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On coherency and other properties of MAXVAR *

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Abstract This paper is concerned with the MAXVAR risk measure on $\mathscr{L}^2$ space. We present an elementary and direct proof of its coherency and averseness. Based on the observation that the MAXVAR measure is a continuous convex combination of the CVaR measure, we provide an explicit formula for the risk envelope of MAXVAR.

Keywords coherent risk measure · risk averse · risk envelope

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1 Introduction

In Cherny and Madan [2] and Cherny and Orlov [3], a new kind of risk measure –"MAXVAR" – is proposed, which is useful in the analysis of large portfolios. Given a probability space $(\Omega, \Sigma, \mathbb{P}_0)$ and a random variable $X \in \mathscr{L}^2(\Omega, \Sigma, \mathbb{P}_0)$, where $\mathscr{L}^2(\Omega, \Sigma, \mathbb{P}_0)$ is the square integrable Lebesgue space ($\mathscr{L}^2$ for short), the MAXVAR is defined as

$$\text{MAXVAR}_n(X) := \mathbb{E}(\max\{X_1, \cdots, X_n\}),$$

where $X_1, \cdots, X_n$ are i.i.d. copies of X. We call $\text{MAXVAR}_n(\cdot)$ the "MAXVAR risk measure".

Note that $\text{MAXVAR}_n(\cdot)$ is always finite on $\mathscr{L}^2$ since $|\text{MAXVAR}_n(X)| \le n\mathbb{E}(|X|) < +\infty$ for any $X \in \mathscr{L}^2$.

In [2,3], the name of "MINVAR risk measure" was used. Since we treat risk measures as a nondecreasing function, we use "MAXVAR risk measure" instead.

* This paper is dedicated to Michel Théra in celebration of his 70th birthday.

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Different from papers [2,3], which considered coherency of MINVAR in $\mathscr{L}^\infty$ space, this paper deals with the $\mathscr{L}^2$ space. Our proof of the coherency of MAXVAR risk measure is direct and independent of [2,3]. Moreover, we show risk averseness of MAXVAR and give an explicit formula for its risk envelope.

In Section 2, we present a simple proof for the coherency of MAXVAR. We show its aversity in Section 3. Section 4 is devoted to the discussion of a continuous representation of MAXVAR and Section 5 provides an explicit formula for its risk envelope.

2 Coherency of MAXVAR

In this section, we show that MAXVAR is a coherent risk measure in basic sense of Rockafellar.

Definition 1 *(Rockafellar [5]) A functional $\mathcal{R} : \mathscr{L}^2 \to (-\infty, +\infty]$ is a coherent risk measure in basic sense if it satisfies*

(A1) $\mathcal{R}(C) = C$ for all constant C;
(A2) ("convexity") $\mathcal{R}(\lambda X+(1-\lambda)Y) \le \lambda\cdot\mathcal{R}(X)+(1-\lambda)\cdot\mathcal{R}(Y)$ for any $X, Y \in \mathscr{L}^2$ and any fixed $0 \le \lambda \le 1$;
(A3) ("monotonicity") $\mathcal{R}(X) \le \mathcal{R}(Y)$ for any $X, Y \in \mathscr{L}^2$ satisfying $X \le Y$;
(A4) ("closedness") If $\|X^k - X\|_2 \to 0$ and $\mathcal{R}(X^k) \le 0$ for all $k \in \mathbb{N}$, then $\mathcal{R}(X) \le 0$;
(A5) ("positive homogeneity") $\mathcal{R}(\lambda X) = \lambda\mathcal{R}(X)$ for any $\lambda > 0$ and $X \in \mathscr{L}^2$.

Theorem 1 $\mathrm{MAXVAR}_n(\cdot)$ *is a coherent risk measure in basic sense.*

Proof **(A1)** is obvious by definition. **(A5)** is also easy to check since if $X_1, \cdots, X_n$ are i.i.d. copies of X and $\lambda > 0$, then $\lambda X_1, \cdots, \lambda X_n$ are i.i.d. copies of λX.

Proof of **(A2)**. We only need to show the following subadditive property of MAXVAR

$$\mathrm{MAXVAR}_n(X+Y) \le \mathrm{MAXVAR}_n(X) + \mathrm{MAXVAR}_n(Y) \quad \forall X, Y. \tag{1}$$

Then (1) and **(A5)** imply **(A2)**. For any $X, Y \in \mathscr{L}^2$, take $(X_1, Y_1), \cdots, (X_n, Y_n)$ as i.i.d. copies of the two dimensional random vector (X, Y). That is, the random vectors $(X_1, Y_1), \cdots, (X_n, Y_n)$ are independent and have the same joint distribution as the random vector (X, Y). Then $X_1, \cdots, X_n$ are i.i.d. copies of X and $Y_1, \cdots, Y_n$ are i.i.d. copies of Y. We next show that $X_1 + Y_1, \cdots, X_n + Y_n$ are i.i.d. copies of $X + Y$.

Since (X_i, Y_i) has the same joint distribution as (X, Y), $i = 1, ..., n$, it follows that $X_i + Y_i$ has the same distribution as $X + Y$. In order to prove that $X_1 + Y_1, \cdots, X_n+Y_n$ are independent, we only need to prove that for any $t_1, \cdots, t_n \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{P}_0\left(X_1+Y_1 \le t_1, \cdots, X_n+Y_n \le t_n\right)\\ =&\mathbb{P}_0\left(X_1+Y_1 \le t_1\right) \cdot \cdots \cdot \mathbb{P}_0\left(X_n+Y_n \le t_n\right).\end{aligned} \tag{2}$$

In fact, since the random vectors $(X_1, Y_1), \cdots, (X_n, Y_n)$ are independent, we have

$$\begin{aligned}&\mathbb{P}_0((X_1, Y_1) \in B_1, \cdots, (X_n, Y_n) \in B_n)\\ =&\mathbb{P}_0((X_1, Y_1) \in B_1) \cdot \cdots \cdot \mathbb{P}_0((X_n, Y_n) \in B_n)\end{aligned} \tag{3}$$

for any Borel sets $B_1, \cdots, B_n \subseteq \mathbb{R}^2$. In particular, if we take

$$B_i = \left\{(x, y) \in \mathbb{R}^2 : \ x + y \leq t_i\right\}$$

for any $1 \leq i \leq n$ in (3), we can get (2). Therefore, $X_1 + Y_1, \cdots, X_n + Y_n$ are independent. Moreover, they are i.i.d. copies of $X + Y$.

Since the definition of MAXVAR does not depend on the choice of the i.i.d. copies, we have

$$\begin{aligned}\mathrm{MAXVAR}_n(X) &= \mathbb{E}(\max\{X_1, \cdots, X_n\}),\\ \mathrm{MAXVAR}_n(Y) &= \mathbb{E}(\max\{Y_1, \cdots, Y_n\}),\\ \mathrm{MAXVAR}_n(X+Y) &= \mathbb{E}(\max\{X_1 + Y_1, \cdots, X_n + Y_n\}).\end{aligned}$$

Furthermore, since

$$\max\{X_1 + Y_1, \cdots, X_n + Y_n\} \leq \max\{X_1, \cdots, X_n\} + \max\{Y_1, \cdots, Y_n\},$$

we get

$$\begin{aligned}\mathrm{MAXVAR}_n(X+Y) &= \mathbb{E}(\max\{X_1 + Y_1, \cdots, X_n + Y_n\})\\ &\leq \mathbb{E}(\max\{X_1, \cdots, X_n\}) + \mathbb{E}(\max\{Y_1, \cdots, Y_n\})\\ &= \mathrm{MAXVAR}_n(X) + \mathrm{MAXVAR}_n(Y).\end{aligned}$$

Proof of **(A3)**. For any $X, Y \in \mathscr{L}^2$ satisfying $X \leq Y$, suppose $X_1, \cdots, X_n$ are i.i.d. copies of X and $Y_1, \cdots, Y_n$ are i.i.d. copies of Y. We can see that $\mathbb{P}_0(X \leq t) \geq \mathbb{P}_0(Y \leq t)$ for any $t \in \mathbb{R}$ since $X \leq Y$. Then we have

$$\begin{aligned}&\mathrm{MAXVAR}_n(X)\\ &= \int_{-\infty}^{0} [\mathbb{P}_0(\max\{X_1, \cdots, X_n\} > t) - 1]\, dt + \int_0^{+\infty} \mathbb{P}_0(\max\{X_1, \cdots, X_n\} > t) dt\\ &= -\int_{-\infty}^{0} (\mathbb{P}_0(X \leq t))^n dt + \int_0^{+\infty} [1 - (\mathbb{P}_0(X \leq t))^n]\, dt\\ &\leq -\int_{-\infty}^{0} (\mathbb{P}_0(Y \leq t))^n dt + \int_0^{+\infty} [1 - (\mathbb{P}_0(Y \leq t))^n]\, dt\\ &= \int_{-\infty}^{0} [\mathbb{P}_0(\max\{Y_1, \cdots, Y_n\} > t) - 1]\, dt + \int_0^{+\infty} \mathbb{P}_0(\max\{Y_1, \cdots, Y_n\} > t) dt\\ &= \mathrm{MAXVAR}_n(Y).\end{aligned}$$

The detail of the first equality is as follows. Denote by

$$F(t) = \mathbb{P}_0(\max\{X_1, \cdots, X_n\} \leq t)$$

the cumulative distribution function of $\max\{X_1, \cdots, X_n\}$. Then

$$\begin{aligned}
\mathbb{E}(\max\{X_1,\cdots,X_n\}) &= \int_{-\infty}^{+\infty} x dF(x) \\
&= -\int_{-\infty}^{0} \left[\int_{x}^{0} dt\right] dF(x) + \int_{0}^{+\infty} \left[\int_{0}^{x} dt\right] dF(x) \\
\text{(by Fubini's Theorem)} &= -\int_{-\infty}^{0} \left[\int_{-\infty}^{t} dF(x)\right] dt + \int_{0}^{+\infty} \left[\int_{t}^{+\infty} dF(x)\right] dt \\
&= -\int_{-\infty}^{0} F(t)dt + \int_{0}^{+\infty} [1-F(t)]dt. \qquad (4)
\end{aligned}$$

And the second equality comes from the fact that $F(t) = (\mathbb{P}_0(X \leq t))^n$.

Proof of **(A4)**. Suppose X^k $(k = 1, 2, \cdots), X \in \mathscr{L}^2$ and $\|X^k - X\|_2 \to 0$ as k tends to infinity. Then $X^k \to X$ in distribution. Denote by $F_k(t)$ the distribution function of X^k $(k = 1, 2, \cdots)$ and by $F(t)$ the distribution of X. Then $\lim_{k\to\infty} F_k(t) = F(t)$ for all continuous points of $F(\cdot)$. It implies that $\lim_{k\to\infty} [F_k(t)]^n = [F(t)]^n$ for all continuous points of $[F(\cdot)]^n$. Note that $[F_k(t)]^n$ is the distribution function of $\max\{X_1^k, \cdots, X_n^k\}$ and $[F(t)]^n$ is the distribution function of $\max\{X_1, \cdots, X_n\}$, where $X_1^k, \cdots, X_n^k$ are i.i.d. copies of X^k $(k = 1, 2, \cdots)$ and $X_1, \cdots, X_n$ are i.i.d. copies of X. Therefore, we have $\max\{X_1^k, \cdots, X_n^k\} \to \max\{X_1, \cdots, X_n\}$ in distribution, and

$$\begin{aligned}
\mathrm{MAXVAR}_n(X^k) &= \mathbb{E}(\max\{X_1^k, \cdots, X_n^k\}) \\
&\to \mathbb{E}(\max\{X_1, \cdots, X_n\}) = \mathrm{MAXVAR}_n(X)
\end{aligned}$$

as k tends to infinity. Thus, if $\mathrm{MAXVAR}_n(X^k) \leq 0$ for all $k = 1, 2, \cdots$ then $\mathrm{MAXVAR}_n(X) \leq 0$. The proof of the theorem is completed.

3 Risk-averseness of MAXVAR

Suppose $\mathcal{R}$ is a functional from $\mathscr{L}^2$ to $(-\infty, +\infty]$. Recall that an *averse* risk measure is defined by axioms (A1), (A2), (A4), (A5) and

(A6) $\mathcal{R}(X) > \mathbb{E}(X)$ for all non-constant X.

We then have the next theorem.

Theorem 2 *If* $n \geq 2$, *then* $\mathrm{MAXVAR}_n(\cdot)$ *is averse.*

Föllmer and Schied [4] proved that if $\mathcal{R}$ is a coherent, law-invariant risk measure in $\mathscr{L}^\infty$ (not $\mathscr{L}^2$) other than $\mathbb{E}(\cdot)$, then $\mathcal{R}$ is averse, where "law-invariant" stands for that $\mathcal{R}(X) = \mathcal{R}(Y)$ whenever X and Y have the same distribution under $\mathbb{P}_0$. Since we are now considering the $\mathscr{L}^2$ case, we cannot use the result in Föllmer and Schied [4] directly. We next give a separate proof.

Proof of Theorem 2 On one hand, for any $X \in \mathscr{L}^2$, let $X_1, \cdots, X_n$ be i.i.d. copies of X. Then we have

$$\mathrm{MAXVAR}_n(X) = \mathbb{E}(\max\{X_1, \cdots, X_n\}) \geq \mathbb{E}(X_1) = \mathbb{E}(X).$$

On the other hand, if $\mathrm{MAXVAR}_n(X) = \mathbb{E}(X) = \mathbb{E}(X_1)$ $(n \geq 2)$, then $\max\{X_1, \cdots, X_n\} = X_1$ almost surely. Similarly, $\max\{X_1, \cdots, X_n\} = X_2$ almost surely. Therefore, $X_1 = X_2$ almost surely. Since X_1 and X_2 are independent, we must have X_1 equals to a constant almost surely, which is equivalent to say X equals to a constant almost surely. Therefore, $\mathrm{MAXVAR}_n(X) > \mathbb{E}(X)$ for non-constant X, which implies that $\mathrm{MAXVAR}_n(\cdot)$ is averse when $n \geq 2$.

Remark. In fact, Theorems 1 and 2 can be obtained as corollaries of Theorem 3 in the next section. See the remark after the proof of Theorem 3 for details. However, we think it is of interest to provide an elementary proof only based on definition of MAXVAR.

4 MAXVAR as a continuous convex combination of CVaR

An important coherent risk measure in basic sense is the conditional value at risk (CVaR) popularized by Rockafellar and Uryasev [6]. Among several equivalent definitions of CVaR, the most familiar one is probably the following.

$$\mathrm{CVaR}_\alpha(X) = \min_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{1-\alpha} \mathbb{E}(X - \beta)_+ \right\}, \tag{5}$$

where $(t)_+ = \max(t, 0)$ and $\alpha \in [0, 1)$. The minimum is attained at $\beta^* = \mathrm{VaR}_\alpha(X)$, and the VaR (“Value-at-Risk”) is defined as

$$\mathrm{VaR}_\alpha(X) := \inf \left\{ \nu \in \mathbb{R} : \mathbb{P}_0(X > \nu) < 1 - \alpha \right\}. \tag{6}$$

In this section, we show that $\mathrm{MAXVAR}_n(\cdot)$ is certain “continuous convex combination” of the CVaR measure in the sense that

$$\mathrm{MAXVAR}_n(\cdot) = \int_0^1 \mathrm{CVaR}_\alpha(\cdot) w_n(\alpha) d\alpha,$$

where $w_n(\alpha)$ $(\alpha \in [0, 1])$ is the “weight function” which satisfies $w_n(\alpha) \geq 0$ on $[0, 1]$ and $\int_0^1 w_n(\alpha) d\alpha = 1$. Specifically, we have the next theorem.

Theorem 3 *For any $X \in \mathscr{L}^2$, we have*

$$\mathrm{MAXVAR}_n(X) = \int_0^1 \mathrm{CVaR}_\alpha(X) w_n(\alpha) d\alpha,$$

where

$$w_n(\alpha) := n(n-1)(1-\alpha)\alpha^{n-2}, \qquad \alpha \in [0, 1]$$

is the weight function.

Remark. It can be easily checked that $w_n(\alpha) \geq 0$ on $[0,1]$, and

$$\int_0^1 w_n(\alpha)d\alpha = n(n-1)\int_0^1 (\alpha^{n-2} - \alpha^{n-1})d\alpha = n(n-1)\left[\frac{1}{n-1} - \frac{1}{n}\right] = 1.$$

Therefore, $w_n(\alpha)$ is indeed a weight function.

Theorem 3 was mentioned in Cherny and Orlov [3] without details. We now give a detailed proof by using the so called "Choquet integral". First, we need a lemma. For any $\alpha \in [0,1)$, define $f_\alpha(\cdot): \Sigma \to [0,1]$ in the following way,

$$\begin{aligned} f_\alpha(A) :&= \begin{cases} \frac{1}{1-\alpha}\mathbb{P}_0(A) & \text{if } \mathbb{P}_0(A) \leq 1-\alpha, \\ 1 & \text{otherwise.} \end{cases} \\ &= g_\alpha[\mathbb{P}_0(A)], \end{aligned}$$

where

$$g_\alpha(x) := \begin{cases} \frac{1}{1-\alpha}x & \text{if } x \in [0, 1-\alpha), \\ 1 & \text{if } x \in [1-\alpha, 1]. \end{cases} \tag{7}$$

We then have the following lemma, which implies that the CVaR measure can be written as the "Choquet integral" with respect to $f_\alpha(\cdot)$.

Lemma 1 *For any $X \in \mathscr{L}^2$ and $\alpha \in [0,1)$, we have*

$$\mathrm{CVaR}_\alpha(X) = \int_{-\infty}^0 [f_\alpha(X > t) - 1]dt + \int_0^{+\infty} f_\alpha(X > t)dt.$$

Proof If $\mathrm{VaR}_\alpha(X) \leq 0$, then

$$\begin{aligned} &\int_{-\infty}^0 [f_\alpha(X > t) - 1]dt + \int_0^{+\infty} f_\alpha(X > t)dt \\ =&\int_{\mathrm{VaR}_\alpha(X)}^0 \left[\frac{1}{1-\alpha}\mathbb{P}_0(X > t) - 1\right]dt + \int_0^{+\infty} \frac{1}{1-\alpha}\mathbb{P}_0(X > t)dt \\ =&\mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\cdot\int_{\mathrm{VaR}_\alpha(X)}^{+\infty} \mathbb{P}_0(X > t)dt \\ =&\mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\cdot\mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] = \mathrm{CVaR}_\alpha(X). \end{aligned}$$

The last step above is due to (5) and (6).

If $\mathrm{VaR}_\alpha(X) > 0$, then

$$\begin{aligned} &\int_{-\infty}^0 [f_\alpha(X > t) - 1]dt + \int_0^{+\infty} f_\alpha(X > t)dt \\ =&\int_0^{\mathrm{VaR}_\alpha(X)} dt + \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} \frac{1}{1-\alpha}\mathbb{P}_0(X > t)dt \\ =&\mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\cdot\int_{\mathrm{VaR}_\alpha(X)}^{+\infty} \mathbb{P}_0(X > t)dt \\ =&\mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\cdot\mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] = \mathrm{CVaR}_\alpha(X), \end{aligned}$$

which completes the proof.

Proof of Theorem 3 Define

$$h(x) := 1 - (1-x)^n, \qquad x \in [0,1].$$

It is not difficult to check that

$$h(x) = \int_0^1 g_\alpha(x) w_n(\alpha) d\alpha, \qquad x \in [0,1], \tag{8}$$

where $g_\alpha(x)$ is as defined in (7). By (4), for any $X \in \mathscr{L}^2$ we have

$$\mathrm{MAXVAR}_n(X) = \int_{-\infty}^0 [h(\mathbb{P}_0(X>t)) - 1]dt + \int_0^{+\infty} h(\mathbb{P}_0(X>t))dt. \tag{9}$$

So by (8), (9) and Lemma 1, together with Fubini's theorem and the fact that $\int_0^1 w_n(\alpha)d\alpha = 1$, we get

$$\begin{aligned}\mathrm{MAXVAR}_n(X) &= \int_{-\infty}^0 \int_0^1 [f_\alpha(X>t) - 1] w_n(\alpha) d\alpha dt + \int_0^{+\infty} \int_0^1 f_\alpha(X>t) w_n(\alpha) d\alpha dt \\ &= \int_0^1 \left[\int_{-\infty}^0 [f_\alpha(X>t) - 1]dt + \int_0^{+\infty} f_\alpha(X>t)dt\right] w_n(\alpha) d\alpha \\ &= \int_0^1 \mathrm{CVaR}_\alpha(X) w_n(\alpha) d\alpha\end{aligned}$$

for any $X \in \mathscr{L}^2$, as desired.

Remark. Theorem 3 says that $\mathrm{MAXVAR}_n(\cdot)$ is a continuous convex combination of the CVaR measure, its coherency in basic sense follows from Proposition 2.1 of Ang et al [1], and its averseness follows from the averseness of the CVaR (Proposition 4.4 of Ang et al [1]) together with the basic property of integral. Therefore, Theorem 3 can actually provide an alternative proof of the coherency and averseness of $\mathrm{MAXVAR}_n(\cdot)$.

5 The risk envelope of MAXVAR

Since

$$\mathrm{MAXVAR}_n(\cdot) = \int_0^1 \mathrm{CVaR}_\alpha(\cdot) w_n(\alpha) d\alpha$$

is a coherent risk measure on $\mathscr{L}^2$, by the dual representation theorem (Rockafellar [5]), there exists a unique, nonempty, convex and closed set $\mathcal{Q}_n \subseteq \mathscr{L}^2$, called "the risk envelope of $\mathrm{MAXVAR}_n(\cdot)$" such that

$$\mathrm{MAXVAR}_n(X) = \sup_{Q \in \mathcal{Q}_n} \mathbb{E}(XQ)$$

for any $X \in \mathscr{L}^2$.

In this section we aim at characterizing the risk envelope of $\mathrm{MAXVAR}_n(\cdot)$". First recall the following well-known result for the discrete convex combination of the CVaR measure, which can be found in Rockafellar [5] and whose proof can be found in Ang et al [1].

Proposition 1 *Let* $\mathcal{R}(\cdot) = \sum_{i=1}^{n} \lambda_i \mathrm{CVaR}_{\alpha_i}(\cdot)$ *with positive weights* λ_i *adding up to* 1*. Then* $\mathcal{R}$ *is a coherent risk measure in the basic sense and its risk envelope is*

$$\left\{ \sum_{i=1}^{n} \lambda_i Q_i : \ 0 \le Q_i \le \frac{1}{1-\alpha_i}, \ \mathbb{E}(Q_i) = 1, \ \forall i = 1, 2, \cdots, n \right\}.$$

A continuous version of Proposition 1 gives the risk envelope of MAXVAR as follows.

Theorem 4 *The risk envelope of MAXVAR is*

$$\mathcal{Q}_n := \mathrm{cl} \left\{ \int_0^1 Q_\alpha w_n(\alpha) d\alpha, \ 0 \le Q_\alpha \le \frac{1}{1-\alpha}, \ \mathbb{E}(Q_\alpha) = 1, \ \forall \alpha \in [0, 1) \right\}, \quad (10)$$

where

$$w_n(\alpha) := n(n-1)(1-\alpha)\alpha^{n-2} \qquad \alpha \in [0, 1]$$

is the weight function (0^0 *is defined to be 1), and "*cl*" stands for the closure in* $\mathscr{L}^2$*.*

Proof Note that the integration "$\int_0^1 Q_\alpha w_n(\alpha) d\alpha$" in (10) is defined pointwise. That is, $Y = \int_0^1 Q_\alpha w_n(\alpha) d\alpha$ means $Y(\omega) = \int_0^1 Q_\alpha(\omega) w_n(\alpha) d\alpha$ for any $\omega \in \Omega$. Since $0 \le Q_\alpha \le \dfrac{1}{1-\alpha}$ for any $\alpha \in [0, 1)$, we have

$$0 \le \int_0^1 Q_\alpha(\omega) w_n(\alpha) d\alpha \le \int_0^1 n(n-1)\alpha^{n-2} d\alpha = n$$

for any $\omega \in \Omega$. Therefore, $\mathcal{Q}_n \subseteq \mathscr{L}^\infty \subseteq \mathscr{L}^2$. In addition, we can check that

$$\begin{aligned} \mathrm{MAXVAR}_n(X) &= \int_0^1 \mathrm{CVaR}_\alpha(X) w_n(\alpha) d\alpha \\ &= \sup \left\{ \mathbb{E}\left(X \int_0^1 Q_\alpha w_n(\alpha) d\alpha \right) : \ \int_0^1 Q_\alpha w_n(\alpha) d\alpha \in \mathcal{Q}_n \right\} \end{aligned} \quad (11)$$

for any $X \in \mathscr{L}^2$. Furthermore, it is easy to check the convexity of $\mathcal{Q}_n$. Since $\mathcal{Q}_n$ is closed in $\mathscr{L}^2$, it follows from the dual representation theorem that Formula (11) implies that (10) is the risk envelope of $\mathrm{MAXVAR}_n(\cdot)$.

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References

1. M. Ang, J. Sun and Q. Yao (2017) On the dual representation of coherent risk measures, *Ann. Oper. Res.* to appear. DOI: 10.1007/s10479-017-2441-3.
2. A. Cherny and D. Madan (2009) New measures for performance evaluation, *Rev. Finan. Stud.* **22**, 2571-2606.
3. A. Cherny and D. Orlov (2011) On two approaches to coherent risk contribution, *Math. Finance* **23**, 557-571.
4. H. Föllmer and A. Schied (2004).*Stochastic Finance (2nd Edition).* Walter de Gruyter, Berlin, Germany.
5. R. T. Rockafellar (2007) Coherent approaches to risk in optimization under uncertainty, *Tutorials in Operations Research, INFORMS*, 38-61.
6. R.T. Rockafellar and S. Uryasev (2000). Optimization of conditional value-at-risk. *Journal of Risk*, **2(3)**, 21-42.