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Quadrature blending for isogeometric analysis

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Abstract

We use blended quadrature rules to reduce the phase error of isogeometric analysis discretizations. To explain the observed behavior and quantify the approximation errors, we use the generalized Pythagorean eigenvalue error theorem to account for quadrature errors on the resulting weak forms [28]. The proposed blended techniques improve the spectral accuracy of isogeometric analysis on uniform and non-uniform meshes for different polynomial orders and continuity of the basis functions. The convergence rate of the optimally blended schemes is increased by two orders with respect to the case when standard quadratures are applied. Our technique can be applied to arbitrary high-order isogeometric elements.

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1 Introduction

The development and use of isogeometric analysis (IGA) for partial differential equations (PDE) has continuously grown over the last decade [7, 8, 15, 16, 19, 20, 22]. Isogeometric analysis can use NURBS (Non-Uniform Rational B-Splines) or T-splines [9, 14, 27]. Thus, IGA uses the framework of computer aided design (CAD) systems to exactly represent complex geometries. This allows to simplify the process of mesh generation and refinement, as well as the information exchange between analysis and design. Isogeometric analysis has several attractive features. The basis functions can have higher continuity across element interfaces and hence the approximated solutions have global continuity of order up to $p-1$, where p is the order of the underlying polynomial. These highly continuous isogeometric solutions are more accurate and robust than standard finite element solutions per degree of freedom, although are more costly per degree of freedom [11–13, 26]. Several efficient implementations of the isogeometric analysis techniques in open source software are available [17, 18, 25, 29].

Dispersion analysis quantifies the approximation errors of a numerical method. The dispersive properties of various numerical methods, such as the finite element method (FEM), the spectral element method (SEM), among others, have been studied in detail in the last decades

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[1, 21]. An accurate and cost-effective scheme can be obtained by employing a weighted average of these methods [24]. In particular, Ainsworth has shown that the optimal blending of FEM and SEM provides two orders of extra accuracy in the phase error [1, 2].

Herein, we describe the errors in the discrete approximations to elliptic eigenvalue problems for the normal modes and frequencies of free structural vibrations. A similar analysis can be applied to the Helmholtz equation (e.g., frequency domain wave propagation). We consider both the eigenvalue and eigenfunction errors. That is, we conduct a global error analysis by describing the errors in the eigenvalues and the eigenfunctions for all modes. The sum of the eigenvalue error and the square of the L_2 eigenfunction error equals the square of the error in the energy norm. To quantify the approximation errors for the case underintegration, we have generalized Strang's Pythagorean theorem to include the effect of inexact integration [10, 28].

2 Problem statement

We first analyse the Laplace eigenvalue problem in one dimension

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} = \lambda u, \quad \text{for } x \in \Omega =]0,1[\\ &u(0) = u(1) = 0, \end{aligned} \tag{1}$$

with homogeneous Dirichlet boundary conditions. The following standard results are specialized for the one-dimensional case to simplify the discussion [10, 22, 28, 30]. The problem has a countable infinite set of eigenvalues $\lambda_j \in \mathrm{R}$ and an associated set of orthonormal eigenfunctions u_j (normalized with respect to the L_2 inner product)

$$0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \leq ... \tag{2}$$

$$(u_j, u_k) = \int_\Omega u_j(x) u_k(x) dx = \delta_{jk}, \tag{3}$$

where δ_{jk} is the Kronecker delta which is equal to 1 when $i = j$ and 0 otherwise. These eigenfunctions form an L_2-orthonormal basis and are orthogonal also in the energy inner product

$$(\mathcal{L} u_j, u_k) = (\lambda_j u_j, u_k) = \lambda_j \delta_{jk}. \tag{4}$$

The standard weak form for the eigenvalue problem is stated as follows: Find all eigenvalues $\lambda_j \in \mathrm{R}$ and eigenfunctions $u_j \in V$ such that, for all $w \in V$,

$$a(w, u_j) = \lambda_j (w, u_j), \tag{5}$$

where

$$a(w, u_j) = \int_\Omega \frac{dw}{dx} \frac{du_j}{dx} dx, \tag{6}$$

and V is a closed subspace of $H^1(\Omega)$. We use the notation of [30] where $(\cdot, \cdot)$ and $a(\cdot, \cdot)$ are symmetric bilinear forms which define the following inner products

$$\|w\|_E^2 = a(w, w), \quad \|w\|^2 = (w, w), \tag{7}$$

for all $v, w \in V$. $H^1(\Omega)$ denotes the Sobolev space of functions

$$H^1(\Omega) = \{ f : \Omega \to R \mid \|f\|_{H^1} < \infty \}, \quad \|f\|_{H^1}^2 = \int_a^b \left[f^2(x) + \left(\frac{d}{dx} f(x) \right)^2 \right] dx. \tag{8}$$

The Galerkin formulation of the eigenvalue problem is the discrete form of (5): Find the discrete eigenvalues $\lambda_j^h \in \mathrm{R}$ and eigenfunctions $u_j^h \in V^h \subset V$ such that, for all $w^h \in V^h \subset V$,

$$a(w^h, u_j^h) = \lambda_j^h (w^h, u_j^h). \tag{9}$$

Because of the minimax principle, all discrete eigenvalues resulting from the finite element method approximate from above the exact eigenvalues

$$\lambda_j^h \geq \lambda_j \quad \text{for all } j. \tag{10}$$

The theorem for the Pythagorean eigenvalue error can be applied in this case [10, 21, 28, 30]. For each discrete mode, the eigenvalue error and the product of the eigenvalue and the square of the eigenfunction error in the L_2-norm sum to the square of the error in the energy norm

$$\lambda_j^h - \lambda_j + \lambda_j \left\| u_j^h - u_j \right\|^2 = \left\| u_j^h - u_j \right\|_E^2. \tag{11}$$

The original eigenvalue problem (1) can be written in matrix form as

$$\mathbf{K}\boldsymbol{\phi_j} = \lambda_j \mathbf{M}\boldsymbol{\phi_j} \tag{12}$$

where

$$\mathbf{M}_{ij} = \int_a^b \phi_i(x)\phi_j(x)dx, \quad \mathbf{K}_{ij} = \int_a^b \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx, \tag{13}$$

with $\phi_i(x)$ piecewise polynomial basis functions. $\mathbf{M}$ and $\mathbf{K}$ are symmetric positive definite banded matrices with the maximum bandwidth equal to p.

We first introduce notation to analyze the effects of quadrature for finite elements, spectral elements, and isogeometric analysis. The integrals in the elemental matrices (13) are evaluated with the help of numerical integration rules. An m-point quadrature rule requires m evaluations of a function $f(x)$ to approximate its weighted integral over an interval $[a, b]$

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{m} w_i f(x_i) + R_m. \tag{14}$$

Here w_i and x_i are the weights and the nodes of the rule, respectively. The quadrature rule is exact for a given function $f(x)$ when the remainder R_m is exactly zero. For example, the standard m-point Gauss-Legendre (GL or Gauss) quadrature is exact for the linear space of polynomials of degree at most $2m - 1$. The quadrature rule employed in the numerical scheme strongly affects the matrix formation and assembly procedures, and, in turn, the overall efficiency of Galerkin-type numerical methods. Further details can be found in [3–6, 10, 23, 28].

Element-level integrals may be approximated using other quadrature rules, for example the Gauss-Lobatto-Legendre (GLL or Lobatto) quadrature rule that is used in the spectral element method (SEM). The Lobatto quadrature evaluated at m nodes is accurate for polynomials up to degree $2m - 3$. However, selecting a rule with $p + 1$ evaluations for a polynomial of order p and collocating the Lagrange nodes with the quadrature positions renders the mass matrix diagonal in 1D, 2D and 3D for arbitrary geometrical mappings.

3 Pythagorean eigenvalue theorem and quadrature errors

In this section we state a generalized Pythagorean eigenvalue error theorem that accounts for modified inner products. We write the approximate solutions for the fully-integrated and the modified inner-product discretizations, respectively, as

$$a(u_j^h, u_j^h) = \lambda_j^h (u_j^h, u_j^h), \tag{15}$$

$$a_h(v_j^h, v_j^h) = \mu_j^h (v_j^h, v_j^h)_h. \tag{16}$$

Here $a_h(\cdot,\cdot)$ and $(\cdot,\cdot)_h$ are the discrete inner products represented by the chosen quadratures. For each j, the set v_j^h and μ_j^h are the discrete eigenpair resulting from these discrete inner product representations. The corresponding eigenfunctions are orthonormal with respect to the appropriate inner product

$$(u_i, u_j) = (u_i^h, u_j^h) = (v_i^h, v_j^h)_h = \delta_{ij}. \tag{17}$$

In the modified inner product case we can write the generalized Pythagorean theorem as follows

$$\left\|u_j - v_j^h\right\|_E^2 = \lambda_j \left[1 - 2(u_j, v_j^h) + 1\right] + a(v_j^h, v_j^h) - \lambda_j, \tag{18}$$

which can be reduced to the following form [28]

$$\left\|u_j - v_j^h\right\|_E^2 = \mu_j^h - \lambda_j + \lambda_j \left\|u_j - v_j^h\right\|_0^2 + \left\|v_j^h\right\|_E^2 - \left\|v_j^h\right\|_{E,h}^2 + \lambda_j \left(1 - \left\|v_j^h\right\|_0^2\right), \tag{19}$$

and can be further simplified to

$$\frac{\left\|u_j - v_j^h\right\|_E^2}{\lambda_j} = \frac{\mu_j^h - \lambda_j}{\lambda_j} + \left\|u_j - v_j^h\right\|_0^2 + \frac{\left\|v_j^h\right\|_E^2 - \left\|v_j^h\right\|_{E,h}^2}{\lambda_j} + \left(1 - \left\|v_j^h\right\|_0^2\right). \tag{20}$$

4 Quadrature blendings for finite elements and isogeometic analysis

The blending of the spectral and finite element methods to achieve an improvement of two orders of accuracy was studied by several authors [2, 24]. This method blends the Gauss quadrature rule, which exactly integrates the bilinear forms to produce the mass and stiffness matrices, with the Lobatto quadrature, which underintegrates them. Ainsworth and Wajid [2] selected the blending parameter to maximize the order of accuracy in the phase error. They proved that the optimal choice for the blending parameter is given by weighting the spectral element and the finite element methods in the ratio p to one scaled by $(p+1)$.

Alternative blendings can lead to much smaller errors in the mid range of the spectrum than the optimal blending. However, only the optimally blended scheme has two additional orders of convergence when compared against the FEM, SEM, or another blending of these two methods. In practice, the blended scheme does not require the assembly of the mass matrices for both schemes and can be realized by simply replacing the standard Gaussian quadrature rule by an alternative one that has the same (or even less) number of quadrature points. Hence, no additional computational cost compared to the Gaussian rule is required. However, the mass matrix in the blended scheme is no longer diagonal like in the case of the SEM. Similar optimal blendings for isogeometric analysis are presented in [10, 28].

5 Numerical examples

Herein, we focus on the one-dimensional problems described in Section 2. The mesh is uniform unless otherwise specified. The exact eigenvalues and eigenfunctions of the 1D problem are $\lambda_j = j^2\pi^2$ and $u_j = \sqrt{2}\sin(j\pi x), j = 1, 2, ...,$ respectively. The approximate eigenvalues λ_j^h are sorted in ascending order and are compared to the corresponding exact counterparts.

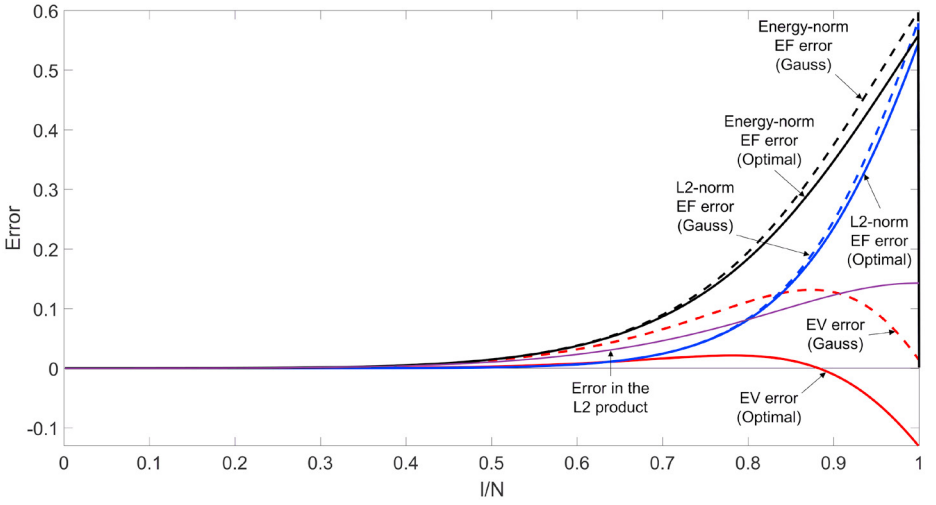

Figure 1: Illustration of the generalized Pythagorean eigenvalue theorem. The optimally blended C^1 quadratic elements are compared with the standard C^1 quadratics.

In the following figures, we present the relative eigenvalue (EV) errors $\frac{\mu_j^h - \lambda_j}{\lambda_j}$, the L_2-norm eigenfunction (EF) errors $\left\|u_j - v_j^h\right\|_0^2$ and, occasionally, the relative energy norm EF errors $\frac{\left\|u_j - v_j^h\right\|_E^2}{\lambda_j}$. This format of error representation clearly illustrates the budget of the generalized Pythagorean eigenvalue theorem (20). Thus, Figure 1 shows the eigenvalue and eigenfunction errors of optimally blended C^1 quadratic elements versus the standard fully-integrated scheme. The optimal scheme allows for a much better approximation of the discrete eigenvalues. The eigenfunctions are either improved a bit, like in this case, or not affected (in higher-order schemes).

Figure 2 illustrates the convergence of the optimally blended C^{p-1} discretization with cubic, quartic and quintic elements. Their blending parameters are explained in [10] and are $\frac{5}{2}$, $\frac{84}{5}$, and 175, respectively.

Figure 3 shows the eigenvalue and eigenfunction errors for cubic elements with different continuity of the basis functions. The quadrature for this scheme was obtained by using $\tau = 3/4$ as a blending parameter. While this value leads to the optimal scheme for C^0 elements, it does not provide two additional convergence orders for cubic elements of higher continuity.

The optimally blended C^0 elements ($\tau = p/(p+1)$) were presented in [2]. We have found that this blending is optimal also for even-order C^1 elements. Figure 4 shows the approximation errors of quartic, quintic and sextic C^1 elements. The $\tau = p/(p+1)$ scheme is optimal for quartic and sextic elements, but not for the quintic ones. The same behavior is observed for higher-order schemes.

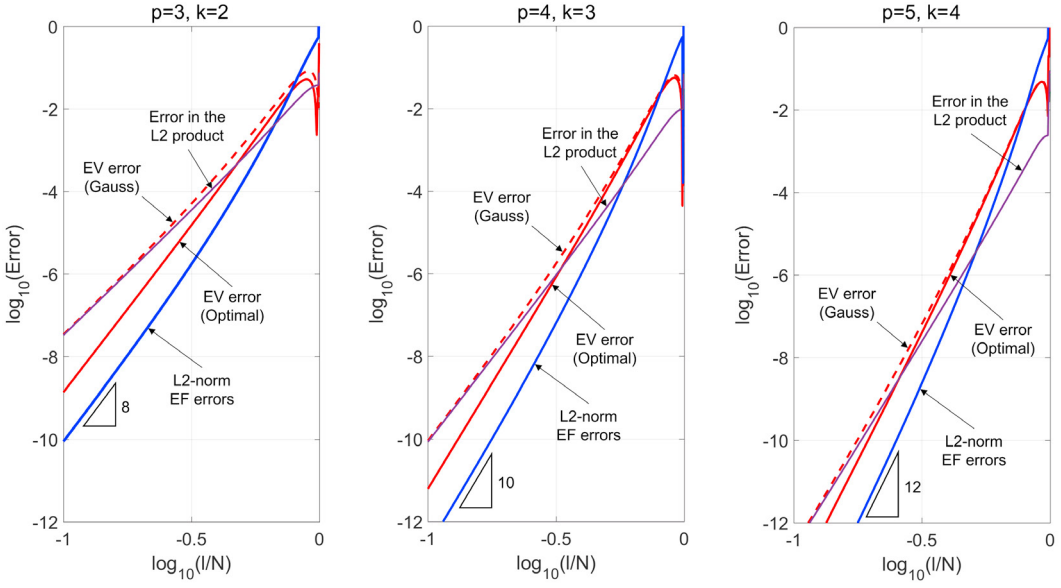

Figure 2: Approximation errors for the optimally blended C^{p-1} elements of order 3 (left), 4 (middle) and 5 (right)

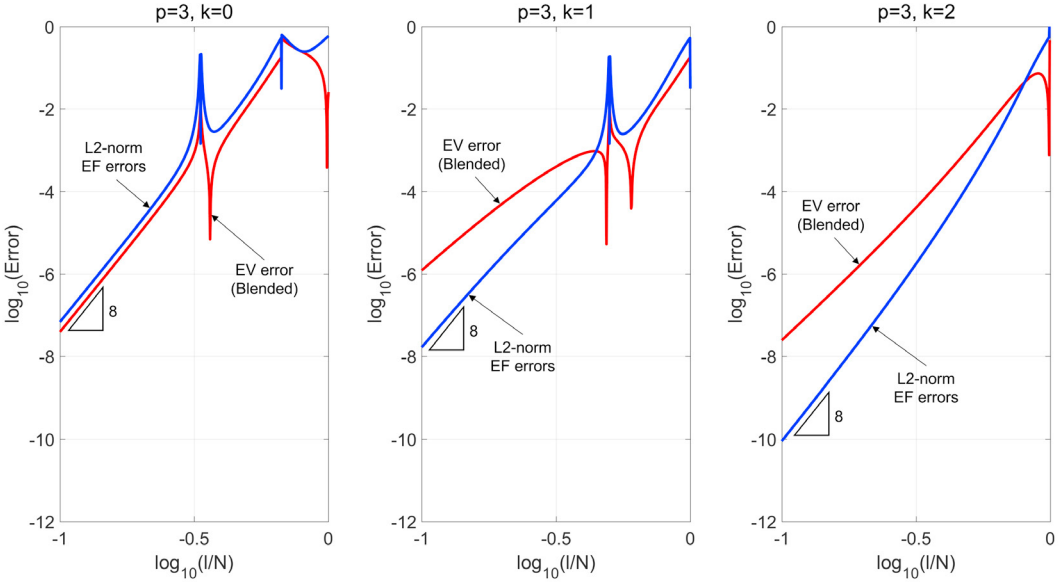

Figure 3: Eigenvalue and L_2-norm eigenfunction errors for cubic elements of continuity 0 (left), 1 (middle) and 2 (right)

In the next examples we consider strongly non-uniform meshes. Figure 5 shows the quadratic C^1 basis functions and approximations errors for a mesh that consists of two types of elements, one of which is 10 times larger than the other. In this case, the basis functions are quite similar

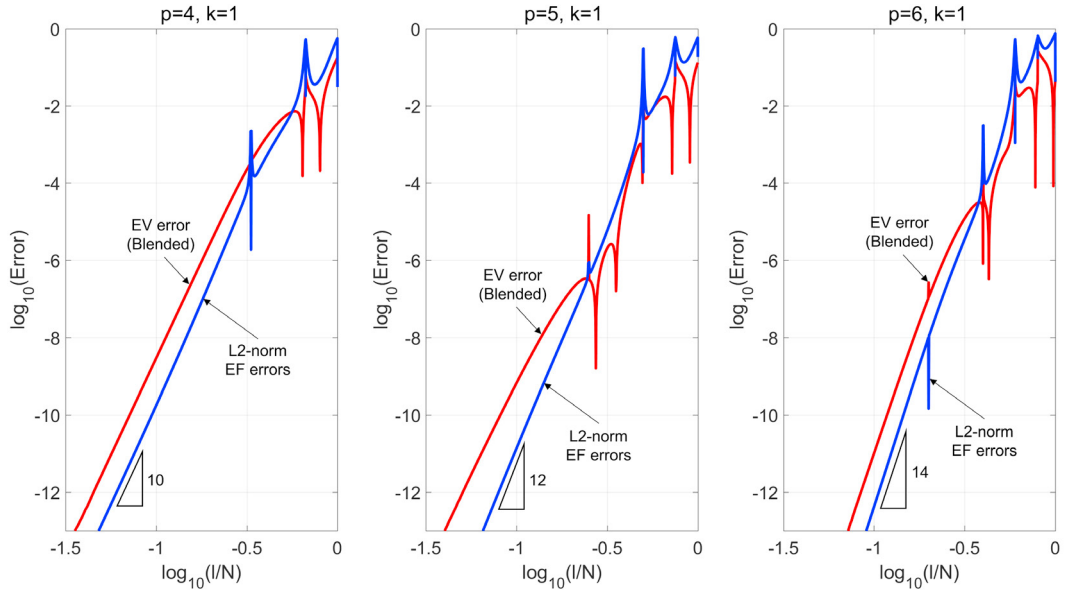

Figure 4: Eigenvalue and L_2-norm eigenfunction errors for C^1 elements of order 4 (left), 5 (middle) and 6 (right)

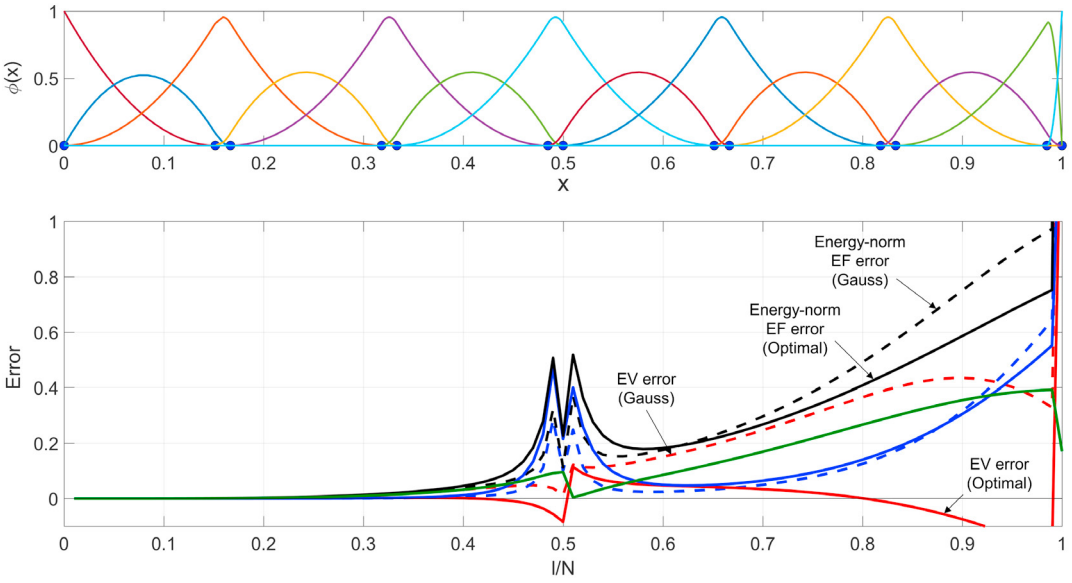

Figure 5: Top: C^1 quadratic basis functions for the non-uniform mesh with the ratio of elements 1:10. Bottom: Eigenvalue and eigenfunction approximation errors for this case.

to the C^0 basis functions on a uniform mesh. The spectrum of this mesh also has the features that are typical for the finite-element case: the large spike in the center and large errors in the optical branch. The optimally blended scheme allows to increase the quality of approximation

of eigenvalues and, to a lesser extent, of eigenfunctions.

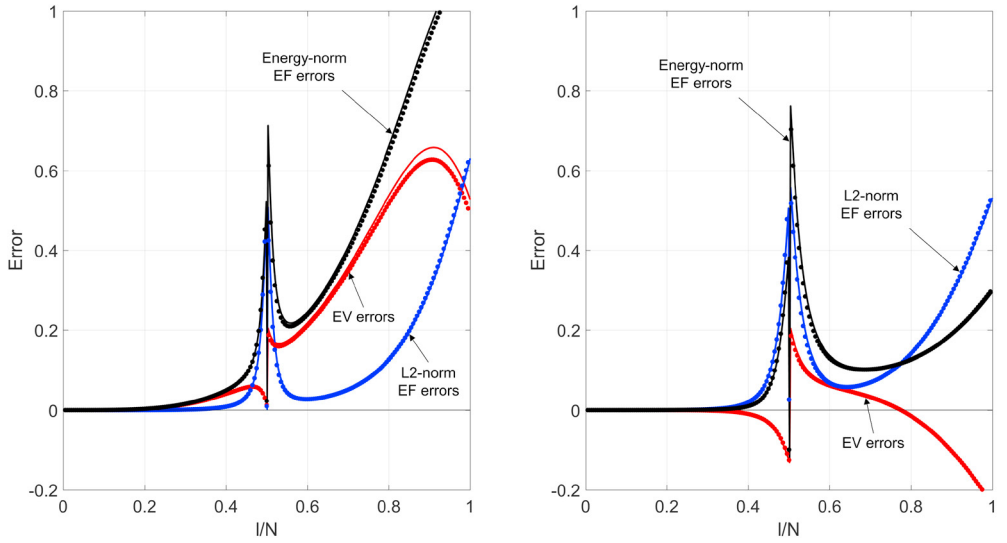

Figure 6: C^0 quadratic elements on a uniform mesh (lines) and C^1 quadratic quadratic elements for the 1:100 non-uniform mesh (circles). Left: standard Gaussian quadrature. Right: the optimally blended scheme.

The spectra for a similar mesh but with a 1:100 ratio of the elements are shown in Figure 6. The basis functions for in this case are similar to the C^0 (finite-element) basis on a uniform mesh. The approximation errors are almost identical in these cases, even though the schemes have different continuity of the basis. The presence of spikes in eigenvalues and eigenfunctions is related to how many different types of basis functions are present, rather than to the continuity of the basis.

6 Conclusions

Using optimally blended quadrature rules we increase the accuracy and robustness of the isogeometric methods. We analyze the dispersion profile of the methods and show that modified inner products can be obtained by generalizing the ideas introduced by Ainsworth and Wajid to the highly-continuous isogeometric discretizations. The new blending schemes for smoother basis functions improve the accuracy of the approximation of the eigenvalues and, in some cases, the eigenfunctions. The optimally blended isogeometric elements are superconvergent by two orders when compared to the standard fully integrated methods.

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