symbol δ_{ij} is Kronecker's delta (the identity tensor).

It is expedient to write the above system of coupled wave equations in matrix form:

$$\begin{bmatrix} L_{ik}^{(1)} & L_{ik}^{(2)} \\ L_{ik}^{(3)} & L_{ik}^{(4)} \end{bmatrix} \cdot \begin{bmatrix} u_k \\ w_k \end{bmatrix} = \mathbf{0}, \quad (10)$$

where we defined the linear differential operators as follows:

$$L_{ik}^{(1)} = \rho \omega^2 \delta_{ik} + \partial_j G[\delta_{jk}\partial_i + \delta_{ik}\partial_j - 2\delta_{ij}\partial_k] + \partial_i H \partial_k, \quad (11)$$

$$L_{ik}^{(2)} = \rho_f \omega^2 \delta_{ik} + \partial_i C \partial_k, \quad (12)$$

$$L_{ik}^{(3)} = L_{ik}^{(2)}, \quad (13)$$

$$L_{ik}^{(4)} = q \omega^2 \delta_{ik} + \partial_i M \partial_k. \quad (14)$$

Note that in inhomogeneous media all the poroelastic parameters are functions of position. In the presence of point sources, the right-hand side of Eq. (10) can be written as

$$-\begin{bmatrix} F_i^0 \delta(r_i - r_i') \\ f_i^0 \delta(r_i - r_i') \end{bmatrix}, \quad (15)$$

where F_k^0 and f_k^0 represent constant forces applied to the bulk and fluid phase, respectively, and $\delta(r_i - r_i')$ denotes the Dirac delta function. The response of system (10) to point sources of the form (15) can be formulated as[25]

$$\begin{bmatrix} u_i \\ w_i \end{bmatrix} = \begin{bmatrix} G_{ik}^F & G_{ik}^f \\ G_{ik}^f & G_{ik}^w \end{bmatrix} \cdot \begin{bmatrix} F_k^0 \\ f_k^0 \end{bmatrix}, \quad (16)$$

where G_{ik}^F, G_{ik}^f, and G_{ik}^w denote the Green's tensors. Thus, the point source response of system (10) in an isotropic unbounded medium is described by three Green's tensors. Explicit expressions for the G_{ik}'s for a homogeneous medium

are given by Pride and Haartsen[25] and reproduced in Appendix A. The wavefields observed at position $\mathbf{r}$ due to arbitrary point sources F_i and f_i, applied at position $\mathbf{r}'$, can be expressed by a convolution equation of the form

$$\begin{bmatrix} u_i(\mathbf{r}) \\ w_i(\mathbf{r}) \end{bmatrix} = \int_{V'} d^3 \mathbf{r}' \begin{bmatrix} G_{ik}^F(\mathbf{r}-\mathbf{r}') & G_{ik}^f(\mathbf{r}-\mathbf{r}') \\ G_{ik}^f(\mathbf{r}-\mathbf{r}') & G_{ik}^w(\mathbf{r}-\mathbf{r}') \end{bmatrix} \cdot \begin{bmatrix} F_k(\mathbf{r}') \\ f_k(\mathbf{r}') \end{bmatrix}. \quad (17)$$

On the basis of Eq. (17), we now derive a wavefield representation in a randomly inhomogeneous medium.

B. The basic scattering equation

In randomly inhomogeneous porous media, all poroelastic parameters can be presented as random fields $X(\mathbf{r})$. To be more specific, we assume that each of these poroelastic parameters is the sum of a constant background value, $\bar{X}$, and a fluctuating part, $\tilde{X}(\mathbf{r})$, so that

$$X = \bar{X} + \tilde{X} = \bar{X}(1 + \varepsilon_X), \quad (18)$$

where $\varepsilon_X = \tilde{X}/\bar{X}$ denotes the relative fluctuations. The average over the ensemble of the realizations (denoted by $\langle \cdot \rangle$) of ε_X is assumed to be zero: $\langle \varepsilon_X \rangle = 0$. The spatial correlation function of two random fields is defined as

$$B_{XX}(\delta\mathbf{r}) = \langle \varepsilon_X(\mathbf{r} + \delta\mathbf{r})\varepsilon_X(\mathbf{r}) \rangle, \quad (19)$$

where the dependence of B on the difference vector $\delta\mathbf{r}$ only is a consequence of the assumption of statistically homogeneous random fields,[16] which we use throughout this paper. The variance of the random process ε_X will be denoted as $B_{XX}(0) = \langle \varepsilon_X^2 \rangle = \sigma_{XX}^2$. Using (18), the differential operators L_{ik} can be also decomposed as[21]

$$L_{ik} = \bar{L}_{ik} + \tilde{L}_{ik}, \quad (20)$$

where the perturbing operator $\tilde{L}_{ik}$ satisfies $\langle \tilde{L}_{ik} \rangle = 0$. As is typical for statistical wave problems, in the following we assume that the constant part $\bar{X}$ and statistical properties of the fluctuations ε_X are known. Therefore, rather than seeking the solution of Eq. (17) for a given realization ε_X, we seek statistical moments of the solution for given statistical properties of fluctuations.

The substitution of (20) into matrix equation (10) yields

$$\begin{bmatrix} \bar{L}_{ik}^{(1)} & \bar{L}_{ik}^{(2)} \\ \bar{L}_{ik}^{(2)} & \bar{L}_{ik}^{(4)} \end{bmatrix} \cdot \begin{bmatrix} u_k \\ w_k \end{bmatrix} = -\begin{bmatrix} \tilde{L}_{ik}^{(1)} & \tilde{L}_{ik}^{(2)} \\ \tilde{L}_{ik}^{(2)} & \tilde{L}_{ik}^{(4)} \end{bmatrix} \cdot \begin{bmatrix} u_k \\ w_k \end{bmatrix}, \quad (21)$$

In the most general case, the perturbing operators $\tilde{L}_{ik}$ contain fluctuations of all poroelastic moduli and densities. The right-hand side of (21) can be thought of as a source term in the homogeneous system (10) due to the presence of inhomogeneities (so-called secondary sources). Thus, Eq. (21) can be understood as an inhomogeneous equation with constant coefficients, whose formal solution can be written by substituting the source term into Eq. (17):

$$\begin{bmatrix} u_i \\ w_i \end{bmatrix} = \begin{bmatrix} u_i^0 \\ w_i^0 \end{bmatrix} + \int_V dV \begin{bmatrix} G_{ij}^F & G_{ij}^f \\ G_{ij}^f & G_{ij}^w \end{bmatrix} \cdot \begin{bmatrix} \tilde{L}_{jk}^{(1)} & \tilde{L}_{jk}^{(2)} \\ \tilde{L}_{jk}^{(2)} & \tilde{L}_{jk}^{(4)} \end{bmatrix} \cdot \begin{bmatrix} u_k \\ w_k \end{bmatrix}. \quad (22)$$

Equation (22) is the basic poroelastic scattering equation. The total wavefields u_i and w_i are composed of wavefields propagating in the homogeneous background medium, u_i^0 and w_i^0, and scattered wavefields (the second term). By definition, u_i^0 and w_i^0 satisfy the homogeneous equation (10). The scattered wavefields are represented by volume integrals whose kernels involve the Green's tensors and the secondary sources. The scattered wavefields vanish if there are no fluctuations in the medium parameters. The integration volume encompasses the inhomogeneous part of the medium, which in our case is the whole 3-D space. We note that Eq. (22) is closely related to the representation integral of the scattered field in an unbounded medium (for the poroelastic formulation of this theorem we refer to Norris[26] and Pride and Haartsen[25]). According to Eq. (17), the wavefields can be represented as a convolution of Green's tensors with the source function. Let us denote Green's tensors for the homogeneous background medium by $^0G_{ik}^{F,f,w}$ and for the inhomogeneous medium by $G_{ik}^{F,f,w}$. Substituting these wavefield representations into Eq. (22), we obtain an equation for the Green's tensors of the inhomogeneous medium,

$$
\begin{bmatrix} G_{im}^F & G_{im}^f \\ G_{im}^f & G_{im}^w \end{bmatrix} = \begin{bmatrix} {}^0G_{im}^F & {}^0G_{im}^f \\ {}^0G_{im}^f & {}^0G_{im}^w \end{bmatrix} + \int_V dV \begin{bmatrix} {}^0G_{ij}^F & {}^0G_{ij}^f \\ {}^0G_{ij}^f & {}^0G_{ij}^w \end{bmatrix}
$$
$$
\cdot \begin{bmatrix} \widetilde{L}_{jk}^{(1)} & \widetilde{L}_{jk}^{(2)} \\ \widetilde{L}_{jk}^{(2)} & \widetilde{L}_{jk}^{(4)} \end{bmatrix} \cdot \begin{bmatrix} G_{km}^F & G_{km}^f \\ G_{km}^f & G_{km}^w \end{bmatrix}. \tag{23}
$$

In order to simplify the equations that follow, we introduce a shorthand notation. The latter equation can be symbolically rewritten as

$$
\mathbf{G} = \mathbf{G}^0 + \int \mathbf{G}^0 \widetilde{\mathbf{L}} \mathbf{G}, \tag{24}
$$

where $\mathbf{G}$, $\mathbf{G}^0$, and $\widetilde{\mathbf{L}}$ represent matrices, whose elements are tensors of rank two, and matrix multiplication rules apply. In (24) we also omitted the integration volume for brevity.

III. FIRST-ORDER STATISTICAL SMOOTHING OF BIOT'S EQUATIONS OF POROELASTICITY

A. Mean Green's tensor

We will now analyze Eq. (24) using a statistical approach. Since the matrix of perturbing operators $\widetilde{\mathbf{L}}$ in Eq. (24) contains fluctuating medium parameters, the resulting matrix of Green's tensors also contains randomly fluctuating elements. Because individual realizations of the random wavefields are never known, it is natural to analyze the statistical moments of $\mathbf{G}$. Solving Eq. (24) by iteration we obtain the scattering series

$$
\mathbf{G} = \mathbf{G}^0 + \int \mathbf{G}^0 \widetilde{\mathbf{L}} \mathbf{G}^0 + \int \int \mathbf{G}^0 \widetilde{\mathbf{L}} \mathbf{G}^0 \widetilde{\mathbf{L}} \mathbf{G}^0 + \int \int \int \cdots. \tag{25}
$$

Averaging this equation by the ensemble of realizations and regrouping the scattering terms yields

$$
\overline{\mathbf{G}} = \mathbf{G}^0 + \int \int \mathbf{G}^0 \mathbf{Q} \overline{\mathbf{G}}, \tag{26}
$$

where $\overline{\mathbf{G}} = \langle \mathbf{G} \rangle$ is the matrix of mean Green's tensors, and $\mathbf{Q}$ is the matrix operator defined as

$$
\mathbf{Q} = \begin{bmatrix} Q_{ik}^{(1)} & Q_{ik}^{(2)} \\ Q_{ik}^{(3)} & Q_{ik}^{(4)} \end{bmatrix} = \left\langle \widetilde{\mathbf{L}} \mathbf{G}^0 \widetilde{\mathbf{L}} + \int \widetilde{\mathbf{L}} \mathbf{G}^0 \widetilde{\mathbf{L}} \mathbf{G}^0 \widetilde{\mathbf{L}} + \int \cdots \right\rangle. \tag{27}
$$

Operator $\mathbf{Q}$ given by Eq. (27) corresponds to the kernel-of-mass operator in the acoustic formulation.[16] The linear integral equation in $\overline{\mathbf{G}}$ [Eq. (26)] is the poroelastic analog of the Dyson equation (see also Gurevich and Lopatnikov,[27] where an analogous equation for the mean field is derived). It is not possible to obtain an exact solution of Eq. (26). A first-order statistical smoothing consists in the first-order truncation of the infinite series expression for the operator $\mathbf{Q}$. Then, we obtain the following approximation for the mean Green's tensor:

$$
\overline{\mathbf{G}} = \mathbf{G}^0 + \int \int \mathbf{G}^0 \langle \widetilde{\mathbf{L}} \mathbf{G}^0 \widetilde{\mathbf{L}} \rangle \overline{\mathbf{G}} \tag{28}
$$

$$
= \mathbf{G}^0 + \int \int \mathbf{G}^0 \mathbf{Q}^B \overline{\mathbf{G}}. \tag{29}
$$

The truncation of the series (27) implies that the first-order statistical smoothing is valid when $|\varepsilon_X| \ll 1$, i.e., when the absolute value of the relative fluctuations of X is a small parameter. Note also that the elements of matrix operator $\mathbf{Q}^B$ only contain terms involving the second statistical moment of the fluctuating parts of the $\widetilde{L}_{ik}$'s, that is, they are of the order $O(\varepsilon^2)$. Higher-order correlations are neglected within the accuracy of the first-order statistical smoothing approximation.

Since Eq. (29) contains a double volume convolution, it is expedient to work with its spatial Fourier transform:

$$
\overline{\mathbf{g}} = \mathbf{g}^0 + (8\pi^3)^2 \mathbf{g}^0 \mathbf{q} \overline{\mathbf{g}}, \tag{30}
$$

where $\overline{\mathbf{g}}$, $\mathbf{g}^0$, and $\mathbf{q}$ denote the spatial Fourier transforms of $\overline{\mathbf{G}}$, $\mathbf{G}^0$, and $\mathbf{Q}^B$, respectively (see Appendix A for a definition of the Fourier transform pair). Equation (30) is of the same form as the equation for the mean Green's function in acoustic random media. However, in contrast to the acoustic case, Eq. (30) is not a simple algebraic equation for the mean Green's tensors, but a system of four tensorial equations for the three unknown mean Green's tensors. Formally, we can express the solution of (30) in the form $\overline{\mathbf{g}} = \mathbf{W}^{-1} \mathbf{g}^0$, where $\mathbf{W}^{-1}$ is the inverse of the matrix $\mathbf{W} = \mathbf{I} - (8\pi^3)^2 \mathbf{g}^0 \mathbf{q}$ with the identity tensor $\mathbf{I}$. However, the computation of $\mathbf{W}^{-1}$ is cumbersome. Instead, we are looking for a more feasible way to compute some elements of $\overline{\mathbf{g}}$. Carrying out the necessary matrix multiplications in (30), we find that this system splits up into two pairs of coupled equations. Since we are only interested in the characteristics of the fast P wave, which are exclusively contained in the Green's tensor $\overline{g}^F$ [see also Eqs. (16) and (A1)], we analyze only those two equations that involve $\overline{g}_{ik}^F$. We obtain

$$
\overline{g}^F = g^F + (8\pi^3)^2 [g^F q^{(1)} \overline{g}^F + g^F q^{(2)} \overline{g}^f + g^f q^{(3)} \overline{g}^F
$$
$$
+ g^f g^{(4)} \overline{g}^f], \tag{31}
$$

$$\bar{g}^f = g^f + (8\pi^3)^2 [g^f q^{(1)} \bar{g}^F + g^f q^{(2)} \bar{g}^f + g^w q^{(3)} \bar{g}^F$$
$$+ g^w q^{(4)} \bar{g}^{f}], \tag{32}$$

where we omitted subscripts for brevity. The quantities g without an upper bar denote the background space Green's tensors. Since all quantities $q^{(i)}$ ($i = 1, ..., 4$) are of the order $O(\varepsilon^2)$, $\bar{g}^f$ is also of the order $O(\varepsilon^2)$. Inserting the expression for $\bar{g}^f$ [Eq. (32)] into Eq. (31) and neglecting terms of higher order than $O(\varepsilon^2)$, we obtain

$$\bar{g}^F = g^F + (8\pi^3)^2 [q^F q^{(1)} \bar{g}^F + g^F q^{(2)} g^f + g^f q^{(3)} \bar{g}^F$$
$$+ g^f q^{(4)} g^{f}]. \tag{33}$$

Equation (33) is an implicit equation for the mean Green's tensor $\bar{g}^F$. Because of its tensorial character, an explicit solution for $\bar{g}^F$ is still difficult to construct. Note, however, that we are not interested in the mean Green's tensor itself but only in a mean P wave number contained in $\bar{g}^F$.

B. Effective wave number of the fast *P*-wave

In order to extract an effective wave number from Eq. (33) we have to introduce further simplifications. Because of the assumption of small fluctuations in the medium parameters ($\varepsilon \ll 1$), we can expect that the fluctuations of the wavefield are also small if the wavelengths are much larger than the size of the inhomogeneities. Then, we can assume that mean Green's tensor $\bar{g}^F_{ik}(\mathbf{K})$ is of the same functional form as a background Green's tensor $g^F_{ik}(\mathbf{K})$ given by Eq. (A8), however, involving some effective P-wave number (and also effective bulk density).

Let us construct a simple case, where most of the Green's tensor components vanish. This can be achieved using the following procedure: We consider an incoming, plane P wave propagating in the x_3 direction (i.e., only the displacement component u_3 is nonzero). The resulting coherent P wave in the inhomogeneous medium will also propagate in the x_3 direction. Therefore, only the tensor components $i = j = 3$ of g^F_{ij} need to be analyzed. Noting that in this case the Green's tensor $g^F_{ik}(\mathbf{K})$ yields the largest contribution for the spatial wave number $K = k_p$, we can approximate the full Green's tensor (A8) by

$$g^F_{33} \approx \frac{-1}{8\pi^3 \rho \omega^2} \left(1 + \frac{K^2}{k_p^2 - K^2} \right). \tag{34}$$

We assume that the mean Green's tensor component is given by

$$\bar{g}^F_{33} \approx \frac{-1}{8\pi^3 \bar{\rho} \omega^2} \left(1 + \frac{K^2}{\bar{k}_p^2 - K^2} \right), \tag{35}$$

where $\bar{k}_p$ is the searched-for effective P wave number. Substituting Eq. (34) for g^F and Eq. (35) for $\bar{g}^F$ into Eq. (33) we obtain, after algebraic manipulations,

$$\bar{k}_p \approx k_p \left(1 + \frac{4\pi^3}{\rho \omega^2} q^{(1)}_{33} \right). \tag{36}$$

Here we neglected terms that contain combinations of the tensor components $q^{(i)}_{33}$. This introduces no additional inac-

curacy because higher-order correlations are neglected within the accuracy of the first-order statistical smoothing $O(\varepsilon^2)$.

The remaining problem is the evaluation of $q^{(1)}_{33}$ in (36), or equivalently, of $Q^{(1)}_{33}$ in space domain. In explicit form, from the first term in the expansion of $\mathbf{Q}$ as given by Eq. (27), we obtain

$$Q^{(1)}_{ik}(\mathbf{r}' - \mathbf{r}'') = \langle \tilde{L}^{(1)}_{ij}(\mathbf{r}') G^F_{jl}(\mathbf{r}' - \mathbf{r}'') \tilde{L}^{(1)}_{lk}(\mathbf{r}'')$$
$$+ 2\tilde{L}^{(1)}_{ij}(\mathbf{r}') G^f_{jl}(\mathbf{r}' - \mathbf{r}'') \tilde{L}^{(2)}_{lk}(\mathbf{r}'')$$
$$+ \tilde{L}^{(2)}_{ij}(\mathbf{r}') G^w_{jl}(\mathbf{r}' - \mathbf{r}'') \tilde{L}^{(2)}_{lk}(\mathbf{r}'') \rangle, \tag{37}$$

where for statistically homogeneous random media both Q_{ik} and G_{ik} depend only on the difference vector $\mathbf{r}' - \mathbf{r}''$. It is interesting to note that in the elastic limit, only the first term of $Q^{(1)}_{ik}$ is nonzero. In the poroelastic case we have to analyze all three terms. Expression (37) involves the perturbing operators $\tilde{L}^{(1)}_{ij}$ and $\tilde{L}^{(2)}_{ij}$ (but not $\tilde{L}^{(4)}_{ij}$). Let us now specify the perturbing operators resulting from (11) and (12),

$$\tilde{L}^{(1)}_{ik} = \tilde{\rho}\omega^2 \delta_{ik} + \partial_j \tilde{G}[\delta_{jk}\partial_i + \delta_{ik}\partial_j - 2\delta_{ij}\partial_k] + \partial_i \tilde{H}\partial_k, \tag{38}$$

$$\tilde{L}^{(2)}_{ik} = \tilde{\rho}_f \omega^2 \delta_{ik} + \partial_i \tilde{C}\partial_k, \tag{39}$$

for a particular situation in which we can find an explicit analytical expression for the right-hand side of Eq. (37). In the following, we neglect fluctuations of the densities ρ and ρ_f. This is possible because of the restriction to low frequencies. It can be shown that incorporation of density fluctuations yields a correction to the background P wave number [the second term in Eq. (36)], which scales with ω^3, whereas the other fluctuations result in a ω^2 dependence, as shown below. This simplification is also in accordance with the 1-D result,[17] where the density fluctuations do not appear in the final expression for the effective P wave number though these fluctuations were not neglected *a priori*. These simplifications yield

$$\tilde{L}^{(1)}_{ik} = \partial_k \tilde{G}\partial_i + \partial_j \delta_{ik} \tilde{G}\partial_j - 2\partial_i \tilde{G}\partial_k + \partial_i \tilde{H}\partial_k, \tag{40}$$

$$\tilde{L}^{(2)}_{ik} = \partial_i \tilde{C}\partial_k. \tag{41}$$

A detailed computation of the three $Q^{(1)}_{ik}$ terms in Eq. (37) using the perturbing operators (40) and (41) is provided in Appendix B. The result in the wave number domain can be represented as

$$q^{(1)}_{33} = q^{HH} + q^{HG} + q^{HC} + q^{GG} + q^{GC} + q^{CC}, \tag{42}$$

where

$$q^{HH} = \frac{1}{8\pi^3} k_p^2 \left(\frac{H^2}{P_d} B_{HH}(0) \right.$$
$$\left. + \frac{C^2}{N} k_{ps}^2 \int_0^\infty r B_{HH}(r) \exp[ik_{ps}r]dr \right), \tag{43}$$

$$q^{HG} = -\frac{1}{3\pi^3} k_p^2 \left(\frac{GH}{P_d} B_{HG}(0) \right.$$

$$\left. + \frac{\alpha^2 MG}{P_d} k_{ps}^2 \int_0^\infty r B_{PM}(r) \exp[ik_{ps}r] dr \right), \quad (44)$$

$$q^{HC} = -\frac{1}{4\pi^3} \frac{C^2}{N} k_p^2 \left(B_{HC}(0) \right.$$

$$\left. + k_{ps}^2 \int_0^\infty r B_{HC}(r) \exp[ik_{ps}r] dr \right), \quad (45)$$

$$q^{GG} = \frac{1}{15\pi^3} \frac{G}{NH^2} k_p^2 \left([4C^2G + 4NHG + NH^2] B_{GG}(0) \right.$$

$$\left. + 4C^2 G k_{ps}^2 \int_0^\infty r B_{GG}(r) \exp[ik_{ps}r] dr \right) \quad (46)$$

$$q^{GC} = \frac{1}{3\pi^3} \frac{\alpha^2 MG}{P_d} k_p^2 \left(B_{GC}(0) \right.$$

$$\left. + k_{ps}^2 \int_0^\infty r B_{GC}(r) \exp[ik_{ps}r] dr \right), \quad (47)$$

$$q^{CC} = +\frac{1}{8\pi^3} \frac{C^2}{N} k_p^2 \left(B_{CC}(0) \right.$$

$$\left. + k_{ps}^2 \int_0^\infty r B_{CC}(r) \exp[ik_{ps}r] dr \right). \quad (48)$$

Here, B_{HH}, B_{HC}, B_{HG}, B_{GG}, B_{GC}, B_{CC} denote the (cross-) correlation functions of the random fields $\tilde{H}$, $\tilde{G}$, and $\tilde{C}$ defined by Eq. (19). In the derivation of the $Q_{ik}^{(1)}$ terms we assumed that the random media realizations are statistically isotropic and therefore the correlation functions depend only on r. The upper bar denoting the background properties is omitted. It is important to note that Eqs. (43)–(48) will provide a correction term to the background P wave number [see Eq. (36)], which exclusively accounts for conversion scattering into Biot's slow wave. The separation of terms in the operator $Q_{ik}^{(1)}$ that describe the scattering process from fast into Biot's slow P wave and all other scattering processes is possible because of the low-frequency assumption [Eq. (3)]. In other words, because of the large separation between the two characteristic frequencies for ordinary elastic scattering and conversion scattering into Biot's slow wave, we can distinguish between the different scattering processes, using the frequency dependency of the corresponding terms in the scattering equation (see also Appendix B).

We will now assume that all correlation functions are of the same functional form and only differ by theirs variances, i.e., $B_{XY} = \sigma_{XY}^2 B(r)$ with $B(0) = 1$ and $B(\infty) = 0$. Substituting then expressions (43)–(48) into Eq. (36) we obtain the final result for the effective P wave number,

$$\bar{k}_p = k_p \left(1 + \Delta_2 + \Delta_1 k_{ps}^2 \int_0^\infty r B(r) \exp[ik_{ps}r] dr \right), \quad (49)$$

with the dimensionless coefficients

$$\Delta_1 = \frac{\alpha^2 M}{2P_d} \left(\sigma_{HH}^2 - 2\sigma_{HC}^2 + \sigma_{CC}^2 + \frac{32}{15} \frac{G^2}{H^2} \sigma_{GG}^2 \right.$$

$$\left. - \frac{8}{3} \frac{G^2}{H} \sigma_{HG}^2 + \frac{8}{3} \frac{G}{H} \sigma_{GC}^2 \right), \quad (50)$$

$$= \frac{\alpha^2 M}{2P_d} \left(\left\langle \left(\varepsilon_H - \frac{4}{3} \frac{G}{H} \varepsilon_G - \varepsilon_C \right)^2 \right\rangle + \frac{16}{45} \frac{G^2}{H^2} \sigma_{GG}^2 \right), \quad (51)$$

$$\Delta_2 = \Delta_1 + \frac{1}{2} \sigma_{HH}^2 - \frac{4}{3} \frac{G}{H} \sigma_{HG}^2 + \left(\frac{4G}{H} + 1 \right) \frac{4}{15} \frac{G}{H} \sigma_{GG}^2. \quad (52)$$

The structure of the effective P wave number can be explained as follows: Due to the presence of random inhomogeneities, there are two terms added to the background wave number k_p. The first term, Δ_2, is frequency independent and consists in a weighted sum of the variances of the random fields $\tilde{H}$, $\tilde{G}$, and $\tilde{C}$. The second term is frequency dependent and contains an integral over the correlation function multiplied by a weighted sum of the variances, Δ_1. It is important to note that the expression for $\bar{k}_p$ describes only the process of conversion scattering from fast into slow P waves. The contribution of purely elastic scattering is left out. The corresponding result would include additional terms involving the correlation functions B_{HH}, B_{GG}, and B_{HG} that describe the elastic scattering (P to P and S waves) and produces the typical Rayleigh frequency dependence for attenuation.[23] In other words, the second and third terms of the effective P wave number (49) correspond to the mechanism of wave-induced fluid flow only. Therefore, an analysis of the properties of $\bar{k}_p$ gives insight into the relationship between the properties of elastic waves and wave-induced flow. By definition, the real part of $\bar{k}_p$ is related to the phase velocity v through $v(\omega) = \omega/\Re\{\bar{k}_p\}$, whereas the imaginary part yields the attenuation coefficient γ: $\gamma(\omega) = \Im\{\bar{k}_p\}$. From the structure of Eq. (49), it can be seen that the phase velocity of the coherent wave in an equivalent medium is smaller than in the background medium. By inspection we find that $\gamma > 0$, that is, the coherent wave is exponentially damped. A detailed analysis of attenuation and velocity dispersion in an equivalent medium is presented in a companion paper.

IV. RANGE OF APPLICABILITY

The derivation of the dispersion relation (49) and the associated results on attenuation and phase velocity dispersion are based on several assumptions that restrict their range of applicability. The main restriction on our results is due to the use of the first-order statistical smoothing approximation. In the acoustic case the applicability condition of this approximation can be written as[16]

$$\sigma_n^2 (ka)^2 \ll 1, \quad (53)$$

where σ_n^2 denotes the variance of the velocity fluctuations, k is the wave number of a wave propagating in the homogeneous background, and a is a characteristic length scale associated with the size of the inhomogeneities. Condition (53) ensures that the correction terms to the background wave

number [similar to those in Eq. (49)] are small. More precisely, this condition was obtained by analyzing the next term in the equation for the kernel-of-mass operator Q and requiring that the difference between the correction terms from the first-order statistical smoothing and that of the next, higher-order approximation is small. It is natural to assume that a similar condition must hold in the poroelastic case.

Using the first-order statistical smoothing the kernel-of-mass operator $\mathbf{Q}$ can be approximated by $\mathbf{Q}^B \approx \langle \widetilde{\mathbf{L}} \mathbf{G}^0 \widetilde{\mathbf{L}} \rangle$ [Eq. (29)]. The next term in the infinite series for $\mathbf{Q}$ reads as

$$\mathbf{Q}^{(2)} = \langle \widetilde{\mathbf{L}} \overline{\mathbf{G}} \widetilde{\mathbf{L}} \rangle, \tag{54}$$

where $\overline{\mathbf{G}}$ is the matrix of the mean Green's tensors found from the first-order statistical smoothing. To compute $\mathbf{Q}^{(2)}$, we assume that there is no multiple scatter of slow P waves into slow P waves and, therefore, the mean Green's tensor component $\overline{g}_{33}^F$ is still of the form (35). Then we can derive from Eq. (33) an equation for the effective P wave number similar to (36), but now involving $q_{33}^{(1)}$ determined from (54). The computation of $q_{33}^{(1)}$ according to (54) is analogous to that shown in the previous section and results in an effective P wave number $\overline{k}_{p2}$:

$$\overline{k}_{p2} \approx k_p + \Delta_2 k_p + \Delta_1 \overline{k}_p k_{ps}^2 \int_0^\infty r B(r) \exp[i k_{ps} r] dr, \tag{55}$$

with $\overline{k}_p$ given by (49). Let us compare this result with the result for the effective wave number using first-order statistical smoothing [Eq. (49)]. Assuming that $\overline{k}_p$ can be represented as a sum of background wave number k_p and a correction term Δk, the difference between (55) and (49) becomes

$$\overline{k}_{p2} - \overline{k}_p = \Delta k \Delta_1 k_{ps}^2 \int_0^\infty r B(r) \exp[i k_{ps} r] dr. \tag{56}$$

A necessary condition for the significance of the correction term using the first-order statistical smoothing Δk is the smallness of the difference (56) compared to Δk:

$$\left| \frac{\overline{k}_{p2} - \overline{k}_p}{\Delta k} \right| \ll 1. \tag{57}$$

Using for instance the correlation function $B(r) = \exp(-|r|/a)$, we obtain, from (56) and (57),

$$\max\{\Delta_1(|k_{ps}|a)^2, \Delta_2\} \ll 1. \tag{58}$$

Relation (58) gives an estimate of the applicability of the first-order statistical smoothing approximation. That is, our results can be used in the case of weak-contrast media and weak wavefield fluctuations.

Physically, the existence of slow P waves is associated with the equilibration of pore pressure that at low frequencies is controlled by the diffusion equation with diffusion length[18] $\lambda_D = \sqrt{\kappa_0 N / \omega \eta}$. Therefore, the interplay between λ_D and the correlation length a defines two different regimes. If $\lambda_D > a$ then the wave-induced pressure disturbance is equilibrated. This relation holds for low frequencies and thus defines the low-frequency or relaxed regime. Conversely, if $\lambda_D < a$ then there is not enough time for the pore

pressure to relax. This unrelaxed case occurs at high frequencies. A characteristic frequency that separates these low- and high-frequency regimes can be written as

$$\omega_c = \frac{\kappa_0 N}{a^2 \eta}. \tag{59}$$

Note that this characteristic frequency is identical to the frequency where maximal attenuation occurs.[24] The existence of both regimes within our wavefield approximation, which is based on the low-frequency approximation of Biot's equations, is only possible if

$$\omega_c \ll \omega_B, \tag{60}$$

which imposes the additional condition for the average size of the mesoscopic inhomogeneities,

$$a^2 \gg \frac{\kappa_0 N}{\eta \omega_B}. \tag{61}$$

Conditions (58) and (61) define the range of applicability of our results. Whereas condition (60) is a necessary condition for the validity of our results, there exists another condition for the observability of wave-induced flow. Only if ω_c is sufficiently different from the characteristic frequency, where elastic scattering of the fast wave modes dominates (i.e., $P \to P$ and $P \to S$), it is possible to distinguish between the two processes. Elastic scattering will dominate at frequency $\omega_S = c/a$, where c is the phase velocity of the fast P-wave mode. Therefore, the observability condition $\omega_S \gtrsim \omega_c$ imposes $a^2 \gtrsim \kappa_0^2 N \rho / (\eta^2 H)$.

V. REDUCTION TO THE 1-D CASE

In order to further substantiate our results, we analyze their connection with the known 1-D result. For a system of randomly layered porous media an effective, complex P wave number was obtained earlier by Gurevich and Lopatnikov;[17] see their equation (56). This 1-D result is also based on the method of (first-order) statistical smoothing. If only the parameter M fluctuates, the 1-D result can be rewritten as

$$\overline{k}_p^{1D} = k_p^* \left[1 + \frac{1}{2} i \frac{\alpha^2 M}{P_d} k_{ps}^* \int_0^\infty dz \, B_{MM}(z) \exp[i k_{ps}^* z] \right], \tag{62}$$

where k_p^* and k_{ps}^* denote effective wave numbers that involve effective parameters H^*, N^*, and effective densities ρ^*, q^*. These effective parameters can be computed according to the so-called poroelastic Backus averaging.[17,18,28] By neglecting terms higher than $O(\epsilon^2)$—which is the overall precision of the weak-fluctuation approximation—we can replace k_{ps}^* with k_{ps}. The effective parameter H^* is computed according to

$$H^* = P_d + \alpha^2 M^* = P_d + \alpha^2 \left\langle \frac{1}{M} \right\rangle^{-1} \approx H \left[1 - \frac{\alpha^2 M}{H} B_{MM}(0) \right]. \tag{63}$$

Then the effective P wave number k_p^* can be expressed as

$$k_p^* = \omega \sqrt{\frac{\rho^*}{H^*}} \approx k_p \left[1 + \frac{1}{2} \frac{\alpha^2 M}{H} B_{MM}(0) \right]. \tag{64}$$

Using (64) Eq. (62) can be written in the form

$$\bar{k}_p^{1D} = k_p \left[1 + \frac{\alpha^2 M}{2H} B_{MM}(0) \right.$$

$$\left. + \frac{1}{2} i \frac{\alpha^2 M}{P_d} k_{ps} \int_0^\infty dz \, B_{MM}(z) \exp[ik_{ps}z] \right]. \tag{65}$$

We will now show that the 1-D result (65) can be recovered from the 3-D result. To do so, we analyze the expression for the Q_{ik} tensor component in the wave number domain q_{33} that is of the form [cf. Eq. (B5)]

$$q_{33} \propto \int d^3\mathbf{r} \, B(\mathbf{r}) \frac{\exp[ik_{ps}R]}{R} \exp[-i\mathbf{K}\cdot\mathbf{r}], \tag{66}$$

where we used only the poroelastic part of $G_{ij,ij}^F$ and $R = |\mathbf{r}|$. Note that in order to obtain Eq. (66) it is not necessary to assume statistical isotropy. We are now considering the following limiting situation.

(a) To degenerate the 3-D random medium into a 1-D random medium we stretch the correlation lengths perpendicular to the direction of wave propagation, $a_\perp$, to infinity so that the correlation function becomes only a function of z with parameter $a_\parallel$, i.e., the correlation length parallel to the direction of wave propagation. Obviously, if the wave propagates mainly in the z direction we can also write the spatial wave vector as $\mathbf{K} \approx (0,0,k_p)^T$.

(b) Since in such a 1-D random medium there are only two directions of wave propagation ($\pm z$), we can use the small-angle approximation[16] (or Fresnel approximation) of the propagator-like term $\exp[ik_{ps}R]/R$:

$$\frac{\exp[ik_{ps}R]}{R} \approx \frac{\exp[ik_{ps}z]}{z} \exp\left[\frac{ik_{ps}r_t^2}{2z} \right], \tag{67}$$

where r_t denotes the absolute value of the transverse coordinate vector $\mathbf{r}_t = (x,y)^T$. Introducing the simplifications proposed in (a) and (b) into Eq. (66) and using cylindrical coordinates $d^3\mathbf{r} = dz \, dr_t \, d\phi \, r_t$, we obtain

$$q_{33} \propto 4\pi \int_0^\infty dz \, B(z) \frac{\exp[ik_{ps}z]}{z} \exp[-ik_p z]$$

$$\times \int_0^\infty dr_t \, r_t \exp\left[\frac{ik_{ps}r_t^2}{2z} \right]. \tag{68}$$

The low-frequency condition $k_p/k_{ps} \ll 1$ means that we can replace the exponential $\exp[-ik_p z]$ by 1. The integral with respect to r always converges because k_{ps} is complex and produces an exponential decreasing multiplier. After performing the integration and substituting the result into Eq. (36), we obtain

$$\bar{k}_p = k_p \left[1 + \frac{\alpha^2 M}{2H} B_{MM}(0) \right.$$

$$\left. + \frac{1}{2} i \frac{\alpha^2 M}{P_d} k_{ps} \int_0^\infty dz \, B_{MM}(z) \exp[ik_{ps}z] \right], \tag{69}$$

which is identical to Eq. (65).

In conclusion, the 1-D effective P wave number can be exactly obtained from the 3-D result. We note that we have only considered the particular case of M fluctuations. However, the approach can be also applied in the case of other parametrizations.

VI. CONCLUSIONS

In the framework of the theory of wave propagation in random media we analyzed the properties of the coherent wave propagating in poroelastic random media. Neglecting the ordinary elastic scattering, we only accounted for conversion scattering from fast P into Biot's slow P wave. This process of conversion scattering is equivalent to the mechanism of pore-pressure relaxation due to wave-induced perturbations. Thus, our results describe the relationship between the dynamic properties of the coherent wavefield and the mechanism of wave-induced fluid flow. In particular, we have derived an explicit expression for the effective P wave number [Eq. (49)] by applying first-order statistical smoothing of Biot's equations of poroelasticity with randomly varying coefficients. This wave number is complex and involves an integral over the correlation properties of the medium fluctuations. From this result it can be seen that the associated phase velocity is smaller than in the homogeneous background medium and that the wave is exponentially damped. We have shown that the previously reported effective P wave number for randomly layered media[17] can be derived from the more general 3-D result.

Our approach is limited to the case of weak-contrast, mesoscopic inhomogeneities [conditions (58) and (61)]. In this paper we focused the analysis to wave propagation in statistically isotropic random media. However, the results can be probably generalized to the case of statistically anisotropic random media. An advantage of the statistical approach is its flexibility to handle complex geometrical distributions of the inhomogeneities. Only the spatial correlation of the fluctuations need to be known in order to compute the dynamic wavefield attributes. A detailed analysis of frequency dependencies of attenuation and dispersion due to wave-induced fluid flow and the potential applicability of the results to real rocks will be the subject of a companion paper.[24]

APPENDIX A: POROELASTIC GREEN'S TENSORS

The complete set of Green's tensors for a homogeneous and isotropic poroelastic continuum—including electroseismic coupling—was derived by Pride and Haartsen.[25] We reproduce only those parts of the Green's tensors, which are related to poroelastic wave propagation. Furthermore, we can simplify these tensors for low frequencies with $|k_p|/|k_{ps}| \ll 1$. We obtain

$$G_{ij}^F(\mathbf{r}-\mathbf{r}_0) = \frac{1}{4\pi\rho\omega^2}\left([k_s^2\delta_{ij}+\partial_i\partial_j]\frac{e^{ik_sR}}{R}-\partial_i\partial_j\frac{e^{ik_pR}}{R}\right)$$
$$-\frac{C^2}{H^2}\frac{1}{4\pi q\omega^2}\partial_i\partial_j\frac{e^{ik_{ps}R}}{R}, \tag{A1}$$

$$G_{ij}^f(\mathbf{r}-\mathbf{r}_0) = \frac{C}{H}\frac{1}{4\pi q\omega^2}\partial_i\partial_j\frac{e^{ik_{ps}R}}{R} \tag{A2}$$

$$G_{ij}^w(\mathbf{r}-\mathbf{r}_0) = -\frac{1}{4\pi q\omega^2}\partial_i\partial_j\frac{e^{ik_{ps}R}}{R}, \tag{A3}$$

where $R=|\mathbf{r}-\mathbf{r}_0|$. In homogeneous and isotropic media the Green's tensors only depend on R. In the low-frequency version of Biot's equations, the wave numbers of fast P, S, and slow P waves are defined as

$$k_p=\omega\sqrt{\frac{\rho}{H}},\quad k_s=\omega\sqrt{\frac{\rho}{G}},\quad k_{ps}=\sqrt{\frac{i\omega\eta}{\kappa_0 N}}=\omega\sqrt{\frac{q}{N}}, \tag{A4}$$

where $N=MP_d/H$. Note that the first three terms of G_{ij}^F are formally identical to the elastodynamic Green's tensor.[29] Indeed, in the elastic limit ($K_d\to K_g$, $\alpha\to 0$ and $\phi\to 0$, $\kappa_0\to 0$), the set of Green's tensors (A1)–(A3) reduces to the single elastodynamic Green's tensor,

$$G_{ij}^{\text{elast}}(\mathbf{r}-\mathbf{r}_0) = \frac{1}{4\pi\rho\omega^2}\left([k_s^2\delta_{ij}+\partial_i\partial_j]\frac{e^{ik_sR}}{R}-\partial_i\partial_j\frac{e^{ik_pR}}{R}\right), \tag{A5}$$

where the P wave number is now given by $k_p=\omega\sqrt{\rho/(K_d+4/3G)}$.

We define the spatial Fourier transform pair in the following way:

$$G_{ij}(\mathbf{r}-\mathbf{r}') = \int d^3\mathbf{K}\, g_{ij}(\mathbf{K})\exp[i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')], \tag{A6}$$

$$g_{ij}(\mathbf{K}) = \frac{1}{(2\pi)^3}\int d^3(\mathbf{r}-\mathbf{r}')G_{ij}(\mathbf{r}-\mathbf{r}')\exp[-i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')]. \tag{A7}$$

In the wave number domain the Green's tensors (A1)–(A3) read as

$$g_{ij}^F(\mathbf{K}) = -\frac{1}{8\pi^3}\frac{1}{\rho\omega^2}\left(\frac{k_s^2\delta_{ij}-K_iK_j}{k_s^2-K^2}+\frac{K_iK_j}{k_p^2-K^2}\right)$$
$$-\frac{1}{8\pi^3}\frac{C^2}{H^2}\frac{1}{q\omega^2}\frac{K_iK_j}{k_{ps}^2-K^2}, \tag{A8}$$

$$g_{ij}^f(\mathbf{K}) = \frac{1}{8\pi^3}\frac{C}{H}\frac{1}{q\omega^2}\frac{K_iK_j}{k_{ps}^2-K^2}, \tag{A9}$$

$$g_{ij}^w(\mathbf{K}) = -\frac{1}{8\pi^3}\frac{1}{q\omega^2}\frac{K_iK_j}{k_{ps}^2-K^2}. \tag{A10}$$

APPENDIX B: COMPUTATION OF THE OPERATOR $Q_{ik}^{(1)}$

In this appendix we compute the kernel-of-mass operator $Q_{ik}^{(1)}(\mathbf{r}'-\mathbf{r}'')$ defined in Eq. (37). First we note that the three terms in Eq. (37) are of the same structure,

$$Q_{ik}(\mathbf{r}'-\mathbf{r}'') = \langle (L_{ij}^{(X)})'\quad G_{jl}^Z(\mathbf{r}'-\mathbf{r}'')\quad (L_{lk}^{(Y)})''\rangle, \tag{B1}$$

where $X=1,2$, $Y=1,2$, $Z=F,f,w$ while single and double primes indicate whether in the differential operator L_{ij} differentiation is with respect to $\mathbf{r}'$ or $\mathbf{r}''$. It is therefore sufficient to present the detailed manipulations only for one term. For example, for the differential operators $(L_{ij}^{(1)})'=\partial_i'\tilde{H}'\partial_j'$ and $(L_{ij}^{(2)})''=\partial_l''\tilde{C}''\partial_k''$, we have

$$Q_{ik}=\langle\partial_i'\tilde{H}'\partial_j'\quad G_{jl}^Z(\mathbf{r}'-\mathbf{r}'')\quad \partial_l''\tilde{C}''\partial_k''\rangle. \tag{B2}$$

We now use partial integrations to shift the spatial derivatives ∂_j' and ∂_l'' to the Green's function [and make use of the property $G^Z(R\to\infty)=0$]. Averaging Eq. (B2) yields

$$Q_{ik}=\partial_i'\quad (B_{HC}(\mathbf{r}'-\mathbf{r}'')\quad G_{jl,jl}^Z(\mathbf{r}'-\mathbf{r}''))\quad \partial_k'', \tag{B3}$$

where we made use of the assumption of statistically homogeneous random fields [Eq. (19)]. The Fourier transform of Q_{ik} is defined as

$$q_{ik}(\mathbf{K}) = \frac{1}{8\pi^3}\int d^3(\mathbf{r}'-\mathbf{r}'')Q_{ik}(\mathbf{r}'-\mathbf{r}'')\exp[-i\mathbf{K}\cdot(\mathbf{r}'-\mathbf{r}'')]. \tag{B4}$$

Noting that spatial derivatives correspond to multiplications in the wave number domain ($\partial_j\leftrightarrow iK_j$), we can express q_{ik} as

$$q_{ik}(\mathbf{K}) = \frac{-K_iK_k}{8\pi^3}\int d^3\mathbf{r}\, B_{HC}(\mathbf{r})G_{jl,jl}^Z(\mathbf{r})\exp[-i\mathbf{K}\cdot\mathbf{r}]. \tag{B5}$$

Next, we need to compute the spatial derivatives of the Green's tensors. The poroelastic part of Green's tensor in Eq. (A1) as well as Green's tensors in Eqs. (A2) and (A3) are of the form $G_{ij}^{\text{poro}}\propto\partial_i\partial_j(e^{ik_{ps}R}/R)$. After differentiation we obtain

$$G_{ij,ij}^{\text{poro}}(\mathbf{r})\propto -k_{ps}^2\left(k_{ps}^2\frac{e^{ik_{ps}R}}{R}+4\pi\delta(\mathbf{r})\right). \tag{B6}$$

Analogously, for the derivatives of the elastic part of Green's tensor (A1) given by equation (A5) we find

$$G_{ij,ij}^{\text{elast}}(\mathbf{r})\propto -k_p^2\left(k_p^2\frac{e^{ik_pR}}{R}+4\pi\delta(\mathbf{r})\right). \tag{B7}$$

Note that the derivatives in (B6) and (B7) are of the same functional form, despite the fact that G_{ij}^{elast} includes S wave contributions. These contributions, however, cancel out when $G_{ij,ij}^{\text{elast}}$ is computed. The parts of the differential operator $L_{ik}^{(1)}$ involving fluctuations of the shear modulus require the computation of the terms $G_{33,33}^{\text{elast}}$ and $G_{3j,3j}^{\text{elast}}$. It can be shown that these derivatives can be expressed in terms of $G_{ij,ij}$:

$$G_{33,33}=G_{ij,ij}\left(\frac{1}{2}\int_0^\pi d\theta\,\sin\theta\cos^4\theta+O(k_p^2R^2)\right). \tag{B8}$$

$$=G_{ij,ij}\left(\frac{1}{5}+O(k_p^2R^2)\right). \tag{B9}$$

Analogously,

$$G_{3j,3j}=G_{ij,ij}(\tfrac{1}{3}+O(k_p^2R^2)). \tag{B10}$$

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Neglecting terms of the order $O(k_p^2 R^2)$ will produce no additional inaccuracy because of the restriction to low frequencies. Because of the same argument, we also neglect the first term in (B7). It is this term that is responsible for dynamic effects of elastic scattering and that is left out in the present analysis.

Inserting expressions (B6) and (B7) into Eq. (B5), introducing spherical coordinates, assuming statistical isotropy $[B(\mathbf{r}) = B(|\mathbf{r}|)]$, and integrating over the angular coordinates, we obtain

$$q_{33} = -\frac{1}{4\pi^3} \frac{C^2}{N} \left(k_p^2 B_{HC}(0) \right.$$

$$\left. + k_p k_{ps}^2 \int_0^\infty dr\, B_{HC}(r) \exp[ik_{ps}r]\sin(k_p r) \right). \quad \text{(B11)}$$

In accordance with our low-frequency assumption, we replace the sine function by its argument and obtain Eq. (45). Repeating these computations for the remaining terms involving correlations and cross-correlation between the random fields of H, G, and C yields Eqs. (43)–(48).

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