Department of Electrical and Computer Engineering

Adaptive Antenna Array Beamforming Using A Concatenation of Recursive Least Square and Least Mean Square Algorithms

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DECLARATION

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for award of any other degree or diploma in any university.

Signature: ………………………………….

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Dedicated to my Parents, my Family and my Wife for their support and affection

ABSTRACT

In recent years, adaptive or smart antennas have become a key component for various wireless applications, such as radar, sonar and cellular mobile communications including worldwide interoperability for microwave access (WiMAX). They lead to an increase in the detection range of radar and sonar systems, and the capacity of mobile radio communication systems. These antennas are used as spatial filters for receiving the desired signals coming from a specific direction or directions, while minimizing the reception of unwanted signals emanating from other directions.

Because of its simplicity and robustness, the LMS algorithm has become one of the most popular adaptive signal processing techniques adopted in many applications, including antenna array beamforming. Over the last three decades, several improvements have been proposed to speed up the convergence of the LMS algorithm. These include the normalized-LMS (NLMS), variable-length LMS algorithm, transform domain algorithms, and more recently the constrained-stability LMS (CSLMS) algorithm and modified robust variable step size LMS (MRVSS) algorithm. Yet another approach for attempting to speed up the convergence of the LMS algorithm without having to sacrifice too much of its error floor performance, is through the use of a variable step size LMS (VSSLMS) algorithm. All the published VSSLMS algorithms make use of an initial large adaptation step size to speed up the convergence. Upon approaching the steady state, smaller step sizes are then introduced to decrease the level of adjustment, hence maintaining a lower error floor. This convergence improvement of the LMS algorithm increases its complexity from $2N$ in the case of LMS algorithm to $9N$ in the case of the MRVSS algorithm, where N is the number of array elements.

An alternative to the LMS algorithm is the RLS algorithm. Although higher complexity is required for the RLS algorithm compared to the LMS algorithm, it can achieve faster convergence, thus, better performance compared to the LMS algorithm. There are also improvements that have been made to the

RLS algorithm families to enhance tracking ability as well as stability. Examples are, the adaptive forgetting factor RLS algorithm (AFF-RLS), variable forgetting factor RLS (VFFRLS) and the extended recursive least squares (EX-KRLS) algorithm. The multiplication complexity of VFFRLS, AFF-RLS and EX-KRLS algorithms are $2.5N^2+3N+20$, $9N^2+7N$, and $15N^3+7N^2+2N+4$ respectively, while the RLS algorithm requires $2.5N^2+3N$.

All the above well known algorithms require an accurate reference signal for their proper operation. In some cases, several additional operating parameters should be specified. For example, MRVSS needs twelve predefined parameters. As a result, its performance highly depends on the input signal.

In this study, two adaptive beamforming algorithms have been proposed. They are called recursive least square - least mean square (RLMS) algorithm, and least mean square - least mean square (LLMS) algorithm. These algorithms have been proposed for meeting future beamforming requirements, such as very high convergence rate, robust to noise and flexible modes of operation. The RLMS algorithm makes use of two individual algorithm stages, based on the RLS and LMS algorithms, connected in tandem via an array image vector. On the other hand, the LLMS algorithm is a simpler version of the RLMS algorithm. It makes use of two LMS algorithm stages instead of the RLS – LMS combination as used in the RLMS algorithm.

Unlike other adaptive beamforming algorithms, for both of these algorithms, the error signal of the second algorithm stage is fed back and combined with the error signal of the first algorithm stage to form an overall error signal for use update the tap weights of the first algorithm stage.

Upon convergence, usually after few iterations, the proposed algorithms can be switched to the self-referencing mode. In this mode, the entire algorithm outputs are swapped, replacing their reference signals. In moving target applications, the array image vector, $\mathcal{F}$, should also be updated to the

new position. This scenario is also studied for both proposed algorithms. A simple and effective method for calculate the required array image vector is also proposed. Moreover, since the RLMS and the LLMS algorithms employ the array image vector in their operation, they can be used to generate fixed beams by pre-setting the values of the array image vector to the specified direction.

The convergence of RLMS and LLMS algorithms is analyzed for two different operation modes; namely with external reference or self-referencing. Array image vector calculations, ranges of step sizes values for stable operation, fixed beam generation, and fixed-point arithmetic have also been studied in this thesis. All of these analyses have been confirmed by computer simulations for different signal conditions. Computer simulation results show that both proposed algorithms are superior in convergence performances to the algorithms, such as the CSLMS, MRVSS, LMS, VFFRLS and RLS algorithms, and are quite insensitive to variations in input SNR and the actual step size values used. Furthermore, RLMS and LLMS algorithms remain stable even when their reference signals are corrupted by additive white Gaussian noise (AWGN). In addition, they are robust when operating in the presence of Rayleigh fading. Finally, the fidelity of the signal at the output of the proposed algorithms beamformers is demonstrated by means of the resultant values of error vector magnitude (EVM), and scatter plots. It is also shown that, the implementation of an eight element uniform linear array using the proposed algorithms with a wordlength of nine bits is sufficient to achieve performance close to that provided by full precision.

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PUBLICATIONS

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2. J. A. Srar, K. S. Chung, and A. Mansour, "Adaptive Array Beamforming Using a Combined LMS-LMS Algorithm," IEEE Trans. on Antennas and Propagation, vol. 58(11), pp. 3545-3557, November 2010.

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4. Jalal SRAR, Kah-Seng CHUNG and Ali MANSOUR, "A new LLMS Algorithm for Antenna Array Beamforming," in 2010 IEEE Wireless Communications and Networking Conference, Sydney, Australia, April 2010.

5. Jalal SRAR, Kah-Seng CHUNG and Ali MANSOUR, "Adaptive Array Beamforming using a Combined LMS-LMS Algorithm," in 2010 IEEE Aerospace Conference, Montana, USA, March 2010.

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ABBREVIATIONS

AFFRLS	Adaptive forgetting factor recursive least square
AOA	Angle of arrival
AOD	Angle of departure
AWGN	Additive white Gaussian noise
BER	Bit error rate
bps	Bit per second
BPSK	Binary phase-shift keying
CCM	Constrained constant modulus
CCM-RLS	Constrained constant modulus-recursive least square
CDMA	Code division multiple access
CM	Constant modulus
CM-MASS	Combined constant modulus and modified adaptive step size algorithm
CM-TAASS	Constant modulus algorithm and time averaging adaptive step size algorithm combination
CMV	Constrained minimum variance algorithm
CMV-LMS	Constrained minimum variance-least mean square
CMV-RLS	Constrained minimum variance-recursive least square
CSLMS	Constraint stability LMS
dB	Decibel
DCT	Discrete cosine transform
DD	Decision directed
DFT	Discrete Fourier transform
DS/CDMA	Direct sequence code division multiple access

DWT	Discrete Walsh transform
EMSE	Steady-state excess mean-square error
EVD	Eigenvalue decomposition
EVM	Error vector magnitude
ESPRIT	Estimation of signal parameters via rotational invariance technique
EX-KRLS	Extended recursive least squares algorithm
FAEST	Fast a posteriori error sequential technique
FEDR	Fast Euclidian direction search
FFT	Fast Fourier transform
FM	Frequency modulation
FRLS	Fast recursive least square
FSK	frequency-shift keying
HoCA	Higher order cumulant algorithm
Hr	Hour
HRLS	Hierarchical recursive least square
Hz	Hertz
IFFT	Inverse fast Fourier transform
KLMS	The Kernal least mean square
Km/h	Kilometre per hour
LLMS	LMS-LMS combination which employs an adaptive array image vector
$LLMS_1$	LMS-LMS combination which employs a fixed array image vector
LMS	Least mean square
LMS_1	First LMS algorithm stage of the LLMS algorithm
LMS_2	Second LMS algorithm stage of the LLMS algorithm

LS	Least-squares
LTE	Long term evolution
Mbits	Mega bits
MCM	Modified constant modulus
MHz	Mega hertz
MIMO	Multiple-input and multiple-output
MMARY	Multi-modulus array
MMSE	Minimum mean square error
MRVSS	Modified robust variable step size
MSC	Most significant coefficient
MSE	Mean square error
MSINR	Maximum signal to interference plus noise signal
MUSIC	Multiple signal classification
MVDR	Minimum variance distortionless response
NAFFRLS	Normalized least mean square adaptive forgetting factor recursive least square
NLMS	Normalized LMS
NSVSSLMS	Normalized square variable step size LMS
OFDM	Orthogonal frequency-division multiplexing
OFDMA	Orthogonal frequency-division multiple access
PSK	Phase-shift keying
QAM	Quadrature amplitude modulation
RAMP	Recursive adaptive matching pursuit
RHS	Right hand side
rms	Root mean square
RLMS	RLS-LMS combination which employs an adaptive array

	image vector
$RLMS_1$	RLS-LMS combination which employs a fixed array image vector
RLS	Recursive least square
RLS-CM	Recursive least square- constant modulus
RVSS	Robust VSSLMS
SD-CM	Steepest-descent-constant modulus
SDD	Soft decision directed
SDMA	Space division multiple access
SINR	Signal-to-noise plus interference ratio
$SINR_o$	Output signal-to-noise plus interference ratio
SIR	Signal to interference ratio
SMI	Sample matrix inversion
SNR	Signal to noise ratio
SVSS	Sign variable step size
TDLMS	Transform-domain LMS
UCA	Uniform circular array
ULA	Uniform linear array
VOA	Variance oriented approach
VSSLMS	Variable step size LMS
WiMAX	Worldwide interoperability for microwave access

LIST OF SYMBOLS AND NOTATIONS

$\lvert\bullet\rvert$	Absolute value operator
θ_d	Angle of arrival of the desired signal
θ_i	Angle of arrival of the interfering signal
G	Array element gain
AF	Array factor
AF_{UCL}	Array factor of the uniform linear array
$\mathcal{F}_{\mathrm{L}}$	Array image vector of the LLMS algorithm
$\mathcal{F}_{\mathrm{R}}$	Array image vector of the RLMS algorithm
A_d	Array vector in the direction of the desired signal
A_i	Array vector in the direction of the interfered signal
ϕ	Azimuth angle
$\tilde{x}$	Average of the input signal samples
$\tilde{P}$	Average power of all symbols involved for the given modulation
f_c	Carrier frequency
$r_{l,ka}$	Coefficients of $\boldsymbol{R}_2$ of the LLMS algorithm
$r_{r,ka}$	Coefficients of $\boldsymbol{R}_{\mathrm{LMS}}$ of the RLMS algorithm
β	Constant between 0 and 1 associated with VSSLMS algorithm
υ	Constant between 0 and 1 associated with MRVSS algorithm
η_e	Constant between 0 and 1 associated with VSSLMS algorithm
q_v	Constant between 0 and 1 associated with NLMS algorithm
γ	Constant between 0 and 1 associated with VSSLMS algorithm
γ_{N}	Convergence factor of NLMS algorithm

$\gamma_{\mathrm{N,min}}$	Convergence factor lower limit of NLMS algorithm
$\gamma_{\mathrm{N,max}}$	Convergence factor upper limit of NLMS algorithm
γ'_{N}	Convergence updated factor of NLMS algorithm
η_{N}	Constant less than or equal to 1-ε associated with NLMS algorithm
η	Constant less than or equal to 1-ε associated with MRVSS algorithm
ε	Constant used in a NLMS algorithm, where its value equals the reciprocal of the number of used snapshots used to estimate the average
ε_{cs}	Constant used to prevent division by zero in CSLMS algorithm
ε_{c}	Constant equal to the reciprocal of the number of used samples employed for the estimation of the array image factor of the RLMS algorithm
$(*)$	Conjugate operator
$\rho_{x_{1,k},V_1}$	Correlation coefficient between $x_{1,k}(t)$ and the error of LMS$_1$ algorithm $\boldsymbol{V}_1(n)$ of the LLMS algorithm
R_{ey}^2	Cross-correlation between the output signal (y) and the output error (e)
$\boldsymbol{R_e}$	Cross-correlation matrix of successive error samples
$\triangleq$	Denotes an equivalent
s_d	Desired signal
$\boldsymbol{\Lambda}_1$	Diagonal matrix of eigenvalues of $\boldsymbol{Q}_1$ of the LLMS algorithm
$\boldsymbol{\Lambda}_{\mathrm{R}}$	Diagonal matrix of eigenvalues of $\boldsymbol{Q}_{\mathrm{R}}$ of the RLMS algorithm
$\boldsymbol{\Lambda}_2$	Diagonal matrix of eigenvalues of $\boldsymbol{R}_2$ of the LLMS algorithm
$diag(\bullet)$	Diagonal of the matrix operator

$\boldsymbol{D}_X$	Diagonal of the matrix
$\boldsymbol{V}_1$	Difference between the estimated and actual tap weights for the LMS$_1$ algorithm stage of the LLMS algorithm
$\boldsymbol{V}_2$	Difference between the estimated and actual tap weights for the LMS$_2$ algorithm stage of the LLMS algorithm
$\boldsymbol{V}_{\mathrm{LMS}}$	Difference between the estimated and actual tap weights for the LMS algorithm stage of the RLMS algorithm
$\boldsymbol{V}_{\mathrm{RLS}}$	Difference between the estimated and optimal tap weights for the RLS stage of the RLMS algorithm
D_{L}	Difference between the current LMS$_1$ stage reference signal sample and the last LMS$_2$ stage error of the LLMS algorithm
D_{R}	Difference between the current RLS stage reference signal sample and the last LMS stage error of the RLMS algorithm
$\boldsymbol{S}$	Differentiation of the inverse of the correlation matrix with respect to the RLS forgetting factor (α_{RLS})
$\boldsymbol{\Psi}$	Differentiation of the weight vector with respect to the RLS forgetting factor (α_{RLS})
f_d	Doppler frequency
R_2	Elements form of $\boldsymbol{R}_2$
E	Eigenvalue of $\boldsymbol{Q}_{\mathrm{R}}$
$\lambda_{\mathrm{L},2}$	Eigenvalue of $\boldsymbol{R}_2$
$\lambda_{\mathrm{R},2}$	Eigenvalue of $\boldsymbol{R}_{\mathrm{LMS}}$
$\mathcal{F}_l$	Elements of the array image factor vector, $\mathcal{F}_{\mathrm{L}}$
$\mathcal{F}_r$	Elements of the array image factor vector, $\mathcal{F}_{\mathrm{R}}$
$\mathcal{D}$	Elements spacing of the uniform linear array
$\mathcal{D}_x$	Elements spacing of the planar array in the x-direction
$\mathcal{D}_y$	Elements spacing of the planar array in the y-direction
$\boldsymbol{q}_1$	Eigenvectors matrix of $\boldsymbol{Q}_1$
$\boldsymbol{q}_{\mathrm{R}}$	Eigenvectors matrix of $\boldsymbol{Q}_{\mathrm{R}}$

$\bar{\xi}$	Ensemble average of the mean square error of the LLMS algorithm
$\bar{\xi}_{\mathrm{RLMS}}$	Ensemble average of the mean square error of the RLMS algorithm
e	Error
δe	Error difference associated with CSLMS algorithm
erf	Error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2/2} dt$
e_1	Error signal of the first LLMS algorithm stage
e_2	Error signal of the second LLMS algorithm stage
e_{RLS}	Error signal of the first RLMS algorithm stage
e_{LMS}	Error signal of the second RLMS algorithm stage
$\hat{\sigma}_e^2$	Estimation of the variance of the output error
$\hat{\sigma}_i^2$	Estimation of the variance of the i^{th} output signal of the transformation block
$E[\bullet]$	Expectation operator
α_{exp}	Exponential forgetting factor constant
$\boldsymbol{X}'$	Finite precision Input signal vector
$\boldsymbol{\alpha}$	Finite precision quantization error vector of the input signal
$\boldsymbol{\alpha}_1$	Finite precision quantization error vector of the input signal associated with the first LMS stage of LLMS algorithm
$\boldsymbol{\alpha}_2$	Finite precision quantization error vector of the input signal associated with the second LMS stage of LLMS algorithm
$\boldsymbol{\alpha}_{\mathrm{LMS}}$	Finite precision quantization error vector of the input signal associated with LMS stage of RLMS algorithm
$\boldsymbol{\alpha}_{\mathrm{RLS}}$	Finite precision quantization error vector of the input signal associated with RLS stage of RLMS algorithm
$\boldsymbol{\rho}_1$	Finite precision quantization error vector of the first LMS stage weights of LLMS algorithm
$\boldsymbol{\rho}_2$	Finite precision quantization error vector of the second LMS

	stage weights of LLMS algorithm
$\boldsymbol{\rho}_{\mathrm{LMS}}$	Finite precision quantization error vector of the LMS stage weights of RLMS algorithm
$\boldsymbol{\rho}_{\mathrm{RLS}}$	Finite precision quantization error vector of the RLS stage weights of RLMS algorithm
$\boldsymbol{\rho}$	Finite precision quantization error vector of the weight
e'_{LLMS}	Finite precision overall LLMS error signal
e'_{RLMS}	Finite precision overall RLMS error signal
η_1	Finite precision truncation and round-off errors associated with the first LMS stage of the LLMS algorithm
η_2	Finite precision truncation and round-off errors associated with the second LMS stage of the LLMS algorithm
η_{LMS}	Finite precision truncation and round-off errors associated with LMS stage of the RLMS algorithm
η_{RLS}	Finite precision truncation and round-off errors associated with RLS stage of the RLMS algorithm
α	Forgetting factor of the gradient algorithms
α_{RLS}	Forgetting factor of the RLS algorithm
$\alpha_{\max}$	Forgetting factor of the RLS algorithm, upper limit
$\alpha_{\min}$	Forgetting factor of the RLS algorithm, lower limit
κ	Gain constant associated with SD-CM algorithm
β_{I}	Gain constant of the MVDR beamforming algorithm
$\boldsymbol{K}$	Gain matrix of RLS algorithm
$(\bullet)^H$	Hermitian of matrix
e_I	Imaginary components of the error
A_S	Input desired signal amplitude
I_m	Input interference signal amplitude
$\boldsymbol{n}$	Input noise vector
$\boldsymbol{Q}$	Input signal correlation matrix

$\boldsymbol{Q}_1$	Input signal correlation matrix of the LLMS algorithm
$\boldsymbol{Q}_\mathrm{R}$	Input signal correlation matrix of the RMS algorithm
$\boldsymbol{R}_\mathrm{LMS}$	Input signal correlation matrix of the LMS algorithm stage of the RLMS algorithm
$\boldsymbol{R}_2$	Input signal correlation matrix of the LMS_2 algorithm stage of the LLMS algorithm
$\boldsymbol{Z}$	Input signal cross-correlation vector
$\boldsymbol{Z}_\mathrm{L}$	Input signal cross-correlation vector of the LLMS algorithm
$\boldsymbol{Z}'_\mathrm{L}$	Input signal cross-correlation vector using self-referencing mode of the LLMS algorithm
$\boldsymbol{Z}_\mathrm{R}$	Input signal cross-correlation vector of the RLMS algorithm
$\boldsymbol{Z}'_\mathrm{R}$	Input signal cross-correlation vector using self-referencing mode of the RLMS algorithm
$\boldsymbol{X}$	Input signal vector, $[x_1, x_2, ..., x_N]$.
$\boldsymbol{X}_1$	Input signal vector of the first stage of the LLMS algorithm
$\boldsymbol{X}_2$	Input signal vector of the second stage of the LLMS algorithm
$\boldsymbol{X}_\mathrm{LMS}$	Input signal vector of the LMS stage of the RLMS algorithm
$\delta\boldsymbol{X}$	Input signal difference of the CSLMS algorithm
s_i	Interfering signal
$[\bullet]_\mathrm{int}$	Integer operator
ψ_d	Inter-element phase shift with respect to the desired signal in radians
ψ_i	Inter-element phase shift with respect to the interfering signal in radians
ψ_c	Inter-element phase shift between the columns of the planar array in radians
ψ_x	Inter-element phase shift between the rows of the planar array in radians
i	Interference signal
$\boldsymbol{P}$	Inverse of the correlation matrix ($\boldsymbol{Q}$)

δ_l^m	Kronecker delta function which is defined as $\delta_l^m = \begin{cases} 0 & l \neq m \\ 1 & l = m \end{cases}$
$\tilde{e}_{\min}$	Lower bound of the time average of the error square signal
$\mu_{\max}$	Maximal allowable step size
$E_{\max}$	Maximum eigenvalue of the LMS_1 stage input signal covariance matrix of the LLMS algorithm
E_{RLS}	Maximum eigenvalue of the RLS stage input signal covariance matrix of the RLMS algorithm
$\Upsilon_{\max}$	Maximum tolerance in the inter-spacing of the array elements
$g_{\max}$	Maximum tolerance of the gain of the array elements
ξ	Mean-square error of an algorithm
ξ_{LLMS}	Mean-square error of the LLMS algorithm
ξ_{RLMS}	Mean-square error of the RLMS algorithm
z	Measurement noise
c_p	Mixing parameter of the affine LMS algorithm
$\xi_{\mathrm{LLMS,min}}$	Minimum mean square error of the LLMS algorithm
$\xi_{\mathrm{RLMS,min}}$	Minimum mean square error of the RLMS algorithm
$\mu_{\min}$	Minimal step size
K_{MSC}	Most significant coefficient index of the FFT output
N	Number of array elements
N_b	Number of bits
N_y	Number of the columns of the planar array elements
N_x	Number of the rows of the planar array elements
k	Number of signal snapshots used for estimating the average
N_{OFDM}	Number of OFDM subcarriers
$(\bullet)_n$	Normalized operator
$\boldsymbol{w}_{opt}$	Optimum weights vector of an algorithm

$\mathbf{W}_{opt1}$	Optimum weights vector of LMS$_1$ stage of the LLMS algorithm
$\mathbf{W}_{opt_{\mathrm{RLS}}}$	Optimum weights vector of RLS stage of the RLMS algorithm
P_d	Output desired signal power
P_i	Output interference signal power
P_n	Output noise power
y	Output signal of an algorithm
y_s	Output signal of the array due to the desired signal
y_1	Output signal of the first entire-output of the affine algorithm
y_{LMS1}	Output signal of the LMS$_1$ algorithm stage of the LLMS algorithm
y_{LLMS}	Output signal of the LLMS algorithm
y_{RLMS}	Output signal of the RLMS algorithm
y_{RLS}	Output signal of the RLS algorithm stage of RLMS algorithm
y_2	Output signal of the second entire-output of the affine algorithm
e_{LLMS}	Overall error signal of the LLMS algorithm
e_{RLMS}	Overall error signal of the RLMS algorithm
μ_l	Positive number related to the LMS step size
c_q	Quantization constant depends on how the inner product of a vector manipulation is implemented
R	Radius of the circular array
R_m	Radius of the QAM signal constellation area
e_R	Real components of the error
$\Re\{\bullet\}$	Real part operator
ε_r	Regularization parameter
d	Reference signal
d_{LMS}	Reference signal for the LMS stage of the RLMS algorithm

d_{RLS}	Reference signal for the RLS stage of the RLMS algorithm
d_1	Reference signal of the first LLMS algorithm stage
d_2	Reference signal of the second LLMS algorithm stage
s_f	Scaling factor
$sign(\bullet)$	Sign function operator
δ	Small positive constant to initiate ***P*** of the RLS algorithm
$\bar{\boldsymbol{g}}$	Smoothed gradient vector
c	Speed of light
$\lVert\bullet\rVert_2^2$	Squared Euclidean norm operator
μ_b	Step size of the bottom LMS stage of the affine LMS algorithm
μ_c	Step size of the combination parameter of the affine LMS algorithm
μ	Step size of the LMS algorithm
μ_1	Step size of the LMS_1 algorithm stage of the LLMS algorithm
μ_2	Step size of the LMS_2 algorithm stage of the LLMS algorithm
μ_{LMS}	Step size of the LMS stage of the RLMS algorithm
μ_{RLS}	Step size of the RLS stage of the RLMS algorithm
μ_t	Step size of the top LMS stage of the affine LMS algorithm
μ_r	Step size ratio constant
$\Delta\theta_{d,3\,\text{dB}}$	The 3 dB beamwidth
x_m	The m^{th} element of the input signal vector
x'_k	The m^{th} element of the outputs of the individual taps of the RLS stage of the RLMS algorithm
n_k	The k^{th} element of $\boldsymbol{n}$
$w_{\text{RLS},m}$	The m^{th} weight of the RLS stage vector of the RLMS algorithm
$w_{1,m}$	The m^{th} weight of the LMS_1 stage vector of the LLMS

	algorithm
$A_{d,k}$	The k^{th} element of $\boldsymbol{A}_d$
t	Time
$\tilde{e}$	Time average of the error square signal
$\tilde{e}_b$	Time average of the instantaneous error signal of the bottom LMS stage of the affine LMS algorithm
$\tilde{e}_t$	Time average of the instantaneous error signal of the top LMS stage of the affine LMS algorithm
τ	Time delay
n	Time index
$\hat{\boldsymbol{D}}$	Time-varying power normalization parameter matrix
g_r	Tolerance of the array elements
Υ	Tolerance of the inter-spacing of the array elements
$trace(.)$	Trace of the matrix $(.)$
x	Transmit signal
$\boldsymbol{U}$	Unitary transformation matrix of the second stage input signal
$\boldsymbol{I}$	Unity matrix
$\tilde{e}_{\max}$	Upper bounds of the time average of the error square signal
ρ	Variable step size adaptation parameter
$\sigma^2_{\boldsymbol{V}_1}$	Variance of $\boldsymbol{V}_1(n)$
σ^2_q	Variance of $\boldsymbol{\alpha}$ and $\boldsymbol{\rho}$ associated with the input signal and weight vectors quantization errors
σ^2_n	Variance of the input noise
σ^2_x	Variance of the input signal
$\sigma^2_{x_{1,k}}$	Variance of the input signal at the k^{th} element of the array
σ^2_1	Variance of the output signal of the LMS_1 stage of the LLMS algorithm
σ^2_{RLS}	Variance of the output signal of the RLS stage of the RLMS

	algorithm
σ_W^2	Variance of the weights
$\sigma_{\mathrm{L},r}^2$	Variance of the weight of the LLMS algorithm
$\sigma_{\mathrm{R},r}^2$	Variance of the weight of the RLMS algorithm
σ_z^2	Variance of the measurement noise
σ_η^2	Variance of the finite precision errors associated with either LMS or RLS stages of the RLMS algorithm
λ	Wavelength of the carrier signal
$\delta\boldsymbol{W}$	Weight difference of the CSLMS algorithm
$\boldsymbol{W}_{tb}$	Weight difference of the entire affine LMS algorithm weights.
$\boldsymbol{W}$	Weight vector
$\boldsymbol{W}_t$	Weight vector of the top LMS stage of the affine LMS algorithm
$\boldsymbol{W}_b$	Weight vector of the bottom LMS stage of the affine LMS algorithm
$\boldsymbol{W}_{0\,\mathrm{LMS}}$	Weight vector modeled by a random walk process of the LMS algorithm stage of the RLMS algorithm
$\boldsymbol{W}_{01}$	Weight vector as modeled by a random walk process of the LMS$_1$ stage of the LLMS algorithm
$\boldsymbol{W}_{0\,2}$	Weight vector as modeled by a random walk process of the LMS$_2$ stage of the LLMS algorithm
$\boldsymbol{W}'$	Weight vector of the Finite precision
$\boldsymbol{W}_1$	Weight vector of the first stage of the LLMS algorithm
$\boldsymbol{W}_{\mathrm{LMS}}$	Weight vector of the LMS stage of the RLMS algorithm
$\boldsymbol{W}_{\mathrm{LLMS}}$	Weight vector of the LLMS algorithm
$\boldsymbol{W}_{\mathrm{RLMS}}$	Weight vector of the RLMS algorithm
$\boldsymbol{W}_2$	Weight vector of the second stage of the LLMS algorithm
$\boldsymbol{W}_{\mathrm{RLS}}$	Weight vector of the RLS stage of the RLMS algorithm
e''_{LMS}	Zero mean measurement noise of the LMS stage of the

	RLMS algorithm
e_1''	Zero mean measurement noise of the LMS_1 stage of the LLMS algorithm
e_2''	Zero mean measurement noise of the LMS_2 stage of the LLMS algorithm
$\boldsymbol{r}_{\text{LMS}}$	Zero mean white iid sequence vector of the LMS stage of the RLMS algorithm
$\boldsymbol{r}_1$	Zero mean white iid sequence vector of the LMS_1 stage of the LLMS algorithm
$\boldsymbol{r}_2$	Zero mean white iid sequence vector of the LMS_2 stage of the LLMS algorithm