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INEQUALITIES FOR THE FUNDAMENTAL ROBIN EIGENVALUE FOR THE LAPLACIAN ON N–DIMENSIONAL RECTANGULAR PARALLELEPIPEDS

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Abstract. Amongst N-dimenstional rectangular parallelepipeds (boxes) of a given volume, that which has the smallest fundamental Robin eigenvalue for the Laplacian is the N-cube. We give an elementary proof of this isoperimetric inequality based on the well-known formulae for the eigenvalues. Also treated are various related inequalities which are amenable to investigation using the formulae for the eigenvalues.

1. Introduction

1.1. Overview

This paper has its origins in earlier papers by the authors. In the more recent of these, in the microfluidics section of an engineering conference [20] (with detailed proofs and related items in a supplement in arXiv [18]), the application led us to prove the $N=2$ case of Theorem 1 in this paper. The main physical application in the older paper [29], heat flow, required $N=3$ but the paper often considered general values of $N \geqslant 2$. In the general setting, Ω is a bounded simply-connected domain in R^N, with piecewise C^1 boundary. We will soon study the special case when Ω is a rectangular parallelepiped, here called a box. (Other words for the same shape include rectangular cuboid, hyper-rectangle, N-interval and N-orthotope.)

Let $\Delta u = \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2}$ denote the N-dimensional Laplacian. We are concerned with the fundamental eigenvalue λ_1 and corresponding positive eigenfunction u_1 satisfying

$$\Delta u_1 + \lambda_1 u_1 = 0 \text{ and } u_1 > 0 \text{ in } \Omega, \qquad \beta \frac{\partial u_1}{\partial n} + u_1 = 0 \text{ on } \partial\Omega\,. \tag{1.1}$$

Here n is the outward normal, $\beta \geqslant 0$, and the second equation is called a ‘Robin boundary condition’. (In the context of fluid flows with slip at the boundary it is called Navier’s boundary condion. In elasticity, it arises with boundaries that are elastically supported, see e.g. [35]. In the context of heat diffusion it is sometimes called Newton’s law of cooling. See [13].)

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The domains Ω we treat are the boxes

$$\Omega = (-a_1, a_1) \times (-a_2, a_2) \times \ldots \times (-a_N, a_N).$$

We sometimes indicate the dependence on the list of a_j by writing $\Omega(\mathbf{a})$. Some of our results, e.g. Theorem 1, (but not Theorems 2, 3 or 4 or Corollary 2) require the volume to be fixed. Then we consider the family of boxes, with volume $(2h)^N$, with $a_j = r_j h$ with all $r_j > 0$ and the product of the r_j equal to 1. We denote the volume of Ω by $|\Omega|$, so that

$$|\Omega(\mathbf{a})| = 2^N \prod_{j=1}^{N} a_j = (2h)^N. \tag{1.2}$$

Our goal throughout the paper is to present results which are valid for $\beta \geqslant 0$, and it is usually the situation that the $\beta = 0$ case is well known. Of our results the easiest to state is the following isoperimetric inequality.

THEOREM 1. Faber-Krahn for boxes with $\beta \geqslant 0$. *Amongst all boxes with a given volume, that which has the smallest fundamental Robin eigenvalue is the cube.*

The classical geometric isoperimetric inequality relating volume and 'perimeter' for boxes, is that there is a $C_\square(N) > 0$ (whose numeric value is unimportant here) such that

$$C_\square(N)|\Omega|^{N-1} \leqslant |\partial\Omega|^N, \qquad \text{and } C_\square(2) = 16,\ C_\square(3) = 36.$$

This combines with the domain monotonicity for boxes, Theorem 2, to yield the following Corollary. With $\beta > 0$ and $N = 2$ we will see (in §1.2) another proof as a corollary of Theorem 4.

COROLLARY 1. *Amongst all boxes with a given 'perimeter', that which has the smallest eigenvalue is the cube.*

Our proofs of Theorem 1 when $\beta > 0$ (including those based on separable convex optimization methods given in our arXiv supplements) are elementary. However our search of the literature has failed to find the result, even with other proofs (and we know that many alternative proofs are possible). We are aware that results can be hard to locate in older literature. Indeed in §3 of [29] we unwittingly rediscovered the formulae for the fundamental Robin eigenfunction for the equilateral triangle, a century and a half after Lamé, and only learnt that it was a rediscovery a decade after our paper. Lamé's work is referenced in this journal in [21].

When $\beta = 0$, the result of Theorem 1 is well-known, and very elementary. Then one has the volume fixed $|\Omega| = 2^N \prod_{j=1}^{N} a_j$ and minimizes the fundamental *Dirichlet* eigenvalue

$$\lambda_1(\beta = 0) = \frac{\pi^2}{4} \sum_{j=1}^{N} \frac{1}{a_j^2}. \tag{1.3}$$

Fixing the volume is equivalent to fixing the product $\prod_{j=1}^{N} 1/a_j^2$, and with this observation, we see that the $\beta = 0$ case is equivalent to the equality case of the AM$\geqslant$GM inequality. (See [14] §2.5 p. 17.) AM abbreviates 'Arithmetic Mean', GM, 'Geometric Mean'. The minimization problem for λ_1, when $\beta = 0$, is similar to that in which one seeks, instead, to minimize the perimeter (again with a constant $C(N)$ which is unimportant here)

$$|\partial\Omega| = C(N)|\Omega| \sum_{j=1}^{N} \frac{1}{a_j}.$$

(Another similar optimization problem solved by the cube is minimizing the polar moment of inertia, I_c, over boxes with given volume.)

Return now to the situation where $\beta \geqslant 0$. The fundamental eigenvalue is given by a formula

$$\lambda_1(\Omega(\mathbf{a})) = \sum_{j=1}^{N} \mu(a_j)^2.$$

The function μ, as given in the transcendental equation (2.2), also depends on β, but in contexts where β is fixed we omit it. When the value of β needs to be indicated, we write $\mu(\beta,a)$ as, for example, $\mu(0,a) = \pi/(2a)$. Our proofs of the isoperimetric result of Theorem 1 follow from properties of the positive, decreasing, convex function μ, or of its inverse. The inverse of μ is an elementary function, denoted ϕ_1 (see equation (3.4), and we have chosen to first establish properties of ϕ and from these properties of μ. See §3. Using only that $\mu(a)$ is convex, indeed log-convex (also called AG-convex), is, by itself, insufficient to establish that the separable convex optimization problem, minimize $\lambda_1(\Omega(\mathbf{a}))$ subject to the volume constraint (1.2), has as its solution that all a_j are equal with $2a_j = |\Omega|^{1/N}$ for all j. For the proof of Theorem 1 additional properties of μ – or of its inverse – are needed. What is actually needed is clear from the following definition and restatement of the theorem.

We will only need to use definitions such as that for GA-convex functions below for positive decreasing convex functions, which, in our application, are C^∞. We remark that in this setting, if inequality (1.4) is satisfied with $k = 2$ it is satisfied for all positive integer k,

DEFINITION 1.

$$g \text{ is GA-convex } \iff g\left(\left(\prod_{j=1}^{k} a_j\right)^{1/k}\right) \leqslant \frac{1}{k}\sum_{j=1}^{k} g(a_j), \tag{1.4}$$

for all positive integers k.

THEOREM 1. (Restated) *The function* $\mu_{(2)}$, *where* $\mu_{(2)}(c) = \mu(c)^2$, *is GA-convex.*

The equation expressing this is that of (1.4), with $k = N$ and with the function g there being replaced by $\mu_{(2)}$.

Other forms of generalised convexity used in this paper, along with GA-convexity, are defined (using $k = 2$ in the definitions, but knowing that they extend to other k) are given in Appendix A. The notation $\mu_{(2)}$ is introduced as, amongst other reasons, differentiation with respect to it is easier to read than attempting to use μ^2.

The same GA-convexity of $\mu_{(2)}$ as in Theorem 1 Restated leads to the following:

COROLLARY 2. *The fundamental eigenvalue* $\lambda_1(\Omega(\mathbf{a})) = \sum_{j=1}^{N} \mu(a_j)^2$ *as given in* §2 *equation* (2.1)*, is a convex function of the* $\ell_j = \log(a_j)$.

Equivalently, with $\mathbf{a}(t) = (a_j(0)^{1-t} a_j(1)^t)$*, then* $\lambda_1(\mathbf{a}(t))$ *satisfies the inequality*

$$\lambda_1(\mathbf{a}(t)) \leqslant (1-t)\lambda_1(\mathbf{a}(0) + t\lambda_1(\mathbf{a}(1)) \text{ for } 0 < t < 1.$$

As mentioned above, when $\beta = 0$, the $\mu(a_j) = \pi/(2a_j)$. Once again the corollary is trivial to prove directly when $\beta = 0$ as then

$$\lambda_1(\beta = 0) = \frac{\pi^2}{4} \sum_{j=1}^{N} \exp(-2\ell_j). \tag{1.5}$$

An outline of our paper is as follows. In §1.2 we note some other results which are familiar in the case $\beta = 0$, and which extend to our boxes with $\beta > 0$. For domains more generally much more has been established when $\beta = 0$, and there are proof techniques which are not available for $\beta > 0$, e.g.Steiner symmetrisation for which [33] will serve as a reference. See §1.3.

The main subject of this paper is the box domain, and we return to this in all subsequent sections. In §2 we give the exact solution for the eigenfunction, and the implicit equation for its eigenvalue. This is developed in §3. We conclude that section with the very short proof of Theorem 1. Further inequalities on the fundamental Robin eigenvalue for boxes we treat are summarised immediately below in §1.2. Their proofs are in §5. In the discussion in §6 we mention some open questions.

For domains more generally, much is known about the fundamental Robin eigenvalue. The corresponding eigenfunction can be taken to be positive, and will be in this paper. The generalization of the original Faber-Krahn inequality is given in [10]. In that context one has an isoperimetric result associated with varying over all bounded domains with a given volume, with the N-dimensional ball as the optimizer. In this paper we vary over boxes with a given volume, again obtaining, in Theorem 1, an 'isoperimetric' inequality, and explaining our description of the theorem as 'Faber-Krahn for rectanges'.

1.2. Other ways the geometry of boxes affects λ_1

In dimensions $N \geqslant 2$, general domain monotonicity, true for $\beta = 0$, isn't true when $\beta > 0$. Rectangles, boxes, however, behave nicely, as noted in the following.

THEOREM 2. *When* $\beta \geqslant 0$*, boxes inherit domain monotonicity from the domain monotonicity that is present when* $N = 1$ *as given by the formula* (2.1)*. That is, if* $\Omega_1 \subseteq \Omega_2$ *then* λ_1 *satisfies the inequality* $\lambda_1(\Omega_1) \geqslant \lambda_1(\Omega_2)$.

Some of the behaviour as one scales the domain is given in the following:

THEOREM 3. *(i) Let* Ω_1 *be a box. Define* $\Omega_t = t\Omega_1$*. Then* $\lambda_1(\Omega_t)$ *is decreasing in* t*, convex in* t *and further, AG-convex and HA-convex.*

(ii) For $\beta = 0$*, for any* Ω_1*, not merely for boxes* $\lambda_1(\Omega_t)$ *is completely monotone.*

Item (i) follows from the corresponding properties for $\mu_{(2)}$ given in Lemma 3. By taking $\Omega_0 = \{0\}$ one may see some similar results, concerning μ rather than $\mu_{(2)}$.

Item (ii) is trivial as it is merely the statement that $1/t^2$ is completely monotone on $t > 0$. We emphasise this statement concerning changes of scale is, when $\beta = 0$, true for any domain Ω. We have included it here to assist in exposition in §6.

When $\beta = 0$ there are many further results for convex domains. Consider the Minkowski sum, defining a $\Omega_t = (1-t)\Omega_0 + t\Omega_1$. Then, when $\beta = 0$, $\lambda_1(\Omega_t)^{-1/2}$ is a concave function of t. Equation (5) of [8] gives

$$\lambda_1(\Omega_t)^{-1/2} \geqslant (1-t)\lambda_1(\Omega_0)^{-1/2} + t\lambda_1(\Omega_1)^{-1/2} \qquad \text{when} \quad \beta = 0. \tag{1.6}$$

While we do not know if there is any result, when $\beta > 0$ for general convex domains, for boxes and $\beta \geqslant 0$ we have a result. Our boxes centred on the origin behave nicely under Minkowski sums. The set of all our boxes centred at the origin is closed under Minkowski sums:

$$\Omega(t) := \Omega((1-t)\mathbf{a} + t\mathbf{b}) = (1-t)\Omega(\mathbf{a}) + t\Omega(\mathbf{b}).$$

(That when $\beta = 0$, $\lambda_1(\Omega_t)^{-1/2}$ is a concave function of t is easily checked for our boxes, When $\beta = 0$ using formula (1.3), one verifies inequality (1.6) by using the $r = -2$ form of the Minkowski inequality as given in [14], §2.11, p. 30.) If the property does extend to general convex domains and $\beta > 0$, some of the ingredients of the $\beta = 0$ proof, e.g. homogeneity of domain functionals, are lost when we fix $\beta > 0$.

THEOREM 4. *Use the notation above for the Minkowski sum of boxes and let the fundamental eigenvalue $\lambda_1(\Omega(t))$ be as given in §2 equation (2.1). For all $\beta \geqslant 0$, $\lambda_1(\Omega(t))^{-1/2}$ a concave function of t. That is*

$$\frac{1}{\sqrt{\lambda_1(\Omega(t))}} \geqslant \frac{1-t}{\sqrt{\lambda_1(\Omega(0))}} + \frac{t}{\sqrt{\lambda_1(\Omega(1))}} \quad \text{for } 0 \leqslant t \leqslant 1.$$

When $N = 2$, Corollary 1 to Theorem 1 follows from Theorem 4 and the observation that if the rectangles Ω_0 and Ω_1, Ω_1 being obtained by rotating Ω_0 through a right angle, then the perimeter of Ω_t is constant in t. Symmetry suggests that the maximum of the concave function $\lambda_1(\Omega(t))^{-1/2}$ will occur at $t = 1/2$, i.e. the square.

For the next theorem, though results in N dimensions are available, for ease of exposition, results presented here and in subsection §5.3 are, unless otherwise indicated, for $N = 2$. Part (ii) of the theorem is not proved in this paper: see [22]. The theorem presents interesting results involving, as well as $\lambda_1(\Omega)$, the polar moment of inertia about the centroid, $I_c(\Omega)$. For our rectangles $(-rh, rh) \times (-h/r, h/r)$,the area is $4h^2$, and

$$I_c(r) = \frac{4}{3}h^4\left(r^2 + \frac{1}{r^2}\right) \qquad \text{while} \qquad \lambda_1(\beta = 0) = \frac{\pi^2}{4h^2}\left(r^2 + \frac{1}{r^2}\right).$$

THEOREM 5. *Define*

$$\mathscr{R}(\Omega) := \frac{\lambda_1(\Omega)|\Omega|^3}{I_c(\Omega)}.$$

(i) $\mathscr{R}(\Omega)$ is maximal among rectangles for the square.

(ii) $\mathscr{R}(\Omega)$ is maximal among triangles for the equilateral triangle, maximal among parallelograms for the square, and maximal among ellipses for the disk.

If we choose to consider families of shapes with the same volume, the results involve the ratio $\lambda_1(\Omega)/I_c(\Omega)$.

This theorem (and more) is proved in [22]. Our inequality (5.1) establishes the $\beta > 0$ version of Theorem 5(i): see §5.3.

Some of the history is given in [23]. In [33] it is noted that, when $\beta = 0$, $\mathscr{R} = 12\pi^2$ is constant for rectangles, which, however, renders the $\beta = 0$ case of Theorem 5(i) somewhat trivial. Tables of $\mathscr{R}(\Omega)$ for various plane shapes, and $\beta = 0$, are given in [33] p. 257. Parallelograms are considered in [32]. Also Hersch [15], equation (5), establishes the result for $\beta = 0$ and $N = 2$, and states it as *Of all parallelograms having given distances between their parallel sides, the rectangle has the largest* λ_1. A variational proof of the result, with $N = 2$ and $\beta = 0$ is straightforward.

There are some subtleties when β is nonzero. We will not be proving (ii) in this paper: see [22].

1.3. A famous question of Polya and Szego [33]

There is a famous question of Polya and Szego [33], p. 159. *Amongst all n-gons of given area, does the regular n-gon have the smallest λ_1?* Symmetrization techniques are used, when $\beta = 0$ and $N = 2$ in [33] pp. 158–159 to establish this to the case when $n = 3$ and $n = 4$ (but the answer is unknown for $n > 4$). Consider $n = 4$. A proof of the Faber-Krahn for quadrilaterals of a give area, that the square minimizes λ_1 has the following steps

(i) Symmetrise the initial quadrilatera with respect to the perpendicular to a diagonal to a kite, symmetric about diagonal d.

Symmetrise the kite about a line perpendicular to d, producing a rhombus.

Symmetrise the rhombus with respect to a perpendicular to one of its sides, producing a rectangle.

(ii) Use the Faber-Krahn for rectangles to show that the square is the minimizer of λ_1.

Our Theorem 1 concerns $\beta > 0$ and, at least is not inconsistent with the possibility that, as is the case with $\beta = 0$, the square is the optimizer over all quadrilaterals with the given area. It provides step (ii) for the case $\beta > 0$.

Consider next $N \geqslant 2$. Symmetrization can be applied when $\beta = 0$ and $N \geqslant 2$ considering all (i.e. not necessarily rectangular) parallelepipeds: see [33] p. 159 item (d). Item (d) states that for any prism (right or oblique) with a given volume and a quadrilateral base, there is a set of successive Steiner symmetrizations which will transform the prism to a cube.

Symmetrization techniques are not directly applicable when $\beta > 0$, and in this paper we restrict our study to rectangular parallelopipeds, here called boxes.

2. The explicit formulae for λ_1 for a box

The function $u=\prod_{j=1}^{N}\cos(\mu(a_j)x_j)$ satisfies

$$\Delta u+\lambda u=0,\qquad \text{with } \lambda=\sum_{j=1}^{N}\mu(a_j)^2. \tag{2.1}$$

(Here we have chosen to look for modes which are symmetric about the axes, which is appropriate for the fundamental mode. In other applications, this need not be appropriate. The general setting is given in [12]§3.1.) The Robin boundary conditions are satisfied if, for all j,

$$\mu(a_j)\tan(a_j\mu(a_j))=\frac{1}{\beta}. \tag{2.2}$$

The function $\mu(c)$ and the geometry determine λ_1: Another organization of the equation is

$$\mu(c)\tan(c\mu(c))=\frac{1}{\beta},\quad \text{or equivalently}\quad \hat{\mu}\tan(\hat{c}\hat{\mu})=1,\quad \text{where } \hat{\mu}=\beta\mu,\ \hat{c}=\frac{c}{\beta}\ . \tag{2.3}$$

The transcendental equations have been widely studied, e.g. [7, 28, 27]. Numerical values, often used for checks, are given in Table 4.20 of [1]. Amongst the various applications are (i) the energy spectrum for the one-dimensional quantum mechanical finite square well, and (though with $c<0$) (ii) (though with $c<0$) zeros of the spherical Bessel function $y_1(x)=j_{-2}(x)$.

We have an interest in the smallest positive solutions,

$$0<\mu(a_j)<\pi/(2a_j),\qquad 0<\hat{\mu}<\pi/(2\hat{c}).$$

As an aside we remark that $\lambda_1(\Omega(\mathbf{a}))$ for the boxis the arithmetic mean of the fundamental eigenvalues of the cubes, i.e. of $\lambda_1(\Omega(a_j\mathbf{1}))$ where $\mathbf{1}$ is the N-vector of 1s.

3. The one-dimensional problem

Determining properties of $\mu(c)$ directly involves some implicit differentiation. For example,

$$\frac{d\mu}{dc}=-\frac{\mu(1+\beta^2\mu^2)}{\beta+c(1+\beta^2\mu^2)},\qquad \frac{d\mu_{(2)}}{dc}=-\frac{2\mu_{(2)}(1+\beta^2\mu_{(2)})}{\beta+c(1+\beta^2\mu_{(2)})}. \tag{3.1}$$

so that, at fixed $\beta>0$, μ decreases as c increases. Also $\mu(c)$ is convex in c:

$$\frac{d^2\mu}{dc^2}=\frac{2\mu(1+\beta^2\mu^2)(2\beta^3\mu^2+c\beta^4\mu^4+2c\beta^2\mu^2+\beta+c)}{(\beta+c(1+\beta^2\mu^2))^3}.$$

One can proceed directly to determine further convexity properties: the first such we found was that μ is log-convex (AG-convex). Of course this implies that $\mu_{(2)}$ is also log-convex (and hence convex).

The function μ has a simple inverse function ϕ_1,

$$\frac{c}{\beta} = \phi_1(\beta\mu) = \frac{1}{\beta\mu}\arctan\left(\frac{1}{\beta\mu}\right).$$

Also define $\phi_2(z) = \phi_1(\sqrt{z}))$. The positive, decreasing functions ϕ are convex, and formulae corresponding to those immediately above for μ are:

$$\frac{d\phi_1}{dz} = -\frac{\phi_1}{z} - \frac{1}{z(1+z^2)}, \qquad \frac{d\phi_2}{dc} = -\frac{\phi_2}{2z} - \frac{1}{2z(1+z)}. \tag{3.2}$$

In notation compatible with that of Appendix B,

$$\frac{d^2\phi_2}{dz^2} = \frac{Q_2(\phi_2(z))}{\phi_2(z)\,(2z(1+z))^2}, \quad \text{with } Q_2(\Phi_2,z) = \Phi_2\left(3(1+z)^2\Phi_2 + 5z + 3\right). \tag{3.3}$$

The numeric details are unimportant here: the notable fact is that ϕ_2 is convex. Indeed, similar calculations can be used to establish further convexity properties – and are indicated in Appendix B. Perhaps it is appropriate to mention the log-convexity of ϕ_2 here as that is the property needed to prove the Faber-Krahn result, Theorem 1.

We chose to obtain properties of ϕ_1 and of ϕ_2 and from these deduce properties of μ using (rather elementary) results from Appendix A. The main results relevant to this paper, and proved in the appendices are given in the next Lemmas. (Shorter and neater proofs of some of the properties, proofs associated with the ϕ functions being completely monotone, and ϕ_2 being Stieltjes, are given in [19].)

LEMMA 1. *Let*

$$\phi_1(z) = \frac{1}{z}\arctan\left(\frac{1}{z}\right), \qquad \phi_2(z) = \frac{1}{\sqrt{z}}\arctan\left(\frac{1}{\sqrt{z}}\right). \tag{3.4}$$

Then both $\phi_1(z)$ and $\phi_2(z)$ are positive, monotonic decreasing, convex functions on $0 < z < \infty$. Both ϕ_1 and ϕ_2 are log-convex (AG-convex), and, even stronger, both are completely monotone.

ϕ_2 is a Stieltjes function so AH-convex (as well as AG-convex) while HA-concave.

ϕ_1 is HA-convex, AG-convex and AH-concave.

COROLLARY 3. *Let μ, $\mu_{(2)}$, ϕ_1 and ϕ_2 be as above.*

The inverse of ϕ_1 is μ and $1/\mu$ is concave (i.e. μ is AH-convex and hence, also, $\log(\mu)$ is convex, μ is AG-convex). μ is aso HA-concave.

The inverse of ϕ_2 is $\mu_{(2)}$ which is log-convex – AG-convex, and HA-convex (hence also GA-convex, i.e. is such that $\mu_{(2)}(\exp(\ell))$ is convex in ℓ). $\mu_{(2)}$ is also AH-concave.

Theorem 1 Restated is included in the latter item in the preceding corollary.

The preceding lemma and corollary are required for several other results of this paper. Definitions are given systematically in Appendix A, but sometimes, as immediately below, repeated in this paper. Further results follow from additional properties of μ, e.g. Theorem 7 follows from properties in Lemma 2.

DEFINITION 2. A function $f:(0,\infty)\to(0,\infty)$ is said to be *GG-convex* iff

$$f(\sqrt{x_0x_1}) \leqslant \sqrt{f(x_0)f(x_1)}.$$

With the inequality reversed it is *GG-concave*.

LEMMA 2. *The functions ϕ_2 and $\mu_{(2)}$ (and ϕ_1 and μ) are GG-concave.*

The convexity properties of the ϕ and of the μ are summarised in the following diagrams. For 'vex' read 'convex': for 'ave' read 'concave'. The diagram for ϕ_2 and $\mu_{(2)}$ is as follows.

$$\begin{array}{ccccc}
\mathrm{AH-vex} & \Longrightarrow & \mathrm{AG-vex} & \Longrightarrow & \mathrm{AA-vex} \\
 & & & & \Uparrow \\
\mathrm{GH-ave} & \Longleftarrow & \mathrm{GG-ave} & & \mathrm{GA-vex} \\
\Downarrow & & \Downarrow & & \\
\mathrm{HH-ave} & \Longleftarrow & \mathrm{HG-ave} & \Longleftarrow & \mathrm{HA-ave} \\
 & & \phi_2 & &
\end{array}
\qquad
\begin{array}{ccccc}
\mathrm{HA-vex} & \Longrightarrow & \mathrm{GA-vex} & \Longrightarrow & \mathrm{AA-vex} \\
 & & & & \Uparrow \\
\mathrm{HG-ave} & \Longleftarrow & \mathrm{GG-ave} & & \mathrm{AG-vex} \\
\Downarrow & & \Downarrow & & \\
\mathrm{HH-ave} & \Longleftarrow & \mathrm{GH-ave} & \Longleftarrow & \mathrm{AH-ave} \\
 & & \mu_{(2)} & &
\end{array}$$

The corresponding results for ϕ_1 and μ are:

$$\begin{array}{ccccc}
\mathrm{AH-ave} & & \mathrm{AG-vex} & \Longrightarrow & \mathrm{AA-vex} \\
\Downarrow & & & & \Uparrow \\
\mathrm{GH-ave} & \Longleftarrow & \mathrm{GG-ave} & & \mathrm{GA-vex} \\
\Downarrow & & \Downarrow & & \Uparrow \\
\mathrm{HH-ave} & \Longleftarrow & \mathrm{HG-ave} & & \mathrm{HA-vex} \\
 & & \phi_1 & &
\end{array}
\qquad
\begin{array}{ccccc}
\mathrm{HA-ave} & & \mathrm{GA-vex} & \Longrightarrow & \mathrm{AA-vex} \\
\Downarrow & & & & \Uparrow \\
\mathrm{HG-ave} & \Longleftarrow & \mathrm{GG-ave} & & \mathrm{AG-vex} \\
\Downarrow & & \Downarrow & & \Uparrow \\
\mathrm{HH-ave} & \Longleftarrow & \mathrm{GH-ave} & & \mathrm{AH-vex} \\
 & & \mu & &
\end{array}$$

From the diagrams above one expects (correctly) that the hardest results to establish by direct calculation would be GA-convexity, GG-concavity and HA-concavity. (We remark, though, that any Stieltjes function is GA-convex and HA-concave, and this applies to ϕ_2.)

- We remark that the properties of μ are the same as in the corresponding diagram for the Stieltjes function ϕ_2. We have no information yet to preclude the possibility that μ is Stieltjes, but as we have no proof that it is even completely monotone, it is too early to speculate.

- Clearly $\mu_{(2)}$ is not Stieltjes. If μ were to be shown to be completely monotone, then so is its square, $\mu_{(2)}$.

As $\mu_{(2)} = \mu^2$ there are some obvious checks. For example, it is clear that the AG, GG- and HG-convexity properties of $\mu_{(2)}$ and of μ must be the same. The convexity properties that differ are AH and HA.

In our proof of Theorem 5(i) we use that $c\mu(c)$ is increasing in c. This is easy to establish. To connect it with earlier papers, we note that [28] defines the function W_t to satisfy $W_t \tan(W_t) = x$ and we have

$$c\mu = W_t\left(\frac{c}{\beta}\right).$$

Routine calculation gives

$$\frac{dW_t}{dx} = \frac{W_t}{x+x^2+W_t^2}, \qquad \frac{d^2W_t}{dx^2} = -\frac{2(1+x)(x^2+W_t(x)^2)}{W_t(x)^2}\left(\frac{dW_t}{dx}\right)^3.$$

Thus W_t is increasing, concave. (This could be obtained with less calculation by noting that $w\tan(w)$ is increasing and convex for $w \in (0, \pi/2)$, and then using properties of inverses similar to that given for decreasing convex functions in Appendix A.)

4. λ_1 for boxes with $|\Omega|$ fixed

4.1. Consequences of μ and $\mu_{(2)}$ being GA-convex

Allowing ourselves some repetition, we believe our neatest proof of Theorem 1 is as follows.

(i) Use the representation (2.1) of λ_1 in terms of $\mu_{(2)}(a_j)$

(ii) Note that ϕ_2 is Stieltjes, and hence AG-convex (or get this directly) and thus $\mu_{(2)}$ is GA-convex. (This is Theorem 1 Restated.)

(iii) Combine the preceding to yield Theorem 1.

Corollary 2 is just another way of describing GA-convexity. If one were to choose this as a starting point it is easy to derive Theorem 1. From the table in Appendix A.1, $\mu_{(2)}(r)$ is GA-convex iff $\mu_{(2)}(\exp(t))$ is convex in t. Define $M(t) = \mu_{(2)}(\exp(t))$. We now illustrate the connection with the Faber-Krahn result with $N = 2$. With the earlier notation where r is related to the aspect ratio of the rectangle (with $r = 1$ for the square),

$$\lambda_1 = \mu_{(2)}(r) + \mu_{(2)}\left(\frac{1}{r}\right) = M(\log(r)) + M(-\log(r)).$$

As each of the M terms is convex, λ_1 is a convex function of $\log(r)$. Also λ_1 is an even function of $\log(r)$. Hence the minimum of λ_1 occurs at $r = 1$, $\log(r) = 0$.

Stronger results than Corollary 2 follow from using the HA-convexity of $\mu_{(2)}$. When $N = 2$ this is indicated in §4.2.

4.2. Simplifications and further results when $N = 2$ and $|\Omega|$ fixed

Define, for $r > 0$, $\Omega(rh, h/r) = (-rh, rh) \times (-h/r, h/r)$. Write $\lambda_1(rh, h/r) = \lambda_1(\Omega(rh, h/r))$. When the orientaion of the rectangle is unimportant, abbreviate these to $\Omega(r)$ and $\lambda_1(r)$. Clearly $\lambda_1(1/r) = \lambda_1(r)$.

Theorem 1 when $N = 2$, that $\mu_{(2)}$ is GA-convex, is that, for $r > 0$,

$$\mu_{(2)}(h) = \mu_{(2)}(\sqrt{rh\frac{h}{r}}) \leqslant \frac{\mu_{(2)}(rh) + \mu_{(2)}(\frac{h}{r})}{2}.$$

Corollary 2, another result from $\mu_{(2)}$ being GA-convex, gives that λ_1 is a convex function of $\log(r)$, and, clearly, an even function of $\log(r)$. So, again, we see that the minimum of $\lambda_1(r)$ is at $r = 1$ as stated in Theorem 1.

The slightly stronger result that $2\mu(h) \leqslant \mu(rh) + \mu(h/r)$ follows from ϕ_1 being AG-convex, so μ is GA-convex. This is the $N = 2$, the inequality $\sum_{j=1}^{N} \hat{\mu}_{a_j} > N\hat{\mu}_{\Box}$ from the end of the preceding subsection: it trivially rewrites to

$$\lambda_1 \geqslant \lambda_{1\Box} + \frac{1}{2}\left(\mu(rh) - \mu(h/r)\right)^2.$$

HA-convexity is a stronger result than GA-convexity, and yields the following, a result that was originally conjectured from the plots of $\lambda_1(r)$ given in [18].

THEOREM 6. *$\lambda_1(r)$ is a convex function of r.*

Proof. The positive, decreasing, convex function $\mu_{(2)}$ is HA-convex, so $\mu_{(2)}(1/c)$ is convex in c. In an obvious notation $\lambda_1(r) = \mu_{(2)}(r) + \mu_{(2)}(1/r)$, and, as both functions on the right are convex, $\lambda_1(r)$ is convex. $\Box$

We remark, but do not use, that the convex function μ is HA-concave, so the function $\mu(r) - \mu(1/r)$ is convex.

Additional results can be found from further properties of ϕ and/or of μ, e.g. μ – and $\mu_{(2)}$ – are *GG*-concave.

Suppose now that the rectangle $\Omega(rh, h/r)$ has $0 < r \leqslant 1$. We now need to consider the orientation of the rectangle, so now denote its fundamental Robin eigenvalue by $\lambda_1(rh, h/r)$. Consider the largest square which can be inscribed in the rectangle (the square $\Omega(hr, hr)$and the smallest square in which the rectangle can be inscribed (the square $\Omega(h/r, h/r)$. The isoperimetric inequality is that $\lambda_1(h, h)$ is bounded above by $\lambda_1(hr, h/r)$ which is the arithmetic mean of $\lambda_1(rh, rh)$ and $\lambda_1(h/r, h/r)$. (This is our old result Theorem 1,)

The GG-concavity of (μ and) $\mu_{(2)}$ yields the following:

THEOREM 7. *$\lambda_1(h, h)$ is bounded from below by the geometric mean of $\lambda_1(rh, rh)$ and $\lambda_1(h/r, h/r)$.*

The key point in the proof is that each of the three terms concerns a λ_1 for a square, and $\lambda_1(c,c) = 2\mu_{(2)}$. Thus the theorem is merely a restatement of the GG-concavity of $\mu_{(2)}$:

$$2\sqrt{\mu_{(2)}(rh)\,\mu_{(2)}(h/r)} = \sqrt{\lambda_1(rh,rh)\,\lambda_1(h/r,h/r)} \leqslant \lambda_1(h,h) = 2\mu_{(2)}(h).$$

We also state this in terms of the a_j which is suggestive of how it might generalize to $N > 2$:

$$\sqrt{\lambda_1(a_1,a_1)\lambda_1(a_2,a_2)} \leqslant \lambda_1(\sqrt{a_1a_2},\sqrt{a_1a_2}) \left(\leqslant \frac{1}{2}(\lambda_1(a_1,a_1)+\lambda_1(a_2,a_2)) \right). \quad (4.1)$$

(The right-hand inequality, in parentheses, is our Faber-Krahn result.) When $\beta = 0$, the left-hand inequality becomes an equality.

5. Proofs of Theorems stated in §1.2

5.1. Proof of Theorem 3, scaling

Part (i) is merely a collection of some of the properties of $\mu_{(2)}$ which are closed under addition. Part (ii) is a consequence of $\lambda_1(\Omega_t) = \lambda_1(\Omega_1)/t^2$.

5.2. Proof of Theorem 4, Minkowski sums

Consider the Minkowski sum

$$\Omega_t = (1-t)\Omega_0 + t\Omega_1.$$

As stated in §1, when $\beta = 0$ it is known that $\lambda_1(\Omega_t)^{-1/2}$ is a concave function of t.

The result at $N=1$, that $1/\mu$ is concave, μ is AH-convex, is given in Corollary 3.

Results more general than the following lemma are proved in [26].

LEMMA 3. *Let the real-valued functions f and g have the same domain of definition $D \subset R^N$ with (i) D convex, (ii) f and g positive and twice continuously differentiable in D, and (iii), with $-1 < \alpha < 0$ both f^α and g^α concave. Then $(f+g)^\alpha$ is concave.*

Proof. Denote the column vector gradient with a D, and the hessian by D^2, and transpose with a superscript T. Define, for any f which is positive and C^2,

$$M(f,\alpha) = f D^2 f + (\alpha-1) Df\,(Df)^T.$$

Note that

$$\text{hessian}(f^\alpha) = \alpha f^{\alpha-2} M(f,\alpha).$$

To establish the result we need to prove that $M(f+g,\alpha)$ is positive (semi-)definite, given $M(f,\alpha)$ and $M(g,\alpha)$ both positive semidefinite.

$$M(f+g,\alpha) = (f+g)D^2(f+g) + (\alpha-1)D(f+g)\,(D(f+g))^T,$$
$$= (f+g)\left(\frac{1}{f}M(f,\alpha) + \frac{1}{g}M(g,\alpha)\right) + (1-\alpha)WW^T,$$

with $W = \sqrt{g/f}\,Df - \sqrt{f/g}\,Dg$. The first two terms of the preceding equation are positive (semi-)definite by the hypotheses of the lemma, and the last is positive semi-definite. Hence $M(f+g,\alpha)$ is positive (semi-)definite, completing the proof. □

Proof of Theorem 4. Since $\mu_{(2)}(a)$ is positive, decreasing with $1/\sqrt{\mu_{(2)}(a)}$ concave, we have, by Lemma 3 with $\alpha = -1/2$, that $\Lambda(\mathbf{a}) = \sum_{j=1}^{N} \mu_{(2)}(a_j)$ is such that $1/\sqrt{\Lambda(\mathbf{a})}$ is concave in the (convex) positive orthant, $\{\mathbf{a}|a_j > 0\}$. □

5.3. Proof of Theorem 5(i)

Our original proof of this result was via variational methods and details are in [18], [19]. We then found a slightly stronger (but uglier) inequality than:

$$\lambda_1(\beta,r) := \lambda_1(\Omega(r)) \leqslant \frac{\lambda_{1\square}}{2}\left(r^2 + \frac{1}{r^2}\right) \qquad \text{where } \lambda_{1\square}(\beta) = \lambda_1(\Omega(1)). \tag{5.1}$$

We remark there is equality in this when $\beta = 0$.

We remark that inequality (5.1) can be viewed as comparing values for functionals at $\beta > 0$ with those for $\beta = 0$. With the notation $\lambda_1(\beta,r)$ as above, inequality (5.1) is

$$\lambda_1(\beta,r) \leqslant \lambda_1(\beta,1)\,\frac{\lambda_1(0,r)}{\lambda_1(0,1)}. \tag{5.2}$$

In terms of μ inequality (5.1) (and other inequalities established earlier) is

$$\left(2\sqrt{\mu_{(2)}(hr) + \mu_{(2)}\left(\frac{h}{r}\right)} \leqslant\ 2\mu_{(2)}(h) \leqslant\right) \mu_{(2)}(hr) + \mu_{(2)}(\frac{h}{r}) \leqslant \mu_{(2)}(h)\left(r^2 + \frac{1}{r^2}\right). \tag{5.3}$$

(The left-most inequality is Theorem 7, and the inequality next to it in parentheses is our Faber-Krahn result.)

We will establish Theorem 5(i) for $\beta > 0$, i.e. will establish inequality (5.3), as a consequence of the GG-concavity of μ (inequality (4.1), Theorem 7) and the inequality in the following lemma.

LEMMA 4.

$$\mu_{(2)}(hr) + \mu_{(2)}\left(\frac{h}{r}\right) \leqslant \sqrt{\mu_{(2)}(rh)\mu_{(2)}(h/r)}\left(r^2 + \frac{1}{r^2}\right). \tag{5.4}$$

Proof. Inequality (5.4) is

$$E(r) = E_- E_+ \leqslant 0 \quad \text{where } E_- = \left(r\mu(hr) - \frac{1}{r}\mu\left(\frac{h}{r}\right)\right),\ E_+ = \left(\frac{1}{r}\mu(hr) - r\mu\left(\frac{h}{r}\right)\right). \tag{5.5}$$

Since $E(1/r) = E(r)$ it suffices to establish the inequality for the case $0 < r < 1$, which we now do.

- Since $c\mu(c)$ is increasing in c, for $0 < r < 1$, $r\mu(hr) < \mu(h) < \mu(h/r)/r$, so the E_- factor of inequality (5.5) is negative.
- Since $\mu(c)$ is decreasing in c, for $0 < r < 1$, $\mu(hr) > \mu(h/r)$ and the factor E_+ of inequality (5.5) is positive.

This proves the lemma. □

For more general Ω, e.g. parallelograms, Theorem 5(i) is easier to prove when $\beta = 0$. It is generalized to Robin boundary conditions (and sums of consecutive eigenvalues) in [22], and further generalized to higher dimensions in [24]. We remark that the rectangle version of their result is the same as our inequality (5.1) (and we note that we have not considered parallelograms in our variational calculations for $\beta > 0$). For details and further results see [11, 25].

6. Conclusion – and open questions

We have, in Theorem 1, established an isoperimetic inequality for the fundamental Robin eigenvalue for boxes. We have also reviewed some related inequalities. In the process we found that working with the explicitly defined Stieltjes function ϕ_2 is overwhelmingly neater than the elementary, but detailed, calculations that arise when working with $\mu_{(2)}$, the inverse of ϕ. Many other inequalities on λ_1, and related functionals, for domains more general than boxes, have been proved and others have been conjectured: see, for example, [11, 25] and the long arXiv article [18].

In [33] it is asked if, *amongst all n-gons of given area, that which has the least λ_1 is the regular n-gon.* When $\beta = 0$ this is, in [33] page 158, proved to be the case, using symmetrisation, when $n = 3$ and $n = 4$. See earlier in this paper, at the end of §1.3. For $\beta > 0$, the question for triangles, $n = 3$, is noted as *Open Problem 1* in [25]. Assuming this is found to be true, it would make sense to ask which classes of quadrilaterals have the property that, at given area, the square has the least λ_1. Our little result, Theorem 1, is that it is true for rectangles, but it is open as to whether it is true for some larger class, e.g. parallelograms or trapeziums.

Returning to boxes we note some open questions. There are indications, mentioned in [18] that $\hat{\mu}(\hat{c})$ might be completely monotone, and the first question is, is it, and, if so, how might it be used. We merely used the log-convexity of ϕ_2 in our proof of Theorem 1, and various other convexity properties in other theorems. However we have the stronger property that ϕ_2 is a Stieltjes function, so completely monotone, and it is reasonable to ask what further properties of $\lambda_1(\Omega)$ can be obtained from this. We have already commented (in Theorem 3(ii)) that when $\beta = 0$, under the scale change, for any Ω_1. $\lambda_1(t\Omega_1)$ is completely monotone. It is an open question as to whether this is true when $\beta > 0$, even for Ω_1 a rectangle.

A. On inverses of positive, decreasing, convex functions

In our application we deal with positive, decreasing convex functions from $(0,\infty)$ onto $(0,\infty)$, An elementary fact that is well-known, and even an example in beginning calculus teaching in connection with implicit differentiation and inverse functions is the following (for which we reference [30] and Proposition 1 of [16]):

FACT. *The inverse of a positive, decreasing convex function is positive, decreasing and convex.*

In our application μ is defined implicitly through a transcendental equation. The inverse of μ is an elementary function denoted ϕ_1. In this appendix we indicate how further convexity properties of ϕ_1 enable one to obtain corresponding convexity properties of μ. As our ϕ_1 and ϕ_2 are completely monotone, and ϕ_2 is Stieltjes some of the convextiy properties follow immediately. For those properties that do not follow from this, ultimately the calculations to determine the further convexity properties of μ are equivalent to those for ϕ.

A.1. Mean convex functions, (p,q)-convex functions

Define, for positive numbers x, y the means

$$M_p(x,y) = \left(\frac{x^p + y^p}{2}\right)^{1/p} \qquad \text{and } M_0(x,y) = \sqrt{xy}.$$

DEFINITION 3. The function f is (p,q)*-convex* ((p,q)*-concave*) if and only if $f(M_p(x,y)) \leqslant M_q(f(x),f(y))$. When (as in our application) f is differentiable, an equivalent definition is that

$$x \mapsto x^{1-p} f'(x)(f(x))^{q-1}$$

is increasing (decreasing). See [3], [4].

Special cases arise sufficiently frequently that there are other notations. There is some literature, notably [2], in connection with 'convexity with respect to means', and the letters A for 'arithmetic', G for 'geometric', and H for 'harmonic' are used to label these. For example, AA-convex is ordinary convexity, AG-convex means log-convex, etc. The following table is standard in the area (but included here for readers with pde specialization)

The set of $(p,1)$-convex functions is obviously closed under addition. $(1,q)$-convexity is related to power-convexity defined and discussed below.

(p,q)	MN		
$(1,1)$	AA	$f(t)$ convex in t decreasing in t	$f\left(\frac{x_0+x_1}{2}\right) \leqslant \frac{f(x_0)+f(x_1)}{2}$
$(1,0)$	AG	$\log(f(t))$ convex in t decreasing in t	$f\left(\frac{x_0+x_1}{2}\right) \leqslant \sqrt{f(x_0)f(x_1)}$
$(1,-1)$	AH	$1/f(t)$ concave in t increasing in t	$f\left(\frac{x_0+x_1}{2}\right) \leqslant \frac{2f(x_0)f(x_1)}{f(x_0)+f(x_1)}$
$(0,1)$	GA	$f(\exp(t))$ convex in t decreasing in t	$f(\sqrt{x_0x_1}) \leqslant \frac{f(x_0)+f(x_1)}{2}$
$(0,0)$	GG	$\log(f(\exp(t)))$ convex in t decreasing in t	$f(\sqrt{x_0x_1}) \leqslant \sqrt{f(x_0)f(x_1)}$
$(0,-1)$	GH	$1/f(\exp(t))$ is concave in t increasing in t	$f(\sqrt{x_0x_1}) \leqslant \frac{2f(x_0)f(x_1)}{f(x_0)+f(x_1)}$
$(-1,1)$	HA	$f(1/t)$ convex in t increasing in t	$f\left(\frac{2x_0x_1}{x_0+x_1}\right) \leqslant \frac{f(x_0)+f(x_1)}{2}$
$(-1,0)$	HG	$\log(f(1/t))$ is convex in t increasing in t	$f\left(\frac{2x_0x_1}{x_0+x_1}\right) \leqslant \sqrt{f(x_0)f(x_1)}$
$(-1,-1)$	HH	$1/f(1/t)$ concave in t decreasing in t	$f\left(\frac{2x_0x_1}{x_0+x_1}\right) \leqslant \frac{2f(x_0)f(x_1)}{f(x_0)+f(x_1)}$

Inclusions of these sets of MN-convex (MN-concave) functions are well established. See [2]. The arrows are to be read, as for example from the entry at right: f AG-convex (log-convex) implies f is AA-convex (ordinarily convex).

The vertical arrows require of the function f that it be positive, decreasing.

$$\begin{array}{ccccc} \mathrm{AH} & \Longrightarrow & \mathrm{AG} & \Longrightarrow & \mathrm{AA} \\ \Uparrow & & \Uparrow & & \Uparrow \\ \mathrm{GH} & \Longrightarrow & \mathrm{GG} & \Longrightarrow & \mathrm{GA} \\ \Uparrow & & \Uparrow & & \Uparrow \\ \mathrm{HH} & \Longrightarrow & \mathrm{HG} & \Longrightarrow & \mathrm{HA} \end{array} \qquad \begin{array}{ccccc} \mathrm{AH} & \Longleftarrow & \mathrm{AG} & \Longleftarrow & \mathrm{AA} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathrm{GH} & \Longleftarrow & \mathrm{GG} & \Longleftarrow & \mathrm{GA} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathrm{HH} & \Longleftarrow & \mathrm{HG} & \Longleftarrow & \mathrm{HA} \end{array}$$

convex concave

A few remarks on the cases when MN-convexity is closed under addition are in order.

- It is clear from the Definition 3 that when $q=1$, the set of $(p,1)$-convex (concave) functions is closed under addition, and obviously remains so when the functions are also positive and decreasing. Thus the AA, GA and HA entries above form (convex) cones.

- There are other sets which form cones, notably the positive decreasing AH-convex functions and AG-convex functions. This case, $p=1$, and $q \leqslant 1$ is often treated in its own right as *power-convex* functions: see Definition 4 below.

- The proof that the positive log-convex, AG-convex, functions form a cone can be adapted to the GG-convex and HG-convex functions. The AGM inequality can be used to establish, for positive numbers $\sqrt{ab}+\sqrt{cd} \leqslant \sqrt{(a+c)(b+d)}$. Applying this in the form

$$\sqrt{f_0(x)f_0(y)}+\sqrt{f_1(x)f_1(y)} \leqslant \sqrt{(f_0+f_1)(x)\,(f_0+f_1)(y)},$$

yields the results.

- See also [31] p. 91 Exercise 2.

DEFINITION 4. A nonnegative function f is said to be q*-th power convex* if, for $q \neq 0$, $qf(x)^q$ is convex, and 0*-power convex* if $\log(f)$ is convex, also called log-convex, or as in [2], AG-convex. See [26].

(When $q<0$, and f is q-th power convex, then $f(x)^q$ is concave.)

If $f \geqslant 0$ is q_0-th power convex then it is q_1-th power convex for $q_1 \geqslant q_0$.

Another property, used here and again in our application, is, from p. 159 of [26]:

If $q \leqslant 1$ then the set of positive, decreasing, convex functions which are q-th power convex is closed under addition. This set is a convex cone in appropriate function spaces.

A.2. Inverses

THEOREM 8. *If a positive, decreasing, convex function f is (p,q)-convex, its inverse g is (q,p)-convex.*

If a positive, decreasing, convex function f is (p,q)-concave, its inverse g is (q,p)-concave.

The proof is straightforward.

In terms of the named means, the result is that If a positive, decreasing, convex function f is MN-convex, its inverse g is NM-convex, and similarly for concavity.

B. Calculations of convexity properties of ϕ_1, ϕ_2, μ, $\mu_{(2)}$

A calculation gives

$$\frac{d}{dx}\left(x^{1-p}\phi_2'(x)(\phi_2(x))^{q-1}\right) = \frac{x^{-1-p}\phi_2^{q-2}}{4(1+x)^2}Q_2(\phi_2,x),$$

where

$$\begin{aligned} Q_2(\Phi_2,x) &= (1+x)^2(2p+q)\Phi_2^2+(2(p+q)(1+x)+x-1)\Phi_2+(q-1), \\ &= (2p+q)\Phi_2^2x^2+(2(2p+q)(\Phi_2+1)+1-2p)\Phi_2 x \\ &\quad +(\Phi_2+1)((2p+q)\Phi_2+q-1). \end{aligned}$$

We establish the convexity/concavity properties by establishing that $Q_2(\phi_2(x),x)$ does not change sign. We do this by considering Φ_2 as an independent variable in $Q_2(\Phi_2,x)$

and investigating this as Φ_2 varies between lower and upper bounds of $\phi_2(x)$, for which we use the interval $[3/(1+3x), 1/x]$.

The upper bound is an obvious consequence of $\arctan(x) < x$. Establishing the lower bound can begin with

$$\frac{d}{dx}\left(\arctan(x) - \frac{x}{\frac{x^2}{3}+1}\right) = \frac{4x^4}{(1+x^2)(3+x^2)^2} \geqslant 0.$$

On integrating the left-hand side from 0 we have

$$\arctan(x) > \frac{x}{\frac{x^2}{3}+1}.$$

From this

$$\phi_2(x) > \frac{1}{\frac{1}{3}+x}.$$

(The weaker inequality $\phi_2(x) > 1/(1+x)$ suffices for all except for testing AH-convexity.)

The values of Q_2 at the end-points of the interval $[3/(1+3x), 1/x]$ are

$$Q_2\left(\frac{3}{1+3x}, x\right) = \frac{4Q_{2-}(p,q)}{(1+3x)^2},$$
$$Q_{2-}(p,q) = 9(p+q)x^2 + 3(5p+4q-1)x + 6p + 4q - 1,$$
$$Q_2\left(\frac{1}{x}, x\right) = \frac{Q_{2+}(p,q)}{x^2},$$
$$Q_{2+}(p,q) = 4(p+q)x^2 + (6p+4q-1)x + (2p+q).$$

(p,q)	MN	$Q_2(\Phi_2, x)$	Notes $[Q_{2-}, Q_{2+}]$
$(1,1)$	AA-convex	$\Phi_2(3(1+x)^2\Phi_2+5x+3)$	all terms in Q_2 positive $[18x^2+24x+9, 8x^2+9x+3]$
$(1,0)$	AG-convex	$2(1+x)^2\Phi_2^2+(3x+1)\Phi_2-1$	Q_2 positive for $\Phi_2 > 1/(1+x)$ $[9x^2+12x+5, 4x^2+5x+2]$
$(1,-1)$	AH-convex	$(1+x)^2\Phi_2^2+(x-1)\Phi_2-2$	sign change below interval $[1, 1+x]$
$(0,1)$	GA-convex	$\Phi_2((1+x)^2\Phi_2+3x+1)$	all terms in Q_2 positive $[9x^2+9x+3, 4x^2+3x+1]$
$(0,0)$	GG-concave	$x\Phi_2-\Phi_2-1$	Use $\phi_2 < 1/x$ in first term $[-3x-1, -x]$
$(0,-1)$	GH-concave	$-(1+x)^2\Phi_2^2-(x+3)\Phi_2-3$	all terms in Q_2 negative $[-9x^2-15x-5, -(x+1)(4x+1)]$
$(-1,1)$	HA-concave	$-(1+x)^2\Phi_2^2-(x-1)\Phi_2$	Use $\phi_2 > 1/(1+x)$ in the first term and $\phi_2 < 1/x$ in the $x\Phi_2$ term $[-6x-3, -3x-1]$
$(-1,0)$	HG-concave	$-2(1+x)^2\Phi_2^2-(x+3)\Phi_2-1$	all terms in Q_2 negative $[-9x^2-18x-7, -4x^2-7x-2]$
$(-1,-1)$	HH-concave	$-3(1+x)^2\Phi_2^2-(3x+5)\Phi_2-2$	all terms in Q_2 negative $[-18x^2-30x-11, -(x+1)(8x+3)]$

The function ϕ_2 is (p,q) convex (concave) iff $Q_2(\phi_2(x),x) > 0$ (resp. $Q_2(\phi_2(x),x) < 0$). We are able to determine the sign of $Q_2(\Phi_2,x)$ over the interval $[3/(1+3x),1/x]$ (and often over a much larger interval of Φ_2, which, however, is irrelevant to our needs). For the sign to remain constant it is, of course, necessary that both Q_{2-} and Q_{2+} have the same sign, and we record these in the table above.

The same process can be applied to Φ_1. For details see Part I of [19]. One result that is used in our application is that ϕ_1 is AH-concave and HA-convex.

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