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Measurable Signal Decoupling with Dynamic Feedforward Compensation and Unknown-Input Observation for Systems with Direct Feedthrough*

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Several feedforward decoupling and estimation problems are treated here in a unified setting, and their exact geometric solution is extended to the general case where the direct feedthrough matrices of all the systems involved are possibly non-zero. To this end, the concepts of self-boundedness and self-hiddenness are generalised and investigated within the general context of non-strictly proper systems. Then, for each problem considered, solvability conditions are provided as well as the explicit structure of the solving compensator or observer.

Keywords: geometric approach; measurable signal decoupling; nonstrictly proper systems; seft bounded and self hidden subspaces; unknown-input observation

1. Introduction

Disturbance decoupling and unknown-input estimation problems have been extensively investigated in the last four decades [1,3,4,9,18,19,23,28,30]; see also the important textbooks [6,25,29]. The basic tool employed for the solution of these two problems is the so-called *geometric approach* to control theory. In this framework, the solvability conditions for these problems are usually expressed in terms of subspace inclusions involving output-nulling and input-containing subspaces, which can be therefore considered as the key tools of the geometric approach.

In this paper, our attention is focused on two well-known decoupling and estimation problems, namely the measurable signal decoupling problem with stability (MSDPs) via dynamic feedforward compensation – sometimes referred to as the *full information* control problem – and the unknown-input observation problem with stability (UIOs), which is the dual of the MSDPs, [4]. Much research effort has been spent in extending the geometric tools and the solvability conditions of the basic decoupling problems to non-strictly proper systems [1,12,24,25]. This extension is important since the models derived from many physical systems often include algebraic relations between inputs and outputs (the so-called feedthrough terms). Moreover, when geometric techniques are employed in the solution of linear-quadratic optimisation problems, the possibility of taking into account direct feedthrough terms enables regular and singular problems to be treated in a unified framework.

In this paper, we are concerned with the issue of generalising the solvability conditions for MSDPs and the UIOs to the case where *all* the feedthrough matrices of the systems involved are possibly non-zero. Moreover, the explicit structure of the decoupling filter for the MSDPs and of the observer for the UIOs are given. Since in the strictly proper case the solvability conditions for these problems can be conveniently expressed in terms of self-bounded and self-hidden subspaces, here an extension of these concepts for non-strictly proper systems is proposed (self-bounded subspaces have been recently defined and

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studied for non-strictly proper systems in [21]; self-hidden subspaces are generalised to systems with direct feedthrough matrices here for the first time). As for the strictly proper case, the use of self-bounded and self-hidden subspaces in the expression of the solvability conditions has several advantages over the use of stabilisability and detectability subspaces. First, by using self-bounded and self-hidden subspaces the computation of eigenspaces, which is often critical for high order systems, is avoided, and the main subspaces used for the determination of the explicit structure of the controller/observer can be found by resorting to the standard routines of the geometric approach. Second, self-bounded and self-hidden subspaces lead to compensators and observers of smaller dimension than those obtained through stabilisability and detectability subspaces. Third, the solution proposed here based on self-bounded and self-hidden subspaces leads to decoupling filters and unknown-input observers with minimal unassignable dynamics, so that the maximum number of poles of the overall system can be placed arbitrarily, [14]. The solution based on self-bounded and self-hidden subspaces is therefore the best in terms of pole assignment.

Numerous problems that may be useful in practice fall in the category of full information decoupling, since in many cases references or disturbances may be accessible for measurement. In these cases, a better performance is achieved by exploiting the measurement of these external inputs by means of feedforward actions. The interest in the MSDPs is also motivated by the fact that such problem is the prototype of a large class of other control problems, such as the model matching problem [12,13,19] and the disturbance decoupling with preview [2,8,27], see Remarks 5.1 and 5.2. On the problem of estimation in presence of unknown inputs there has been a long stream of research, that originated in the late 60s [4] and flourished in the 80s [10,11]. However, this problem still represents a lively research topic, mostly due to its relevance in the context of fault detection, see for example, [20] and the references therein. The solution of this problem will be derived from that of MSDPs by duality in Section 6.

Notation. Throughout this paper, the symbol $\mathbb{R}^{n\times m}$ denotes the space of $n \times m$ real matrices. The image and the null-space of matrix A are denoted by $\mathrm{im}A$ and $\ker A$, while $A^\top$ and $A^\dagger$ denote the transpose and the Moore–Penrose pseudo-inverse of A, respectively. The symbol I_n stands for the $n \times n$ identity matrix, while 0_n denotes the origin of the vector space $\mathbb{R}^n$. If $A : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{J} \subseteq \mathcal{X}$, the restriction of the map A to $\mathcal{J}$ is denoted by $A|\mathcal{J}$. If $\mathcal{X} = \mathcal{Y}$ and $\mathcal{J}$ is A-invariant, the eigenvalues of A restricted to $\mathcal{J}$ are denoted by $\sigma(A|\mathcal{J})$. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are A-invariant subspaces and $\mathcal{J}_1 \subseteq \mathcal{J}_2$, the mapping induced by A on the quotient space $\mathcal{J}_2/\mathcal{J}_1$ is denoted by $A\left|\frac{\mathcal{J}_2}{\mathcal{J}_1}\right.$. Given the matrix $A \in \mathbb{R}^{n\times n}$ and the subspace $\mathcal{B}$ of the linear space $\mathbb{R}^n$, the symbol $< A, \mathcal{B} >$ will stand for the smallest A-invariant subspace of $\mathbb{R}^n$ containing $\mathcal{B}$. In what follows, whether the underlying system evolves in continuous or discrete time is irrelevant and, accordingly, the time index set of any signal is denoted by $\mathbb{T}$, on the understanding that this represents either $\mathbb{R}^+$ in the continuous time or $\mathbb{N}$ in the discrete time. The symbol $\mathbb{C}_g$ denotes either the open left-half complex plane $\mathbb{C}^-$ in the continuous time or the open unit disc $\mathbb{C}^\circ$ in the discrete time.

2. Statement of the Problems

The two problems that are considered in this paper are, as aforementioned, the MSDPs with dynamic feedforward compensation and the UIOs; in both cases, *all* the feedthrough matrices are assumed to be possibly non-zero. We begin by presenting the formulation of the MSDPs: consider a linear time-invariant (LTI) system described by

$$\begin{aligned} \rho x(t) &= Ax(t) + B_1u(t) + B_2w(t), \\ y(t) &= Cx(t) + D_1u(t) + D_2w(t), \end{aligned} \tag{1}$$

where the operator ρ denotes either the time derivative in the continuous time, that is, $\rho x(t) = \dot{x}(t)$, or the unit time shift in the discrete time, that is, $\rho x(t) = x(t+1)$. Also, for all $t \in \mathbb{T}$, $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} = \mathbb{R}^{m_1}$ and $w(t) \in \mathcal{W} = \mathbb{R}^{m_2}$ are inputs and $y(t) \in Y = \mathbb{R}^p$ is the output, while A, B_1, B_2, C, D_1 and D_2 are real constant matrices of suitable dimensions. The signal u is the control input, and is used to influence the dynamical behaviour of the plant. The exogenous input w can be essentially considered as a measurable process noise or a driving disturbance belonging to some function space $\mathbb{W}$ (e.g., in the continuous case the space $\mathbb{W}$ may be taken equal to the class of piecewise continuous functions), to be decoupled from the output y. We identify the system characterised by the quadruple $(A,[B_1 \quad B_2], C,[D_1 \quad D_2])$ with the symbol Δ.

Matrix $A \in \mathbb{R}^{n\times n}$ is assumed to be stable[1], that is, $\sigma(A) \subset \mathbb{C}_g$.

The MSDPs herein considered is stated as the problem of finding a feedforward controller Σ_c connected as in Fig. 1, having full information on the

[1]Note that this condition is necessary as long as a pure feedforward solution is sought. However, it can be easily relaxed to the stabilisability of the pair (A,B). In fact, in this case, a preliminary stabilising state-feedback can be performed, and what follows will be applied to the system thus obtained. In the case where the state of the system is not accessible for measurement, if (A,B) is stabilisable and (A,C) is detectable, the system can be pre-stabilised by the joint action of an asymptotic observer and a static feedback.

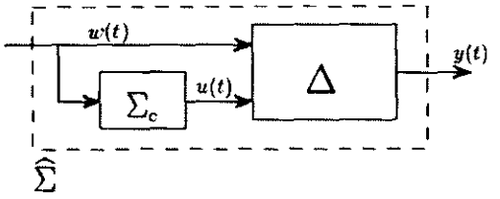

Fig. 1. Block diagram of the full information control problem.

exogenous input w, such that the output y does not depend on the disturbance w. This problem is stated in more precise terms as follows.

Problem 2.1: *Consider Fig. 1. Find an LTI compensator* $\Sigma_c = (A_c, B_c, C_c, D_c)$ *governed by*

$$\begin{cases} \rho\xi_c(t) = A_c\xi_c(t) + B_c w(t) \\ u(t) = C_c\xi_c(t) + D_c w(t) \end{cases} \tag{2}$$

where, for all $t \in \mathbb{T}$, $\xi_c(t) \in X_c = \mathbb{R}^{n_c}$ *is the state of* Σ_c, *such that*

(i) *the output function is not affected by the disturbance w. That is, such that the transfer function matrix*

$$T_{y,w}(\zeta) = [C \quad D_1C_c]\left(\zeta\begin{bmatrix} I_n & 0 \\ 0 & I_{n_c} \end{bmatrix} - \begin{bmatrix} A & B_1C_c \\ 0 & A_c \end{bmatrix}\right)^{-1}\begin{bmatrix} B_2 + B_1D_c \\ B_c \end{bmatrix} + (D_2 + D_1D_c) \tag{3}$$

is zero;

(ii) *the overall system*

$$\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}) \triangleq \left(\begin{bmatrix} A & B_1C_c \\ 0 & A_c \end{bmatrix}, \begin{bmatrix} B_2 + B_1D_c \\ B_c \end{bmatrix}, [C \quad D_1C_c], D_2 + D_1D_c\right)$$

is asymptotically stable, that is, $\sigma(\widehat{A}) = \sigma\left(\begin{bmatrix} A & B_1C_c \\ 0 & A_c \end{bmatrix}\right) \subset \mathbb{C}_g$; *or, equivalently, such that* $\sigma(A_c) \subset \mathbb{C}_g$ *since it has been assumed that* $\sigma(A) \subset \mathbb{C}_g$.

The complex variable ζ in Problem 2.1 represents either the Laplace variable s in the continuous time of the z variable in the discrete time. Notice that *(i)* is equivalent to the requirement of finding a compensator (2) such that the output y of the overall system does not depend on the disturbance w. Requirement *(ii)* guarantees that the compensator is such that for all initial conditions $x(0) \in \mathcal{X}$ and $\xi_c(0) \in \mathcal{X}_c$, the output $y(t)$ converges to zero as t goes to infinity.

The second problem dealt with in this paper is the unknown-input observation, that will be solved by duality arguments. Consider an LTI system described by

$$\begin{aligned} \rho x(t) &= Ax(t) + Bu(t), \\ z(t) &= C_1x(t) + D_1u(t), \\ y(t) &= C_2x(t) + D_2u(t), \end{aligned} \tag{4}$$

where, for all $t \in \mathbb{T}$, $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} = \mathbb{R}^m$ is an input, $y(t) \in \mathcal{Y} = \mathbb{R}^{p_1}$ is the measurement output and $z(t) \in \mathcal{Z} = \mathbb{R}^{p_2}$ is the output to estimate, while A, B, C_1, C_2, D_1 and D_2 are real constant matrices of suitable dimensions. The function u represents a generic input which is not available for measurement: in many applications, the signal u arises as a driving disturbance. Matrix $A \in \mathbb{R}^{n\times n}$ is assumed to be stable, that is, $\sigma(A) \subset \mathbb{C}_g$. We identify the system characterised by the quadruple $(A, B, [C_1^\top \; C_2^\top]^\top, [D_1^\top \; D_2^\top]^\top)$ with the symbol Ω. The unknown-input observation problem with stability consists of finding a stable observer Σ_o for the asymptotic estimation of the output z which exploits the measurement represented by y. It is therefore required that as t goes to infinity, the estimation error $e = z - \tilde{z}$ converges to zero asymptotically.

Problem 2.2: *Consider Fig. 2. Find an LTI unknown-input observer* $\Sigma_o = (A_o, B_o, C_o, D_o)$ *governed by*

$$\begin{cases} \rho\xi_o(t) = A_o\xi_o(t) + B_o y(t) \\ \tilde{z}(t) = C_o\xi_o(t) + D_o y(t) \end{cases} \tag{5}$$

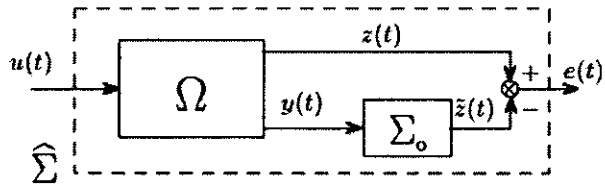

Fig. 2. Block diagram of the unknown input observation scheme.

where, for all $t \in \mathbb{T}$, $\xi_o(t) \in \mathcal{X}_o = \mathbb{R}^{n_o}$ *is the state of the observer, such that*

(i) *the transfer function matrix*

$$T_{e,u}(\zeta) = [C_1 - D_o C_2 \quad -C_o]\left(\zeta\begin{bmatrix} I_n & 0 \\ 0 & I_{n_o}\end{bmatrix} - \begin{bmatrix} A & 0 \\ B_o C_2 & A_o\end{bmatrix}\right)^{-1}\begin{bmatrix} B \\ B_o D_2\end{bmatrix} + (D_1 - D_o D_2) \tag{6}$$

is zero;

(ii) *the overall system*

$$\widehat{\Sigma} = (\widehat{A},\widehat{B},\widehat{C},\widehat{D}) \triangleq \left(\begin{bmatrix} A & 0 \\ B_o C_2 & A_o\end{bmatrix},\begin{bmatrix} B \\ B_o D_2\end{bmatrix},[C_1 - D_o C_2 \quad -C_o], D_1 - D_o D_2\right)$$

is asymptotically stable, that is, $\sigma(\widehat{A}) = \sigma\left(\begin{bmatrix} A & 0 \\ B_o C_2 & A_o\end{bmatrix}\right) \subset \mathbb{C}_g$*; or, equivalently, such that* $\sigma(A_o) \subset \mathbb{C}_g$ *since* $\sigma(A) \subset \mathbb{C}_g$.

Notice that according to the formulation of Problem 2.2 the system Σ_o is indeed an unknown-input observer. In fact, if the transfer function matrix (6) from the input u to the output e is zero and the overall system is stable, then for all initial conditions $x(0) \in \mathcal{X}$ and $\xi_o(0) \in \mathcal{X}_o$ and for all admissible inputs u, the error $e(t)$ converges to zero as t goes to infinity. Since $e(t) = z(t) - \tilde{z}(t)$ for all $t \in \mathbb{T}$, this is equivalent to $\lim_{t\to\infty}(\tilde{z}(t)) = z(t)$, that is, the output of the observer Σ_o converges to the output $z(t)$ to be estimated asymptotically.

It is easily seen that Problems 2.1 and 2.2 are dual to each other. To see this, it is sufficient to notice that the transpose of $T_{y,u}(\zeta)$ defined in (3) equals $T_{e,u}(\zeta)$ in (6) up to the replacement of $(A^\top,B_2^\top,B_1^\top,C^\top,D_2^\top,D_1^\top)$ with (A,C_1,C_2,B,D_1,D_2) and of $(A_c^\top,B_c^\top,C_c^\top,D_c^\top)$ with (A_o,C_o,B_o,D_o).

3. Geometric Background

For the readers' convenience, some fundamental definitions and results of the classic geometric control theory which will be used in the sequel are recalled (for a more detailed discussion on the topics herein introduced we refer to [6,25,29]). In this section, we refer to the quadruple $\Sigma = (A, B, C, D)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. In order to simplify notation, let $\bar{A}_V \triangleq \begin{bmatrix} A \\ C\end{bmatrix}$, $\bar{A}_H \triangleq [A \quad B]$, $\bar{B} \triangleq \begin{bmatrix} B \\ D\end{bmatrix}$ and $\bar{C} \triangleq [C \quad D]$.

An *output-nulling subspace* $\mathcal{V}_\Sigma$ of Σ is defined as a subspace of $\mathbb{R}^n$ satisfying the inclusion

$$\bar{A}_V \mathcal{V}_\Sigma \subseteq (\mathcal{V}_\Sigma \oplus \mathbf{0}_p) + \mathrm{im}\bar{B}. \tag{7}$$

The set $\mathcal{V}(\Sigma)$ of output-nulling subspaces of Σ is closed under subspace addition. Thus, the subspace $\mathcal{V}_\Sigma^\star \triangleq \sum_{\mathcal{V}\in\mathcal{V}(\Sigma)} \mathcal{V} \in \mathcal{V}(\Sigma)$ is the largest output-nulling subspace of Σ. The subspace $\mathcal{V}_\Sigma^\star$ represents the set of all initial states of Σ for which an input function exists such that the corresponding output function is identically zero. Clearly, when D is zero, $\mathcal{V}_\Sigma^\star$ reduces to the maximal (A,B)-controlled invariant subspace contained in the null-space of matrix C, [6,30]. In the following lemma, the most important properties of output-nulling subspaces are recalled.

Lemma 3.1: *The following results hold:*

(i) *The subspace* $\mathcal{V}_\Sigma$ *is output-nulling for* Σ *if and only if a matrix* $F \in \mathbb{R}^{m\times n}$ *exists such that*

$$\begin{bmatrix} A + BF \\ C + DF\end{bmatrix}\mathcal{V}_\Sigma \subseteq \mathcal{V}_\Sigma \oplus \mathbf{0}_p; \tag{8}$$

(ii) *The sequence of subspaces* $(\mathcal{V}_\Sigma^i)_{i\in\mathbb{N}}$ *described by the recurrence*

$$\begin{cases} \mathcal{V}_\Sigma^0 &= \mathbb{R}^n, \\ \mathcal{V}_\Sigma^i &= \bar{A}_V^{-1}((\mathcal{V}_\Sigma^{i-1} \oplus \mathbf{0}_p) + \mathrm{im}\,\bar{B}), \quad i = \mathbb{N}\backslash 0, \end{cases} \tag{9}$$

is monotonically non-increasing. An integer $k \leq n-1$ exists such that $\mathcal{V}_\Sigma^{k+1} = \mathcal{V}_\Sigma^k$. For such integer k the identity $\mathcal{V}_\Sigma^* = \mathcal{V}_\Sigma^k$ holds.

Any matrix F satisfying (8) is usually referred to as a *friend* of the output-nulling subspace $\mathcal{V}_\Sigma$. We denote by $\mathfrak{F}_\Sigma(\mathcal{V}_\Sigma)$ the set of friends of the output-nulling subspace $\mathcal{V}_\Sigma$. As a result of Lemma 3.1, the following corollary holds.

Corollary 3.1: *The r-dimensional subspace* $\mathcal{V}_\Sigma$ *is output-nulling if and only if there exist* $F \in \mathbb{R}^{m\times n}$ *and* $X \in \mathbb{R}^{r\times r}$ *such that*

$$\begin{bmatrix} A+BF \\ C+DF \end{bmatrix} V = \begin{bmatrix} V \\ 0 \end{bmatrix} X \tag{10}$$

where $V \in \mathbb{R}^{n\times r}$ *is a basis of* $\mathcal{V}_\Sigma$ *and* $\sigma(X) = \sigma(A+BF|\mathcal{V}_\Sigma)$.

Now we define the *input-containing subspace* $\mathcal{S}_\Sigma$ of Σ as a subspace of $\mathbb{R}^n$ satisfying the inclusion

$$\bar{A}_H(\mathcal{S}_\Sigma \oplus \mathcal{U} \cap \ker \bar{C}) \subseteq \mathcal{S}_\Sigma. \tag{11}$$

The set $\mathcal{S}(\Sigma)$ of input-containing subspaces of Σ is closed under subspace intersection. As such, the subspace $\mathcal{S}_\Sigma^* \triangleq \bigcap_{\mathcal{S}\in\mathcal{S}(\Sigma)} \mathcal{S} \in \mathcal{S}(\Sigma)$ is the smallest input-containing subspace of Σ.

Lemma 3.2: *The following results hold:*

(i) *The subspace* $\mathcal{S}_\Sigma$ *is input-containing for* Σ *if and only if a matrix* $G \in \mathbb{R}^{n\times p}$ *exists such that*

$$[A+GC \quad B+GD](\mathcal{S}_\Sigma \oplus \mathbb{R}^m) \subseteq \mathcal{S}_\Sigma; \tag{12}$$

(ii) *The sequence of subspaces* $(\mathcal{S}_\Sigma^i)_{i\in\mathbb{N}}$ *described by the recurrence*

$$\begin{cases} \mathcal{S}_\Sigma^0 = \mathbf{0}_n, \\ \mathcal{S}_\Sigma^i = \bar{A}_H\big((\mathcal{S}_\Sigma^{i-1} \oplus \mathcal{U}) \cap \ker \bar{C}\big), \quad i = \mathbb{N}\backslash 0, \end{cases} \tag{13}$$

is monotonically non-decreasing. An integer $k \leq n-1$ exists such that $\mathcal{S}_\Sigma^{k+1} = \mathcal{S}_\Sigma^k$. For such integer k the identity $\mathcal{S}_\Sigma^* = \mathcal{S}_\Sigma^k$ holds.

Any matrix G satisfying (12) is referred to as a *friend* of the input-containing subspace $\mathcal{S}_\Sigma$. We denote by $\mathfrak{S}_\Sigma(\mathcal{S}_\Sigma)$ the set of friends of the output-nulling subspace $\mathcal{S}_\Sigma$. The dual of Corollary 3.1 is as follows.

Corollary 3.2: *The q-dimensional subspace* $\mathcal{S}_\Sigma$ *is input-containing if and only if there exist* $G \in \mathbb{R}^{n\times p}$ *and* $\Lambda \in \mathbb{R}^{(n-q)\times(n-q)}$ *such that*

$$Q[A+GC \quad B+GD] = \Lambda[Q \quad 0] \tag{14}$$

where the full row-rank matrix $Q \in \mathbb{R}^{(n-q)\times n}$ *is such that* $\ker Q = \mathcal{S}_\Sigma$ *and* $\sigma(\Lambda) = \sigma\left(A+GC\middle|\frac{\mathcal{X}}{\mathcal{S}_\Sigma}\right)$.

As in the strictly proper case, any input-containing subspace $\mathcal{S}_\Sigma$ is associated with the existence of an observer, whose input is y, that maintains the information on the canonical projection of the state x on $\mathcal{X}/\mathcal{S}_\Sigma$ (or, in other words, it maintains information on the state of Σ *modulo* $\mathcal{S}_\Sigma$), see [6,25,26]. More precisely, given the input-containing subspace $\mathcal{S}_\Sigma$, an observer ruled by

$$\begin{aligned} \rho h(t) &= Kh(t) + Ly(t), \\ \omega(t) &= h(t), \end{aligned} \tag{15}$$

exists such that if $h(0) = x(0)/\mathcal{S}_\Sigma$, then $h(t) = x(t)/\mathcal{S}_\Sigma$ for all $t \in \mathbb{T}$. To see this, given two matrices G and Λ such that (14) holds, consider the observer (15) with $K = \Lambda$ and $L = -QG$, and define the *error* variable $\varepsilon = Qx - h$. It is easily found that

$$\begin{aligned} \rho\varepsilon(t) &= Q\rho x(t) - \rho h(t) = QAx(t) + QBu(t) \\ &\quad - \Lambda h(t) + QGCx(t) + QGDu(t) \\ &= Q[A+GC \quad B+GD]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} - \Lambda h(t) \\ &= [Q \quad 0]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} - \Lambda h(t) = \Lambda\varepsilon(t) \end{aligned}$$

so that, if $h(0) = x(0)/\mathcal{S}_\Sigma$, that is, if $\varepsilon(0) = 0$, it follows that $\varepsilon(t) = 0$ for all $t \in \mathbb{T}$, implying that $h(t) = x(t)/\mathcal{S}_\Sigma$ for all $t \in \mathbb{T}$. The converse is true as well: suppose the observer (15) maintains information modulo $\mathcal{S}$, where $\mathcal{S}$ is a subspace of $\mathcal{X}$. Let $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathcal{S} \oplus \mathcal{U} \cap \ker \bar{C}$. Let $x(0) = x_0$ and $u(0) = u_0$. Since $x_0 \in \mathcal{S}$, by choosing $h(0) = x_0/\mathcal{S} = 0$ we get $Qx(t) = h(t)$ for all $t \in \mathbb{T}$, which yields $\rho Qx(t) = \rho h(t)$ for all $t \in \mathbb{T}$, that can be explicitly written as

$$Kh(t) + L(Cx(t) + Du(t)) = QAx(t) + QBu(t)$$

In particular, the former holds at $t = 0$, leading to $Q[A \quad B]\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$, since $h(0) = 0$ and by definition $Cx_0 + Du_0 = 0$. It follows that $[A \quad B]\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathcal{S}$. The subspace $\mathcal{S}$ is therefore input-containing, QED. The observer (15) therefore maintains information of the state modulo $\mathcal{S}_\Sigma$. The third fundamental subspace that we need to define is the output-nulling reachability subspace on the output-nulling subspace $\mathcal{V}_\Sigma$, denoted by $\mathcal{R}_{\mathcal{V}_\Sigma}$: let $F \in \mathbb{R}^{m\times n}$ be a friend of the output-nulling subspace $\mathcal{V}_\Sigma$. The *output-nulling reachability subspace* $\mathcal{R}_{\mathcal{V}_\Sigma}$ on $\mathcal{V}_\Sigma$ is the smallest $(A+BF)$-invariant subspace of $\mathbb{R}^n$ containing the

subspace $\mathcal{V}_\Sigma \cap B\ker D$, where $F \in \mathfrak{F}_\Sigma(\mathcal{V}_\Sigma)$. It can be shown that $\mathcal{R}_{\mathcal{V}_\Sigma}$ is independent of the particular friend F in $\mathfrak{F}_\Sigma(\mathcal{V}_\Sigma)$. We denote by $\mathcal{R}^\star_\Sigma$ the output-nulling reachability subspace on $\mathcal{V}^\star_\Sigma$. The relation $\mathcal{R}^\star_\Sigma = \mathcal{V}^\star_\Sigma \cap \mathcal{S}^\star_\Sigma$ holds, [25, Theorem 8.22].

Now, consider an output-nulling subspace $\mathcal{V}_\Sigma$ of Σ and define by $\mathcal{R}_\Sigma$ the reachable subspace on $\mathcal{V}_\Sigma$. For F in $F_\Sigma(\mathcal{V}_\Sigma)$, the eigenvalues of $(A+BF)$ restricted to $\mathcal{V}_\Sigma$, that is, $\sigma(A+BF|\mathcal{V}_\Sigma)$, can be split into two sets: the eigenvalues of $(A+BF|\mathcal{R}_\Sigma)$ are all freely assignable by a suitable choice of the friend F in $\mathfrak{F}_\Sigma(\mathcal{V}_\Sigma)$. The eigenvalues in $\Gamma^{\text{int}}_\Sigma(\mathcal{V}_\Sigma) \triangleq \sigma\left(A+BF\middle|\frac{\mathcal{V}_\Sigma}{\mathcal{R}_\Sigma}\right)$ are fixed, that is, they do not depend on the choice of F, [1]; if these eigenvalues are all in $\mathbb{C}_g$, the output-nulling $\mathcal{V}_\Sigma$ is said to be *internally stabilisable*. Similarly, by denoting with $\mathcal{R}_0$ the smallest A-invariant subspace containing the image of B, the eigenvalues in $\sigma\left(A+BF\middle|\frac{\mathcal{X}}{\mathcal{V}_\Sigma}\right)$ are split into two sets: the eigenvalues of $\left(A+BF\middle|\frac{\mathcal{V}_\Sigma+\mathcal{R}_0}{\mathcal{V}_\Sigma}\right)$ are all freely assignable by a suitable $F \in \mathfrak{F}_\Sigma(\mathcal{V}_\Sigma)$, whereas the eigenvalues in $\Gamma^{\text{ext}}_\Sigma(\mathcal{V}_\Sigma) \triangleq \sigma\left(A+BF\middle|\frac{\mathcal{X}}{\mathcal{V}_\Sigma+\mathcal{R}_0}\right)$ are fixed for all $F \in \mathfrak{F}_\Sigma(\mathcal{V}_\Sigma)$. If the latter are all in $\mathbb{C}_g$, the output-nulling $\mathcal{V}_\Sigma$ is said to be *externally stabilisable*. Hence, the set $\Gamma_\Sigma(\mathcal{V}_\Sigma) \triangleq \Gamma^{\text{int}}_\Sigma(\mathcal{V}_\Sigma) \uplus \Gamma^{\text{ext}}_\Sigma(\mathcal{V}_\Sigma)$ does not depend on the choice of $F \in \mathfrak{F}_\Sigma(\mathcal{V}_\Sigma)$. Note also that the elements of $\Gamma^{\text{int}}_\Sigma(\mathcal{V}^\star_\Sigma)$ are the invariant zeros of Σ. In light of these considerations, it turns out that if the r-dimensional output nulling $\mathcal{V}_\Sigma$ is internally stabilisable, there exist $F \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{r\times r}$ such that (10) holds, where $\mathcal{V} \in \mathbb{R}^{n\times r}$ is a basis of $\mathcal{V}_\Sigma$ and $\sigma(X) = \sigma(A+BF|\mathcal{V}_\Sigma) \subset \mathbb{C}_g$.

Dually, given the input-containing subspace $\mathcal{S}_\Sigma$ and a friend $G \in \mathfrak{S}_\Sigma(\mathcal{S}_\Sigma)$, we define the subspace $\mathcal{Q}_{\mathcal{S}_\Sigma}$ as the largest $(A+GC)$-invariant subspace contained in $\mathcal{S}_\Sigma + C^{-1}\text{im}D$. By duality, it is easy to see that the largest $(A+GC)$-invariant subspace contained in $\mathcal{S}^\star_\Sigma + C^{-1}\text{im}D$, here denoted by $\mathcal{Q}^\star_\Sigma$, is such that $\mathcal{Q}^\star_\Sigma = \mathcal{V}^\star_\Sigma + \mathcal{S}^\star_\Sigma$. For $G \in \mathfrak{S}_\Sigma(\mathcal{S}_\Sigma)$ and by denoting with $\mathcal{Q}_0$ the largest A-invariant contained in the null-space of C, we find $\sigma(A+GC|\mathcal{S}_\Sigma) = \sigma(A+GC|\mathcal{S}_\Sigma \cap \mathcal{Q}_0) \uplus \sigma\left(A+GC\middle|\frac{\mathcal{S}_\Sigma}{\mathcal{S}_\Sigma\cap\mathcal{Q}_0}\right)$, where $\Xi^{\text{int}}_\Sigma(\mathcal{S}_\Sigma) \triangleq \sigma(A+GC|\mathcal{S}_\Sigma \cap \mathcal{Q}_0)$ are fixed and $\sigma\left(A+GC\middle|\frac{\mathcal{S}_\Sigma}{\mathcal{S}_\Sigma\cap\mathcal{Q}_0}\right)$ are free for all $G \in S_\Sigma(\mathcal{S}_\Sigma)$; if $\Xi^{\text{int}}_\Sigma(\mathcal{S}_\Sigma) \subset \mathbb{C}_g$, $\mathcal{S}_\Sigma$ is said to be internally stabilisable. Similarly, $\sigma\left(A+GC\middle|\frac{\mathcal{X}}{\mathcal{S}_\Sigma}\right) = \sigma\left(A+GC\middle|\frac{\mathcal{Q}_\Sigma}{\mathcal{S}_\Sigma}\right) \uplus \sigma\left(A+GC\middle|\frac{\mathcal{X}}{\mathcal{Q}_\Sigma}\right)$, where $\Xi^{\text{ext}}_\Sigma(\mathcal{S}_\Sigma) = \sigma\left(A+GC\middle|\frac{\mathcal{Q}_\Sigma}{\mathcal{S}_\Sigma}\right)$ are fixed while $\sigma\left(A+GC\middle|\frac{\mathcal{X}}{\mathcal{Q}_\Sigma}\right)$ are free for all $G \in \mathfrak{S}_\Sigma(\mathcal{S}_\Sigma)$; if $\Xi^{\text{ext}}_\Sigma(\mathcal{S}_\Sigma) \subset \mathbb{C}_g$, $\mathcal{S}_\Sigma$ is said to be externally stabilisable. As a result of this, if the q-dimensional input-containing subspace $\mathcal{S}_\Sigma$ is externally stabilisable, there exist $G \in \mathbb{R}^{n\times p}$ and $\Lambda \in \mathbb{R}^{(n-q)\times(n-q)}$ such that (14) holds and $\sigma(\Lambda) = \sigma(A+GC\left|\frac{\mathcal{X}}{\mathcal{S}_\Sigma}\right.) \subset C_g$. With this choice of Λ, the associated observer (15) is therefore stable, and equation $\rho\varepsilon(t) = \Lambda\varepsilon(t)$ implies that now for any input function u and for any pair of initial conditions $x(0)$ and $h(0)$, we have $\lim_{t\to\infty} h(t) = x(t)/\mathcal{S}_\Sigma$. As such, not only does the observer maintain information of $x(t)/\mathcal{S}_\Sigma$, but it can also *recover* information modulo $\mathcal{S}_\Sigma$. This incomplete estimate may be fully satisfactory if, for instance, it is not necessary to know the whole state, but only a given linear function $Hx(t)$ of it: in this case, the estimate of this function is complete if and only if $\ker Q = \mathcal{S}_\Sigma \subseteq \ker H$. In fact, in this case a matrix K exists such that $H = KQ$, so that the knowledge of $Qx(t)$ provides full knowledge of $Hx(t)$.

If we design the observer (15) by using $\mathcal{Q}_\Sigma$, the estimation error ε converges to zero with arbitrary dynamic, since $\sigma(\Lambda) = \sigma\left(A+GC\middle|\frac{\mathcal{X}}{\mathcal{S}_\Sigma}\right)$ are all freely assignable with a suitable choice of $G \in \mathfrak{S}_\Sigma(\mathcal{Q}_\Sigma)$.

4. Self-Bounded and Self-Hidden Subspaces

Now, the concept of self-bounded controlled invariance defined in [5] is extended to systems with direct feedthrough.

Definition 4.1: *The output-nulling subspace $\mathcal{V}_\Sigma$ of Σ is self-bounded if $\mathcal{V}^\star_\Sigma \cap B\ker D \subseteq \mathcal{V}_\Sigma$.*

Clearly, by definition both $\mathcal{V}^\star_\Sigma$ and $\mathcal{R}^\star_\Sigma$ are self-bounded, as they both contain $\mathcal{V}^\star_\Sigma \cap B\ker D$. Unlike $\mathcal{V}(\Sigma)$, the set

$$\Phi(\Sigma) \triangleq \{\mathcal{V}_\Sigma \in \mathcal{V}(\Sigma) | \mathcal{V}_\Sigma \supseteq \mathcal{V}^\star_\Sigma \cap B\ker D\}$$

of self-bounded output-nulling subspaces of Σ admits both a maximal and a minimal element. In fact, $\Phi(\Sigma)$ is closed under subspace addition and intersection as shown in [21]. Now, given $\mathcal{V}_1$, $\mathcal{V}_2 \in \Phi(\Sigma)$, it is easily seen that their sum $\mathcal{V}_1 + \mathcal{V}_2$ is the smallest element of $\Phi(\Sigma)$ containing both $\mathcal{V}_1$ and $\mathcal{V}_2$, and $\mathcal{V}_1 \cap \mathcal{V}_2$ is the largest element of $\Phi(\Sigma)$ contained in both $\mathcal{V}_1$ and $\mathcal{V}_2$. Hence, $(\Phi(\Sigma), +, \cap; \subseteq)$ is a lattice. As such, it admits a maximum element, which is $\mathcal{V}^\star_\Sigma$, and a minimum element, which is $\mathcal{R}^\star_\Sigma$. By duality, the concept of self-hidden conditioned invariance defined in [5] is extended to systems with direct feedthrough.

Definition 4.2: *The input-containing subspace $\mathcal{S}_\Sigma$ of Σ is self-hidden if $\mathcal{S}_\Sigma^\star + C^{-1}\mathrm{im}D \supseteq \mathcal{S}_\Sigma$.*

Clearly, by definition both $\mathcal{Q}_\Sigma^\star$ and $\mathcal{S}_\Sigma^\star$ are self-hidden. The set

$$\Psi(\Sigma) \triangleq \{\mathcal{S}_\Sigma \in \mathcal{S}(\Sigma) | \mathcal{S}_\Sigma^\star + C^{-1}\mathrm{im}D \supseteq \mathcal{S}_\Sigma\}$$

of self-hidden input-containing subspaces of Σ is closed under subspace addition and intersection. Therefore, it admits a largest and a smallest elements, which are $\mathcal{Q}_\Sigma^\star$ and $\mathcal{S}_\Sigma^\star$, respectively.

Lemma 4.1. *Let $\mathcal{V}_\Sigma', \mathcal{V}_\Sigma'' \in \Phi(\Sigma)$. Then, $\mathcal{V}_\Sigma' \subseteq \mathcal{V}_\Sigma''$ implies $\mathfrak{F}_\Sigma(\mathcal{V}_\Sigma'') \subseteq \mathfrak{F}_\Sigma(\mathcal{V}_\Sigma')$. Dually, given $\mathcal{S}_\Sigma', \mathcal{S}_\Sigma'' \in \Psi(\Sigma)$, $\mathcal{S}_\Sigma' \subseteq \mathcal{S}_\Sigma''$ implies $\mathfrak{S}_\Sigma(\mathcal{S}_\Sigma') \subseteq \mathfrak{S}_\Sigma(\mathcal{S}_\Sigma')$.*

A proof of the first part of this lemma – which is a generalisation of Properties 4.1.7 and 4.1.9 in [6] – can be found in [21]. The second part can be proved by duality: in fact, it is not difficult to see that by defining the dual of Σ as the system described by the quadruple $\Sigma^\top = (A^\top, C^\top, B^\top, D^\top)$, then $\mathcal{V}_\Sigma \in \mathcal{V}(\Sigma)$ if and only if $\mathcal{V}_\Sigma^\perp \in \mathcal{S}(\Sigma^\top)$ and $\mathcal{V}_\Sigma \in \Psi(\Sigma)$ if and only if $\mathcal{V}_\Sigma^\perp \in \Psi(\Sigma^\top)$. As a consequence of Lemma 4.1, given the friend F of $\mathcal{V}_\Sigma^\star$, for any self-bounded subspace $\mathcal{V}_\Sigma \in \Phi(\Sigma)$ the map F is a friend of $\mathcal{V}_\Sigma$. In the dual setting, given the friend G of $\mathcal{S}_\Sigma^\star$, for any self-hidden subspace $\mathcal{S}_\Sigma \in \Phi(\Sigma)$ the map G is a friend of $\mathcal{S}_\Sigma$.

5. Solution of MSDPS with Feedforward Compensation

Before presenting the solution of Problem 2.2, some useful results on self-bounded output-nulling subspaces are introduced, which are the extension of the Properties 4.2.1 and 4.2.2 in [6, pp. 220-221] to non-purely dynamical systems. Let $\bar{B}_1 \triangleq \begin{bmatrix} B_1 \\ D_1 \end{bmatrix}$ and $\bar{B}_2 \triangleq \begin{bmatrix} B_2 \\ D_2 \end{bmatrix}$, and recall that we have defined Δ as the quadruple $(A, [B_1 \quad B_2], C, [D_1 \quad D_2])$. Moreover, let Σ be described by (A, B_1, C, D_1).

Lemma 5.1: *Let $\mathrm{im}\bar{B}_2 \subseteq (\mathcal{V}_\Sigma^\star \oplus \mathbf{0}_p) + \mathrm{im}\bar{B}_1$. The following facts hold:*

(i) $\mathcal{V}_\Sigma^\star = \mathcal{V}_\Delta^\star$;
(ii) $\Phi(\Delta) \subseteq \Phi(\Sigma)$;
(iii) *For all $\mathcal{V}_\Delta \in \Phi(\Delta)$ there holds* $\mathrm{im}\bar{B}_2 \subseteq (\mathcal{V}_\Delta \oplus \mathbf{0}_p) + \mathrm{im}\bar{B}_1$;
(iv) *If an internally stabilisable output-nulling subspace $\mathcal{V}_\Sigma \in \mathcal{V}(\Sigma)$ exists such that* $\mathrm{im}\bar{B}_2 \subseteq (\mathcal{V}_\Sigma \oplus 0_p) + \mathrm{im}\bar{B}_1$, *then the subspace $\mathcal{R}_\Delta^\star = \min \Phi(\Delta)$ is internally stabilisable.*

The statement *(iv)* is the extension of a well-known property that was first presented as a conjecture by Basile and Marro in [5], and then proved by Schumacher in [22] in the case when both D and G are zero. The proof of these properties for non-strictly proper systems can be found in [21]. Notice that by virtue of *(iii)*, in the case where $\mathrm{im}\bar{B}_2 \subseteq (\mathcal{V}_\Sigma^\star \oplus \mathbf{0}_p) + \mathrm{im}\bar{B}_1$ holds, the more stringent inclusion $\mathrm{im}\bar{B}_2 \subseteq (\mathcal{R}_\Delta^\star \oplus \mathbf{0}_p) + \mathrm{im}\bar{B}_1$ holds, as well, since $\mathcal{R}_\Delta^\star$ is an element of $\Phi(\Delta)$. As a result, if $\mathrm{im}\bar{B}_2 \subseteq (\mathcal{V}_\Sigma^\star \oplus \mathbf{0}_p) + \mathrm{im}\bar{B}_1$ holds, two matrices of suitable dimensions Π_1 and Π_2 exist such that

$$\bar{B}_2 = \begin{bmatrix} R \\ 0 \end{bmatrix} \Pi_1 + \bar{B}_1 \Pi_2 \tag{16}$$

where R is a basis matrix of $\mathcal{R}_\Delta^\star$. Equation (16) is linear, so that the set of all matrices Π_1 and Π_2 satisfying (16) are parameterised by the expression

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = \begin{bmatrix} R & B_1 \\ 0 & D_1 \end{bmatrix}^\dagger \begin{bmatrix} B_2 \\ D_2 \end{bmatrix} + KZ \tag{17}$$

where K is a basis matrix of the null-space of the matrix $\begin{bmatrix} R & B_1 \\ 0 & D_1 \end{bmatrix}$ and Z is an arbitrary matrix of suitable dimensions. Hence, the pair of matrices (Π_1, Π_2) computed by means of (17) is not unique in general, unless $\mathcal{R}_\Delta^\star \cap B_1 \ker D_1 = \mathbf{0}_n$.

The following theorem provides the necessary and sufficient solvability conditions for Problem 2.1, as well as the explicit structure of the decoupling filter Σ_c.

Theorem 5.1: *Problem 2.1 admits solutions if and only if following two conditions hold:*

(i) $\mathrm{im}\,\bar{B}_2 \subseteq \mathcal{V}_\Sigma^\star \oplus \mathbf{0}_p + \mathrm{im}\,\bar{B}_1$;
(ii) *$\mathcal{R}_\Delta^\star$ is internally stabilisable.*

In the case where $\dim(\mathcal{R}_\Delta^\star) > 0$, let R be a basis of $\mathcal{R}_\Delta^\star$. Let also $F \in \mathfrak{F}_\Sigma(\mathcal{R}_\Delta^\star)$ and X be such that $\sigma(A + BF) \subset \mathbb{C}_g$ and

$$\begin{bmatrix} A + B_1 F \\ C + D_1 F \end{bmatrix} R = \begin{bmatrix} R \\ 0 \end{bmatrix} X, \tag{18}$$

so that $\sigma(X) \subset \mathbb{C}_g$. Let (Π_1, Π_2) be such that (16) holds. A compensator Σ_c, solving Problem 2.1 is described by the quadruple

$$(A_c, B_c, C_c, D_c) = (X, \Pi_1, FR, -\Pi_2). \tag{19}$$

If $\mathcal{R}_\Delta^\star = \mathbf{0}_n$, the decoupling filter reduces to static unit $D_c = -\Pi_2$.

Proof: We first prove sufficiency of conditions *(i)–(ii)* by showing that when $\dim(\mathcal{R}^*_\Delta) > 0$ the compensator given by (19) indeed solves Problem 2.1. First, notice that by *(ii)* the subspace $\mathcal{R}^*_\Delta$ is internally stabilisable, and, since A is assumed to be stable, then the pair (A,B_1) is stabilisable, so that R^*_Δ is externally stabilisable. Hence, a matrix $F \in \mathfrak{F}_\Sigma(\mathcal{R}^*_\Delta)$ such that $\sigma(A + B_1F) \subset \mathbb{C}_g$ indeed exists. The overall system $\widehat{\Sigma}$ from the input w to the output y is described by the quadruple $(\widehat{A},\widehat{B},\widehat{C},\widehat{D})$, where $\widehat{A} = \begin{bmatrix} A & B_1FR \\ 0 & X \end{bmatrix}$, $\widehat{B} = \begin{bmatrix} B_2 - B_1\Pi_2 \\ \Pi_1 \end{bmatrix}$, $\widehat{C} = [\,C \quad D_1FR\,]$, $\widehat{D} = D_2 - D_1\Pi_2$. We need to show that the transfer function $G_{\widehat{\Sigma}}(\zeta) = \widehat{C}(\zeta I_{n+n_c} - \widehat{A})^{-1}\widehat{B} + \widehat{D}$ is zero. To this end, it suffices to show that $\widehat{D} = 0$ and that the reachable subspace from the origin of $\widehat{\Sigma}$ is contained in the null-space of $\widehat{C}$, [30]. From (16), we find $D_2 = D_1\Pi_2$, so that $\widehat{D}$ is zero. Now, we show that $\widehat{C}\widehat{A}^k\widehat{B} = 0$ for all $k \geq 0$; by using (16) and (18), we find that

$$\widehat{C}\widehat{A}^k\widehat{B} = [\,C \quad D_1FR\,]\begin{bmatrix} A^k & \sum_{i=0}^{k-1} A^iB_1FRX^{k-i-1} \\ 0 & X^k \end{bmatrix} \begin{bmatrix} B_2 - B_1\Pi_2 \\ \Pi_1 \end{bmatrix} \tag{20}$$

$$= CA^k(R\Pi_1 + B\Pi_2) - CA^kB_1\Pi_2 + C\sum_{i=0}^{k-1} A^iB_1FRX^{k-i-1}\Pi_1 - CRX^k\Pi_1 \tag{21}$$

$$= CA^kR\Pi_1 + C\sum_{i=0}^{k-1}(A^iRX^{k-1} - A^{i+1}RX^{k-i-1})\Pi_1 - CRX^k\Pi_1 \tag{22}$$

which is zero for all $k \geq 0$ since $\sum_{i=0}^{k-1}(A^iRX^{k-1} - A^{i+1}RX^{k-i-1}) = RX^k - A^kR$. It follows that $\widehat{C}\,\mathrm{im}\begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \widehat{A}^2\widehat{B} & \dots \widehat{A}^{n-1}\widehat{B} \end{bmatrix} = \mathbf{0}_p$, so that $G_{\widehat{\Sigma}}(\zeta)$ is zero. In the case where $\mathcal{R}^*_\Delta = \mathbf{0}_n$, by (16) we find that $\widehat{B}_2 = \widehat{B}_1\Pi_2$. Using the control law $u(t) = -\Pi_2 w(t)$, we get

$$\begin{bmatrix} \rho x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} Ax(t) + (B_2 - B_1\Pi_2)w(t) \\ Cx(t) + (D_2 - D_1\Pi_2)w(t) \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} x(t)$$

whose associated transfer function matrix is clearly zero. The system matrix of the closed-loop system is strictly stable since $\sigma(X) \subset \mathbb{C}_g$.

Now we prove necessity of conditions *(i)–(ii)*. Let Σ_c be a solution to Problem 2.2. Since $G(\zeta)$ is zero for all $\zeta \in \mathbb{C}$, it follows that $D_2 + D_1D_c = 0$. Now, let $\mathcal{H}$ be the reachable subspace from the origin of the overall system $\widehat{\Sigma}$, that is,

$$\mathcal{H} \triangleq < \begin{bmatrix} A & B_1C_c \\ 0 & A_c \end{bmatrix}, \begin{bmatrix} B_2 + B_1D_c \\ B_c \end{bmatrix} > . \tag{23}$$

Since $\widehat{\Sigma}$ is stable, that is, $\sigma(\widehat{A}) \subset \mathbb{C}_g$, it follows that $\sigma(\widehat{A}|\mathcal{H}) \subset \mathbb{C}_g$ and $\sigma\left(\widehat{A}\Big|^{\mathcal{X}\oplus\mathcal{X}_c}_{\mathcal{H}}\right) \subset \mathbb{C}_g$, that is, $\mathcal{H}$ is an internally and externally stable $\widehat{A}$-invariant subspace.

Since it is assumed that the transfer function $G(\zeta)$ is zero, it follows that $\mathcal{H} \subseteq \ker[\,C \quad D_1,C_c\,]$. Now, consider a basis matrix of $\mathcal{H}$ partitioned as $H = \mathrm{im}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$, where the columns of H_1 span a subspace in $\mathcal{X}$ and those of H_2 span a subspace in $\mathcal{X}_c$. We recall that the projection $\mathfrak{P}(\mathcal{H})$ of $\mathcal{H}$ on the state space $\mathcal{X}$ of the plant Σ is defined as

$$\mathfrak{P}(\mathcal{H}) \triangleq \left\{ x \in X \,|\, \exists z \in \mathcal{X}_c : \begin{bmatrix} x \\ z \end{bmatrix} \in \mathcal{H} \right\}.$$

It is easy to check that $\mathrm{im}H_1 = \mathfrak{P}(H)$. From the $\hat{A}$-invariance of $\mathcal{H}$ and from the inclusion $\mathrm{im}\hat{B} \subseteq \mathcal{H} \subseteq \ker\hat{C}$, two matrices L and Y exist such that the following identities hold:

$$\begin{bmatrix} A & B_1C_c \\ 0 & A_c \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} L, \tag{24}$$

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} Y = \begin{bmatrix} B_2 + B_1D_c \\ B_c \end{bmatrix}, \tag{25}$$

$$[\,C \quad D_1C_c\,]\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = 0. \tag{26}$$

Moreover, $\sigma(L) = \sigma(\hat{A}|\mathcal{H}) \subset \mathbb{C}_g$. From (25), it follows that $\mathfrak{P}(\mathcal{H}) \supseteq \mathrm{im}(B_2 + B_1D_c)$. Combining the latter with $D_2 + D_1D_c = 0$ yields $\mathrm{im}\begin{bmatrix} B_2 + B_1D_c \\ D_2 + D_1D_c \end{bmatrix} \subseteq \mathfrak{P}(\mathcal{H}) \oplus \mathbf{0}_p$, which in turn implies

$$\mathrm{im}\begin{bmatrix} B_2 \\ D_2 \end{bmatrix} \subseteq (\mathfrak{P}(\mathcal{H}) \oplus \mathbf{0}_p) + \mathrm{im}\begin{bmatrix} B_1 \\ D_1 \end{bmatrix}. \tag{27}$$

Now we show that $\mathfrak{P}(\mathcal{H}) \subseteq \mathcal{V}_\Sigma^*$. To this end, we prove that $\mathfrak{P}(\mathcal{H})$ is an internally stabilisable output-nulling subspace of Σ. Combining the first row of (24) with (26) yields $\begin{bmatrix} A \\ C \end{bmatrix} H_1 = \begin{bmatrix} H_1 \\ 0 \end{bmatrix} L - \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} C_c H_2$, so that, since $\mathrm{im} H_1 = \mathfrak{P}(\mathcal{H})$, we get

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathfrak{P}(\mathcal{H}) = (\mathfrak{P}(\mathcal{H}) \oplus \mathbf{0}_p) + \mathrm{im} \begin{bmatrix} B_1 \\ D_1 \end{bmatrix}.$$

Hence, $\mathfrak{P}(\mathcal{H}) \in \mathcal{V}(\Sigma)$, so that $\mathfrak{P}(\mathcal{H}) \subseteq \mathcal{V}_\Sigma^*$. Furthermore, since as already observed $\sigma(L) \subset \mathbb{C}_g$, $\mathfrak{P}(\mathcal{H})$ is an internally stabilisable output-nulling subspace for Σ. By Lemma 5.1, *(iv)*, it follows that $\mathcal{R}_\Delta^*$ is internally stabilisable, as well.

By using the decoupling filter (19) described in Theorem 5.1, the set of eigenvalues of the overall system $\hat{\Sigma}$ is $\sigma(A) \uplus \sigma(X)$, where $\sigma(X) = \sigma(A + BF|\mathcal{R}_\Delta^*) \subset \mathbb{C}_g$, for a suitable $F \in \mathfrak{F}_\Sigma(\mathcal{R}_\Delta^*)$, and the order of the compensator equals the dimension of the $\mathcal{R}_\Delta^*$, that is, the smallest element of $\Phi(\Delta)$. The exploitation of self-bounded subspaces in stating the solvability conditions and for the derivation of the explicit structure of the compensator has several advantages over the use of the stabilisability subspace $\mathcal{V}_g^*$ often used in the literature, [24,30]. First, since $\mathcal{V}_g^* \supseteq \mathcal{R}_\Delta^*$, the dimension of the compensator devised here is smaller than that following from the use of $\mathcal{V}_g^*$. Second, from a computational point of view checking the conditions *(i)–(ii)* in Theorem 5.1 is much easier than checking the corresponding condition $\mathrm{im}\bar{B}_2 \subseteq (\mathcal{V}_g^* \oplus \mathbf{0}_p) + \mathrm{im}\bar{B}_1$ involving $\mathcal{V}_g^*$. In fact, while $\mathcal{R}_\Delta^*$ can be computed as the intersection $\mathcal{V}_\Delta^* \cap \mathcal{S}_\Delta^* = \mathcal{V}_\Sigma^* \cap \mathcal{S}_\Delta^*$, finding a base for $\mathcal{V}_g^*$ requires eigenspace computation, [7], which often leads to a heavy computational burden. Third, the use of $\mathcal{R}_\Delta^*$ ensures that when the MSDP with stability is solvable, a maximal set of eigenvalues of the overall system exists which is present for any solution (these eigenvalues are usually referred to as the fixed poles of the decoupling problem), and at least one feedback matrix F exists such that all the remaining eigenvalues can be assigned arbitrarily. From the results in [14] it turns out that the fixed poles of the MSDPs are given by the union (with repetition) of the eigenvalues of A and $\Gamma_\Sigma(\mathcal{R}_\Delta^*)$. The generalisation of this result for non-strictly proper systems can be found in [21].

Remark 5.1: The solution herein presented for the MSDPs can be used to solve the decoupling of previewed input signals in the discrete case, see [2,8,17,27] for the strictly proper case. Consider a discrete-time system described by (1), where, with respect to Problem 2.1, some extra information is available on the disturbance to be rejected w. More precisely, now not only is the signal w available for measurement, but it is supposed to be known in advance with a preview time $N > 0$, see Fig. 3. As such, if at time t the input $w(t)$ is applied to the system, the compensator has access to the future value $w(t+N)$, and hence also to $w(t-N+1), w(t-N+2), \ldots, w(t)$. It is easily seen from Fig. 3 that the N-delay stage accounts for the pre-knowledge of the signal $w(t)$ so that the compensator Σ_c exploits the preview information on $w(t)$ represented by $w_p(t) = w(t+N)$. It follows that the previewed signal decoupling can be solved by solving a MSDPs, where now the plant is given by the series connection of the N-delay and of the system Δ'. If we consider any realisation (A_d, B_d, C_d) of the N-delay, the solution of this problem is the one given in Theorem 5.1, where now Δ is described by the matrices $A^\Delta = \begin{bmatrix} A & B_2 C_d \\ 0 & A_d \end{bmatrix}$, $B_1^\Delta = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, $B_2^\Delta = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$, $C^\Delta = [C \quad D_2 C_d]$, $D_1^\Delta = D_1$, $D_2^\Delta = 0$. By substitution of the matrices $(A, B_1, B_2, C, D_1, D_2)$ with $(A^\Delta, B_1^\Delta, B_2^\Delta, C^\Delta, D_1^\Delta, 0)$ in Theorem 5.1, it follows that the problem admits solution if and only if

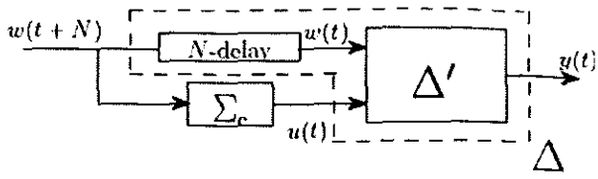

Fig. 3. Block diagram of the previewed signal decoupling scheme.

(i) $\mathrm{im}B_2^\Delta \subseteq \mathcal{V}_\Sigma^* + B_1^\Delta \ker D_1^\Delta$;
(ii) $\mathcal{R}_\Delta^*$ is internally stabilisable.

where now $\Sigma = (A^\Delta, B_1^\Delta, C^\Delta, D_1^\Delta)$. The simplified form of the structural condition *(i)* with respect to that presented in Theorem 5.1 is due to D_2^Δ being zero. If conditions *(i)–(ii)* are satisfied, the inner structure of the compensator Σ_c is given in Theorem 5.1 with the obvious substitutions. Clearly, as N increases, condition *(i)* becomes more likely to be satisfied. In other words, the more information on the disturbance w is made available to the controller, the easier it becomes for the controller to reject such disturbance.

Remark 5.2: Another important problem that can be easily turned into a MSDPs is the so-called model matching, see [12,13,16,19]. Given a system $\Sigma = (A,B,C,D)$ along with a model $\Sigma_m = (A_m, B_m, C_m, D_m)$ having the same output spaces, the exact model matching consists of finding a compensator $\Sigma_c = (A_c, B_c, C_c, D_c)$ such that the input/output behaviour of

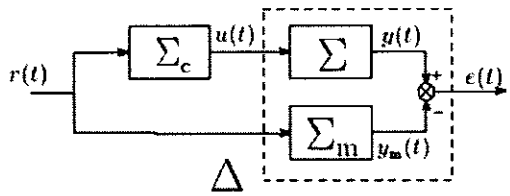

Fig. 4. Block diagram for model matching.

the series connection between Σ and Σ_c equals that of the given model Σ_m, or, equivalently, such that the difference e between the output of the original system Σ and that of the model Σ_m is identically zero, see Fig. 4. This problem can be turned into a MSDPs where Δ is described by the matrices $A^\Delta = \begin{bmatrix} A & 0 \\ 0 & A_m \end{bmatrix}$, $B_1^\Delta = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $B_2^\Delta = \begin{bmatrix} 0 \\ B_m \end{bmatrix}$, $C^\Delta = [\, C \;\; -C_m \,]$, $D_1^\Delta = D$, $D_2^\Delta = -D_m$. The problem admits solution if and only if the conditions in Theorem 5.1 hold, where the matrices (A,B_1,B_2,C,D_1,D_2) have to be replaced by $(A^\Delta,B_1^\Delta,B_2^\Delta,C^\Delta,D_1^\Delta,D_2^\Delta)$.

6. Solution of the UIOs

As aforementioned, Problems 2.1 and 2.2 are dual to each other. Hence, the counterpart of the results presented so far for the solution of the MSDPs are presented here in the dual context of UIOs without proofs. Let $\Sigma = (A,B,C_2,D_2)$. Moreover, let $\bar{C}_1 \triangleq [C_1 \;\; D_1]$ and $\bar{C}_2 \triangleq [C_2 \;\; D_2]$.

Lemma 6.1: *Let* $\ker \bar{C}_1 \supseteq (\mathcal{S}_\Sigma^\star \oplus \mathcal{U}) \cap \ker \bar{C}_2$. *The following facts hold:*

(i) $\mathcal{S}_\Sigma^\star = \mathcal{S}_\Omega^\star$;
(ii) $\Psi(\Sigma) \subseteq \Psi(\Omega)$;
(iii) *For all* $\mathcal{S}_\Omega \in \Psi(\Omega)$ *there holds* $\ker \bar{C}_1 \supseteq (\mathcal{S}_\Omega \oplus \mathcal{U}) \cap \ker \bar{C}_2$;
(iv) *If an externally stabilisable input-containing subspace* $\mathcal{S}_\Sigma \in \mathcal{S}(\Sigma)$ *exists such that* $\ker \bar{C}_1 \supseteq (\mathcal{S}_\Sigma \oplus \mathcal{U}) \cap \ker \bar{C}_2$, *then the subspace* $\mathcal{Q}_\Omega^\star = \max \Psi(\Omega)$ *is externally stabilisable.*

The proof of this lemma follows from that of Lemma 5.1 by duality. In the case where $\ker \bar{C}_1 \supseteq (\mathcal{S}_\Sigma^\star \oplus \mathcal{U}) \cap \ker \bar{C}_2$ holds, the more stringent inclusion $\ker \bar{C}_1 \supseteq (\mathcal{Q}_\Omega^\star \oplus \mathcal{U}) \cap \ker \bar{C}_2$ holds, since $\mathcal{Q}_\Omega^\star$ is an element of $\Psi(\Omega)$. As a consequence, two matrices of suitable dimensions Π_1 and Π_2 exist such that

$$\bar{C}_1 = \Pi_1 [\, Q \;\; 0 \,] + \Pi_2 \bar{C}_2 \tag{28}$$

where Q is a full row-rank matrix such that $\ker Q = \mathcal{Q}_\Omega^\star$. The set of matrices Π_1 and Π_2 satisfying (28) is parameterised by the expression

$$[\, \Pi_1 \;\; \Pi_2 \,] = [\, C_1 \;\; D_1 \,] \begin{bmatrix} Q & 0 \\ C_2 & D_2 \end{bmatrix}^\dagger + KZ$$

where the rows of Z are linearly independent and span the null-space of the matrix $\begin{bmatrix} Q^\top & C_2^\top \\ 0 & D_2^\top \end{bmatrix}$, while K is an arbitrary matrix of suitable size. Hence, the pair of matrices (Π_1,Π_2) satisfying (28) is unique if and only if $C_2\mathcal{Q}_\Omega^\star + \mathrm{im}D_2 = \mathbb{R}^{p_2}$ or, equivalently, if and only if $\mathcal{Q}_\Omega^\star + C_2^{-1}\mathrm{im}D_2 = \mathbb{R}^n$.

The following theorem is the dual of Theorem 5.1, and provides solvability conditions for Problem 2.2, as well as the explicit structure of the unknown-input observer Σ_o.

Theorem 6.1: *Problem 2.2 admits solutions if and only if*

(i) $\ker \bar{C}_1 \supseteq (\mathcal{S}_\Sigma^\star \oplus \mathcal{U}) \cap \ker \bar{C}_2$;
(ii) $\mathcal{Q}_\Omega^\star$ *is externally stabilisable.*

Let (i)-(ii) hold. If $\dim \Omega < n$, *let* Q *be a full row-rank matrix such that* $\ker Q = \mathcal{Q}_\Omega^\star$. *Let also* $G \in \mathfrak{S}_\Sigma(\mathcal{Q}_\Omega^\star)$ *and* Λ *be such that* $\sigma(A + GC) \subset \mathbb{C}_g$ *and*

$$Q[\, A + GC_2 \;\; B + GD_2 \,] = \Lambda [\, Q \;\; 0 \,] \tag{29}$$

so that $\sigma(\Lambda) \subset \mathbb{C}_g$. *Let* (Π_1,Π_2) *be the such that (28) holds. An observer* Σ_o, *solving Problem 2.2 is described by the quadruple* $(A_o,B_o,C_o,D_o) = (\Lambda, -QG,\Pi_1,\Pi_2)$. *If* $\mathcal{Q}_\Omega^\star = \mathbb{R}^n$, *the observer reduces to the static unit* $D_o = \Pi_2$.

The proof follows by applying duality arguments to the involved systems and subspaces, [6].

Remark 6.1: The solution proposed for UIOs can be exploited to solve the problem of smoothing with fixed lag for discrete systems, where preview information shows up in the delay between the measurement and the generation of the estimate. Consider the discrete-time case and suppose that in the system described by (4) the task it to provide an estimation of $z_d(t) = z(t - N)$, see Fig. 5. The N-delay stage now accounts for the delay tolerated for the estimation of z, so that $z_d(t) = z(t - N)$ represents the available latency in the estimation problem. It follows that the fixed-lag smoothing can be tackled by solving a UIOs, where now the plant is given by the series connection of the N-delay and of the system Ω'. More precisely, if we consider a realisation (A_d,B_d,C_d) of the N-delay, it follows that the solution of this problem is the one given in Theorem 6.1, where now Ω is described by the

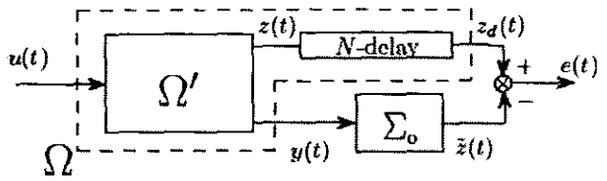

Fig. 5. Block diagram of the fixed-lag smoothing scheme.

matrices $A^{\Omega} = \begin{bmatrix} A & 0 \\ B_d C_1 & A_d \end{bmatrix}$, $B^{\Omega} = \begin{bmatrix} B \\ B_d D_1 \end{bmatrix}$, $C_1^{\Omega} = [0 \;\; C_d]$, $D_1^{\Omega} = 0$, $C_2^{\Omega} = [C_2 \;\; 0]$, $D_2^{\Omega} = D_2$. In fact, by substitution of the matrices (A,B,C_1,D_1,C_2,D_2) with $(A^{\Omega},B^{\Omega},C_1^{\Omega},0,C_2^{\Omega},D_2^{\Omega})$, it turns out that the problem admits solution if and only if

(i) $\ker C_1^{\Omega} \supseteq \mathcal{S}_{\Omega}^{\star} \cap (C_2^{\Omega})^{-1} \mathrm{im} D_2^{\Omega}$;
(ii) $\mathcal{Q}_{\Omega}^{\star}$ is externally stabilisable,

where now $\Sigma = (A^{\Omega},B^{\Omega},C_2^{\Omega},D_2^{\Omega})$. If conditions *(i)–(ii)* are satisfied, the inner structure of the observer Σ_o is given in Theorem 6.1 with the obvious substitutions.

7. Conclusions

By extending the notions of self-bounded and self-hidden subspaces, the solution of several exact control and estimation problems has been provided in the general case where all the systems involved are possibly non-strictly proper. For all these problems, the solvability conditions have been expressed by *(i)* a geometric inclusion involving output-nulling and input-containing subspaces; *(ii)* a so-called stability condition, on a self-bounded subspace in the decoupling problems and on a self-hidden subspace in the estimation problems. The use of self-bounded and self-hidden subspaces enables the decoupling filter for MSDPs and the unknown-input observer for UIOs with the minimal unassignable dynamics to be explicitly derived through easily implementable procedures that do not require eigenspace computations.

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