Science and Mathematics Education Centre

Melody of Functions and Graphs: Improving Senior Secondary Mathematics Students' Understanding of the Concept of Function by the Integration of Mathematics and Music

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DECLARATION

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made.

This thesis contains no material which has been accepted for award of any other degree or diploma in any university.

The research presented and reported in this thesis was conducted in accordance with the National Health and Medical Research Council National Statement on Ethical Conduct in Human Research (2007) – updated March 2014. The proposed research study received human research ethics approval from the Curtin University Human Research Ethics Committee (EC00262), Approval Number # SMEC-55-14.

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Date: 19 October 2018

ABSTRACT

This thesis describes an action research study explicitly targeting the improvement of the author's senior secondary students' understanding of the concept of function. It involves the active integration of mathematics and music through using nine relevant, author-designed analogies between different aspects of function and the corresponding aspects of music.

The importance of the problem is reflected by two major facts: (1) the practical fact, that proper, wider understanding of the concept of function is crucial in the process of learning mathematics, and (2) the theoretical fact, that improving senior secondary mathematics students' understanding of the concept of function through the integration of mathematics and music has not been widely researched. A range of previous research results show that music and music-enriched instruction or traditional music education may improve students' mathematical achievements in general terms, and the possible effects of music on pre-school, primary school, and middle school students' general mathematics scores have been widely researched. The present study has targeted the improvement of senior secondary students' understanding of the concept of function through the active integration of mathematics and music, a topic on which there has been a paucity of research.

Two research questions have been addressed. The first is the main focus of the study: (RQ-1) "Does the active integration of mathematics and music by using these nine analogies between different aspects of function and the corresponding aspects of music make a statistically significant improvement in my senior secondary mathematics students' understanding of function?" The second question is auxiliary: (RQ-2) "Does this type of integration of mathematics and music make a statistically significant improvement in my senior secondary mathematics students' attitudes and beliefs regarding mathematics?"

Because the study was designed to improve senior secondary (Year 11) students' understanding of different aspects of function, it does not concern the effects of music on mathematics achievements in general terms, nor it is about passive listening to music in mathematics lessons. The study involved the active integration of

mathematics and music in the form of nine author-designed conceptual analogies between function and music, using relevant software tools, explicitly targeting six well defined aspects of function. The study concentrated on the improvement of the understanding of all six aspects of the functions topic in the Australian Curriculum 2014. These aspects of function are described in the Functions section of Topic 1 (Functions and Graphs) of Unit 1 of the Year 11 Mathematical Methods course defined as follows: (1) Understand the concept of function as a mapping between sets; (2) Use function notation, domain and range, independent and dependent variables; (3) Understand the concept of the graph of a function; (4) Examine translations and the graphs of $y = f(x) + a$ and $y = f(x + b)$; (5) Examine dilations and the graphs of $y = c \times f(x)$ and $y = f(k \times x)$; and (6) Recognise the distinction between functions and relations, and the vertical line test.

It is shown that the active integration of mathematics and music via the author's nine analogies (between aspects of function and aspects of music) significantly improves secondary mathematics students' understanding of function aspects (1) and (4) above.

In order to measure improvement of the understanding of these six aspects of function, this study has combined mathematical-logical, visual and musical intelligences, aiming to determine any positive effects of this integration which could be employed by practitioners. A sample of forty-four Year 11 Mathematical Methods (16-17 years old) students at a college in Adelaide participated in this action research intervention. The instrument used to measure the improvement of students' understanding of different aspects of function was a textbook-based (Haese Mathematics, 2015) comprehensive mathematics test designed by the author. The study had a dual focus because both the theoretical preparations and the practical applications and outcomes of the research were considered. A post-positivist paradigm, quasi-experimental research design was used, along with multiple research methods, experimental- and control-groups, dependent and independent variables, and pre- and post-testing.

Regarding the main research question, the findings suggest that the active integration of mathematics and music through using the designed nine analogies between the concept of function and music made a statistically significant improvement in the sample of senior secondary (Year 11) mathematics students' understanding of function. The auxiliary research question proved statistically not significant, possibly due to the relative short (90 minutes) duration of treatment involved in this research. The project's dual focus on theory and practice has generated new theoretical and practical knowledge to add to the paucity of research on this topic at the secondary level. While the findings of this action research study are confined to the sample, its contributions to the existing data base concern both the theoretical considerations and definitions, along with the practical applications of the nine relevant analogies between different aspects of the concept of function and music at senior secondary (Year 11) mathematics level, all of which might be employed with success by teachers addressing the Curriculum.

It is suggested that the positive results of the study have significant implications for the teaching of mathematics at senior secondary level, namely to reinforce important mathematical concepts by the active integration of other disciplines.

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I consider that the events in my life are the main background factors which inclined me towards mathematics and music and – as a result – have influenced this study. Because I am the product of these mathematical and musical influences and the study is my product, I believe that it is important to consider and acknowledge them.

As a student I have had eminent mathematics teachers at all three levels of schooling: my primary, secondary and university level mathematics teachers and professors opened the ways towards mathematics and guided me on the road towards an entirely beautiful world, the world of mathematics. Because they made an imprint on me, I consider and acknowledge that their work has an indirect impact on this thesis, and I am thankful for the work of these great mathematics teachers and professors. Mr Béla Balázs, my primary mathematics teacher for 4 years, and Mr Péter Márton (1935-2008), my secondary mathematics teacher for 4 years, formed the first very important shapes of my mathematical thinking. My university professors – to name just a few – have made a decisive impact on my mathematical thinking during my four years pure mathematics studies at the Faculty of Mathematics and Computer Sciences, Department of Mathematics at the Babeș-Bolyai University of Cluj-Napoca: Dr Imre Virág (1931-2015), Dr József Kolumbán, Dr Márton Balázs (1929-2016), Dr Béla Orbán (1929-2016), Dr Gábor Goldner (1940-2011), Dr Ioan A. Rus, Dr Petru T. Mocanu (1931-2016), Dr Marian Țarină (1932-1992), Dr Ioan Purdea, and the advisor and supervisor of my Bachelor of Mathematics Degree Examination Thesis, the legendary Dr Gyula I. Maurer (1927-2012), the co-founder and editor-in-chief of the mathematical journal "*Mathematica Pannonica*" (http://mathematica-pannonica.ttk.pte.hu/), founded in 1990 in Hungary, Austria and Italy, devoted to the publication of high quality research papers on pure and applied mathematics. Thank you for the mathematics.

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I am thankful to the creators of the instruments, who have expressed their positive consent to use the instruments in my research: Haese Mathematics Pty Ltd, Adelaide, South Australia; Dr Barry Fraser, Curtin University, Perth, Western Australia; Dr Braden Phillips, The University of Adelaide, South Australia; Mr Habib Amir, Crystal Magic Studio Ltd, London, England; and Mr Harald Meyer, Harald Meyer Mobile Software, Austria. Thank you for the instruments.

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I would like to dedicate this study to my parents Emma Nagy and Domokos Nagy, to the family of my sister Emese Molnos, to the family of my brother Domokos Nagy, and to my wife, Mária-Magdolna Nagy, and my son, Tamás Robert Nagy. Thank you for the life.

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Chapter 1

INTRODUCTION TO THE STUDY

1.1 Background to the Study

In this section I write briefly about myself, my family background, my mathematics and music teachers, and my mathematics teaching practice. I also write about the idea, the objectives and the design of the present study as well as about the participating students, the secondary college where the research was performed in 2016 and the study's significance.

1.1.1 About Myself, My Family Background, and My Teachers

I am a secondary mathematics specialist teacher and have taught mathematics for 16 years. The first four years of these, between 1982-1986, were at a Hungarian secondary college in my native town of Székelykeresztúr in Transylvania (Romania) teaching in Hungarian and Romanian languages. The last 12 years, from 2006-2018, involved teaching in English at seven different secondary colleges in Adelaide, South Australia. Between 1987-2006 I was working as a mathematician-statistician and computer software developer in Romania (1987-1989) speaking in Hungarian and Romanian languages; in Austria (1990-2000) speaking in German, and in Australia (2001-2005) in English. In those countries I designed different mathematical-statistical models and computer software algorithms and applied these models and algorithms in various mainframe-, mini-, micro-, and portable computer solutions in the fields of medicine, banking, agriculture, industry and commerce. During this time I was a regular reader and problem solver of two Hungarian computer journals, *Mikroszámítógép Magazin* (Microcomputer Magazine) and *Computerworld-Számítástechnika* (Computerworld - Computing Technology). My solutions of different problems – mainly mathematics-centered algorithms – have been published in these journals. I was the prize winner of an extensive (one year long) problem solving competition, organised by the *Mikroszámítógép Magazin* computer journal.

I grew up in a family where sciences and mathematics, literature and languages, arts and music were cultivated and appreciated. My music lover mother, Emma Nagy (born Szabó, 1933) is a retired secondary teacher of languages who speaks and reads many languages fluently and who published a language-related book in 2012. My father, Domokos Nagy (1928-1996) was an agronomic engineer who has applied different agricultural production optimisation strategies through using accessible technologies and relevant mathematical theoretical models. I also would like to mention my grandfathers, Mózes Nagy (1883-1960), a bee-keeper and honey merchant, who was good in everyday applied arithmetic and measurements, and Kálmán Szabó (1903-1976), a church organ player and folk music collector for many decades whose comprehensive folk music collection book of around 1000 folk music pieces – edited by my uncle, Attila K. Szabó (1937-2018), teacher of pedagogy and psychology and author of four books and numerous articles – is expected to be published in 2018.

I live in Adelaide, Australia, in a happy family with my wife, Mária-Magdolna Nagy, and our son Tamás Robert Nagy.

1.1.2 The Role of Mathematics and Music in My Life

Both mathematics and music have fascinated me for many years.

During the eight years of my elementary and primary schooling, my personal feeling was that I learned mathematics, for example the concept of fraction and the mode of operations with fractions, much more easily and quickly than the other students in the class. Obviously, at that time this observation was not formulated as I would formulate it today, but now I think that a skill transfer effect occurred, namely that the existing musical skill and knowledge of how to use musical notes such as the whole note ($^1/_1$), the minim or half note ($^1/_2$), the crotchet or quarter note ($^1/_4$), the ♪ (quaver or eighth note, $^1/_8$), the semiquaver or sixteenth note ($^1/_{16}$) and so on, made it obvious and simple for me to understand, use and operate the symbols $^1/_2$, $^1/_4$, $^1/_8$, $^1/_{16}$ and so on fractions. For example, the musical skill and knowledge of how to use dotted notes, which means to lengthen the duration of the note by one-half of its original duration (e.g. the dotted quaver or eighth note ♪. means the time duration

$^1/_8 + ^1/_{16} = ^3/_{16}$) made simple for me to understand addition of fractions having different denominators.

During the four years of my secondary schooling, when I won a county-level secondary school mathematics competition and a county-level secondary school piano competition, I think that the precise requirements of playing a classical piano piece and the precise requirements of solving a mathematics problem were in perfect accordance, influenced each other positively and have shaped both my mathematical thinking and my musical feelings.

During the four years of my university studies at the Babeș-Bolyai University, Cluj-Napoca, Romania, when I was a mathematics student in the Faculty of Mathematics, I concentrated much more on mathematics subjects such as Mathematical Analysis, Algebra and Linear Programming, Analytical Geometry, Differential Equations and Equations with Partial Derivatives, Informatics, Geometry of Surfaces, Mechanics, Functional Analysis, Numerical Analysis, Complex Analysis, Mathematics Teaching Methodology, Physics, Astronomy, Statistics, Operational Research, Lie Groups, Topological and Differential Variations, Category Theory, and others, but my interest in music did not cease. I was a regular attender of the weekly classical music concerts of the Philharmonic State Orchestra of Cluj-Napoca. I also attended some courses as a non-enrolled, external student at the "Gheorghe Dima" Music Academy of Cluj-Napoca, while Dr István Angi allowed me to regularly attend his Aesthetics of Music course in 1979-1980. I also attended many of Dr Ede Terényi's Harmony, Counterpoint and Composition presentations and, as a listener, several of Dr Ionel Pantea's Canto and Interpretation lessons.

I have enjoyed the wonderful world of mathematics. When I was a secondary student I regularly attended the Students' Mathematics Group, organised by the mathematics teachers at the college, where we, the students, presented and solved interesting mathematics problems every week. I remember that there were challenges such as: (1) proving Pythagoras' theorem in different independent ways (there are many independent proofs, including secondary school appropriate geometrical, algebraic and trigonometrical proofs); (2) connecting algebra and geometry by observing relationships and the correspondence between algebraic equations and geometrical lines and curves (I would name them today as visual analogies or visual representations of mathematical-logical constructs); (3) proving logical compound statements using predicate calculus; (4) dealing with mathematical expressions, identities, equations and inequalities in Physics; (5) presenting the life and work of important mathematicians; (6) writing about irrational numbers, complex numbers, and famous mathematical constants such as the Archimedes' constant $\boldsymbol{\pi}$, Pythagoras' constant $\sqrt{\mathbf{2}} = 2^{0.5}$, Euler's constant $\boldsymbol{e}$, the (at that time) hard to visualise imaginary unit i, and the golden ratio $\boldsymbol{\varphi}$; (7) presenting mathematical curiosities such as Fibonacci series in nature, Golden section in arts, number systems from base 2 to 16, perfect numbers, operations with Roman numerals and so on.

When I was a university mathematics student I was fascinated by learning about the multi-dimensional generalisations of the previously known concepts and also by having the opportunity to learn in a structured and concentrated way a huge amount of high level mathematics, observing the power of both pure mathematics and applied mathematics. I remember that after learning the multi-dimensional linear programming algebraic algorithm, named the simplex algorithm, used to solve n-dimensional optimisation problems calculating the maximum or minimum of an objective function which involves frequent change of the basis of the considered n-dimensional vector space, it was “natural” to conclude that the simplex algorithm is a sophisticated complex theory while the complex analysis in 2D is a relatively simple(x) theory with the $z = a + bi$ approach... I was fascinated by and enjoyed the power of the theory of the abstract algebraic structures of groups, rings, fields and lattices and their applications, of the theory of differential equations and their applications, and of the theory of the non-Euclidean geometries and their applications. I also remember that when I was reading pure mathematics abstract books or course materials on algebraic structures, analytical geometry, topology, differential equations, non-Euclidean geometry, Lie-groups, and category theory, I was often thinking about how it would be possible to connect the studied abstract mathematical content with non-mathematical analogies. I also remember my enthusiasm, as a university mathematics student, when I read for the first time the books written by Zoltán P. Dienes (1916-2014) which connect mathematics with games, music, language, and dance, or the book titled “*Poetica Matematică*” by Solomon Marcus (1925-2016) which connects mathematics with poetry.

I have enjoyed the world of music, both passively, as a regular music listener and classical music concert attender and actively, as a private piano student and amateur pianist playing solo piano pieces by J. S. Bach, Haydn, Mozart, Beethoven, Chopin, Liszt, Bartók, and other composers to audiences of 50-350 people at different local- and county-level music competitions and festive occasions, then as a hobby piano player and a band member.

1.1.3 My Mathematics Teaching Practice

I am a practitioner – a secondary mathematics teacher whose goal is to improve his students' understanding of mathematics continuously. In order to achieve this goal I strive to develop and improve my own mathematics teaching practice. My frequently used mathematics teaching method is to explain abstract mathematical concepts by using different conventional and non-conventional examples and analogies. The conventional, or routine examples and analogies, which usually exist in the textbooks, relate mathematical concepts to routine situations such as geometrical measurements, financial calculations, physical processes, biological populations, everyday situations, and others. The non-conventional, or non-routine mathematical examples and analogies, which usually don't exist in the textbooks, relate mathematics to non-routine situations, like music, poetry, painting, sculpture, architecture, algorithms in nature and human creation, and others. Both the conventional and non-conventional types require a certain level of abstract thinking, and they develop students' mathematical thinking and understanding in different ways. In order to improve my students' understanding of abstract mathematical concepts, I present, discuss and demonstrate different mathematical concepts, properties, theorems and theories by employing regularly a combination of conventional and non-conventional examples and analogies, using students' different human intelligences. These include mainly logical/mathematical, visual/spatial, verbal/linguistic, musical/rhythmic, interpersonal but also bodily/kinaesthetic intelligences as described in Howard Gardner's (1983) Multiple Intelligences theory. This theory generalised the previous concept of human intelligence. Contrary to the previous approach, when intelligence was considered being verbal/linguistic and logical/mathematical only, this theory originally defined seven intelligences or set of abilities, talents or mental skills, as follows:

(1) verbal/linguistic, (2) logical/mathematical, (3) visual/spatial, (4) body/kinaesthetic, (5) musical/rhythmic, (6) interpersonal, and (7) intrapersonal. Later two additional intelligences, the naturalist and the existential intelligences have been added to these original seven intelligences. It is unlikely that new intelligences will be added to this set of nine intelligences (abilities, talents or mental skills) in the future. Edwards (2009) cites the father of multiple intelligences, Howard Gardner, who in an interview in 2009 stated that:

> After 25 years, I've added only one intelligence (the naturalist intelligence) and am considering another (the existential intelligence). And so the prospect of a geometric progression seems most unlikely. In the short run, I don't think we (as a species) are going to discover or evolve new intelligences. What happens, instead, is that already existing intelligences get mobilized for new purposes.
>
> (Howard Gardner, 2009, as cited in Edwards, 2009, p. 2)

In this study I have mobilised my Year 11 senior secondary mathematics students' intelligences – including their general human musical intelligence – for the purpose to improve their understanding of function.

Davis, Christodoulou, Seider, and Gardner (2012) summarise Gardner’s eight intelligences as described in the following Table 1.1.

Table 1.1

Gardner's Eight Intelligences

Intelligences	Description
Linguistic	An ability to analyze information and create products involving oral and written language such as speeches, books, and memos.
Logical-Mathematical	An ability to develop equations and proofs, make calculations, and solve abstract problems.
Spatial	An ability to recognize and manipulate large-scale and fine-grained spatial images.
Musical	An ability to produce, remember, and make meaning of different patterns of sound.
Naturalist	An ability to identify and distinguish among different types of plants, animals, and weather formations that are found in the natural world.
Bodily-Kinaesthetic	An ability to use one's own body to create products or solve problems.
Interpersonal	An ability to recognize and understand other people's moods, desires, motivations, and intentions.
Intrapersonal	An ability to recognize and understand his or her own moods, desires, motivations, and intentions.

Note. Adapted from "The Theory of Multiple Intelligences," by K. Davis, J. Christodoulou, S. Seider, and H. Gardner, 2012, pp. 6-7.

Armstrong (2009) in his book, prefaced by Howard Gardner, states about the logical-mathematical intelligence, that "includes sensitivity to logical patterns and relationships, statements and propositions (if-then, cause-effect), functions, and other related abstractions" (p. 6). We can observe that sensitivity to logical patterns and to the abstract concept of function have a crucial role in logical-mathematical intelligence, and on other hand the ability to make meaning of patterns of sound is crucial in musical intelligence. Armstrong (2009) summarises certain key points of the Multiple Intelligences model, such as: (1) each person possesses all eight intelligences, (2) most people can develop each intelligence to an adequate level of competency, (3) intelligences usually work together in complex ways, and (4) there are many ways to be intelligent within each category. Armstrong (2009) describes how the MI theory can be used in practice in primary and junior high school by presenting different examples in the forms of lessons and programs, targeting different objectives, for example to teach students to recognize circles; to help students master the multiplication facts for the 7s; to reinforce the concept of what it means to 'multiply'; and to explain the function of x in an equation.

In this study I have mobilised the general human musical intelligence of my Year 11 mathematics students – their ability to make meaning of different patterns of sound – for the purpose to improve their sensitivity to the abstract concept of function. The human intelligences are weighted individually for every person and their intensity is also changing during the life of an individual. Because learners prefer different intelligences or combinations of intelligences in order to learn a concept or a skill, Gardner's (1983) Multiple Intelligences theory can be applied in educational practice in an individualised mode (Armstrong, 2009; Gardner, 1993; Phillips, 2009). This has an impact on education in general and on mathematics education in particular. Some articles describe different modes of using multiple intelligences in mathematics education. These include learning mathematics through a given particular intelligence, for example, body/kinaesthetic (Bingham, 2007), musical/rhythmic (Nisbet & Bain, 1998, 2000); and learning through a combination of intelligences (Armstrong, 2009; Munro, 1994; Wahl, 1998; Willis & Johnson, 2001).

Bingham (2007) has researched how to teach function transformations through the integration of algebra and modern dance by using students' logical-mathematical and kinaesthetic intelligences. Munro (1994) presents a model how to accommodate individual learning of mathematics through multiple intelligences. Wahl (1998) gives lesson plans/activities in order to teach mathematics using different intelligences up to year 10 level. Willis and Johnson (2001) describe how to use all of these intelligences to teach multiplication. Sain (1986) states that there is no “royal road” to mathematics, which is absolutely confirmed by both teachers and learners of mathematics. There is no royal road towards mathematics, but we know that there are roads, which are different. These roads towards mathematics can be very diverse, depending upon a range of factors. In my mathematics teaching practice I always try to find the best possible road, in the form of teaching approaches and methods, to reach and improve my students' understanding of mathematics. The original aim of this study was not about carrying out an abstract theoretical thesis but had a much more practical aspect: to carry out an investigation to test my idea regarding applying new strategies in order to improve the quality of my work as a secondary mathematics teacher. It was imperative to have the auditable, objective evidence of the improvement of the quality of the theoretical and practical aspects of my mathematics teaching work. But what does it mean to improve the quality of my work as a secondary mathematics teacher? Obviously, the improvement of the quality of my own teaching practice can be measured only by measuring the improvement of my students' understandings and achievements. The teaching work of a secondary mathematics teacher is a very complex issue. The final product: students' mathematical understanding and achievement is the result of a wide range of subjective and objective factors and variables. Some of these factors and variables are under the control of the mathematics teacher while others are not.

1.1.4 Active Integration of Mathematics and Music

Active integration of mathematics and music means using relevant mathematical examples together with corresponding musical analogies, targeting and representing the studied aspects of mathematics in mathematical-logical, visual-graphical, and audible-musical multi-modal, embodied forms. In contrast, the passive integration of mathematics and music means listening to music without explicitly targeting any connections or structural analogies between mathematics and the listened music piece, then measuring mathematical outcomes. A typical relevant example of passive integration of mathematics and music is the “Mozart effect” study, which concluded that listening to W. A. Mozart’s “*Sonata for Two Pianos in D Major (K. 448)”* temporally improves students' short term spatial-temporal reasoning (Rauscher, Shaw & Ky, 1993). The active integration of mathematics and music is much more mathematical while the passive integration is much more close to Psychology and Neurobiology. Zhan's (2002) meta-study regarding passive integration of music and mathematics states that:

> Music targets one specific area of the brain to stimulate the use of spatial-temporal reasoning, which is useful in mathematical thinking. However, as to the question of whether or not music is the magical portion that will elevate anyone's ability to do math, the answer unfortunately . . .would be no.
>
> (Zhan, 2002, p. 1)

1.1.5 Selection of the Suitable Research Design

Educational research is very diverse: there are many approaches to the theoretical definitions of the concept of research itself, and as a result there are many research types or categories. Every research project has to have clearly defined goals or objectives, which usually are stated as research questions, as well as a comprehensive research plan which includes actions, methods, data collection and analysis strategies to achieve these objectives. The research is highly imprinted by the research paradigm, which includes the philosophical foundations of the research, the ontology (what is considered as an existing truth, what is reality?) and the epistemology (what is considered as knowledge, how can we know the reality?). The different approaches to research are known as research methodology or research design. Crotty (1998) describes how research epistemology, theoretical perspective (the considered approach to know something), methodology (what is to do in order to create knowledge) and methods (what do we use) relate to each other. As described by Crotty (1998), the research types depend upon epistemology (objectivism, constructivism, subjectivism) of the research, the theoretical perspective of the research (positivism and post-positivism, interpretivism, critical inquiry, feminism, postmodernism, and others), the methodology or design of the research (experimental research, survey research, ethnography, heuristic inquiry, action research, grounded theory, multiple case studies, and others), and methods (sampling, measurement and scaling, questionnaire, observation, interview, focus group, case study, statistical analysis, data reduction, theme identification, and others).

There are also other classifications of research methodology or research design. Creswell (2008) classifies methodology as either quantitative, qualitative or mixed method research. The goal of a traditional research project within the positivist and post-positivist paradigm, epistemological view of objectivism, experimental or quasi-experimental methodology, using experimental and control groups, dependent and controlled independent variables, tests and questionnaires to collect data, and statistical analysis of the collected data, is to establish a likely correlation or cause-and-effect relationship between the variables (Creswell, 2008). This goal implies a large sample and a sound control of the independent variables.

All types of research methodology or design have their advantages and limitations. I considered several possible methodologies – paradigms which are realistic and give valid answers to the research questions. The following four methodologies were considered initially: experimental research, quasi-experimental research, case studies, and action research. I dwelt on the advantages and limitations of these four research methodologies regarding the study. Due to the nature of this research, namely, to involve the active integration of mathematics and music at senior secondary (Year 11) mathematics level, the most important pinpoints of a classical traditional educational research project, such as: working with a large sample, controlling a range of important independent variables, considering the potential confounding variables, performing the treatment for at least the half of the sample, collecting relevant data, performing an adequate comprehensive statistical analysis, and so on, appeared not to be a realistic, do-able option. If any of these requirements was missing or incomplete in the planned research, then the potential risk to commit a type of statistical error, either by incorrectly rejecting a true null hypothesis or by incorrectly accepting a false null hypothesis, appeared to be very high, also posing possible questions regarding both the internal validity and reliability of the research. Based on these considerations, and in order to perform a fully correct, proper and relevant research, to assure the high level of validity and reliability of the research, I came to the conclusion to avoid carrying out a traditional experimental or quasi-experimental study. The main argument against performing a case study is the fact that I was inclined towards performing a piece of research within the frame of the epistemological view of objectivism, which by nature underpins the post-positivist paradigm. This mode of thinking underpins the use the quantitative method of statistical analysis by collecting quantitative data through tests and questionnaires and performing a relevant, proper statistical analysis of the collected data.

As a final conclusion, the idea of an experimental or quasi-experimental research and the idea of a case study was rejected. This very important conclusion has narrowed both the goals and the type of the research and inclined me towards researching how I could improve the quality of my own practice of teaching functions, by researching the improvement of my secondary students' understanding of function within the frame of an action research study.

Laverty (2016) summarises the main reasons for pursuing educational research as follows:

- Examine your classroom practice through a systematic process of inquiry.
- Record successes and failures with the goal of improving student learning and teaching practice.
- Reflect on findings in relation to existing educational research literature.
- Validate your teaching practice and build theory relating to educational approaches.
- Share and disseminate experiences to build upon what we know about teaching and learning processes.

(Laverty, 2016, p. 1)

During my study I have fulfilled these five reasons: I examined my classroom practice in a systematic process of inquiry; recorded successes (my students' understanding improved on four mathematical variables) and a failure (my students achieved less on one mathematical variable) with the goal of improving both my students' understanding of the concept of function and my teaching practice of function; reflected on my findings in relation to the existing relevant literature; validated my teaching practice and built a knowledge regarding this methodology of teaching functions at senior secondary level; and shared my experiences.

Regarding the form of the research, three very important major facts have underlined my belief that opting for an action research mode was the best possible option for this study:

(1) the fact that researching how to improve the quality of my own teaching practice does not require a large sample;

(2) the fact that it is possible to control the main independent variables (group, gender, learning environment, mathematics studies, music studies) during the research; and

(3) the fact that using nine self-designed musical analogies of different aspects of function, the three relevant external software tools and the other instruments (pre- and post-tests and questionnaires) in the classroom was possible without any additional approval procedures at my secondary college. As an external researcher, using these tools during Year 11 Mathematical Methods lessons at other secondary schools would have been difficult due to the fairly bureaucratic approval procedures at different levels within the State educational system. This is fully understandable.

1.1.6 Development of the Research Proposal

Crotty (1998) suggests that in developing a research proposal two particular questions are very important. According to Crotty (1998),

> In developing a research proposal, we need to put considerable effort into answering two questions in particular. First, what methodologies and methods will we be employing in the research we propose to do? Second, how do we justify this choice and use of methodologies and methods? The answer to the second question lies with the purposes of our research - in other words, with the research question that our piece of inquiry is seeking to answer. 1t is obvious enough that we need a process capable of fulfilling those purposes and answering that question.
>
> (Crotty, 1998, p. 2)

Considering the above theoretical approaches to research and also my own preferences and way of thinking, during the planning of activities and practices to guide how to perform this research, I set up the positivistic paradigm in terms of ontology, epistemology, methodology and methods. In the case of my study the most suitable and preferred paradigm, the theoretical frame of a post-positivist approach has been chosen. This has determined:

(1) the ontological view of the existence of a single truth,
(2) the epistemological view of objectivism,
(3) the action research methodology, and

(4) the use of multiple methods: collecting quantitative data by using pre- and post-tests, and employing statistical analysis.

This setting is justified by the fact that within this framework the research questions should be answered satisfactorily.

As a mathematics teacher working with Year 11 students, I considered how could I improve the quality of my own practice of teaching functions and as a result improve my students' understanding of function. Based on my previous experiences and on the reported positive results of research articles (An, Capraro, & Tillman, 2013, Bingham, 2007; Cheek & Smith, 1998; Dienes, 1987; Evans, 2009; Harris, 2005; Harris 2007; Johnson & Edelson, 2003; Kelstrom, 1998; Munro, 1994; Nisbet & Bain, 1998; Nisbet & Bain, 2000; Rauscher, Shaw & Ky, 1993; Rauscher, 2003; Schumacher, Altenmüller, Deutsch & Vitouch, 2006; Spychiger, 1999a; Spychiger, 1999b; Still & Bobis, 2005; Vaughn, 2000; Vaughn & Winner, 2000; Wahl, 1998; Willis & Johnson, 2001), I came to the conclusion that using a non-conventional approach by actively integrating mathematics and music may help to achieve these goals.

Consequently this study presents my efforts to use a non-conventional, self-developed method. I was “(1) taking action to improve” my students' understanding of function by using a non-conventional method, “and (2) conducting research to explain and justify the process of taking action”. (McNiff & Whitehead, 2010, p. 196). I believe that the research has created new knowledge regarding the practice of improving students' understanding of function which can contribute to an innovative theory of teaching functions and improving the understanding of function through the integration of mathematics and music. This study explains and justifies the researcher-practitioner's effort, gives an account of what has been done, summarizes the results and places the generated theoretical and practical knowledge in a future perspective.

I have used a non-conventional method designed by myself, appealing on my Year 11 Mathematical Methods students' logical-mathematical, musical, and visual-graphical intelligences in order to improve their understanding of function.

Considering these elements of Gardner's (1983) Multiple Intelligences (MI) theory helped me to expand my teaching repertoire. As Armstrong (2009) describes:

> MI theory provides a way for all teachers to reflect upon their best teaching methods and to understand why these methods work (or why they work well for some students but not for others). It also helps teachers expand their current teaching repertoire to include a broader range of methods, materials, and techniques for reaching an ever wider and more diverse range of learners.
>
> (Armstrong, 2009, p. 56)

The importance of understanding of functions in mathematics is crucial. The function concept can be represented in different ways such as algebraic, tabular, graphical, and verbal, to name the classical traditional representation modes only. Janvier (1985) states that the word “representation” has three different meanings. Firstly, “representation means some material organisation of symbols such as diagram, graph, schema which refers to other entities or 'modelises' various mental processes” (Janvier, 1985, p. 4). The second meaning of the word “representation” is much more abstract than the first one, stating that representation is “a certain organisation of knowledge in the human mental 'system' or in the long-term memory” (p. 4). The third meaning is a special case of the second one, stating that representation is a mental image. Different representations of the function concept can refer to all of these three meanings of the word ‘representation’.

Hitt (1998) states that “a central goal of mathematics teaching is thus taken to be that the students be able to pass from one representation to another without falling into contradictions” (p. 125). In this study I made an effort to present musical representations of different aspects of function teaching my Year 11 mathematics students to pass from traditional algebraic, graphical, and tabular representations of function to musical representation without falling into contradiction, without confusing existing knowledge, but in contrary, empowering their existing knowledge through these musical embodiments.

A range of research articles report possible positive effects of music on mathematical abilities of students of different age. Despite these facts, the active integration of mathematics and music in senior secondary mathematics classes through relevant musical analogies to improve students' understanding of function has not been widely researched. There was no previous or similar research available which would allow me to follow and adapt its approach and structure. I have designed, performed and reported this study accordingly to my thinking, preferences and possibilities. I think that this is the reason that accomplishing this study was a pleasure for me. I hope that other researchers and interested secondary mathematics teachers will consider this study as a valuable resource in the future.

1.1.7 The Objectives of the Study

Within the frame of an action research study I conducted an intervention with my Stage 1 (Year 11) Mathematical Methods students to see if I could enhance their understanding of function, a very important element in the mathematics curriculum. The objectives of the study were to evaluate the effects of integration of mathematics and music on senior secondary (Year 11) mathematics students' (1) understanding of the concept of function, and (2) attitudes and beliefs regarding mathematics. The study focused on finding out if the active integration of mathematics and music had a significant effect on senior secondary (Year 11) mathematics students' understanding of a specially targeted mathematical concept, the concept of function, as well as on students' attitudes and beliefs regarding mathematics.

The first objective of the study, regarding the concept of function has six aspects, which constitute the six mathematical dependent variables of the study. These are in perfect accordance with the six definitions of the Australian Curriculum (AC) 2014 regarding the Functions section of Topic 1 (Functions and graphs) of Unit 1 of the Year 11 Mathematical Methods course described in the Table 1.2 below.

Table 1.2

The Six Mathematical Dependent Variables of the Study

AC Code	Definitions of the Six Mathematical Dependent Variables
ACMMM022	Understand the concept of function as a mapping between sets.
ACMMM023	Use function notation, domain and range, independent and dependent variables.
ACMMM024	Understand the concept of the graph of a function.
ACMMM025	Examine translations and the graphs of $y = f(x) + a$ and $y = f(x + b)$.
ACMMM026	Examine dilations and the graphs of $y = cf(x)$ and $y = f(kx)$.
ACMMM027	Recognise the distinction between functions and relations, and the vertical line test.

(Australian Curriculum, 2014)

The students studied the topic of functions six months earlier than the pre-test of this study. Their knowledge of function faded in time and it became important to review the topic in order to assure further successful learning of mathematics. Students' previous study, their existing knowledge of the topic inclined me towards performing an action research with the goal of improving my Year 11 Mathematical Methods students' existing understanding of function. Students' prior learning have had an effect on their mathematics pre-test and post-test. My goal was to measure the difference between the two tests after the treatment, the improvement of their understanding of function after the treatment, which had its effect on students' post-test in the experimental group. I consider that there would not have been a statistically significant difference between the two mathematics tests of the experimental group without the treatment, as there was no statistically significant difference in the case of the control group. Based on this consideration I attribute the statistically significant improvement of understanding of function of the experimental group to the treatment. The problem addressed was to find out the measure of change (increase or decrease) of students' understanding if I performed a revision of the topic using a non-conventional teaching method during a normal 90 minute double lesson. The chosen non-conventional teaching method was the active integration of mathematics and music through the use of nine author-designed musical analogies of different aspects of the function concept (described later). The six aspects of the function concept, or the six learning outcomes on function, considered as mathematically dependent variables of this action research study, were used in accordance with their respective Australian Curriculum 2014 codes:

ACMMM022, ACMMM023, ACMMM024, ACMMM025, ACMMM026 and ACMMM027, as described in Table 1.2 above.

1.2 Research Questions

The study has targeted the following two research questions:

1. Does the active integration of mathematics and music by using these nine analogies between different aspects of function and the corresponding aspects of music make a statistically significant improvement in my senior secondary mathematics students' understanding of function?

2. Does this type of integration of mathematics and music make a statistically significant improvement in my senior secondary mathematics students' attitudes and beliefs regarding mathematics?

The main research aim and purpose of this action research project, as expressed also in the title of the study, is the first question; the second question being considered as an auxiliary question. As stated previously, the first question has been formulated as a consequence or as a mode of measuring the original aim of this research project. The original aim of this action research project can be considered as the initial action research question of this study, namely, how I could improve my own teaching practice of function. The original challenge was to observe what I was doing and how I could improve the quality of my practice. This approach inclined me to formulate the above main research question and to consider a self-study of my own actions in the form of an action research as opposed to a traditional research. As McNiff and Whitehead (2010) state:

> In traditional social science research, the researcher stands outside the research situation and observes what other people are doing. They therefore adopt an outsider or spectator perspective. In action research, the researcher becomes the centre of the research. The focus is on the improvement of personal learning, so they adopt an insider, self-study perspective.
>
> (McNiff & Whitehead, 2010, p. 11)

I was not standing outside of my research situation to observe what other people were doing, adopting an outsider or spectator perspective, but rather myself, my actions and my ideas regarding the design of the nine relevant analogies and the whole research became the focus of the research. The effectiveness of these actions and introduced ideas in this study was measured in terms of any increase in my Year 11 mathematics students' understanding of function.

1.3 Rationale for the Study

If we consider the educational aspects of the possible positive effects of integrating mathematics and music, then we can observe on one side that in pre-schools and children’s educational TV shows, a new knowledge or story usually is presented through integrating verbal language, visual models, music and dance, which all are different intelligences in Gardner’s (1983) Multiple Intelligences theory. Conversely, this type of integrated mode of teaching by using a range of combined human intelligences, or by integration of different subjects, appears to be rarely used in schools and colleges, especially as the “efforts to integrate mathematics with music are rare” (Kleiman, 1991, p. 1). Regarding mathematics education, this means that in schools and colleges a new mathematical concept, that is, the concept of function, is usually introduced through using mainly the logical-mathematical intelligence channel. Other intelligence channels, excluding the verbal-linguistic and the visual-spatial ones, are rarely used. Because students possess different mixtures of skills, integrated teaching approaches can be employed to empower the understanding of mathematics by activating different intelligences. Students’ individual learning preferences can be addressed by a range of relevant teaching methods, and in the development of these methods Gardner's (1983) Multiple Intelligences theory is highly applicable and useful also in practice (Gardner, 1993).

Because students learn in different ways, using different intelligences, often more intelligences simultaneously, Munro (1994) suggests the followings for the teaching of mathematics:

> **Present mathematics ideas in a range of ways**. When teachers are teaching an idea, they can vary how they present it; in words either visually or auditorily, in pictures or as a series of actions. These things are under the control of teachers; they can select the input format.
>
> **Cueing students to think about the idea in different ways**. Teachers can't control how students think about the ideas being learnt. Even if ideas are presented visually, this does not mean that the students will visualize images.
>
> **Switching ways of thinking about ideas**. Building ideas in one representational format doesn't suit all students. Many students can be helped to learn an idea when they are cued to think of it in different ways.
>
> **Encouraging students to monitor how they learn best and to understand their preferred ways of learning**. Students can learn to understand and value their own approach to mathematics learning, to see that they can learn successfully.
>
> (Munro, 1994, p. 9)

My study has considered Gardner’s (1983) Multiple Intelligences theory, and Munro's (1994) suggestions above, mainly the first. The study has been designed in an effort to integrate mathematics with music, present the concept of function in four different ways (embodiments), and use multiple intelligences such as mathematical-logical (through algebraic and tabular representations of functions), musical (through audible representations of different aspects of functions by relevant author-designed musical analogies) and visual-spatial (through graphical representations of functions).

Wahl (1998) presents 14 activities and over 90 sub-activities of teaching mathematics up to school year level 10 through the original seven basic intelligences of Gardner's (1983) Multiple Intelligences theory. These activities are well presented, adaptable to students' various knowledge levels, empower both individual and group work, and are mathematically meaningful up to school year level 10. These activities, as models, were a source of inspiration for my study, which was designed to be well presented, adaptable, empowering both individual and group work, and mathematically meaningful at Year 11 mathematics level.

I believe that this research study has expanded the theoretical and practical knowledge accumulated in Wahl's (1998) publication by targeting a higher school year level, Year 11 mathematics, and the concept of function through the active integration of mathematics and music. This study also addressed the mathematical-logical, visual-graphical, verbal and inter-personal intelligences explicitly, which together with the musical intelligence made a total of five out of the original seven basic intelligences of Gardner's (1983) theory. The two not explicitly addressed intelligences being the kinaesthetic and the intra-personal intelligences.

My study has considered the power of the relationships between mathematics and music in a unique, particular way, through using nine relevant musical analogies in order to describe in a non-traditional, musical-auditory way the six different aspects of the function concept explicitly requested by the Year 11 Mathematical Methods Australian Curriculum 2014.

The Hungarian born world-famous mathematician and cognitive psychologist Zoltán P. Dienes (1916-2014), who created the field of Psychomathematics, the psychology of mathematics learning, worked around the globe, in England, France, Germany, Italy, Australia (mathematics education professor at The University of Adelaide in the late 1950s and early 1960s), New Guinea, USA, Chile, Brazil, Argentina and Canada, has had an important contribution to the 20th century's mathematics education. Dienes (1987) has developed a sequence of different structured activities, named lessons, integrating music and literature to teach both explicit- and hidden mathematics, confirming that learning real mathematics has to be an enjoyable process, and has used innovative games, songs, poetry and dance to teach abstract mathematical concepts. I was inspired by this approach to mathematics teaching, being curious how these ideas might apply in practice. I was inclined towards finding answers to questions such as how does it work, how effective it is using games, music and dance during mathematics lessons?

In order to have a deeper inside look into these type of questions, to check how this works in my mathematics teaching practice, I initiated, designed and coordinated a non-conventional study, titled "*Dancematics*" (Nagy, 2008), involving a small group of four Year 11 mathematics and dance students, aiming for the integration of mathematics and dance. The "*Dancematics*" project was presented by a student at the South Australian Mathematics Talent Quest 2008 (SAMTQ 2008), organised by The Mathematical Association of South Australia in 2008, and has been awarded a "Judges Commendation for Independent Initiative in Exploring Maths in Dance". At that time I was not prepared to perform a research study by posing an appropriate research question and collecting and analysing relevant data. Despite the fact that there are a range of auditable documents which have been prepared by the "*Dancematics*" project, there are recorded videos of the mathematics presentation by the student on a DVD, and the success of the project at SAMTQ 2008 was published in the Term 4 Newsletter 2008 of The Mathematical Association of South Australia, I, as the coordinator of the project, did not consider it a proper research study.

The creator of the Six-stage Theory of Learning Mathematics, Dienes (2004) identified an analogy between interpreting music and teaching mathematics, and also posed a question regarding finding other ways to teach the beauties of mathematics, stating that:

> The concert pianist makes gigantic efforts to interpret the amazing complexity of a Beethoven sonata to the lay public by decoding an abstruse symbol system and transforming it into an enjoyable artistic experience. Why could we not find ways to interpret the beauties of mathematics to the lay public through the methods more accessible to the general public?
>
> (Dienes, 2004, p. 12)

An earlier dated, but a similar statement is in Dienes (1966): "Just as music is accepted as a valuable part of the general education of all, with the few excelling, so in mathematics everyone can reach enjoyment, receiving its long-valued stimulation and satisfaction" (p. 14).

Dienes observed that “the reason mathematics is boring in schools is because no real mathematics is taught in schools” (Dienes, 25/04/2006, as cited in Sriraman & Lesh, 2007, p. 67).

Bálint (2015) summarised Dienes' work and states that the “new mathematics” is the mathematics of structures. Teaching the new mathematics is not about teaching a sequence of knowledge but to help students to develop their authentic mathematical mode of thinking, to form mathematical structures mentally. Dienes has developed games which made possible to observe, experiment, understand and represent these structures. Representing these structures is possible through multiple embodiments of the abstract relationships.

I made an effort in this study to find a way to interpret the beauties of mathematics, to teach “real mathematics” through a method which I believe was really appreciated by the participating students: using structural analogies between mathematics and music. The present study was inspired by Dienes’ (1960, 1966, 1987, 2000a, 2000b, 2001, 2004) general work, including the above mentioned analogy between interpreting music and teaching mathematics and the possibility to reach students' enjoyment in mathematics. Instead of developing a game, which would have been a 3D model, I have developed analogies which show similar structures between function and music. My approach was one of many possible embodiments, using music. I have embodied the multiple facets of a mathematical abstract concept in music. I believe that this study fully confirms Dienes’ (2004) findings and statements in a particular way.

The American cognitive psychologist Jerome S. Bruner (1915–2016), who made fundamental contributions to cognitive learning theory in educational psychology in the 20th century, emphasises the role and importance of both the teacher and the mode of teaching. Bruner (1960) states, that:

> The teacher is not only a communicator but a model. Somebody who does not see anything beautiful or powerful about mathematics is not likely to ignite others with a sense of the intrinsic excitement of the subject. A teacher who will not or cannot give play to his own intuitiveness is not likely to be effective in encouraging intuition in his students.
>
> (Bruner, 1960, p. 90)

I have seen and appreciate the beauty and power of mathematics since decades and I think that I have ignited many of my students with a sense of excitement of mathematics and have encouraged the intuition of many of my students effectively. Bruner (1960) states that "The effort to give visible embodiment to ideas in mathematics is of the same order as the laboratory work" (p. 81). I made an effort to give both visible and audible embodiments of different aspects of function in this study and I confirm Bruner's statement. The effort to give visible and audible embodiments was a very pleasant and exciting work, similar to a pleasant and exciting laboratory work.

Learning mathematics is a long, complex process which can be aided, in part, by some non-mathematical approaches. One of the possible non-mathematical aids is using different skill transfer effects between music and mathematics (McKeachie, 1987). My study has been designed to measure, firstly, the change (increase or decrease) of the cognitive skills of understanding of function, and secondly, the change of non-cognitive skills of beliefs and attitudes towards mathematics at senior secondary level, when the process is aided by skill transfer effects between music and mathematics. In order to perform this action research, I designed and presented nine musical analogies – musical embodiments – of different aspects of function using "a multiple embodiment approach – introducing the function concept in a variety of settings with the hope of effecting transfer of learning" (Eisenberg, 2002, p. 141).

Regarding the possible positive effects of music on mathematics achievement, Kelstrom (1998) stated that these have been poorly employed in the curriculum. Rauscher (2003) concludes that "music may act as a catalyst for cognitive abilities in other disciplines" (p. 4). I believe that my study confirms Rauscher's (2003) statement in a particular way, with regards to senior secondary mathematics students' cognitive abilities concerning the mathematical concept of function.

In their study about common misperceptions in learning mathematics and music and the interactions of these disciplines, Malone, Leong and Lamb (2001, p. 6) state that “music provides the reality and physical experiences corresponding to mathematical abstractions”. I believe that my study totally confirms Malone et al.'s (2001) observation because my study did show through analogies that music provides the reality and physical audio experiences corresponding to mathematical abstractions of different aspects of function.

Using analogies in teaching can be beneficial in observing inter-concept similarities, understanding new concepts and in improving attitudes (Treagust, 2001). I believe that, by targeting the concept of function, using analogies and observing inter-concept similarities between different aspects of the mathematical concept of function and the similar aspects of the musical concept of melody, my study also confirms and fully supports Treagust’s (2001) findings.

1.4 Significance of the Study

This study has the following four levels of potential significance: (1) for secondary mathematics teachers, and may have be important for the Australian Curriculum; (2) for the body of research on senior mathematics teachers' knowledge and practice of active integration of mathematics and music; (3) for future students, their understanding of function and maybe their enjoyment of mathematics; and (4) for my knowledge and practice of secondary mathematics teaching.

First, the study has a potential significance for secondary mathematics teachers, and may have be important for the Australian Curriculum. The designed mathematical-musical analogies, the created relevant graphs and tables, the considered software tools and musical examples, the whole study, form a concise and precise mode of active integration of mathematics and music in order to improve senior secondary (Year 11) students' understanding of function at senior secondary level throughout Australia and around the world. The study has also the advantage that the interested secondary mathematics teachers do not need to be musically educated. The study's approach can be used in any secondary school, regardless of whether students are musically educated or not. The theoretical knowledge created by this study is that the teaching method using this type of active integration of mathematics and music improves senior secondary students' understanding of function and that the practical knowledge created by the study is the vehicle for how this can be achieved. Both the theoretical and practical knowledge created by the study, together with the above named parts of the study are capable of being used successfully by secondary mathematics teachers and researchers. The study has the potential to be considered by the Australian Curriculum which, in order to enhance students' broader way of thinking, encourages the use of diverse analogies and creative subject integrations.

Second, I believe that this study has the potential to contribute to the body of research on senior mathematics teachers' knowledge and practice of active integration of mathematics and music and ways of presenting non-conventional analogies in senior secondary mathematics classrooms. As well, the study presents a way of improving students' understanding of function. This study's way of presentation also may contribute to positively change students' beliefs and attitudes regarding mathematics, however, this question usually needs consistent teacher effort and creativity over a long period of time.

Third, I believe that the study has significance for future students, their understanding of function and maybe their enjoyment of mathematics. The statistical evidence suggests that the performed treatment made a statistically significant improvement in my students' understanding of the concept of function. I believe that the students also have enjoyed the activities, the musical examples, and the presented musical-mathematical analogies. Due to ethical reasons the treatment was not recorded, but the students, as a sign of their enjoyment, reacted enthusiastically during the intervention. Students' statistically significant improvement of understanding of function as well as students’ positive reactions during the intervention are evidences of the positive effects of the study on my students’ mathematical knowledge and feelings regarding mathematics.

Finally, the study gave a new meaning to my knowledge and practice of secondary mathematics teaching work. It has deepened my knowledge of teaching the concept of function and improved my practice of presenting and applying non-conventional methods in the secondary mathematics classroom. The planned actions, the practical steps, the designed relevant mathematical-musical analogies, the selected informative musical pieces, the design of the mathematics test and its marking scheme, the three computer software tools introduced, and the statistical analysis methods of this study have re-empowered and re-confirmed the fact that teaching mathematics is a wonderful activity, which can connect successfully very different worlds such as abstract mathematical-logical constructs and beautiful musical pieces in a natural, understandable and productive way. The study gave me new knowledge of my teaching practice regarding how to improve my secondary students' understanding of function. The study has empowered the way of my thinking, regarding how mathematical concepts can be taught by using non-traditional, non-conventional analogies.

1.5 Overview of the Study

This thesis has five chapters as follows. Chapter 1: Introduction to the study; Chapter 2: Literature review; Chapter 3: Methodology; Chapter 4: Results, Analysis and Discussion; and Chapter 5: Conclusions.

Chapter 1 presents preliminary information to the study regarding my background, the role of mathematics and music in my life, and my teaching practice, which all have indirectly determined this study. I also give the definition of the concept of active integration of mathematics and music as it is interpreted in the study, followed with the description of the selection process of a suitable research design, the formulation of the research proposal and the objectives of the study. The two research questions, the rationale and significance of the study concludes Chapter 1.

Chapter 2 commences with a structured overview of the considered research articles, categorized as interventional, correlational or meta-studies. Chapter 2 then describes the relevant comprehensive literature of the multifaceted relationships between mathematics and music, historically, from ancient times. Chapter 2 presents the literature of the effects of music on mathematical skills of students of different ages. The chapter also provides the literature of the history and importance of the function concept, together with the literature of the learning problems related to function. A short review of the relevant literature regarding the integration of mathematics with other forms of art, like dance and poetry, are also presented. Also described briefly is the literature regarding different non-musical approaches to improve students' understanding of function. The relevant literature is presented in such a way to allow the reader to observe how this study fits in and adds to the literature base along with the significance of the study within the literature base.

Chapter 3 describes the framework of the study, including the paradigm, methodology, ethics and validity considerations. The six mathematical dependent variables of the study, the six aspects of function, are presented as they are described in the Australian Curriculum 2014. Chapter 3 describes the instruments used:

(1) A mathematics test, which has been designed by me based on my students' textbook, Haese Mathematics (2015), and on the requirements of the Australian Curriculum 2014;

(2) Three questionnaires with proven validity and reliability:
- Mathematics Related Beliefs Questionnaire (Op't Eynde & De Corte, 2003),
- Test Of Mathematics Related Attitudes (Fraser, 1981; Taylor, 2004), and
- What Is Happening In This Class (Fraser, 1998);

(3) Three software tools:
- Virtual Piano (2015) by Crystal Magic Studio Ltd, London, England,
- Original Sine (2014) by The University of Adelaide, Australia, and
- Music Speed Changer (2012) by Harald Meyer Mobile Software, Austria.

(4) Nine author-designed musical analogies of different aspects of function which constitute the core idea of this study.

Chapter 3 also presents the data collection methods and the statistical data analysis considerations.

Chapter 4, Results, Analysis, and Discussion, describes the results of the research study, presents the tables generated by the statistical analysis tests and discusses these results. The Null Hypotheses of the study are stated, together with the alpha values, set a priori. Then the results of the mathematics tests of both groups are presented and commented. The requirements of the t-test statistical analysis are discussed, then the statistical test results of the mathematics tests are presented, including the results of the (1) Independent t-test for two samples, (2) Analysis of the group statistics, (3) Analysis of the mathematics achievement of the experimental group, (4) Analysis of the mathematics achievement of the control group. These statistics are followed by the presentation and statistical analysis of the collected non-cognitive data, which includes: (1) the main statistics of the non-cognitive scales of the experimental group, and (2) the main statistics of the non-cognitive scales of the control group. After these statistics, Chapter 4 contains the discussion of these results and, as a result, the statements of rejection and acceptance of the Null Hypotheses.

Chapter 5 describes the conclusions and implications of the research study. The chapter provides a summary and advantages of the designed and actively integrated nine relevant analogies. The chapter also summarizes the links to previous research studies and also describes the limitations of the study. Chapter 5 concludes with recommendations for further research and final conclusions.

Chapter 2
LITERATURE REVIEW

2.1 Introduction. Structured Overview of the Research Articles

In order to have a structured overview of the literature I have categorized the articles as interventional studies, correlational studies, and meta-studies. The sub-categories are: general and specific. The difference between these two being whether general mathematics knowledge and skills, or concept specific knowledge and skills were researched. The present study is a specific interventional study. It is interventional because an intervention, a treatment is involved and it is specific because the concept of function is targeted specifically.

Because this study is about improving senior secondary (Year 11) mathematics students' understanding of function through active integration of mathematics and music, there is a need to briefly review the literature regarding the:

- history and the importance of the concept of function,
- improvement of secondary students' understanding of function in a traditional way,
- relationships between mathematics and music, and the effects of music on mathematics,
- integration of mathematics and music in mathematics classrooms, and
- improvement of secondary students' understanding of mathematics through music.

Table 2.1 presents a list of the interventional studies considered which targeted the concept of function. The articles are listed in chronological order, grouped by mathematics level (primary, secondary, tertiary) and music integration (yes, no).

Table 2.1

Interventional Studies Targeting Mathematical Concepts

Article reference	Maths level	Mathematical concept / topic	Music integration	Sample size	Methodology/ Comment
Nisbet & Bain (1998, 2000)	Primary, aged 10-11	Line graphs	Yes	51	Quasi-experimental
Willis & Johnson (2001)	Primary	Multiplication	Yes	N/A	Mixed
Johnson & Edelson (2003)	Primary	Patterns, sorting, ratio, fractions	Yes	N/A	Experimental, Mixed
Evans (2009)	Pre-school, aged 3-7	Recognising numbers, counting, different shapes	Yes	18	Mixed
An et al. (2013)	Primary	General abilities of modelling, strategy and application	Yes	46	Quasi-experimental
Vinner, (1983)	Secondary	Concept image	No	N/A	Case study
Borba & Confrey (1996)	Secondary1 6 years old	Function transformations	No	1	Case study. Computer based multi-representation.
Hitt (1998)	Secondary	Function	No	30	Quantitative. Secondary Mathematics Teachers.
Zazkis et al. (2003)	Secondary	Horizontal translation by -3 of the quadratic function $y = x^2$	No	41	Quasi-experimental. 10 students, 15 teachers, and 16 pre-service teachers.
Sever & Yerushalmy (2007)	Secondary	Function dilation	No	2	Case study. Qualitative.
Bingham (2007)	Secondary	Function transformations	No	7	Qualitative research
Anabousy et al (2014)	Secondary Year 9	Function transformations	No	19	Quantitative. Geogebra software.
Steketee (2015)	Secondary	Function	No	N/A	Mixed
Thomas (2003)	Tertiary	Function	No	34	Case study
Evangelidou et al. (2004)	Tertiary	Function	No	164	Qualitative
Lage & Gaisman (2006)	Tertiary	Function transformations	No	158	Mixed
Faulkenberry (2011)	Tertiary	Function transformations	No	26	Mixed

Table 2.2 below presents a list of the correlational studies considered which analysed the correlation between music and mathematics without any explicit integration of mathematics and music. The articles are listed in chronological order.

Table 2.2

Correlational Studies Regarding Music and Mathematics

Article reference	**Mathematics level**	**Correlation**	**Sample size**	**Methodology / Comment**
Rauscher, Shaw & Ky (1993)	Secondary	Music ↔ Spatial temporal reasoning intelligence	36	Psychological tests. Mozart effect study.
Nisbet & Bain (1998, 2000)	Primary, aged 10-11	Melodies ↔ Line graphs	51	Experimental.
Cheek & Smith (1998)	Secondary, Year 9	Music ↔ Mathematics	113	Iowa test of basic mathematics skills.
Vaughn (2000)	Primary, Secondary	Music ↔ Mathematics	N/A	Three meta-analyses.
Vaughn & Winner (2000)	Secondary	Arts ↔ Mathematics Arts ↔ Language	535,818 - 3,345,430	Analysis of USA SAT scores. Meta-analyses.
Still & Bobis (2005)	Primary	Music ↔ Mathematics	N/A	Case study on primary teachers.
Harris (2005, 2007)	Pre-school, aged 3-6	Music ↔ Mathematics	200	Experimental design. Montessori method.

2.2 Relationships Between Mathematics and Music

Since more than 2500 years, from Pythagoras' time, a considerable amount of mainly anecdotal evidence and beliefs have accumulated suggesting that there are deep relationships between mathematics and music. Because mathematics and music are two universal languages, two ways of expression of patterns of our human rational thoughts and emotional feelings, we can observe that these relationships are very complex.

The two universal languages, mathematics and music, are expressing concepts in different ways. The challenge to thinkers (mathematicians, musicians) is to observe the analogies between these different expressions. The Polish mathematician, Stefan Banach (1892-1945), the founder of the modern functional analysis, one of the most important mathematicians of the 20th century, stated that “Good mathematicians see analogies. Great mathematicians see analogies between analogies”. The mathematical concepts of function and relation have their musical analogies. I have seen (and heard) nine musical analogies of these important mathematical concepts and have presented them in this study. I am not sure when I will see (and hear) analogies of these analogies...

Still and Bobis (2005) observe that:

> Mathematical qualities are also inherent in other aspects of music, such as rhythm, tempo and melody. When it comes to the written recording of both of the disciplines, the use of clefs, quavers, stave and bar lines are the internationally-recognised symbols for music, whereas, for mathematics it is numbers, signs of equality/inequality and algebraic notation.
>
> (Still & Bobis, 2005, p. 712)

Visintin (2017) gives a summary of the four basic elements or aspects of music:

> In western music one usually distinguishes four basic elements: (i) the melody = the sequence of sounds that gives a musical meaning to a composition; (i) the harmony = the notes that are superposed to the melody (i.e., the accompanying notes); (This provides a sort of background for melodic developments.) (iii) the rhythm = the time structure of the composition. (iv) the timbre = the character or quality of a musical sound, given by the musical instrument or voice that produces that sound. (As was observed earlier, this corresponds on how the energy is distributed among the multiples of the fundamental frequencies.) Loosely speaking, the melody is sequential, whereas the harmony is simultaneous.
>
> (Visintin, 2017, p. 5)

Out of the five aspects of music mentioned by Still and Bobis (2005) and Visintin (2017) – melody, harmony, rhythm, tempo, and timbre – my study has considered the mathematical expression modes of melody, harmony, and tempo which are written in internationally recognised symbols of mathematics, and can contribute to improve students' appreciation of both mathematics and music. Using the other two aspects of music – rhythm and timbre – in order to improve students' mathematical understanding would be a recommended good topic for future research.

Due to the nature of mathematics and music, in order to study the relationships between them we often need the help of Physics, especially Acoustics which has a major role in studying these relationships. The physical concepts of frequency and amplitude are just two very important examples to observe the importance of Physics in this topic.

2.2.1 Ancient Times. Pythagoras, Plato, Aristoxenus, Euclid

> "Mathematics is God" (Pythagoras)
>
> "Music gives a soul to the universe, wings to the mind, flight to the imagination, and life to everything" (Plato)

The ancient Greek philosophers have classified the main forms of mathematics, the collection of the topics or objects of thinking, as discrete and continuous. They have considered music as the discrete relative form of mathematics, while arithmetic was considered as the discrete absolute form of mathematics. This classification of mathematics has considered geometry as the continuous static form of mathematics while astronomy as the continuous moving form of mathematics (Garland & Kahn, 1995). Later, in the middle ages, also grammar, rhetoric and logic were considered as forms or branches of mathematics, in this way arithmetic, music, geometry, astronomy, grammar, rhetoric and logic became the "seven liberal arts" (Hartfeldt, Eid, & Henning, 2002). We can see that arithmetic and music were considered as having relationships, being different types of expression of a common knowledge.

Our understanding is that the ancient Greeks, Pythagoras and his follower thinkers, without knowing the concept of frequency, have observed and explored first the relationships between mathematics and music by performing their experiments and describing the relations between the lengths of a vibrating string and the corresponding musical sounds. Pythagoras (c. 570-495 BC), known for his mathematical theorem regarding the relation between the lengths of the sides of any right angled triangle, has observed that the pitch of a musical sound is inverse proportional with the length of the vibrating string producing the sound. Pythagoras and his follower thinkers, named in common the Pythagoreans, stated and believed that "all is number". The Pythagoreans, who made their influence on thinkers like Euclid (the "father of geometry", who wrote the "*Elements*", and also wrote texts on music), Plato, Aristotle and through them on the Western philosophy, logic and mathematics, have observed that halving the length of a vibrating string always gives a musical sound which is an octave higher than the original musical sound. The Pythagoreans have explored which ratios of a unit length string produce pleasant (consonant, as opposed to non-pleasant or dissonant) harmonies with the original sound produced by the vibrating unit length string. They have observed and stated that these pleasant (consonant) harmonies are produced by the ratios 1:1, 4:3, 3:2, 2:1. The first number in these ratios is expressing the number of equal parts of the unit string, the second number is expressing how many parts we vibrate. For example, 4:3 means that we divide the unit string into 4 equal parts (which was easy to do more than 2500 years ago as well!) and vibrate not the whole unit string, but just 3 parts out of 4. If we think about the bounds of a guitar then we can understand the idea of this approach. The same idea is applied also for the other Pythagorean ratios. These four ratios express harmonies corresponding to, as we know them today, the unison, perfect fourth, perfect fifth, and octave harmonies or harmonic intervals. There are also other ratios which correspond to pleasant (consonant) harmonies, e.g. the ratio 5:4 corresponds to the "major third" pleasant (consonant) harmony.

The Pythagoreans, based on their belief that "all is number", being interested also in number mysticism, have extended their discoveries regarding the analogies between mathematical ratios and musical harmonies to the whole universe. They wanted to express the whole universe by numbers, stating different analogies, e.g. measuring or estimating the positions and distances between different stars and attributing them corresponding musical sounds and harmonies, imagining in this way a type of universal music, the music of spheres.

We can see the importance of different analogies, which have challenged and improved the knowledge, understanding and imagination of these ancient thinkers. This study has been designed based on self-developed analogies between mathematics and music with the goal to challenge and improve my senior secondary (Year 11) mathematics students' knowledge, understanding and imagination of the concept of function.

As mentioned above, music was classified as the discrete relative form of mathematics and arithmetic as the discrete absolute form of mathematics. This classification can be exemplified as follows. The arithmetic ratios 1:1, 4:3, 3:2, 2:1 are discrete absolute objects of thinking, while the corresponding pleasant (consonant) musical harmonies: unison, perfect fourth, perfect fifth, and octave are discrete relative. The statement that the ratios 1:1, 4:3, 3:2, 2:1 are discrete absolute, is evident, as it is the statement that the musical harmonies are discrete. We only need to observe that the musical harmonies are relative. This is explained, observing that all of these musical harmonies can be build relative to any musical sound. For example, all the following couples of musical sounds are perfect fifth musical harmonies: (C, G), (D, A), (E, B), (F, C), (G, D), (A, E).

The perfect fifth pleasant (consonant) musical harmony, also known as the “pure quint”, represented by the arithmetic ratio of 3:2, has been used to tune different stringed instruments (violins, harps, guitars, clavichords, pianos, and so on) for many centuries. This mode, known as “Pythagorean tuning”, was a natural and easy way of tuning chorded instruments by ear. Harp players, for example, needed to know how to tune a series of perfect fifths only. Pythagoras has extended this mode by using also the octave pleasant (consonant) musical harmony, represented by the arithmetic ratio of 2:1. Both musical harmonies, the perfect fifth and the octave offered a natural and easy way to tune stringed instruments by ear. But a theoretical and as a result a practical problem occurred, namely that starting from any sound, no sequence of perfect fifths built on that sound fits exactly any sequence of perfect octaves built on that sound. The smallest difference is between twelve perfect fifths and seven octaves, which are very close to each other, but still there is a small difference between them. This difference is named the Pythagorean comma or the Pythagorean interval. This discrepancy was discovered by Pythagoras himself and also the Pythagoreans worked on this problem, but they have not succeeded in solving it. The practical consequence of this deficiency was that musical instruments had to be tuned differently for each musical key. This is hard to believe today, but at that time if the music pieces were in different music keys (scales), e.g. in C-major key and in F-major key, then in order to play them on a given instrument, that instrument had to be tuned differently, two times, accordingly to the considered music keys (scales). The theoretical (mathematical) and as a result the practical (tuning of instruments) problem was solved about 2000 years later with the development of mathematics and the introduction of the well-tempered tuning mode. Before the development of the theoretical and practical aspects of the well-tempered or chromatic tuning mode there were a range of development phases, for example the just intonation mode, which were interesting intellectual endeavours, but these are not the objects of this study.

As the Pythagoreans have made a unique correspondence between mathematical fractions and the pleasant (consonant) musical harmonies, we can state that the Pythagoreans made the first active integration of mathematics and music by designing the first known musical analogies of the mathematical concept of ratio. My study presents a different type of active integration of mathematics and music through nine self-designed musical analogies of different aspects of the concept of function.

If we use the musical analogies of fraction, created by the Pythagoreans, and go one step further, considering the multiplication of fractions, then we can observe the musical analogy of the mathematical operation of multiplication of fractions. For example, the product of the mathematical ratios 4:3 and 3:2, corresponding to the perfect fourth and the perfect fifth musical pleasant (consonant) harmonies, is $(4:3) \times (3:2) = 4:2 = 2:1$. We can observe, that the result, the ratio 2:1, is expressing a musical harmony. As a result, the musical analogy of multiplying these two fractions and simplifying the result, and obtaining a different fraction, is considering a perfect fourth harmony then extending it by a perfect fifth harmony, which is resulting in a different harmony, an octave harmony! The musical analogy of the operation of multiplication of fractions is extending a musical harmonic interval by another musical harmonic interval, obtaining a musical harmony of two sounds. This can be easily demonstrated on a piano.

Based on the observations above, the arithmetical analogy of the Pythagorean comma, the problem of the Pythagorean tuning mode, observed by Pythagoras himself, can be given as follows. A sequence of twelve perfect fifths is represented mathematically by the product $^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 \times {}^3/_2 = ({}^3/_2)^{12} = 129.7463379$. The sequence of seven octaves is represented mathematically by the product $^2/_1 \times {}^2/_1 \times {}^2/_1 \times {}^2/_1 \times {}^2/_1 \times {}^2/_1 \times {}^2/_1 = ({}^2/_1)^7 = 128$. We can observe that $({}^3/_2)^{12} \neq ({}^2/_1)^7$, because $({}^3/_2)^{12} = 129.7463379 \neq 128 = ({}^2/_1)^7$. This inequality is expressing arithmetically the discrepancy of the tuning mode introduced by Pythagoras. Obviously this, by itself, is also an analogy between mathematics and music, even if it presents a non-wanted discrepancy.

These type of analogies between mathematics and music can be created and presented in primary and middle school mathematics classrooms. The present study targeted to design, create and present musical analogies not of fractions or operations with fractions, which are primary and middle school level mathematics topics, but of different aspects of the concept of function, which are secondary school level mathematics topics.

Despite the fact that the Pythagoreans have not used the concept of frequency, we know today, that these ratios express the ratios of frequencies of the sounds forming these consonant harmonies. The 1:1 ratio is expressing that frequency of the first sound equals the frequency of the second sound. The same musical sound is produced two times, which means that we have the unison harmony, e.g. we have the (C, C) couple of sounds. The 4:3 ratio means that 4 × frequency(sound-1) = 3 × frequency(sound-2), or in words, that we produce two musical sounds, where the second musical sound has a higher frequency, obtained by multiplying the frequency of the first musical sound by $^4/_3$, e.g. we have the couple C and $^4/_3$ × frequency of C = F, or the (C, F) couple of musical sounds, the perfect fourth consonant harmony. The 3:2 ratio means that 3 × frequency(sound-1) = 2 × frequency(sound-2), or in words, we produce two musical sounds, where the second musical sound has a higher frequency, obtained by multiplying the frequency of the first musical sound by $^3/_2$, e.g. we have the couple C and $^3/_2 \times$ frequency(C) = G, or the (C, G) couple of musical sounds, the perfect fifth consonant harmony.

Plato (428-348 BC), the great philosopher and founder of the first university institution in Europe, the Academy in Athens, being influenced by the Pythagoreans also has considered that music is a form of mathematics and believed that mathematics is the key to the universe.

Aristoxenus (375-335 BC) was the first ancient philosopher who criticized the Pythagorean view regarding music, arguing that music is not a part of mathematics but it is an autonomous discipline. Aristoxenus is considered being the father of musicology. Kontossi and Raducanu (2010) state that:

> In the *Harmonics Elements* Aristoxenus discusses music in a scientific way. He creates an independent science for the field of harmonics and divides musical knowledge into distinct subjects. He brings together all the elements of earlier scholarship, which he organizes and judges. Applying the Aristotelian scientific doctrine to the subject he defines the elements of its science, he gives a complete description of musical phenomena, setting out from the simplest of entities (musical sound) and proceeding to increasingly complex combinations of intervals and 'systems', thus justifying his significant role as an innovator of the discipline of musicology.
>
> (Kontossi & Raducanu, 2010, p. 43)

2.2.2 Middle Ages. Boethius, Fibonacci

The historical Middle Ages or Medieval Period is considered to be the time period between the 5th and 15th century, started with the fall of the Roman Empire and ended with the emerge of the Renaissance. During these centuries the Chinese, Indian and Arabic mathematicians were the most important thinkers. The European thinkers have been much more interested in philosophy, theology, metaphysics and literature. Leonardo da Pisa's (son of Bonacci, or Fibonacci, born in 1170) book, titled "*Liber Abaci*", published in 1202, revised in 1228, containing arithmetic and algebra, the knowledge of the Indian and Arabic mathematicians, and Fibonacci's other book, titled “*Practica Geometriæ*“, published in 1220, containing geometry, together with Fibonacci's own mathematical contributions, constituted the most important sources of mathematics in Europe for three centuries.

> European knowledge and study of arithmetic, geometry, astronomy and music was limited mainly to Boethius’ translations of some of the works of ancient Greek masters such as Nicomachus and Euclid. Europe’s first great medieval mathematician was the Italian Leonardo of Pisa, better known by his nickname Fibonacci. Although best known for the so-called Fibonacci Sequence of numbers, perhaps his most important contribution to European mathematics was his role in spreading the use of the Hindu-Arabic numeral system throughout Europe early in the 13th Century, which soon made the Roman numeral system obsolete, and opened the way for great advances in European mathematics.
>
> (Source: http://www.storyofmathematics.com/medieval.html)

In the early Middle Ages the main musical expression form was monophonic, having a single melody line. The characteristic sacred vocal music, such as Gregorian chants, and other songs were sung unaccompanied, due to strict church restrictions. Later, parallel melody lines were added to the Gregorian chants, and the musical expression form became polyphonic, sung by church choirs and accompanied by instruments. These two forms of music, monophonic and polyphonic are used in my study to present the musical analogies of the mathematical concepts of function and relation. I have designed the musical analogy of the Vertical Line Test for Function, a required mathematical knowledge in Year 11 Mathematical Methods curriculum, see Analogy 3 in Chapter 3, and the participant students were asked to test these two aspects of music.

2.2.3 Modern Times. Galilei, Kepler, Mersenne, Descartes, Euler, Bach

Galileo Galilei's father, Vincenzo Galilei, an Italian mathematician, musician, composer and music theorist, designed a tuning system of equal intervals in 1581. Zhu Zaiyu, a Chinese prince and musicologist, wrote an equal temperament system in 1596. Equal temperament was approximated at a certain extent, but the theory and practice of tuning stringed musical instruments using the Pythagorean tuning method has been changed in the 17th century.

Johannes Kepler (1571-1630), in his series of five books, titled "*Harmonices Mundi*", published in 1619, describes the harmony of the world starting with a mathematical music theory, then generalising to cosmology of heavens and the earth. Hartfeldt et al. (2002) state that Kepler formed the mathematical-cosmological foundation of music theory through defining the ratios of the consonant (pleasant) harmony intervals based on constructability of regular polygons, then observing the same ratios in the planetary orbits. Based on these common ratios, describing the harmony of the world, Kepler defined analogies between music and the motion of planets, and gave a mathematical formula which attempts to describe the music of spheres. The importance of analogies in the history of human thinking can be observed.

Galileo Galilei (1564-1642), the father of the scientific revolution, who had the view that "the universe is written in the language of mathematics", following his fathers' interest, also has studied the properties of vibrations and the relation between the sound pitch and the frequency of a vibrating string. Galileo Galilei published his book, titled "*Il Saggiatore*" (in Italian. *The Assayer*) in Rome in 1623. In this book Galileo Galilei put the fundamentals of the scientific methodology, wrote about the universe in the language of mathematics, and also about the harmony of the universe, which was partly prepared by Johannes Kepler's ideas formulated in "*Harmonices Mundi*" in 1619. It can be observed that the correlation between the frequency, expressed as a number, a mathematical concept, and the sound pitch, a musical concept, observed and described by Galileo Galilei, constitutes an analogy between mathematics and music, involving also physics and acoustics. This analogy has been used in my study intensively and it was a great help to build other relevant analogies between mathematics and music which were used in my research.

The French theologian, philosopher, physicist, musician and mathematician Marin Mersenne (1588-1648), known for prime numbers of the form 2^n-1, for some natural numbers *n*, named Mersenne primes, in his music theory book, titled "*Traité de l'harmonie universelle. Oú est contenu la Musique Theorique & Pratique des*

Anciens & Modernes, avec les causes de ses effets. Enrichie de Raisons prises de la Philosophie & des Mathematiques", known under the alternative title of "*Harmonie Universelle*", first published 1627 then in 1636, gave a new, modern meaning to the topic of musical acoustics. Mersenne's book, being a comprehensive music theory book, discussing different aspects of music, also connecting to theology, physics, philosophy, and mathematics, as expressed also in the title of the book, can be considered as a modern book presenting the relationships between mathematics and music. Mersenne expressed in an equation the relation between the frequency of a vibrating string, the length of the string, the stretching force of the string and the mass of the string. The formula can be found in Benson (2008, p. 91). This was the first of the many important steps towards a solid mathematical music theory, and towards a new method of tuning instruments, having culminated later in the equal-tempered or well-tempered tuning mode.

Rene Descartes (1596-1650), who unified algebra and geometry, claimed that mathematics is universal, has the power to unify all the existing human knowledge. Descartes' statement is based on different observed analogies, including analogies between mathematics and music. Descartes wrote a book, in Latin, titled "*Compendium Musicae*" (Compendium of Music). Descartes stated as cited in DeMarco (1999), that:

> In our search for the direct road towards truth, we should busy ourselves with no object about which we cannot attain a certitude equal to that of the demonstrations of arithmetic and geometry.
>
> (Descartes, as cited in DeMarco, 1999, p. 36)

In his article, DeMarco (1999) states that "there is an evident Pythagorean flavour in Descartes' *Compendium.* The impression is given that the significance of music is to provide mathematical images for the mind" (p. 38). For Descartes, music was an object, a challenge for the intellect. The goal of my study implied the use of music to provide the mathematical images of function for the minds of my students. The feelings and the mood of humans induced by music cannot be described mathematically, but the structure and the elements of music can be represented in a mathematical way, which is a real challenge for the intellect. This study has challenged my students' intellect and their imagination in this non-conventional way.

The mathematicians Daniel Bernoulli (1700-1782) and Leonhard Euler (1707-1783) worked together studying the physical properties of sounds of different instruments, describing them mathematically. Euler did show interest also in mathematical music theory. He was one of the first mathematicians in modern era who published a book on this topic. Euler's book, titled ”*Tentamen novae theoriae musicae*” was published in 1739, about 100 years after Mersenne's book. The anecdote tells, that this book has had little general success, because it contained too much mathematics for musicians and too much music theory for mathematicians. However, the great minds, like Bernoulli, the family of Johann Sebastian Bach, d'Alembert (1717-1783), Fourier (1768-1830), and others, have appreciated Euler's book.

Euler described mathematically some aesthetical aspects of music, including harmony proportions, rhythms, musical forms, and pleasentness of harmonies. Sándor (2009) in his article presents the details of a function introduced by Euler. This function, named by Euler, the ”*Gradus suavitatis*” (in translation: “degree of suavity” or “degree of pleasantness”) function can be applied to measure the grade of the pleasantness of a music harmony. Euler approached this psychological question by thinking that musical pleasure is based on hearing the proportions, representing harmonies. He has considered also the time-length and the musical pitch. Euler used different ratios, representing Pythagorean musical harmonic intervals as arguments of this function, and obtained the values given in Table 2.3 following.

Table 2.3

Values of Euler's Function of Pleasantness of Musical Harmonies

Harmony	**Octave**	**Fifth**	**Fourth**	**Major third**	**Minor third**	**Major second**	**Minor second**	**Tritone**
Ratio	½	$^2/_3$	¾	$^4/_5$	$^5/_6$	$^9/_{10}$	$^{15}/_{16}$	$^{32}/_{45}$
Function Value	E(½) = 2	E($^2/_3$) = 4	E(¾) = 5	E($^4/_5$) = 7	E($^5/_6$) = 8	E($^9/_{10}$) = 10	E($^{15}/_{16}$) = 11	E($^{32}/_{45}$) = 14

The interpretation of the values of Euler's function is that the function value is inversely proportional to the pleasantness of the harmony. If the value of Euler's

”*Gradus suavitatis*”-function is smaller, then the corresponding harmony is more pleasant. This means, for example, that the most pleasant harmony out of these eight harmonies considered above is the octave harmony, having the lowest “*gradus suavitatis*”, pleasantness degree of E(½) = 2, and the less pleasant harmony out of these eight harmonies considered above, represented by the ½, $^{2}/_{3}$, ¾, $^{4}/_{5}$, $^{5}/_{6}$, $^{9}/_{10}$, $^{15}/_{16}$, and $^{32}/_{45}$ ratios is the tritone harmony, having the highest pleasantness grade of $E(^{32}/_{45}) = 14$. This function of Euler was not presented to the participant secondary mathematics students of this study, because it requires a high level mathematical knowledge. It is an interesting, amazing fact that Euler, a leading mathematician, was thinking about a mathematical music theory, and created a function to express the grade of pleasantness of musical harmonies in 18th century.

Helmholtz, the great scientist of the 19th century, characterised by Farge (2005) as “one of the rare scientists able to be creative in domains, as diverse as medicine, anatomy, physiology, acoustics, fluid mechanics, physics, music, aesthetics, epistemology…” (p. 1), gave a function to describe the unpleasantness of musical harmonies. If $\boldsymbol{x}$ is the number of beats (produced by the interference of two sounds having fairly close frequencies) then Helmholtz's function value **h(x)** gives the degree of unpleasantness of the harmony, where:

$$\mathbf{h(x) = ax / (30 + x^2)^2}.$$

Johann Sebastian Bach (1685-1750), one of the greatest composers of all time, studied the mathematical problem which solved the practical question of the well-tempered tuning mode of instruments. J. S. Bach had extensive knowledge not “just” playing, but also planning and constructing organs and clavichords (the early versions of today's pianos). He was interested in acoustics and its mathematical-theoretical backgrounds.

The progress in mathematics and physics made possible to develop a new practice mode of tuning instruments which overcame the deficit of the Pythagorean tuning mode. This mode divided an octave into 12 equal semi-tones. As a result of the work of Vincenzo Galilei, Galileo Galilei, Zhu Zaiyu, Mersenne, Euler and other mathematicians and physicists, in this new mode of tuning instruments, the physical concept of frequency was considered, rather than the length of a string, and a very important requirement was achieved, namely that any two keys 12 half-tones apart formed an octave harmony, which was not the case with the Pythagorean tuning mode. This was possible only by assuring that the ratio of the frequencies of any two consecutive half-tones to be the same constant. This constant, which should be the favourite number of all musicians, surprisingly is an irrational number, $2^{1/12}$, the 12^{th} root of 2, having the first ten digits 1.059463094. To be honest, as I am, I think that this irrational number anyway is the favourite number of all musicians, maybe without being mathematically aware about this fact, but by enjoying the music created through the introduction of this number into the practice of tuning instruments. Maybe also this observation is expressing in a way the complexity of the relationships between mathematics and music...

This irrational number, $2^{1/12}$, is used in the well-tempered tuning mode to calculate the frequencies of all musical notes. All keys are tuned in such way that the ratio of the frequencies of any two consecutive keys is $2^{1/12}$. In this way it is assured that starting from any key, after 12 keys the frequency is doubled and an octave harmony is created, eliminating in this way the deficiency of the Pythagorean tuning mode. Mathematically, starting from any key-1, with the frequency f(key-1), the frequency of the 12^{th} key will be given by $f(\text{key-}12) = f(\text{key-}1) \times (2^{1/12})^{12} = f(\text{key-}1) \times 2$, which means that the frequency of the starting key-1 is doubled, as a result these two keys, key-1 and key-12, produce an octave harmony. This was a desire of musicians for more than two thousand years.

The piano has 88 keys. In well-tempered mode the frequency f(n) of the n^{th} key on the piano is calculated by using the following exponential function:

$$\mathbf{f(n) = 440 \times (2^{1/12})^{(n-49)},\ n\ \varepsilon\ \{1, \dots , 88\}}$$

The 49^{th} key on the piano, the A_4 note is a distinguished one, being the tuning reference note. This key is tuned with the help of a tuning fork to the frequency of 440 Hz. Benson (2002) gives the historical background how this frequency was internationally adapted as the modern concert pitch:

> Historically, this was adopted as the U.S.A. Standard Pitch in 1925, and in May 1939 an international conference in London agreed that this should be adopted as the modern concert pitch. Before that time, a variety of standard frequencies were used. For example, in the time of Mozart, the note A had a value closer to 422 Hz, a little under a semitone flat to modern ears. Before this time, in the Baroque and earlier, there was even more variation. For example, in Tudor Britain, secular vocal pitch was much the same as modern concert pitch, while domestic keyboard pitch was about three semitones lower and church music pitch was more than two semitones higher.
>
> (Benson, 2002, p. 16)

We can see from the function presented above that if we consider the 49^{th} key, then we have n = 49 and $f(49) = 440 \times (2^{1/12})^{(49-49)} = 440 \times (2^{1/12})^0 = 440 \times 1 = 440$ Hz. We also can observe that if we consider a key which is to the right from the 49^{th} key on the piano, the A_4 note, then the corresponding frequency f(n) is higher than 440 Hz, and if we consider a key which is to the left from the 49^{th} key on the piano, the A_4 note, then the corresponding frequency f(n) is lower than 440 Hz. Mathematically: if $n > 49$ then $f(n) > f(49) = 440$ Hz, because we have a positive power, $n – 49 > 0$, and if $n < 49$ then $f(n) < f(49) = 440$ Hz, because we have a negative power, $n – 49 < 0$.

We calculate the frequency of the 1^{st} key on the piano using the above exponential function as follows: $f(1) = 440 \times (2^{1/12})^{(1-49)} = 440 \times (2^{1/12})^{-48} = 440 \times 2^{-48/12} = 440 \times 2^{-4}$ $= 440 \times (½)^4 = 440 \times {}^1/_{16} = 440 / 16 = 27.50$ Hz.

We can observe that the sequence of the frequencies of the keys of a piano form a geometric sequence, also known as geometric progression, where the first term is f(1) = 27.50 and the common ratio is $2^{1/12}$.

This fact, which offers a musical analogy to the concept of geometrical sequence, can be used in senior secondary mathematics lessons effectively when students learn about geometric sequences. The topic of geometric sequences is incorporated in the Australian Curriculum 2014 in “Topic 2: Arithmetic and geometric sequences and series” of the senior secondary mathematics curriculum as follows:

- "recognise and use the recursive definition of a geometric sequence: $t_{n+1} = rt_n$ (ACMMM072)"
- "use the formula $t_n = r^{n-1}t_1$ for the general term of a geometric sequence and recognise its exponential nature (ACMMM073)"

Applying the notations of the Australian Curriculum 2014 to our knowledge regarding the well-tempered tuning of piano keys we have $r = 2^{1/12}$ and $t_1 = 27.50$ Hz. With these values, a musical analogy of recognising and using the recursive definition of the geometric sequence, $t_{n+1} = rt_n$ (ACMMM072) can be given. Because the ratio of the frequencies of any two consecutive piano keys is $r = 2^{1/12}$, we clearly have $t_2 = 2^{1/12} \times t_1$, $t_3 = 2^{1/12} \times t_2$, $t_4 = 2^{1/12} \times t_3$, $t_5 = 2^{1/12} \times t_4$, and so on, in general terms $t_{n+1} = 2^{1/12} \times t_n$. The recursive definition of a geometric sequence can be clearly recognised and used with the help of this musical example. An example of using the formula $t_n = r^{n-1}t_1$ for the general term of a geometric sequence and recognising its exponential nature (ACMMM073) also can be given. Because we have $t_1 = 27.50$, $t_2 = 2^{1/12} \times 27.50$, $t_3 = 2^{1/12} \times t_2 = (2^{1/12}) \times (2^{1/12}) \times 27.50 = (2^{1/12})^2 \times 27.50$, $t_4 = (2^{1/12})^3 \times 27.50$, and so on, the general formula for the n^{th} term of the sequence being $t_n = (2^{1/12})^{n-1} \times 27.50$, the formula can be used to calculate and state the frequency of the n^{th} key of the piano. As a geometric sequence is a $N \rightarrow R$ function, the domain being the set of natural numbers (N) and the range being the set of real numbers (R), the examples given above have been used in this study.

The conundrum of the Pythagorean tuning mode has been solved. The solution was not possible within the Pythagorean paradigm, but only by creating a new musical paradigm, a "well-tempered" mode, backed by a beautiful generalisation of the mathematical music theory behind the problem. We show this solution, mathematically, as follows. We know that in the well-tempered mode the octave is divided into 12 equal notes, and the frequency of any note is calculated by multiplying the frequency of the previous note by the irrational number $2^{1/12}$. This means that:

- considering any note, having the frequency f, then the octave harmony will be formed with the 12^{th} note, having the frequency $f \times (2^{1/12})^{12} = f \times 2^{12\times1/12} = f \times 2^1 = f \times 2 = 2f$,

– the ratio between the frequencies of any two notes forming an octave is 2f / f = 2, meaning that doubling the frequency gives the octave harmonic interval,
– considering any note, having the frequency f, then the quint harmony will be formed with the 7th note, having the frequency $f \times (2^{1/12})^7 = f \times 2^{7\times 1/12} = f \times 2^{7/12}$,
– the ratio of the frequencies of any two notes forming a quint harmony is $f \times 2^{7/12} / f = 2^{7/12}$.

If we consider twelve stacked quint harmonies, with a starting note having the frequency f, then the end-note, which is 12 × 7 = 85 notes apart, having the frequency $f \times (2^{1/12})^{85}$. We can conclude that the ratio of the frequencies of the two end-notes of twelve stacked quint harmonies is $f \times (2^{1/12})^{12\times 7} / f = (2^{1/12})^{12\times 7} = 2^{12\times 7/12} = 2^7 = 128$.

If we consider seven stacked octave harmonies, with a starting note having the frequency f, then the end-note, which is 7 × 12 = 85 notes apart, having the frequency $f \times (2^{1/12})^{85}$. We can conclude that the ratio of the frequencies of the two end-notes of seven stacked octave harmonies is $f \times (2^{1/12})^{7\times 12} / f = (2^{1/12})^{7\times 12} = 2^{7\times 12/12} = 2^7 = 128$.

The equality of the two results above, 128 = 128, shows that within the well-tempered mode twelve stacked quint harmonies perfectly fit seven stacked octave harmonies. This means that within this well-tempered mode the conundrum of the Pythagorean mode does not exist, it has been eliminated. This is what we wanted to demonstrate. Q.E.D.

Not all of the above ideas and the solution were presented to the participant students of this study due to the shortness of time availability, but I have presented the frequency table of the piano keys to the students of the experimental group during the treatment (see Appendix N). The presented frequency table of the piano keys by itself is a function given in tabular form, where the domain is the set of piano key numbers and the range is the set of piano frequencies described above. The algebraic form of this function was given previously: $f(n) = 440 \times (2^{1/12})^{(n-49)}$, for every n ε {1, … , 88}. We can observe from both tabular and algebraic form of the function, that this is a bijective function, also named as one-to-one function, because any two different piano key numbers have different corresponding frequencies.

We have a bijective (one-to-one) function, and we know that every bijective (one-to-one) function has an inverse function. We can state – based on the logical process of syllogism – that also this function, being a bijective function, has an inverse function. The domain of the inverse function is the set of the piano key frequencies and the range is the set of the piano key numbers. The connection between the independent variable (frequency **f**) and the corresponding dependent variable (piano key number **n**) can be obtained as follows. It is known from the above considerations that

$$\mathbf{f = 440 \times (2^{1/12})^{(n-49)}}$$

After dividing both sides of this equality by 440 we obtain the following equivalent form:

$$\mathbf{f / 440 = (2^{1/12})^{(n-49)}}$$

After simplifying the power of a power, we obtain the following equivalent form:

$$\mathbf{f / 440 = 2^{(n-49)/12}}$$

After forming the logarithm of base 2 of both sides we obtain the following equivalent form:

$$\mathbf{\log_2(f / 440) = \log_2 2^{(n-49)/12}}$$

After using the property of the logarithm we obtain the following equivalent form:

$$\mathbf{\log_2(f / 440) = (n-49) / 12}$$

After multiplying both sides by 12 we obtain the following equivalent form:

$$\mathbf{12 \times \log_2(f / 440) = n - 49}$$

After adding 49 to both sides we obtain the following equivalent form:

$$\mathbf{n = 12 \times \log_2(f / 440) + 49}$$

As a result, if we consider a given piano key frequency **f**, then the corresponding piano key number **n** can be obtained by the formula deducted above, $n = 12 \times \log_2(f / 440) + 49$. For example, considering the frequency f = 440 Hz of the A_4 piano note, the "normal A", then the corresponding piano key number can be calculated based on the logarithmic function given above as $n = 12 \times \log_2(440 / 440) + 49 = 12 \times \log_2(1) + 49 = 12 \times 0 + 49 = 0 + 49 = 49$. The obtained 49th piano key, indeed, will produce the sound having 440 Hz. This piano note has a distinguished role because it is used as a reference for tuning all other keys of a piano. The presented calculation of the logarithmic inverse function of the considered exponential function constitutes a musical analogy of the concept of inverse function, and I believe that contributes to students' understanding of both exponential functions and logarithmic functions.

The ideas above and the given solution can be presented in secondary mathematics classrooms if the students have covered the relevant mathematical topics, such as exponential functions, geometric progressions, indices, index laws, inverse functions, and logarithmic functions. I plan to present these to my students in the future, as Dienes (1966) describes:

> Of course, the differences between the notes in the scale are multiplicative differences as far as the frequencies are concerned. And so this should be an interesting example to practise the idea of powers and roots. To obtain the frequency of a note a semitone above a certain note you have to multiply the frequency of this note by the twelfth root of two, because going up an octave we have to multiply the frequency by two, and the semitone is one-twelfth of an octave.
>
> (Dienes, 1966, p. 87)

J. S. Bach has written a great musical work to demonstrate the beauty of the well-tempered mode. J. S. Bach's great work, "*The Well-Tempered Clavichord*", a collection of brilliant baroque music pieces, forty-eight preludes and fugues, covering all twenty-four major and minor tonalities, was written nearly 300 years ago in 1722, and dedicated in the manuscript "for the benefit and use of musical young persons who are desirous to learn, also for enjoyment of those who are already skilled in this study" (Bach, 1925). J. S. Bach's work, "*The Well-Tempered Clavichord*" circulated in manuscripts for a long time, four of them in J. S. Bach's own handwriting (one of them can be seen in the British Museum), because the work was first published in printed editions by Simrock and Hoffmeister & Kühnel (now Peters) in 1800 and 1801.

Some people state that J. S. Bach was a "mathematical composer". I personally don't share this view despite the fact that in J. S. Bach's adored music obviously there are a range of patterns, structures, inversions, algorithms, and so on, which can be described mathematically. J. S. Bach's genius, including his musical imagination, brilliant technique, composing style, his soul, and his unbelievable knowledge and speed to write musical sheets, can't be simplified to describe him just as a "mathematical composer".

The counterpoint music style has been extensively used in Baroque music era, considered to be between 1600-1750. It is a type of music when many independent melodies are played simultaneously. The different melodies are played together but not with the scope to serve as accompaniment for a main melody, because these are truly autonomous melodies played simultaneously. In order to compose a good melody, the composer has to be inspired and talented. We can see the order of inspirational complexity in order to compose many independent melodies which played simultaneously sound beautiful. In his composing style, J. S. Bach has used counterpoint intensively.

In this study one of the self-designed musical analogies of mathematical function is the counterpoint (see Section 3.3.4, Analogy 4 on page 116), presenting the structural similarities between representing the graphs of more than one functions on the same coordinate system and the musical counterpoint, which is described in details in Chapter 3. The chosen musical example to illustrate this analogy is J. S. Bach's beautiful piece of music, "*Arioso from Cantata, BWV 156*".

A musical canon is that type of music when the same melody is played, but starting at different time-points and/or playing at different musical pitch. A canon can be described mathematically by using a periodic function f and its horizontal translations (representing the same melody at the same musical pitch, but starting at a different time-point), vertical translations (representing the same melody played at a different musical pitch), horizontal dilations (representing the same melody but played faster or slower), vertical dilations (representing the same melody played differently), or a combination of these translations. Phillips (2018) gives a graphical example how a periodical function and its translation can represent a musical canon:

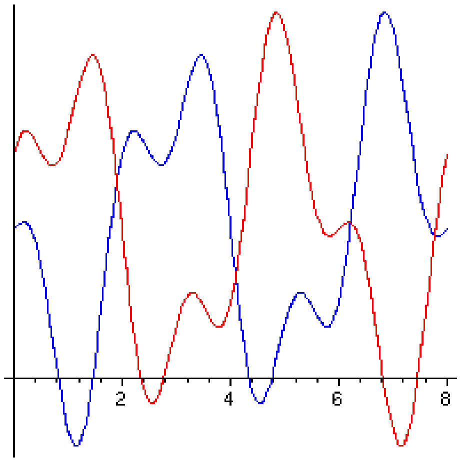

Figure 2.1: Periodic Functions Representing a Musical Canon.
Reprinted from “Math and the Musical Offering,” by T. Phillips,
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The red graph represents a periodic function $f(t)$ of period 8, and the blue graph is g(t) = f(t - 2), obtained by the horizontal translation of the function f(t) by the horizontal translation constant b = -2. Phillips (2018) in his online article, published by the American Mathematical Society, explains how functions can represent mathematically some of J. S. Bach's musical canons. Phillips (2018) states that:

> Johann Sebastian Bach's '*Musical Offering'* contains ten canons. In each of these canons a musical line is played twice (or four times in Canon 10). The second versionis always transformed with respect to the first by shifting in time, but it may also be shifted in pitch, turned upside-down, stretched, or played backwards. Each of these transformations occurs in the mathematics of elementary functions; they are examples of how new functions can be made out of old and of how a function can be tailored to fit a new situation. We will look at some simple transformations and see how they are exemplified in the first five of the *Musical Offering* canons.
>
> (Phillips, 2018)

Phillips (2018) also gives examples of functions which can be applied to represent the first five of the J. S. Bach's "*Musical Offering*" canons as follows:

- Canon 1: $g(t) = f(18 - t)$
- Canon 2: $g(t) = f(t - 1)$
- Canon 3: $g(t) = -f(t - 0.5) + K$
- Canon 4: $g(t) = -f[(t – 0.5) / 2] + L$
- Canon 5: $g(t) = f(t - 1) + H$

If we check the types of the above function transformations, then we can observe that these types are variations and combinations of function translations described in the Australian Curriculum 2014, which are the objects of this study. These are: horizontal translations (Canons 1, 2), horizontal and vertical translations, and horizontal and vertical dilations (Canon 3, 4), and horizontal and vertical translations (Canon 5). All these types of function transformations were covered in this study and the corresponding self-designed musical analogies were presented at Year 11 Mathematical Methods level. A musical canon is a special form of the musical counterpoint, presented in this study through the Analogy 4 which explains how the graphs of two or more functions represented on the same coordinate system can be modelled as a musical counterpoint. These facts demonstrate that the present study connected well to the general trend of research carried out in this topic.

Both mathematics and music express the already named different abstract patterns of our human rational thoughts and emotional feelings in a specific symbolic language. Different patterns of audio sounds are created by music. The musical sheet notes of these audio patterns form an analogous structure, a different, written abstract pattern. The power of mathematics is in general a logical study of different abstract patterns and structures. Harkleroad (2006), considering mathematics and music as being a duet, states that:

> Math and music do have much in common. At heart, abstract patterns form the stock-in-trade for both. To express these patterns, each field has developed its own symbolic language, used the world over regardless of

> nationality. And the two areas, although in different ways, combine the intellectual and the aesthetic in a wonderful blend.
>
> (Harkleroad, 2006, p. 1)

Benson's (2008) comprehensive book, titled "*Music: A Mathematical Offering*", obviously paraphrasing the title of J. S. Bach's "*Musical Offering*" work, is an excellent source of exploring the relations between mathematics and music. Benson (2008) connects sound's physical and perceptual attributes in a natural way, as amplitude and loudness, frequency and pitch, spectrum and timbre, and duration and length. My study has presented these type of connections in forms of analogies between mathematics and music.

Wright (2009), a mathematician and musician, in his university course, titled "*Mathematics and Music*", offered at Washington University of St. Louis, states that:

> It has been observed that mathematics is the most abstract of the sciences, music the most abstract of the arts. Mathematics attempts to understand conceptual and logicaltruth and appreciates the intrinsic beauty of such. Music evokes mood and emotion by the audio medium of tones and rhythms without appealing to circumstantial means of eliciting such innate human reactions. Therefore, it is not surprising that the symbiosis of the two disciplines is an age old story. The Greek mathematician Pythagoras noted the integral relationships between frequencies of musical tones in a consonant interval; the 18th century musician J. S. Bach studied the mathematical problem of finding a practical way to tune keyboard instruments. In today's world it is not at all unusual to encounter individuals who have at least some interest in both subjects.
>
> (Wright, 2009, p. v)

Indeed, in many aspects mathematics and music are related, showing structural analogies. A wide range of forms is characteristic for both mathematics and music. Both of them can have very simple forms, which can be presented to pre-school aged children, and both can be very sophisticated and complicated, hard to understand or to interpret also for highly educated specialists. In order to master them, both mathematics and music require a long time to study. Both mathematics and music have intellectual and aesthetic aspects, which make mathematics and music, and the relationships between these two universal languages so beautiful. This makes possible to think about different analogies between mathematics and music and to find how to use these analogies adequately. These may be used to describe a range of music related phenomena, from basic concepts of music to high level musical activities such as music compositions. At the beginning of the 20th century, Joseph Schillinger (1895 – 1943), a music teacher, composer, and music theorist, who created a mathematical method of musical composition, named on his honour "*The Schillinger System of Musical Composition*", has described mathematically different aspects of music such as melody, harmony, counterpoint, rhythm, form and semantics. He has created analogies between mathematics and different art forms and has influenced a range of artists, including the composer George Gershwin (1898 – 1937). After decades of study and research, Schillinger wrote a pioneer book, titled "*The Mathematical Basis of the Arts",* which was published after his death in 1943. Schillinger (1943) describes mathematically different art forms and states that "scientific analysis of musical composition reveals that all the processes involved in the creation of a musical composition may be represented by elementary mathematical procedures" (p. 15). Analogies between mathematics and music – targeting music compositions – are used today, at the beginning of the 21st century as well. The paper of Hed, Gjerdingen, and Levin (2012) "draws useful analogies between the mathematics of subdivision schemes and the hierarchical structures of music compositions" (p. 17). It can be observed that the structural analogies between mathematics and music may be used as powerful instruments.

My study presents nine author-designed analogies between different aspects of the concept of function and the corresponding aspects of music, and employs these analogies in order to improve senior secondary students' understanding of function. The analogies presented in this study refer different aspects of function to melody, harmony, counterpoint, transposition of music, musical repetition, and tempo of music. Obviously, in many other aspects music and mathematics are very different. Music can be enjoyed without any previous education, but mathematics can't, except in geometric design. Music usually is a lifelong integral part of the majority of people but mathematics it is not.

Kelstrom (1998) gives a summary of the multiple facets of music, which may be used in education, stating that:

> Music education is more than learning to sing or play an instrument. It is more than entertaining or pleasing an audience. It is more than a pleasant diversion of recreation. Music is a science. It is a mental discipline. It is an art. It has a mathematical foundation. It is a language. It is a physical activity. Any subject that combines science, discipline, language, math, physical activity and art must not only be worthwhile, but absolutely essential to the education of our children. Music incorporates every other area of study in some way. What a tool to be used in education!
>
> (Kelstrom, 1998, p. 41)

The following sub-sections describe the possible positive effects of this "tool to be used in education" Kelstrom (1998, p. 41), music, on students' mathematics skills as reflected in the literature, and connect the findings to my study.

2.3 Effects of Music on Students' General Mathematical Skills

Both the relationships between mathematics and music, and the effects of music on students' general mathematical skills have been researched in a range of different ways. Research has suggested that music can enhance students' academic outcomes in mathematics (An et al., 2013; Evans, 2009; Harris, 2005, 2007; Nisbet & Bain, 1998, 2000; Rauscher, 2003; Schumacher et al., 2006; Spychiger, 2001; Still & Bobis, 2005; Vaughn, 2000).

The majority of the relevant research studies have been performed mainly involving pre-school, primary and middle school students, targeting corresponding levels of mathematics. The participants of the present study were senior secondary, Year 11 Mathematical Methods students (16-17 years old students) and the study has targeted a specific mathematical concept, the concept of function, through active integration of mathematics and music. The study has not changed the mathematical format, wording or content of the relevant parts of the function section of the Year 11

Mathematical Methods curriculum, and has used them as they are described in the Australian Curriculum 2014.

Wahl (1998) states that:

> A vast amount of data supports the fact that music influences learning. It can have its effects preceding study, accompanying work, or by actually being the subject of the study, like fractional notes and rhythmic patterns.
>
> (Wahl, 1998, p. 18)

My study has not been concerned regarding the possible influences of music on learning when music is played preceding mathematical work or during the mathematical work as a background accompaniment, because, as stated previously, these are considered as being passive integrations of mathematics and music, and the passive integration modes were not the objects of this study.

Wahl's (1998) recommendations how to integrate music and mathematics during mathematics lessons are as follows. The words in bold are emphasised in Wahl (1998):

> 1. Play 3-10 minutes of **'conductive' music before undertaking learning tasks**. [Start with Mozart then try some other classical, baroque, and romantic composers as your confidence grows].
> 2. Use **soundbreaks** of a few minutes of music between activities to pick up tired energy and low moods (the theme from 'Flashdance'), activate imagination (Paul Winter's 'Earthbeat'), or focus intent (Bach's 'Well-Tempered Klavier'.)
> 3. **Use clapping, singing, humming, rhythmic movement, rhythmic words (raps), or jingles** to help students recall and recite concepts, tables and procedures.
> 4. Ask groups of students to **compose songs or raps** that capture or abbreviate a particular concept.
> 5. Use **musical language and metaphors** to describe maths ideas.
>
> (Wahl, 1998, pp. 18-19)

I considered the possibility to use these recommendations in my study, but decided to use the 5th recommendation only. Because I categorise the 1st recommendation as being a passive integration of mathematics and music, this has not been considered in my study. The 2nd recommendation, using soundbreaks of a few minutes of any other music between activities, also has not been considered due to the nature of the present study. The 3rd recommendation above has not been used, because the goals of the present study were totally different ones: instead of memorising, recalling or reciting concepts, tables and procedures, the improvement of students' conceptual understanding of function has been targeted. Regarding the 4th recommendation, I was inclined to consider this recommendation in my study for a short time initially, but decided not to use due to two reasons. First, the time factor, that there were 90 minutes only to perform the treatment lesson with the experimental group. Asking students to compose songs or raps that capture or abbreviate the concept of function wouldn't fit into this time frame. Second, songs or raps composed by students with the goal to capture for example different aspects of the function concept may increase students' mood, but also may deepen students' misconceptions regarding functions if these songs and raps are not adequate, and are not checked by the teacher first.

I considered the recommendation 5 above to be valuable for this study. In this study a range of musical language and metaphors have been used to describe different aspects of function at Year 11 mathematics level. The used musical language and metaphors were carefully selected to be understandable for every student. Some of them were explained briefly in a very accessible mode, appealing to Year 11 mathematics students' general human non-trained music intelligence and level of abstract mathematical-logical thinking, to understand the designed nine musical analogies.

An et al. (2013) have researched the effects of mathematics and music integration on elementary students' mathematical abilities of modelling, strategy and application. Two classes (N = 46) participated in the research for five weeks, and the results suggest that the integration of mathematics and music have had positive effects on elementary students' mathematical abilities. An et al. (2013) state that traditional mathematics teaching methods, e.g. assigning the same problem to every student, teaching from the textbook, insisting on one way to explain concepts and solve problems, neglecting students' individual conceptual understanding is the main reason for low mathematical achievement and for the mathematical anxiety.

I made an effort to research the effects of active integration of mathematics and music at senior secondary (Year 11) level, using two classes (*N* = 44), non-traditional methods, explaining different aspects of the concept of function in a non-traditional way, actively integrating music into the mathematics lesson. I also confirm the observation of An et al. (2013), where the authors state that:

> Music is an ideal form of art to be integrated in mathematics instruction. The links between music and mathematics are very rich and include melody, rhythm, intervals, scales, harmony, tuning, and temperaments. These musical concepts are related to the mathematical concepts of proportions and numerical relations, integers, logarithms and arithmetical operations and the content areas of algebra, probability, trigonometry, and geometry.
>
> (An et al., 2013, p. 2)

This study has used the links between the mathematical concept of function and the corresponding musical concept, the melody, as well as the mathematical concept of relation and the corresponding musical concept, the harmony.

As An et al. (2013) also observe, the integration of mathematics and music in mathematics lessons has the potential to improve students' attitudes towards learning mathematics and to improve students' understanding of mathematics. The present study has two research questions, concentrating on the main one, the improvement of students' understanding of function, this main goal expressed also in the title of the study. The question of improving students' attitudes towards learning mathematics was considered as an auxiliary question due to logistics reasons regarding the shortness of the intervention. I am aware of the fact that the question of improving students' attitudes and beliefs regarding mathematics is an important problem, but researching this in a comprehensive mode was not possible within the available frames of this study, due to logistics, which has not permitted to extend the research for a longer time. But, interestingly, also this study can report a statistically significant improvement of students' perception of the “Enjoyment of mathematics lessons” scale (containing 10 items) of the Test Of Mathematics Related Attitudes (TOMRA) questionnaire (Fraser, 1981; Taylor, 2004), which has been used in pre-test and post-test form. I would not make any long-term, general conclusion based on this particular result.

An et al. (2013) state that their study was based on a theoretical framework that includes Gardner's (1983) Multiple intelligences theory. The present study also was partly based on Gardner's (1983) Multiple Intelligences theory and the corresponding practice (Gardner, 1993), targeting bi-directional skill transfers between mathematics and music. The skill transfers were modelled and encouraged in this study by the presented nine relevant analogies between function and music.

Evans (2009) integrated music and movement to teach mathematics for 4-7 years old kindergarten children (N = 18). The children received a formal mathematics instruction for 30-45 minutes every day. Throughout the day music and songs were played as background and also played to teach certain mathematics skills. The children listened to music and participated in different music- and movement activities before formal mathematics instructions were given. The songs played were related to the mathematical concepts learned by the kindergarten children. The covered and tested mathematics concepts were: number recognition 1-20, writing numbers 1-20, rote counting numbers 1-100, days of the week, months of the year, and shape recognition (square, triangle, rectangle, and circle). Evans (2009) reported that music and movement activities positively affected kindergarten children's mathematics outcomes.

I am aware about the importance of introducing age appropriate music and mathematics activities to children during their early childhood. I also know that they are fairly often applied in kindergartens, but unfortunately they are not so often applied in primary schools and rarely applied in secondary schools. My study was an effort to improve my knowledge and practice of teaching senior secondary mathematics through active integration of mathematics and music, but may serve as a model to interested senior secondary mathematics teachers or may be considered to be incorporated in the relevant curriculum in the future.

Kelstrom (1998) notes that the possible positive effects of music on students' cognitive development is mainly disregarded in schools and subject curricula, and states that "the contribution music makes to the academic achievements of students is ignored" (p. 42).

Dienes applied the Six-stage Theory of Learning Mathematics for different mathematical concepts like integers (Dienes, 2000b) and rational numbers (Dienes, 2001). Dienes' (2000b, 2001) approaches were a source of inspiration for my study which has applied these at senior secondary level, explicitly targeting the important function concept.

Different skill transfers are very important components of the pedagogical-psychological processes of learning, decision making, designing and explaining. The philosophical roots of the concept of skill transfer has been known since Plato – as the ancient Greek thinkers believed that learning mathematics helps develop general cognitive skills – but was researched mainly in the 20th century (Molnár, 2002). Since about the 1970s, as an implication of the available knowledge in cognitive psychology, different aspects of how music affects cognitive processes and perceptions have been widely studied and as a result a new research domain, the research of skill transfers has been established (Schumacher et al., 2006). The effects of music are grouped in two main categories: short-term and long-term effects. The short-term effects are studied mainly by Psychology and the long-term effects by Philosophy, Pedagogy and specialist Educational sciences. In general terms these effects are not invoked by music only, but they constitute a complex outcome of a range of environmental, social-cultural and individual factors (Schumacher et al., 2006). During skill transfer both cognitive and non-cognitive skills may be affected. An important question in the research of skill transfer is the ratio of resultant cognitive and non-cognitive skills. Rauscher (2004) states that:

> Several concerns remain unaddressed: Little is known regarding the exact aspects of music instruction that contribute to the transfer effects. Also, further longitudinal studies are needed to determine the duration of these effects. Another concern is that currently available tests of reading and math

> achievements may not be sufficiently sensitive to the complexity of language and mathematical learning potentially affected by music instruction.
>
> (Rauscher, 2004, p. 120)

McKeachie (1987) too, states that “transfer depends upon domain-specific knowledge, general cognitive skills, and goals and values” (p. 711). McKeachie (1987) notes that “we should consider transfer of motivation as well as transfer of learned cognitions if we are to understand transfer of learning in educational situations” (p. 707). McKeachie’s (1987) and Rauscher's (2004) observations cited above regarding the nature and mode of skill transfer have suggested to me the idea to also consider measuring students' mathematics related beliefs and attitudes, which is reflected in the second – auxiliary – research question.

From many teachers’ experiences, and research also confirms that there is no “royal road” to mathematics (Sain, 1986; Marsh, 1999). Learning mathematics is a long, complex process which can be aided, in part, by some non-traditional, non-standard, non-mathematical approaches. One of the possible non-mathematical aids is using different skill transfer effects between music and mathematics (McKeachie, 1987).

Vaughn (2000) conducted three meta-analyses of a total of 25 studies on relations between mathematics and music, from elementary to college level, which were published between 1950-1999. Firstly, she made the meta-analysis of eight correlational studies which were conducted on investigating the correlation between voluntary study of music and mathematics results. The general outcome of these eight studies showed a modest positive correlation between voluntary study of music and mathematics results. Secondly, she made the meta-analysis of five experimental studies which were conducted on researching if music instruction causes the improvement of mathematics understanding. This second meta-analysis concluded that music instruction improves mathematics. Thirdly, she made the meta-analysis of twelve experimental studies which were conducted on researching if music played in the background improves mathematics test scores. The third meta-analysis concluded that there was a very small positive effect on mathematics test results when music was played in the background. After the three meta-analyses of these 25 studies, Vaughn's (2000) general conclusion is formulated in her article's title as well, namely, that she has found a general “modest support for the oft claimed relationship” between mathematics and music.

I was inspired by Vaughn's (2000) classification and approach of relevant research studies and have used it to categorise the articles considered in this research as experimental (interventional) studies and correlational studies. The experimental studies involve an intervention, a type of integration of mathematics and music. The correlational studies concentrate on different aspect of correlations between mathematics and music, without an intervention. In the case of the experimental studies I also have categorised the mode of the integration as follows. The integration is active, when the intervention occurs in such a way that the student has to do an activity which explicitly connects the two subjects. An example of an experimental study involving the active integration of mathematics and music is Nisbet and Bain's (2000) study, where students were asked to listen to melodies and match them with corresponding line graphs. The present study is another example of an experimental study involving the active integration of mathematics and music because students were asked to actively observe nine structural analogies between function and music. The integration is passive, when the intervention occurs in such a way that the student listens to music before or during a mathematics test, but there is nothing else to do which explicitly connects the two subjects. A famous example of an experimental study involving the passive integration of mathematics and music is the “Mozart effect” study (Rauscher et al., 1993).

In their correlation study, Vaughn and Winner (2000) have analysed different aspects of 10-12 available years (1987-1998) of SAT scores of large numbers of year 9-12 high school students in the United States (N = 535,818 to N = 3,345,430), researching the correlation between students' composite verbal and math SAT scores and students' combined art (music, visual art, dance, and drama) scores. After using different statistical analysis methods (one-way ANOVA tests, t-tests, independent meta-analyses, and small meta-analyses) of the relevant SAT scores data, the main conclusion of the research is that “the verbal and math SAT scores of students taking any form of art, irrespective of the number of years, are significantly higher than for students who take no art.” (Vaughn & Winner, 2000, p. 83). The researchers have analysed the data also respective to the number of art years, concluding another important observation of the report, the result of the trend analysis of scores of students with zero, one, two or three years of art study, namely that students' composed verbal and math SAT scores increased gradually in the first 3 years of art study and then jumped sharply at 4 years of studying art. There was no statistically significant difference between the composite verbal and math SAT scores of students with 4 years of art experience and students with more than 4 years of art experience. The researchers also have separated students' verbal SAT scores and students' mathematics SAT scores. The relevant statistical analysis shows that similarly to the combined verbal and math effect sizes, and separated verbal effect sizes, the math effect size increased sharply at 4 years of art experience. Comparing the separated math and verbal effect sizes, Vaughn and Winner (2000) have observed that the math effect size is considerably smaller than the verbal effect size. This is explained by the fact that the art forms and the related art activities in high schools involve language. The authors also have examined 10 years of SAT scores data regarding the correlations of different separate art subjects scores and mathematics scores. Vaughn and Winner (2000) considered the following categories of art subjects:

> No course work in arts; acting or the production of a play; drama or theatre appreciation; studio art and design; art history or art appreciation; dance; music history, theory, or appreciation; music, instrumental or vocal performance; photography or film making.
>
> (Winner, 2000, p. 82)

In the final statistics the art category of photography/film making has not been included. Interestingly to me, that out of the remaining eight art categories the best correlated art category with mathematics became the category of 'acting or the production of a play'. Having a very close correlation statistic to the first placed art category, the second and the third best correlated art categories with mathematics were the considered two musical categories: 'music history, theory, or appreciation', and 'music, instrumental or vocal performance'.

My observation is that because the objective of the research of Vaughn and Winner (2000) was to find the correlation between high school students' verbal and mathematics SAT scores and art SAT scores, these results demonstrate mainly the possible silent and general skill- and attitude- transfers between different types of art and mathematics. These possible skill- and attitude- transfers are silent, in terms of not being the case of effective integration of an art subject with mathematics, but studying both an art subject and mathematics in the school. These possible skill- and attitude-transfers are general, because they are not expressing the effects of specific aspects of an art subject on specific aspect of mathematics.

In my research, described in this study, I have addressed some possible explicit and specific skill- and attitude- transfers between mathematics and music. I name them explicit, because the research was designed to study the effects of active integration of mathematics and music, explicitly focusing on improving secondary students understanding of function. I consider that in my research the addressed possible skill- and attitude- transfers are specific, because the research was designed to address specific aspects of the function concept, appealing through self-designed, student-age and mathematics-level appropriate musical analogies to senior secondary students' general music intelligence.

A range of research evidence shows that kindergarten and primary students who receive music education achieve better mathematical results than their peers who do not receive music education (Harris, 2005).

Integrating mathematics and music in the classroom is widely researched at early childhood, kindergarten, pre-school and primary school level. Johnson and Edelson (2003) describe different modes how mathematics and music can be combined at elementary classroom level. Their article is an excellent summary of the related activities, such as recognizing, describing and translating patterns, serial ordering, sorting, classifying, combination and ratio activities, as well as learning fractions. Johnson and Edelson (2003) mention that one of the reason of using the integration of mathematics and music is that there are children possessing better musical intelligence than logical-mathematical intelligence. This observation is in perfect correlation with Gardner’s (1983) Multiple Intelligences theory. Thus we can conclude from the research evidence that some aspects of mathematics and music do have relationships and structural analogies. These correlations, relationships and structural analogies can serve as indicators for teachers and learners of mathematics and potentially can be used to improve the mathematical learning outcome. (Harris, 2005; Harris, 2007; Kelstrom, 1998; Nisbet & Bain, 1998; Nisbet & Bain, 2000; Spychiger, 1999a; Spychiger, 1999b). I have used in this study the educational advantages of different correlations, relationships and structural analogies between mathematics and music in order to improve my senior secondary (Year 11) mathematics students' understanding of the concept of function.

Nisbet and Bain (1998, 2000) claim that students are able to match melodies with their visual representations, namely with line graphs and music notations, and this is positively related to both mathematical and musical ability. Nisbet and Bain (1998, 2000) state that further research is required to find out how audio and visual representations of mathematical functions can assist mathematics students, including visually impaired students, in understanding functions in algebra, calculus and statistics. Nisbet indicated his interest in this study's plan to address a particular aspect of this question (S. Nisbet, personal communication, 11th November 2010). My study can be considered as an extension of Nisbet and Bain's (1998, 2000) studies as it has used analogies between mathematical and musical representations of function, matching different aspects of function to the corresponding aspects of music. The participants in this study observed, saw and heard nine structural analogies between different function operations and their corresponding musical representations. The outcomes of my study are in perfect accordance with Nisbet and Bain's (1998, 2000) findings.

Harkleroad (2006) describes mathematically various music tuning systems (Pythagorean, just intonation, equal tempering), which could be applied in primary (8-12 years old) and junior secondary (13-15 years old) level mathematics classes in order to improve the learning outcome of the concepts of natural, rational and irrational numbers. Harkleroad presents the existing 48 elementary musical operations (musical transpositions, inversions and retrograde operations) and how these musical operations form a group, an abstract algebraic structure. This could be applied in a high level secondary specialist mathematics class or at university level to improve the learning outcome of the concept of group. My study presents self-designed senior secondary level structural analogies between function translations and musical transpositions.

Loy (2006) writes about the relationships between mathematics and music, presenting a type of mathematical music theory, using high level mathematical concepts and techniques. Zhang (2009) presents a framework of atonal music theory represented by the abstract mathematical group theory. Zhang (2009) introduces her work as follows:

> In 2002, a music theorist by the name of Julian Hook published a paper in the Journal of Music Theory titled, "Uniform Triadic Transformations." In this paper, Hook generalized some existing music theoretical concepts and greatly improved their notation. Hook's UTTs formed a group with interesting algebraic properties. This paper will first give the reader a review of all necessary group theory to understand the discussion of Hook's UTTs. Then it will review music theory (atonal theory in particular) and its evolution to the UTTs. Finally, it will discuss the UTTs themselves and conclude with some musical applications.
>
> (Zhang, 2009, p. 3)

Townsend (2011) presents music theory in the frame of abstract group theory. One of the musical examples used by Townsend (2011) is "*Canon in D*" by J. Pachelbel. The same music piece is used in this study – in a different context – when presenting a musical analogy of horizontal translation of function, g(x) = f(x + c), as a musical repetition or re-statement. See Analogy 6 in Section 3.3.6, Chapter 3.

Andreatta, Ehresmann, Guitart, and Mazzola (2013) interpret the *Six Variations* of Beethoven's "*Piano Sonata Op. 109"* within the frame of mathematical category theory, as a category-oriented process. They propose the following categorical modelling "A_1: melodic, A_2: rhythmical, A_3: contrapuntal; A_4: permutational, A_5: permutational with thirds and A_6: colimit" (p. 26).

The works of Harkleroad (2006), Loy (2006), Zhang (2009), Townsend (2011), and Andreatta et al. (2013) are a pleasure to read if the reader has the knowledge and understanding of both the involved high level mathematics and the music theoretical concepts. Due to the level of mathematics used, the ideas of these works cannot be presented in these forms to secondary mathematics students, unlike the ideas of the present study, which targeted Year 11 students, secondary level mathematics, functions, as described in the Australian Curriculum 2014, using relevant secondary level mathematics and basic music concepts perceptions. These can be considered as general human music intelligence factors, such as being able to observe sound pitch, volume, music transformation, repetition and tempo, and presenting relationships and analogies between function and music in a mode which is accessible to senior secondary mathematics students.

The phenomenon named “Mozart effect” states that listening to Mozart’s “*Sonata for Two Pianos in D Major (K. 448)*” temporally improves the spatial-temporal reasoning (Rauscher et al., 1993). The Mozart effect has been researched and reported in different studies with varied results, including controversial outcomes. Rauscher & Shaw (1998) state that “some researchers, however, have not reproduced these result, casting some doubt on the generalizability of the effect” (p. 835). For instance, McKelvie and Low (2002), claim that the Mozart effect does not exist: listening to the mentioned Mozart sonata does not improve significantly student's short term spatial ability (McKelvie & Low, 2002). The Mozart effect is a typical example of passive integration of mathematics and music, where the effects of listening to this beautiful Mozart composition have been researched, unlike my study, which targeted explicitly the active integration of mathematics and music, where accessible musical structural analogies serve to describe well defined mathematical aspects of the function concept through using students' general human music intelligence.

Montessori (1912) developed a complex educational program, more than 100 years ago, which has been very successful since that time. Montessori’s educational program includes education of the senses (vision, taste, smell, and so on); education of the intellect, education in language, education in mathematics; education in music; education in manual work, art and building; education in agricultural work (culture of plants and animals); muscular education (gymnastics) including education of the child’s diet, and the child’s preparation for the social life (Montessori, 1912). Montessori's comprehensive educational program is highly integrative, integrating a range of subjects in different modes, appealing to the complex education of students' all intelligences. My study has actively integrated mathematics and music in a particular way, has appealed to students' intellect and intelligences through the presented analogies which have described the same aspects of a concept in many different ways (embodiments): mathematically, visually, and musically.

Harris (2005) showed that 3-5 years old children who received music-enriched instruction through the Montessori Mozart Programme had significantly higher mathematics scores than children in a control group. Furthermore, after comparing music-enriched Montessori instruction and traditional Montessori instruction, Harris (2007) observed statistically significant differences in mathematics scores between students who received traditional Montessori instruction and students who received music-enriched Montessori instruction. Harris (2007) states that:

> It appears that students who received music-enriched Montessori instruction had higher levels of mathematics achievement than students who received traditional Montessori instruction.
>
> (Harris, 2007, p. 33)

In general terms, learning a new concept is a construct in students' understanding. Thompson (1994) emphasises that we cannot directly research the way of understanding of a single, targeted mathematical concept. There are many other concepts, for example, expression, operation, quantity, variable, equation, and so on, still under development in students' understanding, and this dynamic construct has an influence on the understanding of the targeted concept. (Thompson, 1994). This fact is in correlation with Piaget's theory of cognitive development (Wood, Smith, & Grossniklaus, 2001), where the processes of assimilation (adapting the new concept into a pre-existing cognitive schema), accommodation (changing pre-existing concepts to understand the new concept) and equilibration (balancing assimilation and accommodation in order to create stable understanding) play a central role during learning of a new concept. Piaget's theory of cognitive development (retrieved from http://psych.colorado.edu/~colunga/P4684/piaget.pdf) has applications in education. Wood et al. (2001) describe the four stages of cognitive development defined by Piaget as follows: (1) sensorimotor stage (birth - 2 years old), (2) preoperational stage (2-7 years old), (3) concrete operations stage (7-12 years old), and (4) formal operations stage (12+ years old). These cognitive developmental stages cannot be skipped, the cognitive development is not discontinuous, every human has to go through every stage in this order of stages, however, the speed of cognitive development is highly individual for every person depending on a range of factors. Piaget's theory of cognitive development can be applied in mathematics education (Ojose, 2008). Every developmental stage ties developmentally appropriate mathematics instruction. My study, involving 16-17 year old students, addressed the appropriate mathematics instructions of the formal operation stage, when students operate with abstract concepts such as function and relation, and are able to understand different analogies between similar structures of the concepts of mathematics and music. The characteristic mathematical level of the formal operation cognitive developmental stage is when students “typically begins to develop abstract thought patterns where reasoning is executed using pure symbols” (Ojose, 2008, p. 28). The formal operation stage involves pure mathematical reasoning skills which refer to the mental process of generalising and evaluating of logical arguments and abstract concepts, and include clarification, inference, evaluation, and application. Ojose (2008) has described these as follows:

- Clarification. Clarification requires students to identify and analyze elements of a problem, allowing them to decipher the information needed in solving a problem. By encouraging students to extract relevant information from a problem statement, teachers can help students enhance their mathematical understanding.
- Inference. Students at this stage are developmentally ready to make inductive and deductive inferences in mathematics. Deductive inferences involve reasoning from general concepts to specific instances. On the other hand, inductive inferences are based on extracting similarities and differences among specific objects and events and arriving at generalizations.
- Evaluation. Evaluation involves using criteria to judge the adequacy of a problem solution. For example, the student can follow a predetermined rubric to judge the correctness of his solution to a problem. Evaluation leads to formulating hypotheses about future events, assuming one’s problem solving is correct thus far.
- Application. Application involves students connecting mathematical concepts to real-life situations. (Ojose, 2008, p. 28)

My study has encouraged students through the mathematical-musical analogies to clarify different aspects of function, identify and analyse the corresponding elements of similar concepts in mathematics and music. Both the deductive and inductive inferences were present in the study. The deductive inference, reasoning from general concept to specific instance, was present in the study, for example, in the case of horizontal dilation of function. Students have observed that the musical analogy of horizontal translation of function is change of the tempo (speed) of music. From the general mathematical concept, $g(x) = f(c \times x)$ with $c > 0$, students have deducted the specific analogous musical instances of faster, equal, or slower tempo of music, corresponding to $c > 1$, $c = 1$, or $c < 1$. The inductive interference, which is based on extracting similarities and differences among specific objects and events and arriving at generalizations, was present in my study as a result of the nature of the analogies, which presented logical similarities and physical differences between mathematical and musical objects and expressed the generalizations of the concepts.

This study set out to improve secondary students' understanding of function through integration of mathematics and music, and I believe that the study challenged and helped participant students' intrinsic cognitive processes of assimilation, accommodation and equilibration.

Gardner (1983) has underlined that some composers have shown interest in mathematics related activities and a range of mathematicians have been interested in music. A number of famous scientists and mathematicians have acknowledged the deep connections and analogies between mathematics and music: Pythagoras, Euclid, Eratosthenes, Ptolemy, Galileo, Mersenne, Descartes, Huygens, Leibniz, Bernoulli, Euler, Bolyai, Jacobi, Sylvester, Helmholtz, Poincaré, and Einstein, to name the greatest of all time. Some of them have described mathematics in terms of music, or music in terms of mathematics. The English mathematician, James J. Sylvester (1814-1897), the founder of the American Journal of Mathematics, who took singing lessons from Gounod and wrote poems as well, has described mathematics as “the music of reason” and music as “the mathematics of sense”. “May not Music be described as the Mathematic of sense, Mathematic as Music of the reason? Thus the musician *feels* Mathematic, the mathematician *thinks* Music" as cited in Bell (1953, pp. 445-446). The German mathematician and philosopher, Gottfried Wilhelm

Leibniz (1646-1716), described music as being the subliminal counting of the human soul. Leibniz wrote in a letter to Goldbach on 17 April 1712 in Latin, as cited and translated in Schopenhauer (1969, p. 256, n. 46): „Musica est exercitium arithmeticae occultum nescientis se numerare animi" (Music is an unconscious exercise in arithmetic in which the mind does not know it is counting). The Hungarian mathematician, János Bolyai (1802-1860), one of the founders of the non-Euclidean geometry, and who was a virtuoso violin player as well, has written a mathematical music theory. Bolyai's nearly 200 years old work can be found in Benkő (1975). The German polyhistor Hermann von Helmholtz (1821-1894) considered music and mathematics as the two most immaterial, fleeting and delicate sources of impression on human mind. Farge (2005) cites Helmholtz as follows:

> I have always been attracted by the wonderful, highly interesting mystery: it is precisely in the doctrine of tones, in the physical and technical foundations of music, which of all arts appears to be the most immaterial, fleeting, and delicate source of incalculable and indescribable impression on our mind, that the science of the purest and most consistent though, mathematics, has proven so fruitful.
>
> (Helmholtz, as cited in Farge, 2005, p. 8)

The Romanian composer, George Enescu (1881-1955), Yehudi Menuhin's violin teacher, stated that "Music is related to mathematics. But the great musicians were not mathematicians, if you want, they were, but unconsciously. Wagner was an unconscious mathematician" (Benkő, 1975, p. 22). The composer George Barati (1913-1996), director of the Honolulu Symphony and Opera, who corresponded with Albert Einstein, has written an article about his perceptions regarding the relationships between music and mathematics (Barati, 1966). The Russian composer, Igor Stravinsky stated that "Musical form is close to mathematics – not perhaps to mathematics itself, but certainly to something like mathematical thinking and relationship" (Source: http://www.math.utep.edu/Faculty/lesser/M&Mquotes.html).

The concept of function is a very important concept in mathematics. There is evidence that students have difficulties learning different aspects of function. Eisenberg (2002) states that function "it proves to be one of the most difficult concepts to master in the learning sequence of school mathematics" (p. 140). Students who don't understand functions cannot master advanced mathematical concepts, as a result they cannot develop higher order mathematical thinking, which is critical not only to their mathematical thinking. Thus the process of understanding of the concept of function as well as identifying function related misconceptions has been widely researched (Brown, 2009; Clement, 2001; Evangelidou, Spyrou, Elia, & Gagatsis, 2004; Faulkenberry, 2011; Hitt, 1998; Kalchman & Fuson, 2001; Laverne, 1971; Panaoura, Michael-Chrysanthou, Gagatsis, Elia, & Philippou, 2017; Tall & Vinner, 1981; Thomas, 2003; Vinner, 1983; Zazkis, Liljedahl, & Gadowsky, 2003).

In the process of learning functions students construct a personal cognitive concept image of function. Thompson (1994) states that "a predominant image evoked in students by the word 'function' is of two written expressions separated by an equal sign" (p. 5). A very important question in learning functions is how does students' constructed subjective, personal concept image of function relate to the objective, formal concept definition of function (Vinner, 1983; Tall & Vinner 1981). Functions may be thought of in different ways through multiple representations or embodiments – such as algebraic, geometric, and tabular function representations, to name the most traditional and conventional representation modes – that define the mode and the manner in which students construct their own personal concept image of function. Tall and Vinner (1981) state that:

> The function may be thought of as an action which maps *a* in *A* to *f(a)* in *B*, or as a graph, or a table of values. All or none of these aspects may be in an individual's concept image. But a teacher may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure.
>
> (Tall & Vinner, 1981, p. 153)

Research shows (Vinner, 1983) that secondary students' concept image of function is different to the concept definition of function which causes cognitive conflicts.

My study relates to the studies of Tall and Vinner (1981), and Vinner (1983) because it targets the improvement of my students' concept image of function through presenting functions not “as an action which maps a in A to f(a) in B, or as a graph, or a table of values” (Tall & Vinner, 1981, p. 153), but in a non-conventional way, through musical representations, in this mode contributing to improve my students' concept image of function.

Thomas' (2003) research study, performed at The University of Auckland, considers teacher trainees' understanding of function with respect to algebraic, graphical, ordered pair, and tabular representations. The participants were asked to decide whether or not a given representation is a function, giving a reason for their answer. Thomas (2003) reports that trainee teachers displayed “gaps in their understanding of function” (p. 4-292). Also the collected qualitative data confirm this statement. There were teacher trainee comments such as: “I suppose I realised how unsure I am about what makes a function a function" and "Thanks for reminding me what a function is. But I still couldn't remember what it is exactly” (p. 4-297). Thomas (2003) states that:

> For many teachers the graphical representation of function is becoming dominant to such an extent that it could hinder a growth in inter-representational understanding. Certainly the teachers, and hence their students, would benefit from development of stronger inter-representational thinking about function.
>
> (Thomas, 2003, p. 4-298)

My study relates to Thomas' (2003) study by considering algebraic, graphical, and tabular representations of function and presenting the corresponding musical representations (embodiments) of function, in this way improving my students' inter-representational thinking about function.

Evangelidou et al. (2004) also have outlined university students’ misunderstandings of the concept of function. Evangelidou et al. (2004) have emphasized the importance of proper understanding of the function concept.

Kalchman and Fuson (2001) state that:

> The topic of functions has been widely recognized as being central and foundational to mathematics in general. Literature indicates, however, that students of all ages have difficulty mastering the topic using traditional instruction approaches. The roles of numeric and spatial understanding in this domain are critical given that a concern among mathematics educators is that students have difficulty not only with moving among representations of a function (e.g., table, graph, equation, verbal description), but also with understanding how and why the function concept is 'representable' in tables, graphs, and equations. Each of these representations embodies both spatial and numeric aspects of any function.
>
> (Kalchman & Fuson, 2001, p. 195)

Different research articles describe the ways of learning the concept of function. These ways or methods include mainly integrated logical-mathematical (algebraic) and visual-graphical (geometric) approaches. An important aspect of learning mathematics is identifying the sources of lack of knowledge and common misconceptions. If these are identified then appropriate pedagogical methods can be employed to overcame them. Mathematics teachers' understanding of function is also researched (Hitt, 1998; Zazkis et al., 2003). Some lack of knowledge and understanding of function is reported in the case of secondary mathematics teachers as well.

Hitt (1998) studied the errors committed by some secondary mathematics teachers and their understanding of the concept of function, comparing these with the errors committed by secondary mathematics students. The thirty secondary mathematics teachers in Hitt's (1998) sample were beginning a postgraduate course on mathematics education. They were asked to answer fourteen questionnaires regarding different function representations, two questionnaires per week for seven consecutive weeks, working individually for one hour per questionnaire. Hitt (1998) reports that “like students at secondary school, the teachers exhibit errors when they confound the subconcepts of the concept of function (Domain, Image Set)” (p. 129). The participating secondary mathematics teachers had major difficulties with questions regarding composite functions. Hitt (1998) reports that when teachers were asked to give two examples of real variable functions such that the composite function $(f \circ f)(x) = f(f(x)) = 1$ for every $x \varepsilon R$, then “not one teacher gave a correct response, correctly constructing two functions with this property” (p. 129). Out of thirty there were twenty one correct responses to the first function only, by giving the first function $f_1(x) = 1$ for all $x \varepsilon R$, which is an evident and banal example of a function such that $f_1(f_1(x)) = 1$ for every $x \varepsilon R$. Hitt (1998) reports that twelve teachers in their answer to the second function basically have repeated their first function f_1 in a different form as being f_2, such as $f_2(x) = \sin^2(x) + \cos^2(x)$, which is obviously incorrect, because in this case f_2 and f_1 are identical, as $f_2(x) = \sin^2(x) + \cos^2(x) = 1 = f_1(x)$. Hitt (1998) is not presenting a requested second function, a non-evident and non-banal example of a correct response to this question. My response to this question would have been the following two functions:

f_1: $R \rightarrow R$, $f_1(x) = 1$ if $x < 1$, and $f_1(x) = 1 / x$ if x is more than or equal to 1.

<u>Proof</u>: (i) If $x < 1$ then by definition $f_1(x) = 1$, as a result $f_1(f_1(x)) = f_1(1) = 1$.

(ii) Otherwise $f_1(x) = 1/x < 1$, as a result $f_1(f_1(x)) = f_1(1/x) = 1$.

Based on (i) and (ii) above we can state that $f_1(f_1(x)) = 1$ for all $x \varepsilon R$, Q.E.D.

f_2: $R \rightarrow R$, $f_2(x) = 1$ if x is less than or equal to 1, and $f_2(x) = x / (x + 1)$ if $x > 1$.

<u>Proof</u>: (i) If $x < 1$ or $x = 1$ then $f_2(f_2(x)) = f_2(1) = 1$.

(ii) If $x > 1$ then $f_2(x) = x/(x+1) < 1$, as a result $f_2(f_2(x)) = f_2(x/(x+1)) = 1$.

Based on (i) and (ii) above we can state that $f_2(f_2(x)) = 1$ for all $x \varepsilon R$, Q.E.D.

We can see that the two functions given above are different to each other, for example, f_1 is continuous on R while f_2 is discontinuous at the point $x = 1$, because $\lim_{x\rightarrow 1} f_2(x) = f_2(1) = 1$ and $\lim_{1\leftarrow x} f_2(x) = \lim_{1\leftarrow x} [x / (x+1)] = ½$, so the left and right limits of $f_2(x)$ at $x = 1$ are different.

An interesting aspect of Hitt's (1998) article is the result regarding teachers' teaching preferences of the definition of the concept of function. Teachers' teaching preferences of the definition of the concept of function has a major effect on students' understanding of function. The majority of participating secondary mathematics teachers preferred the definition of function through the algebraic rule of correspondence or through the set of ordered pairs. The definition of the function concept through the concept of variable was not favoured by the participating teachers. In my teaching practice I present to the students all the standard definitions of function, together with various examples. Hitt (1998) concludes the article with the summary stating that "the results show that in unusual learning situations, this group of teachers does not coherently articulate between various systems of representation involved with the concept of function, due to various difficulties" (p. 134). I think that this is due to the fact that some secondary mathematics teachers were not adequately taught by their teachers on one side, and they have not observed their own misconceptions, or they have not corrected their already identified and known misconceptions on other side. I think that both cases are a potential danger to the future of mathematics education. In my eyes no secondary mathematics learning situation in terms of function has to be unusual for a secondary mathematics teacher who is teaching functions, such an important mathematical concept. Also, to articulate between various systems of representation of the concept of function without any difficulties should be expected from every secondary mathematics teacher who is in charge to teach functions to secondary mathematics students.

Zazkis et al. (2003) have studied pre-service secondary teachers (N = 15), practising secondary teachers (N = 16), and year 11 and 12 senior secondary students (N = 10) knowledge and explanations of horizontal translation of the graph of the simplest quadratic function, $y = x^2$. Zazkis et al. (2003) have addressed the participants' expectations and intuitive description of the relationship between the graph described by the quadratic function $y = x^2$, and the graph described by the function $y = (x - 3)^2$, obtained by a horizontal translation of the function $y = x^2$. The results of Zazkis et al. (2003) confirm that "the horizontal shift of the parabola is, at least initially, inconsistent with expectations and counter intuitive to most participants" (p. 437).

I also have observed similar phenomena regarding function transformations during my mathematics teaching practice, and I agree that many senior secondary students have problems with these type of simple horizontal translations of functions. Usually students think that, for example, the graph of the translated function $y = f(x - 3)$ is to the left of the graph of the original function, $y = f(x)$. The reason is not surprising:

lack of proper conceptual understanding of function. In the present study I have addressed also the horizontal translation of function, with the help of a musical analogy, see Analogy 6, section 3.3.6 in Chapter 3, stating that the horizontal translation of a function y = f(t) by a constant *b*, which is y = *f(t+b)*, has an analogy in music: the musical repetition. If the translation constant is negative, b < 0, e.g. b = -3, as it is in the case of the study of Zazkis et al. (2003), then the graph of the translated function will be shifted -b = 3 steps horizontally to the right of the graph of the original function. This can be explained naturally, through music, if we think in terms of the above mentioned musical analogy. If we have a melody, represented by a function f(t), then the translated function g(t) = f(t-3) is nothing else but the repetition of the original melody which was played e.g. 3 minutes ago! Obviously, the repeated music piece g(t) = f(t-3) comes after the original music piece f(t), which means mathematically, that the graph of the function g(t) = f(t-3) is "to the right" of the graph of the function f(t).

As mentioned above, Zazkis et al. (2003) have studied thirty one secondary mathematics teachers' and pre-service secondary mathematics teachers' understanding of horizontal translation of function through a particular question. Zazkis et al. (2003) have interviewed the participant secondary teachers and pre-service secondary teachers (N = 31) asking them to explain the question why the graph of the translated function y = f(x – 3), in this case $y = (x-3)^2$, is to the right of the graph of the original function, y = f(x), in this case $y = x^2$, and the authors (Zazkis et al., 2003) have summarized teachers' responses as follows:

> A majority of the interviewed teachers (18 out of 31) referred to the 'rule of horizontal translation'. According to this 'rule', $y = (x-3)^2$ has the same shape as $y = x^2$ but is located three units to the right. Thus, teachers' reliance on rules is indicative of students' reliance on memorizing these rules. To reinforce memorization and to explain function translation to their students several practicing teachers have formulated 'the law of opposites'. They indicated that having 'the law' helped their students in 'getting it right'. This, of course, raises issues with regard to the purpose and value of mathematics education. If the purpose is to 'get it right' on an exam, then introducing such a law has its merit. However, if the purpose is to teach mathematical

> thinking, then the creation of such a law jeopardizes the consistency of mathematical structure and directs students' attention to memorization rather than to explanation.
>
> (Zazkis et al., 2003, p. 442)

I also have observed similar signs during my teaching practice, which signs – I think – are the results of the reproductive thinking mode, which is approaching mathematics as a collection of definitions, formulae, theorems and problem-solving-steps to be memorised and reproduced. I am aware of the fact that real mathematics is not about memorising formulae, rules, laws and problem-solving-steps. As English (2007) emphasised, mathematics is the study of structure. The most important goal of mathematics teaching is to develop students' skills to detect and observe patterns in similar or seemingly not similar situations, as opposed to develop reproductive thinking focused on memorising.

English (2007) gives the summary of development stages of mathematics education, which is in close relation with cognitive psychology. These stages are:

- "Meaningful Learning" (1930s and 1940s); learning with understanding (Brownell)
- "Gestaltists" (1950s); productive thinking (Van Engen, Wertheimer)
- "New Mathematics" (1960s); multiple embodiments from concrete to symbolic (Bruner, Dienes)

English (2007) states that there were many developments in mathematics education in subsequent years, which include:

- a decreased emphasis on constructivism as the dominant paradigm,
- new developments in the learning sciences, in particular, a focus on complexity theory,
- an increased focus on mathematical reasoning and interdisciplinary modelling,

- a significant increase in research on the mathematics needed in various work place, settings and the implications for mathematics education,
- a broadening of theoretical perspectives, including an increased focus on social-cultural-political aspects of mathematics education,
- developments in research methodology, in particular, a focus on design research, and
- the increased sophistication and availability of technology.

Faulkenberry (2011, p. 49) states that "conceptual knowledge refers to knowledge of the underlying structure of mathematics" and emphasises the importance of conceptual understanding instead of memorising and applying rules. Faulkenberry (2011) states that:

> Transformations of functions is a topic that is often taught by memorizing and applying rules without an understanding of the underlying concept. The National Mathematics Advisory Panel (2008) report states that many students do not understand the procedures for transforming functions or why they are done the way they are.
>
> (Faulkenberry, 2011, pp. 49-50)

My study presented transformations of functions in a special way, through musical analogies, which does not appeal on memorising and applying rules, but on helping my students to broaden their conceptual understanding of the underlying concept and their understanding of the procedures of function transformations. It is important to mention the fact, which also was a positive feedback regarding my teaching practice and the chosen musical analogies of function transformations, that in my study the largest improvement, 21.59%, of my students' understanding of six different aspects of function was achieved on function transformations. See the details in Chapter 4.

Because mathematics and music have a range of similar patterns, and I did not want my students to memorise or recall different formulae, laws and rules regarding functions but rather to observe and detect analogue structures in a seemingly not similar subject, I came to the conclusion to develop relevant analogies between function and music and perform this action research to improve my students' conceptual understanding of function.

English (2007) states that “students’ development of powerful models should be regarded as among the most significant goals of mathematics education” (p. 121). The designed nine mathematical-musical analogies of my study provided nine examples of powerful mathematical models for students. My study used an integrated mathematical-musical approach through presenting analogous structures between function and music, targeting senior secondary (Year 11) mathematics students' conceptual understanding of function, a topic that has not been widely researched. In my study the goal was to provide my students with powerful models in order to improve their conceptual understanding of function and – partly – to bring about a possible change in my students’ attitudes and beliefs regarding mathematics. Again, neither phenomena appears to have been widely researched in this way.

My study was not a correlational study, but an active interventional study: It was not about the effects of music or music education on students' mathematics outcome, and not about passive listening to music before or during mathematics lessons. There was a very strong emphasis on active integration of mathematics and music through presenting structural analogies between functions and music. These structural analogies constitute powerful musical models of the six aspects of function described in Australian Curriculum 2014.

2.4 Brief History and Importance of Function in Mathematics Education

My study has addressed the concept of function, a very important mathematical concept. Eisenberg (2002) states that “the function concept has become one of the fundamental ideas of modem mathematics, permeating virtually all the areas of the subject” (p. 140). Kleiner (1989) describes the circa 4000 year history of the development of the concept of function. The first 3700 years can be considered as anticipation, preparing the way to evolve the development of the modern concept in the last 300 years, parallel with the development of mathematical analysis and calculus to the form as we know and accept the function concept today. Kleiner

(1989) cites the author W. L. Schaaf, who in his work, titled "*Mathematics and World History*", published in 1930, stated that:

> The keynote of Western culture is the function concept, a notion not even remotely hinted at by any earlier culture. And the function concept is anything but an extension or elaboration of previous number concepts — it is ather a complete emancipation from such notions.
>
> (Schaaf, 1930, as cited in Kleiner, 1989, p. 1)

This observation explains that the concept of function shaped and formed the way of thinking of a whole human culture, because the development of the concept had a major influence on the ways of development of different mathematical subjects and mathematics itself as a whole. The development of mathematics definitely had a major role on both describing physical processes and influencing technical development of the human society in the last 300-350 years. Because the "nature is written in the language of mathematics" (Galileo Galilei), the concept of function has had a distinguished role in discovering and describing nature's rules.

The word "function" was introduced by Leibniz in 1692, who observed that the tangent to a curve depends upon the curve, in other words the tangent is a function of the curve. In the case of this function, using the latter introduced modern concepts as we know them today: the domain is the set of all points of a curve, the independent variable or the argument of the function is a point on the curve, the range is the set of tangent lines, and the dependent variable or the image of the argument is the unique tangent to the given curve at the given point of the curve.

Kleiner (1989) cites the definition of function given by Euler in 1755:

> If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others. If, therefore, x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions…
>
> (Euler, 1755, as cited in Kleiner, 1989, p. 7)

Kleiner (1989) also cites the definition of function given by the French mathematician Jean-Baptiste Joseph Fourier (1768 - 1830):

> In general, the function f(x) represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa *x*, there are an equal number of ordinates f(x). All have actual numerical values, either positive or negative or null. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity.
>
> (Fourier, as cited in Kleiner, 1989, p. 8)

Fourier did not state explicitly that for every abscissa x we correspond an unique ordinate f(x) but Fourier's above expression of "there are an equal number of ordinates f(x)" implies that Fourier was thinking about a function as we define a bijective function today. Kleiner (1989) states that "Fourier's work raised the analytic (algebraic) expression of a function to at least an equal footing with its geometric representation (as a curve). His work had a fundamental and far-reaching impact on subsequent developments in mathematics" (p. 8). Fourier's mathematical work had an impact on mathematical description of music theory as well. Visintin (2017) in the article about Fourier series and music theory describes how Fourier series can be used to describe the harmonics (also named overtones) of a frequency which are sinusoidal functions, multiples of the frequency considered, and explain why we hear differently the same sound (frequency) produced by two different instruments. For example, as Visintin (2017) states, "the clarinet produces mainly odd-numbered harmonics, whereas the guitar produces even as well as odd harmonics. This is due to the structure of these instruments" (p. 3). We can see how important is the theoretical knowledge and physical applications of function in music theory and practice.

There are different approaches how to define the concept of function. All of these different definitions of function are mathematically correct and equivalent but raise different questions and challenges in mathematics education. The standard definition of function – used since the introduction of Georg Cantor's (1845 - 1918) set theory at the beginning of the 20th century – is stated by Thompson (1994) as follows:

> The current standard definition of function highlights correspondence over variation — elements in one set correspond to elements in another so that each element in the first corresponds to exactly one element in the second. Since the 1930's this ordered pair notion of function has been taken as the 'official' definition of function, largely because it solved many problems introduced historically by people like Fourier who wished to define functions by a limit process. The ordered-pair definition has received strong criticism on pedagogical grounds that it can be meaningful only to people who recognize the problems it solves, but not to a student who is new to the idea of function.
>
> (Thompson, 1994, p. 10)

In terms of pedagogy, Eisenberg (2002) gives different ways how the concept of function can be introduced to students:

> At the definition level the function concept can be introduced in a variety of contexts, through arrow diagrams, tables, algebraic description, as a black input-output box, as ordered pairs, etcetera. Of all of these approaches the pedagogically weakest and nonintuitive one seems to be the approach using ordered pairs. Here, a function is defined as a certain sort of set; one which is made up of ordered pairs in which no two ordered pairs have the same first element and different second elements. This seemingly innocent definition proved to conjure up all kinds of logistic and epistemological problems, which incredibly, were often addressed explicitly in some school curricula.
>
> (Eisenberg, 2002, p. 141)

Maurer and Virág (1976) give the abstract algebraic definition of the concept of relation with *n* variables as follows. A **relation** defined on the sets $A_1, A_2, \ldots, A_n$ ($n \varepsilon N$) is the system $(A_1, A_2, \ldots, A_n, S)$, where S is a subset of the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$. Notation: $\rho_S = (A_1, A_2, \ldots, A_n, S)$. If $n = 2$ then ρ_S is a binary relation (or a relation with two variables). If $n = 3$ then ρ_S is a ternary relation (or a relation with three variables), (Maurer & Virág, 1976, p. 29). If we consider a binary relation, $\rho_S = (A, B, S)$ and *a* is an element of the set A ($a \varepsilon A$) then the section of the relation

ρ_S with respect to the element *a* is defined being the set $\rho_S[a]$ containing all the elements *b* of the set B (*b* ε B) having the property that (a, b) ε S.

$$\boldsymbol{\rho_S[a] = \{b \, \varepsilon \, B| \, (a, b) \, \varepsilon \, S\}}$$

Maurer and Virág (1976) give the abstract algebraic definition of the concept of function as follows. The binary relation ρ = (A, B, S) is a **function** if for every *a* ε A, the section $\rho_S[a]$ contains one and only one element *b* ε B (Maurer & Virág, 1976, p. 41). At secondary mathematics level the definition of the function concept is given in a different form, stating that a function is a correspondence between two sets A and B connecting every element *x* ε A with one and only one element *y* ε B. We use the notations f: A → B, f(x) = y, where x ε A, and we name the set A the domain and the set B the range of the function *f*. The graph of the function f is the set G = {(x, f(x))| x ε A}. If we compare this definition to the general definition given by Maurer and Virág (1976) as described above then we can observe that they are in accordance because all three sets A, B and S of a binary relation ρ = (A, B, S) are present. The first set A is the domain, the second set B is the range, and the third set S, the subset of the Cartesian product A x B, is the set S = G, the graph of the function *f*. We can see that the mathematical concept of function is a specific form of the mathematical concept of relation. Because the sets A and B can be any type of sets, the mathematical concepts of relation and function can be presented in many different ways. In my study I have considered that the set A is the set of time and the set B is the set of musical frequencies, amplitudes, and piano-keys. Based on these considerations I have designed nine musical analogies of different aspects of function and relation at senior secondary mathematics level, which are presented in this study. See Analogies 1 - 9 in sections 3.3.1 - 3.3.9 in Chapter 3.

Analogy 2 (see the section 3.3.2 in Chapter 3) presents the concept of musical harmony as an embodiment of the concept of mathematical relation. The abstract algebraic definition of the concept of relation on the sets R^+, A_1, A_2, …, A_n (n ε N) offers this interesting musical analogy. If R^+ represents the set of time and the sets A_1, A_2, …, A_n (n ε N) represent the musical sounds played by *n* (n ε N) instruments then a mathematical relation $\rho_S = (R^+, A_1, A_2, \ldots, A_n, S)$ represents a musical symphony played by these *n* (n ε N) instruments. Analogy 3 (see the section 3.3.3 in

Chapter 3) constitutes the musical equivalence of the Vertical Line Test for Function taught at secondary level mathematics, which is used to decide if a given visual graph represents a function or a relation.

Balázs and Kolumbán (1978) define the concept of *m* dimensional vector function having *n* variables as follows. Let A be a non-empty subset of the n-dimensional Euclidean space, $A \subset R^n$. If we attribute for every element *x* of the set A one and only one element $y = f(x)$ of the set R^m, through a given process, then we say that we have defined a **vector function** with *n* real variables and we denote it by $f: A \rightarrow R^m$. Observations: (1) if n = m = 1 then we speak about a real valued function having one real variable, (2) if m = 1 then we speak about a real function having *n* real variables (Balázs & Kolumbán, 1978, p. 95).

We can observe that both definitions of function given above define precisely the general concept of multi-dimensional function with *n* variables, which are generalisations of the function concept taught at secondary level. The definition given by Maurer and Virág (1976) is a top-down definition because the concept of function is defined as a particular case of the more general concept of relation, and it is introduced based on previously defined concepts of binary relation and of section of the binary relation with respect to an element. The definition given by Balázs and Kolumbán (1978) is a bottom-up definition because the concept of function is defined based on set theoretical concepts only, without using any additional higher order mathematical concept.

The concept of function is defined at secondary level differently compared to the above definitions, because the function with one variable is defined. The Year 11 Mathematical Methods mathematics textbook (Haese Mathematics, 2015), used by the participants of this study, is presenting a top-down function definition mode. The concept of binary relation is defined first – without stating that the relation defined is a binary relation – as follows: "a **relation** is any set of points which connect two variables" (p. 62), then the function concept is defined as being a special relation: "a **function** is a relation in which no two different ordered pairs have the same x-coordinate or first component" (p. 63). Secondary students can conclude that every function is a relation, but relations are more than functions. They may also observe that the set of functions is a subset of relations. After learning the definition of the concept of function (where both the independent and dependent variables are one dimensional values, the function usually being f: $R \rightarrow R$), the secondary students learn about function notation, graph of function, function transformations such as horizontal and vertical translations and dilations, as well as making difference between function and relation by using a visual-graphical test, the Vertical Line Test for Function, as described in Australian Curriculum 2014. A range of articles report that secondary students have different types of difficulties and misconceptions regarding some aspects of function (Anabousy, Daher, Baya'a, & Abu-Naja, 2014; Eisenberg, 2002; Faulkenberry, 2011; Hitt, 1998; Kalchman & Fuson, 2001; Steketee, 2015; Zazkis et al., 2003). I made an effort in this study to help my students to overcame these difficulties in a non-standard but natural way: by active integration of mathematics and music through nine self-designed musical analogies regarding different aspects of function. In many mathematics books, including textbooks and relevant articles, functions and graphs are connected. The logical-mathematical abstract constructs of functions are presented by their visual graphs, in this way helping a better understanding. The algebraic and visual "symbol systems both contribute to and confound the development of understanding" (Leinhardt, Zaslavsky, & Stein, 1990, p. 3). In my study I have introduced the third symbol system, which is musical. Leinhardt et al. (1990) state that:

> The bridge between functions and graphs is also interesting because the intellectual landscape, so to speak, looks different from each side of the bridge — if graphs are used to explicate functions, the sense of function (and graph) is quite different from what is presented the other way around.
>
> (Leinhardt et al., 1990, p. 3)

The first part of the title of this thesis, "Melody of Functions and Graphs", is expressing my goal to present musically some concepts which usually are presented algebraically and graphically. I think that this idea is another bridge towards understanding functions, because if music is used to explicate functions then the sense of function (and graph) is different.

In their article, Anabousy, Daher, Baya'a, and Abu-Naja (2014), exploring Grade 9 students' learning of function transformation by using the “Geogebra” software, state that the students succeeded to work with function transformations in their algebraic and graphic representations, but they have had difficulties in their verbal representations, as:

> The participating students had some difficulties working verbally with function transformations, especially when the reflection transformation was involved. This result points at the need of students' involvement with three representations to conceive deeply and accurately the different themes of the topic, namely, the algebraic, the graphic and the verbal representations.
>
> (Anabousy et al., 2014, p. 97)

Anabousy et al. (2014) concentrate on three intelligences described in Gardner's (1983) Multiple Intelligences theory, namely: mathematical-logical, visual-spatial and verbal intelligences. My study relates to Anabousy et al. (2014) by extending the range of representations of function transformations with musical representations, using the musical intelligence in addition to the other intelligence channels.

Different representations of function are important in Janvier's model, named the “star model for understanding functions”, a traditional model to learn the concept of function. Janvier (1985) states that “we showed that the concept of function was only unique from an axiomatic perspective. In other words, it could be broken into several non-overlapping semantic domains, such as variable, transformation, sequence, isomorphism” (p. 2). Janvier's star model for understanding functions does not consider musical representation modes of function. My study has considered musical representations of different aspects of function, improving my Year 11 mathematics students' understanding of function in this way.

The geometric transformations of rotation, transfer, and mirroring in the curriculum are not connected to functions, but in reality they are geometric functions. Steketee (2015) discusses the potential value of using technology-supported geometric transformations, referred as “*Geometric Functions*” to introduce and develop function concepts. This geometrical approach can help to develop students' intuitive understanding of variables and function and overcame possible misconceptions. Steketee (2015) states that:

- Students have difficulties with many function-related concepts (variable, function, domain, range, relative rate of change, composition, and inverses).
- Students need to experience a variety of functions to form a robust conception.
- Though other examples may be given, the conventional approach quickly settles down to R → R functions. But many important functions do not merely map real numbers to real numbers.

(Steketee, 2015, p. 618)

I agree with the three above observations, and my research also was based on this type of observations. My statements below will show how my study relates to Steketee's (2015) study:

- Students having difficulties with function-related concepts use words like function, expression, variable, equation, argument, value, identity, and so on, not in a proper way, not in context, often interchanging them which is a serious sign of major difficulties with mathematics. During my mathematics teaching career I have observed a range of students having these type of difficulties.
- Students need to experience a variety of functions. My study has been designed to present different aspects of functions in the form of music, offering students to enlarge their conception of function. I can refer to them as “Music Functions”, as an analogy to “Geometric Functions” above.
- Due to the fact that the conventional approach of teaching and learning functions quickly settles down to R → R functions, my study has used a non-conventional approach by presenting functions musically, which are not the traditional R → R functions, in this way challenging and improving my students' traditional understanding of the function concept.

Borba and Confrey (1996) report about a case study regarding function transformations as follows:

> A case study of a 16-year-old student working on transformations of functions in a computer-based, multi-representational environment. The didactic approach to reflections, translations and stretches began with visualisation exercises, and then was extended to investigate the implications of visual changes in data points, and subsequently, in algebraic symbolism.
>
> (Borba & Confrey, 1996, p. 319)

Borba and Confrey (1996) have not used music integration, but they have targeted the concepts of function translations and dilations (“stretches”), which both are targeted also in my study. Their steps of visualisation, data points observation and algebraic observation were present in my study as well, but through a different intelligence channel (Gardner, 1983). For example, listening to a melody played in two different speeds, observing the corresponding parameter changes ($c < 1$ or $c > 1$), then investigating the algebraic and graphic forms of the corresponding function dilation. See Section 3.3.9 Analogy 9 in Chapter 3. Borba and Confrey's (1996) and also my approach are examples of different embodiments (as interpreted by Bruner and Dienes) of the same concept of horizontal dilation of functions.

Sever & Yerushalmy (2007) in their case study of two calculus students' conceptions regarding function dilation stress that “developing competence in solving function related problems, means being proficient in multiple linked representations of functions (graphic, algebraic and tabular) and learning to move freely between them” (p. 1509). My study includes the traditional graphic, algebraic, and tabular representations and the non-traditional musical representation of functions, helping students to become proficient in these types of function representations and to move freely between them.

Borba and Confrey (1996) have used a non-traditional method to improve the understanding of function translations, because “traditionally, transformations of functions have been taught with a strong emphasis on algebraic symbolism” (p. 319).

I have used a different non-traditional method to research a similar topic due to the same reason 20 years after Borba and Confrey's (1996) study.

This concludes the Literature Review. The following chapter addresses the study's Methodology.

Chapter 3

METHODOLOGY

3.1 Introduction

In order to perform the study a range of actions were necessary, starting from the design-phase. In this phase there was a need to read, analyse, and synthetize the existing literature and apply the relevant theoretical and practical knowledge of the topic adequately. Also, it was crucial to perform the research in a professional way, finding answers to the research questions and informing the reader within the framework of an appropriate research report – one that can be checked, applied, modified and re-performed. It was a very important consideration – not to be too theoretical, but suitable, applicable and accessible by mathematics teachers and senior secondary mathematics students regardless of their musical skills and knowledge.

I first describe in Section 3.2 the framework of the study, including issues related to the paradigm, methodology, ethics and validity of the research. Second, in Section 3.3, the nine self-designed analogies between the concept of function and music are described. These nine relevant analogies were selected, considered and defined by myself and they explicitly target the Australian Curriculum's requirements regarding Year 11 Mathematical Methods students' understanding of function. They have been used in this study as tools to improve students' understanding of function. Finally, in Section 3.4, the used research methods are described**.**

3.1.1 Obtaining Research Approvals

The ethics approval to perform this educational research has been obtained Curtin University, being valid for 4 years, between 07/08/2014 - 06/08/2018. The obtained ethics protocol approval number of this research is SMEC-55-14 and the memorandum letter is dated 11 August 2014. See Appendix J.

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In order to perform a formal educational research in South Australian schools, it is compulsory to obtain approval from the school sector governing authorities, school principals, subject learning area coordinators, and subject teachers. It is also compulsory to obtain written consent from the participant students and their parents. I obtained approval from Catholic Education South Australia on 23 July 2015. See Appendix K – Research Approval from Catholic Education South Australia.

3.1.2 Inviting Participants

Informed Consent Forms (see Appendix L) and Participant Information Sheets (see Appendix M) were given to all 47 Year 11 Mathematical Methods students in the secondary college where I was a contract mathematics teacher at that time, and 46 signed Informed Consent Forms (see Appendix L) were returned, acknowledging through their parents'/guardians' signatures that they had read and understood the Participant Information Sheet and agreed to participate voluntarily in the study. Of the 47 Year 11 Mathematical Methods students at the college one student only did not return the signed Informed Consent Form. Out of the 46 students who returned their signed Informed Consent Forms, two students (one from both groups) did not write their post-tests. Their pre-test data is not considered in this study, which includes the finished pre-tests and post-tests data collected from 44 students.

3.2 Framework of the Study

3.2.1 Paradigm, methodology, ethics and validity

The research methodology took the form of an action research study designed to observe if I could improve my students' understanding of the function concept. An experimental and a control group were randomly assigned out of the existing two Year 11 Mathematical Methods classes at a co-educational, multicultural, mixed socio-economic metropolitan secondary 9-12 college in South Australia in 2016. Forty-four students participated in the study, distributed equally between the experimental and control groups. The research was ethical and fair, showing respect toward all participants, without advantaging or disadvantaging any of the participants as described in Creswell (2008). The participant students attended their regular mathematics lessons, they were not asked to attend any additional session or to perform any extra activity. The procedures used in this research were considered part of normal school activities. The importance of internal validity is stressed in Trochim (2006) as “for studies that assess the effects of social programs or interventions, internal validity is perhaps the primary consideration” (Trochim, 2006, Internal Validity, Retrieved from https://socialresearchmethods.net/kb/intval.php).

The internal validity of my research was assured by the high degree of comparability of the two groups before and during the research. The group distribution of the 44 participant students was equal because in both groups there were 22 students. This fact also has assured that the ratio of the group sizes was not greater than 1.5 (being 22/22 = 1 < 1.5), a condition in order to perform a statistical independent t-test. The two groups were comparable in that there were no significant differences in mathematical ability, music studies, gender, mathematics-related beliefs and attitudes, learning environment perceptions, and taught curriculum. The performed MANOVA test revealed that there was no statistically significant difference in terms of music study, $F(36, 1) = 191.85$, $p = .057$, Pillai's Trace = 1.000, Wilks' Lambda = .000, Hotelling's Trace = 6906.43.

The importance of this key internal validity issue, namely the degree to which the groups are comparable before the study, is stated by Trochim (2006) as follows:

> A multiple-group design typically involves at least two groups and before-after measurement. Most often, one group receives the program or treatment while the other does not and constitutes the "control" or comparison group. But sometimes one group gets the program and the other gets either the standard program or another program you would like to compare. In this case, you would be comparing two programs for their relative outcomes. Typically, you would construct a multiple-group design so that you could compare the groups directly. In such designs, the key internal validity issue is the degree to which the groups are comparable before the study. If they are comparable, and the only difference between them is the program, posttest differences can be attributed to the program.
>
> (Trochim, 2006, Multiple Group Threats, Retrieved from https://socialresearchmethods.net/kb/intmult.php)

In my study the two groups were comparable before the study in terms of the students' mathematical ability and previous mathematics results, as demonstrated by the college records. In both groups there were high achievers, good achievers, satisfactory level achievers and non-satisfactory level achiever students. This assured that the dependent mathematical variables were approximately normally distributed in both groups – a requirement in order to perform the t-test analysis. Students' mathematical knowledge, in particular students' knowledge of different aspects of function were equivalent, with the MANOVA statistical analysis test of the mathematics pre-tests confirming this fact, as described in detail in Chapter 4.

The two groups were considered comparable in terms of students' music studies, because in both groups there were students with music studies – 10 students in the experimental group and six students in the control group, and were students with no music studies – 12 students in the experimental group and 16 students in the control group. The two groups were comparable in terms of students' gender also – in the experimental group there were seven females and 15 males, and in the control group there were 10 females and 12 males.

In my study the control group has received the traditional program and the experimental group has received the treatment, the non-traditional program regarding the revision of the function topic, and both groups have improved their knowledge of function. The difference between the two group's improvements is that the experimental group's improvement is statistically significant (p = .011) and the control group's improvement is statistically not significant (p = .192). Trochim (2006) states that “the key question in internal validity is whether observed changes can be attributed to your program or intervention (i.e., the cause) and not to other possible causes (sometimes described as 'alternative explanations' for the outcome)” (Trochim, 2006, Internal Validity, Retrieved from https://socialresearchmethods.net/kb/intval.php). Because both groups had the same probability to be influenced by other possible causes before the treatment day, and because I taught both groups for 90 minutes each on the treatment day, meaning that they had the same level of teacher attention and support, I believe that the observed difference between the two group's improvements of understanding of function can be attributed to the treatment and not to other possible causes.

The presence of different non-cognitive factors of personality attributes, such as students' mathematics related beliefs, attitudes, and learning environment perceptions (named as non-cognitive factors) had the potential to have an important impact on students' mathematics test results. In order to monitor these types of students' non-cognitive factors, relevant data has been collected before and after the treatment by using three questionnaires with proven validity and reliability. The non-cognitive data, collected with the help of a total of 123 five-point, Likert scale items, are grouped in scales or variables, as follows: two scales of students' perceptions regarding their mathematics-related beliefs, four scales of students' perceptions regarding their mathematics-related attitudes, and seven scales of students' perceptions regarding their learning environment. The details of these three questionnaires, Mathematics Related Beliefs Questionnaire (MRBQ) (Op't Eynde & De Corte, 2003), Test Of Mathematics Related Attitudes (TOMRA) (Fraser, 1981; Taylor, 2004), and What Is Happening In This Class (WIHIC) (Fraser, 1998) are presented in Chapter 4. The statistical analysis of data collected by these three questionnaires show that before the treatment the two groups of participating students were fairly equivalent with respect to these non-cognitive factors because out thirteen scales, there were no statistically significant differences on eleven of the scales, and only two scales showed statistically significant different perceptions – the first on the “Teacher as Facilitator” scale (p = .009), and the second on the “Student Self-efficacy” scale (p = .020). These details establish the equivalence of the two groups with respect to the participants' non-cognitive factors.

The students of both groups had two different mathematics teachers prior to the intervention, which explains students' different perceptions regarding the “Teacher as Facilitator” scale. Both groups had learned the topic of function six months before the pre-test studying under the two teachers, essentially in the same way because the mathematics curriculum, textbook, exercises and tests were the same for both groups at the college. The students attended the same secondary college, had access to the same school facilities, had been taught the same Year 11 Mathematical Methods mathematics curriculum program, and had used the same mathematics textbook. These facts also contribute to the high level of equivalence of the two groups.

The research was conducted within the post-positivist paradigm, using the scientific method, quasi-experimental, two-group, pre- and post-tests research design, and involved the collection of quantitative data. Three different research methods were used: questionnaires, measurements and statistical analyses. The independent variables include: gender, mathematics knowledge, music training, mathematics-related beliefs and attitudes, and learning environment descriptors.

3.2.2 The Collected Data

In order to answer the research questions and to monitor the level of equivalence of the two groups, quantitative data was collected from each participant student by using a mathematics test and three questionnaires. The collected data represents two different categories of information: (1) cognitive: students' knowledge of function and (2) non-cognitive: students' perceptions regarding their own mathematical attitudes and beliefs and their learning environment.

The six mathematical dependent variables (see Table 1.2 on page 17) express the extent of understanding of different aspects of the topic of function as defined in the Australian Curriculum 2014. To identify these six variables in the study, I have used the codes ACMMM022, ACMMM023, ACMMM024, ACMMM025, ACMMM026 and ACMMM027 as they are used in the Australian Curriculum 2014.

3.2.3 The Mathematics Test

The first category of data was collected by measurement, using a relevant mathematics test I constructed, based on the students' mathematics book, Haese Mathematics (2015), and on the relevant objectives of the Australian Curriculum 2014. The ten main questions, having 36 sub-questions with a total of 60 marks, of the mathematics pre-test and post-test were selected exclusively from the students' textbook, Haese Mathematics (2015). I used these mathematics questions with the written permission of the publisher (see Appendix E). The mathematics test questions measured the extent of the understanding of the six aspects of the concept of function (the six mathematical dependent variables) and they were marked and the achieved percentages calculated as described in the following Table 3.1.

Table 3.1

Marks and Percentages of the Six Mathematical Dependent Variables

AC Mathematics Variable's code	Variable	Maximum marks	Achievable percentages
ACMMM022	Function definition	11	18.33%
ACMMM023	Function notation	9	15.00%
ACMMM024	Graph of a function	10	16.67%
ACMMM025	Function translations	12	20.00%
ACMMM026	Function dilations	6	10.00%
ACMMM027	Relations	12	20.00%
Total		60	100.00%

The achievable percentages, corresponding to the maximum marks, of different aspects of function vary between 10% - 20%, assuring the homogeneity of their distribution. The mathematics pre-test and the mathematics post-test were identical. The definitions of the codes of the dependent variables are listed in Table 1.2 on page 17 and the mathematics test questions can be found in Appendix A. The following Table 3.2 summarises how the questions of the mathematics test (see Appendix A) were assigned to address the six mathematical dependent variables.

Table 3.2

Mathematics Test Questions Addressing the Mathematical Variables

Code of the Mathematical Variable	Name of the Mathematical Variable	Mathematics Test Questions and Sub-questions Addressing the Variable
ACMMM022	Function definition	Q. 1/a, b, c; Q. 5/a; Q. 7.
ACMMM023	Function notation	Q. 2/a_1, a_2, b; Q. 5/b; Q. 6/a, b, c, d.
ACMMM024	Graph of a function	Q. 4/a, b, c, d.
ACMMM025	Function translations	Q. 8/a, b, c; Q. 9/a, b.
ACMMM026	Function dilations	Q. 10/a, b.
ACMMM027	Relations	Q. 3/a, b, c, d, e, f, g, h, i, j, k, l.

The mathematics tests (see Appendix A) have been marked by me based on a pre-defined marking scheme which is summarised in the following Table 3.3.

Table 3.3

Marking Scheme of the Mathematics Test

Question\Marks	0 mark	1 mark	2 marks	3 marks	4 marks
Question 1/a, b, c.	False answer or no answer given.	Good answer. Full reason not given.	Good answer. Full reason given.	N/A	N/A
Question 2/a_1, a_2.	False answer.	Good answer.	N/A	N/A	N/A
Question 3/a – l.	False answer.	Good answer.	N/A	N/A	N/A
Question 4/a.	False answer or no answer given.	Good answer. Full reason not given.	Good answer. Full reason given.	N/A	N/A
Question 4/b.	False answer or no answer given.	One value stated, no interpretation.	One value stated with interpretation, OR both values stated without interpretation.	Both values stated with partial interpretation.	Both values stated with full interpre-tation.
Question 4/c, d.	False answer or no answer given.	One value stated.	Both values stated.	N/A	N/A
Question 5/a.	False answer or no answer given.	Good answer. Full explanation not given.	Good answer. Full explanation given.	N/A	N/A
Question 5/b.	False answer or no answer given.	One set stated correctly.	Both sets stated correctly.	N/A	N/A
Question 6/a – d.	False answer.	Good answer.	N/A	N/A	N/A
Question 7.	False answer or no answer given.	Good answer. No explanation given.	Good answer. Partial explanation given.	Good answer. Full explanation given.	N/A
Question 8/a, b, c.	No graph or wrong graph drawn.	Shape of the translated graph correct. Both x- and y-coordinates incorrect.	Shape of the translated graph correct. One coordinate correct.	Shape of the translated graph correct. Both coordinates correct.	N/A
Question 9/a, b.	False answer or no answer given.	Argument or free term correct.	Argument and free term correct.	N/A	N/A
Question 10/a, b.	No graph or wrong graph drawn.	Shape of the dilated graph correct. Both x- and y-coordinates incorrect.	Shape of the dilated graph correct. One coordinate correct.	Shape of the dilated graph correct. Both coordinates correct.	N/A

An important question in action research is the reliability and validity of the instruments used (McNiff & Whitehead, 2010). I made efforts to establish the reliability and validity of the mathematics test. The validity of the mathematics test refers to the suitability of the test to measure the goal of the study: improvement of students' understanding of the targeted six aspects of function. The mathematics test was designed by me, using questions exclusively from the studied Stage 1 Mathematical Methods mathematics textbook, Haese Mathematics (2015). The mathematics test contained 10 questions, a total of 36 sub-questions, each addressing a particular aspect of function. These questions and their sub-questions have covered the complete range of the expected knowledge at senior secondary level of mathematics education of these six aspects of function in a proportional mode, assuring a high level of validity of the mathematics test.

The reliability of the mathematics test refers to the credibility of the data collected with this test. The mathematics tests were completed by individual students, without being influenced by each other or using any additional material such as textbooks or notes. During their tests there were two teachers in the classroom, myself and a cooperating colleague teacher, monitoring students' work. I was aware of the fact that if the same test is used as pre-test and post-test, then the memory effect can occur. The memory effect is the phenomenon when the participants complete their post-test by still having the memories of their completed pre-test. I was thinking initially to eliminate the memory effect by using two different tests, but this would have had posed additional questions regarding the equivalence of the two different tests. Different pre- and post- tests have to have positive and very high correlations. To assure a suitable correlation factor, the two tests had to be tested with a different sample, which was not possible, and as a result I decided to use the same mathematics test for both pre-test and post-test. In my study the possibility of the occurrence of the memory effect was eliminated by administering the two tests at distant dates. The tests were administered with a time difference of two months: the pre-test on 20th September 2016 and the post-test on 18th November 2016. The mathematics test contained a range of different functions, graphs, expressions and numerical values, as a result, I believe that students had no memories of these questions and their solutions after two months of busy secondary school life. The treatment intervention has been performed on 15th November 2016. During the treatment none of the questions of the mathematics test was reviewed or solved.

3.2.4 The Questionnaires

Data in the second category, describing different non-cognitive indicators, were collected by using three questionnaires with proven validity and reliability:

(1) What Is Happening In This Class (Fraser, 1998), WIHIC, see Appendix B.

(2) Mathematics Related Beliefs Questionnaire (Op't Eynde & De Corte, 2003), MRBQ, see Appendix C.

(3) Test of Mathematics-Related Attitudes (Fraser, 1981; Taylor, 2004), TOMRA, see Appendix D.

(1) The WIHIC Questionnaire

The codes, scales, item numbers, and the score scheme of the WIHIC questionnaire (Fraser, 1998) are shown in Table 3.4 below.

Table 3.4

Evaluation Scheme of the WIHIC Questionnaire

Code	**Scale**	**Item number**	**Score scheme**
SC	Student cohesiveness	1 - 8	Direct (1-2-3-4-5)
TS	Teacher support	9 - 16	Direct (1-2-3-4-5)
INVMT	Involvement	17 - 24	Direct (1-2-3-4-5)
INVST	Investigation	25 - 32	Direct (1-2-3-4-5)
TO	Task orientation	33 - 40	Direct (1-2-3-4-5)
C	Cooperation	41 - 48	Direct (1-2-3-4-5)
E	Equity	49 - 56	Direct (1-2-3-4-5)

All of 56 items of the WIHIC Questionnaire (Fraser, 1998) are direct-formulated items, and were scored by using the direct score scheme as follows: Almost never = 1, Seldom = 2, Sometimes = 3, Often = 4, Almost always = 5.

(2) The MRBQ Questionnaire

The codes, scales, item numbers, and score schemes of the MRBQ questionnaire (Op't Eynde & De Corte, 2003) are shown in the Table 3.5 below.

Table 3.5

Evaluation Scheme of the MRBQ Questionnaire

Code	Scale	Item number	Score scheme
T	Teacher as facilitator	1 - 11	Direct (5-4-3-2-1)
S	Student self-efficacy	12 - 23	Direct (5-4-3-2-1)
F	Functional necessity	24 - 36	Reverse (1-2-3-4-5)
R	Real world relevance / application	37 - 47	Direct (5-4-3-2-1)

All direct-formulated items (1-23, 37-47) of the MRBQ questionnaire (Op't Eynde & De Corte, 2003) were scored by using the direct score scheme, assigning numerical values to students' responses as follows: Strongly agree = 5, Agree = 4, Not sure = 3, Disagree = 2, Strongly disagree = 1.

All reverse-formulated items (24-36) of the MRBQ questionnaire (Op't Eynde & De Corte, 2003) were scored by using the reverse score scheme, assigning numerical values to students' responses as follows: Strongly agree = 1, Agree = 2, Not sure = 3, Disagree = 4, Strongly disagree = 5.

(3) The TOMRA Questionnaire

The codes, scales, item numbers, and score schemes of the TOMRA questionnaire (Fraser, 1981; Taylor, 2004) are shown in the Table 3.6 below.

Table 3.6

Evaluation Scheme of the TOMRA Questionnaire

Code	Scale	Item number	Score scheme
E	Enjoyment of mathematics lessons	1, 3, 5, 7, 9, 11, 13, 15, 17, 19.	Direct 1, 5, 9, 13, 17. Reverse 3, 7, 11, 15, 19.
I	Attitude to mathematics inquiry	2, 4, 6, 8, 10, 12, 14, 16, 18, 20.	Direct 2, 6, 10, 14, 18. Reverse 4, 8, 12, 16, 20.

The 10 direct-formulated items (1, 2, 5, 6, 9, 10, 13, 14, 17, 18) of the TOMRA questionnaire (Fraser, 1981; Taylor, 2004) were scored by using the direct score scheme, assigning numerical values to students' responses as follows: Strongly agree = 5, Agree = 4, Not sure = 3, Disagree = 2, Strongly disagree = 1.

The 10 reverse-formulated items (3, 4, 7, 8, 11, 12, 15, 16, 19, 20) of the TOMRA questionnaire (Fraser, 1981; Taylor, 2004) were scored by using the reverse score scheme, assigning numerical values to students' responses as follows: Strongly agree = 1, Agree = 2, Not sure = 3, Disagree = 4, Strongly disagree = 5.

For all of these three questionnaires the pre-test and post-test items were identical. These data were collected in order to measure and monitor different aspects of the equivalence of the experimental and control groups. These two categories of quantitative data were collected. Having no qualitative data is offering new channels for future research.

3.2.5 The Software Tools

The following three software tools have been used during the treatment: Original Sine (2014), Virtual Piano (2015), and Music Speed Changer (2012).

(1) Original Sine (2014)

The Original Sine software was developed by the School of Electrical and Electronic Engineering, The University of Adelaide, Australia.

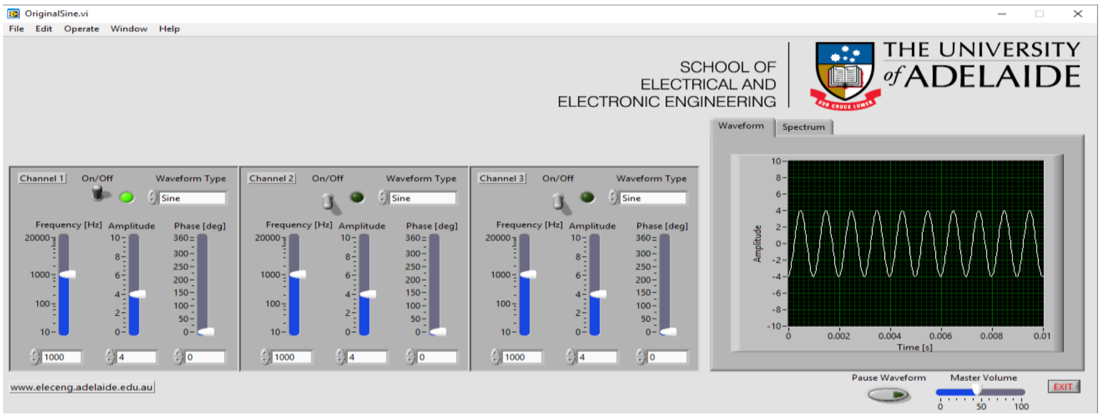

Figure 3.S: The Original Sine Software.

The Original Sine (2014) software tool allows to explore the frequency, amplitude, phase, and wave form of sound. The first three parameters can be simply modified by their sliders or entering numerical values, and the last one can be changed by selecting the wave form (sine, square, triangle, white noise) from a menu list. The software produces the audio sound and displays the corresponding visual graph of the sound accordingly to the parameter setup, which means that the user can hear and see the effects of the parameter setup simultaneously. Specifying different values for frequency (0 – 20,000 Hz) and amplitude (0 - 10) such as F = 440 and A = 6.5 requires logical/mathematical intelligence, listening to the produced sound requires musical intelligence and observing the corresponding graph requires visual intelligence. In this way the software connects the logical/mathematical, musical and visual intelligences and offers triple representations (embodiments) of the concept of sound. The Original Sine (2014) software offers another great feature which was used in my study. It is possible to combine up to three different sounds by using the sound channels. This makes possible to listen to and observe more than one sound simultaneously, a harmony, which is the musical analogy of the mathematical concept of relation, considered in this study (see Analogy 2). These features of the software were the decisive ones that I have incorporated this great tool in my study.

The Original Sine (2014) software can also be used to experiment the superposition of different sound waves. If we activate the four channels of the software and set four different frequencies and amplitudes for the four sound sources then the software combines the four sound outputs by adding them together. This is the superposition of the four channels. In this way it is possible to produce constructive sound waves which are in phase being synchronised, so the sound waves rise and fall at the same time, or destructive sound waves which are out of phase being not synchronised, so one sound wave rises and the other falls at the same time. The students in the experimental group have interacted with the Original Sine (2014) computer software during the treatment lesson. The students observed that changing the frequency of the sound affects the pitch of the sound and changing the amplitude of the sound affects the loudness of the sound. See Section 3.4.2, Activity Two in Chapter 3 on page 123.

The Original Sine (2014) software has been used in this study with the written permission of the developer. See Appendix H.

(2) Virtual Piano (2015)

Figure 3.V: The Virtual Piano Software.

The Virtual Piano (2015) online software (Fig. 3.V above) has been developed by Crystal Magic Studio Ltd, London, England. It is a great free product, which simulates brilliantly the sounds of a five-octave piano. The piano keys of the Virtual Piano (2015) software can be played by using the assigned keys of the computer keyboard, or by using the touch screen monitor of the computer, or by clicking on the virtual piano key. Virtual Piano (2015) states about this great product that it is:

> The original and most loved Virtual Piano – worldwide. Established in 2006, Virtual Piano is now played by more than 19 million people a year. This free to use platform enables you to play the piano through your computer keyboard, without the need to download or install an app. The best part is that you don't need prior knowledge of the music notation. The Virtual Piano music sheets use plain English alphabet and simple semantics, so you can enjoy the experience of playing the piano instantly. Since its inception, Virtual Piano has been used as a learning tool in the world's most prestigious schools – it has helped young children to get a feel for music – it has been the stepping stone for some of the world's greatest artists. Virtual Piano is fast becoming a form of expression and communication between different cultures and regions of the world – crossing language, space and time. Our vision is to spread the joy of playing the piano to every corner of the globe. Our goal is to engage and inspire people of all ages and abilities, to nurture a passion for music.
>
> (Virtual Piano. (2015). About. Retrieved from www.virtualpiano.net/About)

Because all students have a Microsoft tablet and internet access at the college, they were able to start the Virtual Piano (2015) software and explore piano sounds within seconds. The Virtual Piano (2015) software has enabled students to experiment with melodies (musical representations or embodiments of functions) and harmonies (musical representations or embodiments of relations) in an accessible, easy and natural way. Because previous music or piano studies were not requirements of my study, and some students have never used a piano before, in order to observe how to produce a musical melody or harmony on a piano the students in the experimental group have experienced with the Virtual Piano (2015) online application during the treatment lesson. See Activity Three and Activity Four in Chapter 3. Choosing the Virtual Piano (2015) software to use in this study was a very good decision. The Virtual Piano software has been used in this study with the confirmation and written permission of the developer. See Appendix F.

(3) **Music Speed Changer (2012)**

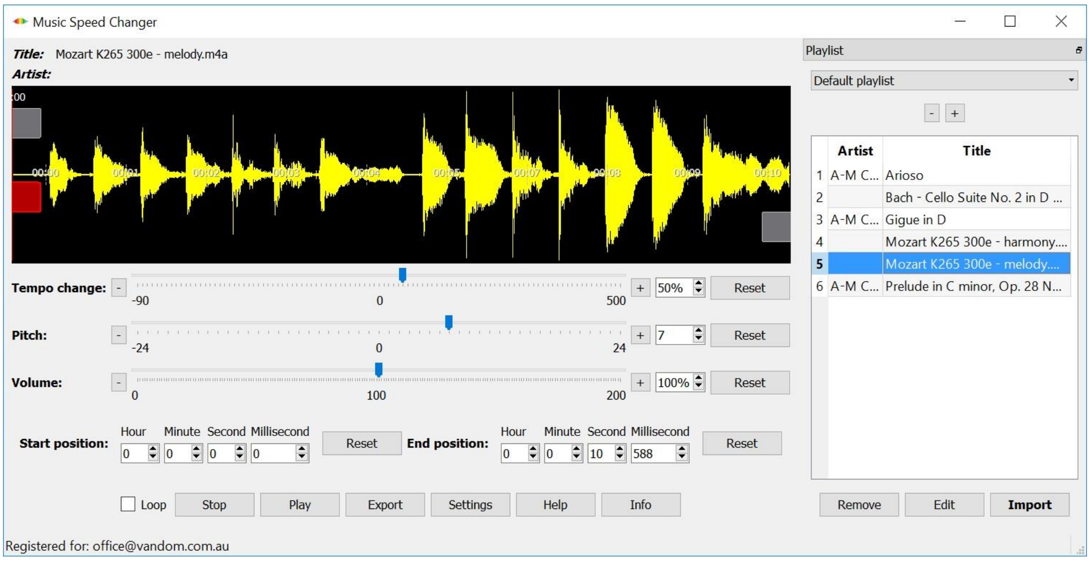

Figure 3.M: The Music Speed Changer Software.

The Music Speed Changer (2012) software (Fig. 3.M above) offers to change the tempo (speed), pitch, and volume of music independently in real-time. These aspects of music are described by the corresponding aspects of function in my study. Changing any of these aspects of the music with the help of this software is nothing else but entering relevant constant numbers into the software, which entered values are considered by the software as parameters of the corresponding function, and as a result the affected aspect of music changes accordingly. To observe how to change the tempo (speed), pitch, and volume of a melody, the Music Speed Changer (2012) software has been used. A licence to use the Music Speed Changer (2012) software has been bought by me, the students have not installed this software on their tablets but they have observed the use of the software demonstrated by me. The software has been installed on both my home Windows 10 PC and my Windows 10 based college tablet computer with the written permission of the developer. See Appendix G.

3.2.6 Statistical Analysis Considerations

In order to statistically analyse the data, three statistical analysis tests have been considered: GLM (General Linear Model), MANOVA (Multivariate Analysis of Variance) and the independent t-test.

Schlomer (2008) states that t-tests:

> Offer an opportunity to compare two groups on scores such as differences between boys and girls or between children in different school grades. A t-test is a type of inferential statistic, that is, an analysis that goes beyond just describing the numbers provided by data from a sample but seeks to draw conclusions about these numbers among populations. To do this, the t-test analyzes the difference between the two means (a.k.a. two averages) derived from the different group scores. T-tests tell the researcher if the difference between two means is larger than would be expected by chance (i.e. statistically significant).
>
> (Schlomer, 2008, p. 2)

My goal was to find out if the difference between the means of students' improvements of understanding of function in the two separate groups was statistically significant or not. I believe that the independent t-test was the most appropriate statistical analysis tool for this purpose. The t-test is a parametric statistical test. Stangroom (2018) recommends its use and also the non-parametric alternative of the t-test, the Mann-Whitney U test, by stating that if the data is not normally distributed, then the use of the Mann-Whitney U test is recommended.

If there is a difference between the means of students' improvements of understanding of function, then an important question is to address and interpret the effect size appropriately. Cohen (1988) explains the use of the phrase 'effect size' as follows:

> Without intending any necessary implication of causality, it is convenient to use the phrase 'effect size' to mean 'the degree to which the phenomenon is present in the population,' or 'the degree to which the null hypothesis is false.' Whatever the manner of representation of a phenomenon in a particular research in the present treatment, the null hypothesis always means that the effect size is zero.
>
> (Cohen, 1988, pp. 9-10)

In the case of my study it means that the effect size is expressing the grade to which this type of active integration of mathematics and music improves my senior secondary mathematics students understanding of function. In other words, if the Null Hypothesis is false, then I will not only reject the Null Hypothesis but will also state the degree of rejection, the effect size in the sample.

Becker (2000) states that “effect size (ES) is a name given to a family of indices that measure the magnitude of a treatment effect. Unlike significance tests, these indices are independent of sample size” (p. 1). This means that the magnitude of the treatment effect of this study is expected to be similar also when the sample size is larger than the sample size of this study.

The following information regarding an independent t-test for two samples are from Laerd Statistics (2013). I relate the general assumptions and requirements of an independent t-test for two samples to my study, showing how the data of the study fits into the requirements of a t-test:

The independent t-test, also called the two sample t-test, independent-samples t-test or student's t-test, is an inferential statistical test that determines whether there is a statistically significant difference between the means in two unrelated groups.

The null hypothesis for the independent t-test is that the population means from the two unrelated groups are equal:

$$\mathbf{H_0: \mu_1 = \mu_2}$$

In most cases, we are looking to see if we can show that we can reject the null hypothesis and accept the alternative hypothesis, which is that the population means are not equal:

$$\mathbf{H_A: \mu_1 \neq \mu_2}$$

To do this, we need to set a significance level (also called alpha) that allows us to either reject or accept the alternative hypothesis. Most commonly, this value is set at 0.05.

(Laerd Statistics, 2013)

Landau and Everitt (2004) state that:

The independent samples t-test is used to test the null hypothesis that the means of two populations are the same, $\mathbf{H_0: \mu_1 = \mu_2}$, when a sample of observations from each population is available. The observations made on the sample members must all be independent of each other.

(Landau & Everitt, 2004, p. 39)

The main null hypothesis, $H1_0$, of my study for the performed independent t-test was that the means of the increase of students' understanding of the concept of function of the experimental and control groups are equal. The corresponding alternative hypothesis, $H1_A$, was that the means of the increase of students' understanding of the concept of function of the experimental and control groups are not equal. The significance level – the alpha value – has been set a priori at $\alpha = 0.05$, which assures a corresponding 95% confidence interval of means.

In order to run an independent t-test, there is a need for:

- one independent, categorical variable that has two levels or groups, and
- one continuous dependent variable.

My study has satisfied these requirements. In this study the independent, categorical variable was the student group, having the two groups of students: the experimental group and the control group. The continuous dependent variable was the groups' increase of understanding of the concept of function, expressed in marks, in the range of 0-60.

Laerd Statistics (2013) states that:

> Unrelated groups, also called unpaired groups or independent groups, are groups in which the cases (e.g., participants) in each group are different. Often we are investigating differences in individuals, which means that when comparing two groups, an individual in one group cannot also be a member of the other group and vice versa.

Schlomer (2008) describes the independent samples t-test as follows:

> The independent samples t-test is used to compare two groups whose means are not dependent on one another. In other words, when the participants in each group are independent from each other and actually comprise two separate groups of individuals, who do not have any linkages to particular members of the other group (in contrast to dependent samples).
>
> (Schlomer, 2008, p. 3)

In my study, the two considered groups were unrelated – unpaired, independent – groups, because the participant students in each group were different, and all participant students were members in one group only, either the experimental group or the control group.

Regarding the assumption of normality of the dependent variable Laerd Statistics (2013) states that:

> The independent t-test requires that the dependent variable is approximately normally distributed within each group. However, the t-test is described as a robust test with respect to the assumption of normality. This means that some deviation away from normality does not have a large influence on Type I error rates. The exception to this is if the ratio of the smallest to largest group size is greater than 1.5 (largest compared to smallest).
>
> (Laerd Statistics, 2013)

The main dependent variable of my study – the increase of students' understanding of the concept of function – was approximately normally distributed within both experimental and control group, because in both groups there were high achievers, good achievers, satisfactory level achievers and non-satisfactory level achiever mathematics students. The assumption of homogeneity of variance has been tested using Levene's Test of Equality of Variances, produced in SPSS Statistics when running the independent t-test statistical analysis. Also, the two groups were equal in population, as in both groups there were 22 students assuring that the ratio of the group sizes being 22/22 = 1, less than 1.5, satisfying the observation above regarding the normality condition of performing a t-test.

Three instruments were used to collect non-cognitive data: my students' perceptions regarding their attitudes and beliefs towards mathematics, and regarding their learning environment; a total of thirteen variables. These three instruments, the MRBQ, TOMRA and WIHIC questionnaires use Likert-type scores.

3.3 The Nine Author-Designed Analogies Between Function and Music

An analogy is like a metaphor, I would say, if I would define – incorrectly – the concept of "analogy" by using … an analogy. But we know (from mathematics) that defining a concept in terms of itself is not a valid definition, so I define analogy as being an observation of similarities between two different objects, concepts, ideas or other entities. If we observe the similarities between two things then we use an analogy, maybe informally, without being aware of it. Analogies are very diverse in nature as there are analogies between two countries' climate, two languages, two cars, two art works, two school subjects, two mathematical proofs, two person's thinking modes, knowledge, lifestyles, behaviours, beliefs, attitudes, and so on. Analogies can be created and expressed also in many different ways: using words (a metaphor), drawing, painting, sketching, taking a 2D picture, creating a 3D model, and so on.

The importance of analogies is stated by Bruner (1960) as follows:

> It is difficult to believe that general heuristic rules - the use of analogy, the appeal to symmetry, the examination of limiting conditions, the visualization of the solution - when they have been used frequently will be anything but a support to intuitive thinking.
>
> (Bruner, 1960, p. 64)

As mentioned earlier, the mathematician Stefan Banach (1892-1945) stated that "Good mathematicians see analogies. Great mathematicians see analogies between analogies". Analogies are powerful instruments in education at all levels, in all subjects, teachers use metaphors and analogies often. Research confirms that teaching with analogies is the most effective when students are familiar with the analogy used (Harrison & Treagust, 2006).

In order to address the six required learning outcomes of function of the Australian Curriculum 2014, Year 11 Mathematical Methods, I have identified, considered and designed nine relevant analogies between function and music and have presented and used them as tools during the treatment of the experimental group. These analogies can be used economically and in a valid and reliable way.

3.3.1 Analogy 1 Mathematical Function ≈ Musical Melody

The students had observed that if they press one key each time on the Virtual Piano (2015) and if t denotes the time and $f(t)$ the frequency of the sound at the time t, then the played melody is the audio/musical representation of the function $y = f(t)$.

We can represent a melody by a function, and this constitutes the analogy between the function and the melody. For example, the well known *W.A. Mozart: Variations* *in C, K.265(300e)*, originally an ancient French folk song, the "*Twinkle, twinkle, little star*" melody is represented as a time-frequency function in Figure 3.1 below.

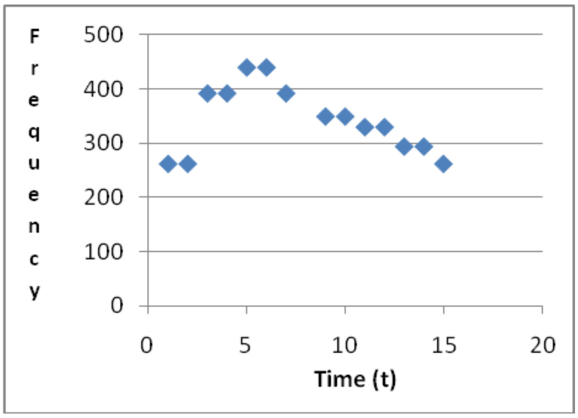

Figure 3.1: Mathematical function ≈ Musical melody.

In general terms, if a sequence of single keys are pressed, one key only at every time point, then we have one and only one sound at any given time, we produce a musical melody which can be described by a function as follows.

Because one key cannot produce two or more different sounds, playing a melody by pressing a sequence of keys, but one key only at every time point, means associating single sound frequencies to every time point. Associating for every time point one and only one sound frequency means mathematically that we construct a function by mapping these two sets of time points and sounds.

Musically we play a melody, mathematically we construct a function. It is very important that in a melody there is one and only one musical sound at a time point. This assures the analogy between the concept of mathematical function and the concept of musical melody. If we listen to a melody, then we "hear" a function, where the domain of the function is the set of the time points and the range of the function is the set of the musical sound frequencies.

The independent variable is the time and the dependent variable is the sound frequency. If we denote the independent variable by ***t***, representing the time, varying in a given time frame domain, and the dependent variable by **f(t)**, representing the unique frequency of the sound at the time ***t***, being well defined at every time ***t***, then the musical melody – the sequence of the frequency values – is described by the mathematical function **y = f(t)**.

As a result, every musical melody has a corresponding unique mathematical function, and every mathematical function has a corresponding unique musical melody. This unique correspondence constitutes the structural analogy between function and melody, and it is the key to build other analogies between different aspects of function and the corresponding aspects of music.

I observe here that the above described unique correspondence between melodies and functions creates a bijective function between the set of functions and the set of melodies. The independent variable of this newly created function is **f**, representing any mathematical function, and the dependent variable is **m(f)**, representing the melody corresponding to the function **f**. As a result, also this newly created function has a corresponding melody, produced with the help of functions representing different melodies. This is not a secondary level mathematics question, but a question to be treated within Category Theory.

3.3.2 Analogy 2 *Mathematical Relation ≈ Musical Harmony*

If two or more piano keys are pressed at the same time, then we play two or more frequencies at that time. This is a musical harmony. Because multiple frequencies are allocated to a time point, the harmony can be represented by using a mathematical relation only. This constitutes the structural analogy between the concepts of mathematical relation and musical harmony. Figure 3.2 shows a harmonization of the considered melody in the form of a graph representing a mathematical relation.

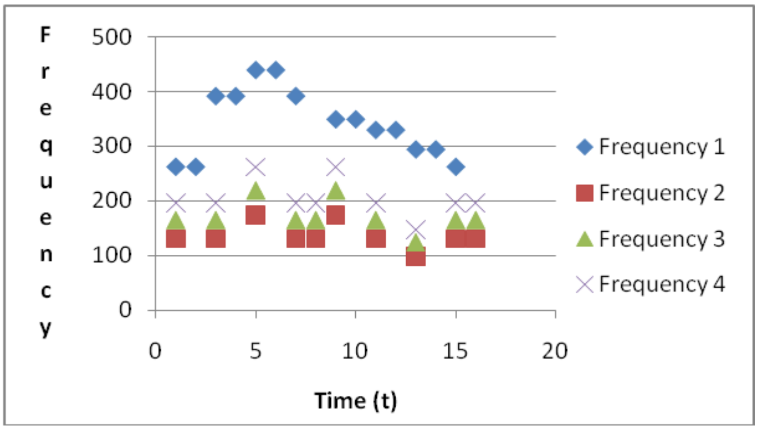

Figure 3.2: Mathematical Relation ≈ Musical Harmony.

3.3.3 Analogy 3

Vertical Line Test for Function ≈ Test for Two Aspects of Music

The musical analogy of the *Vertical line test for function,* a mathematical test to decide if a given graph represents a function or a relation, is the *Test for two aspects of music,* an audio test to decide if a given musical piece is a melody or a harmony. In the latter case if there is at least one time point when the given music piece has more than one musical sound then that music piece is a harmony otherwise it is a melody.

The students have listened to the "*Prelude in C-minor Op. 28 No. 20*" piano piece by F. Chopin and to the first part of "*Suite No. 2 in D-Minor BWV 1008*" cello piece by J. S. Bach, and have checked the provided corresponding music sheets (see Appendix O and Appendix Q). They have empowered their knowledge regarding the *Vertical line test for function* concluding that the first music piece is a harmony which can be represented by a mathematical relation only, and the second piece is a melody, representable by a function. The mathematics pre-test showed that the students' knowledge and understanding regarding the *Vertical line test for function* was very high. They were able to apply the *Vertical line test for function* correctly for a range of different graphs, being able to decide if a given mathematical graph represents a mathematical function or a mathematical relation. The students were able to transfer their existing knowledge of the *Vertical line test for function* to a different, non-mathematical situation by performing an audio test to decide if the listened beautiful piece of music is a melody or a harmony. The students' feedback came within a few seconds, stating the correct answers that the first piece of music is a harmony, corresponding not to a mathematical function but to a mathematical relation, and the second piece of music is a melody, corresponding to a mathematical function.

In order to transfer an existing mathematical knowledge or skill to an analogous musical knowledge or skill, there is a need for a certain level of abstract thinking, as a result, because the students have achieved this transformation of thinking with the help of these musical analogies and examples, I think that can be stated that these musical analogies and examples have enhanced students' abstract thinking.

3.3.4 Analogy 4

Graphs of Multiple Functions ≈ Musical Counterpoint

The students have listened to and have checked the sheet of J. S. Bach's beautiful piece of music, "*Arioso from Cantata, BWV 156*" (see Appendix P), which is a harmony, but a special one: there are two melodies sounding at the same time. There is no other harmonization in this music. This type of harmony is a musical counterpoint. The mathematical analogy of the musical counterpoint is plotting multiple graphs of functions on the same axes. The considered functions are independent to each other, but they are figured on the same coordinate system. Graphing two functions on the same coordinate system is a traditional, common mode in mathematics education to teach and learn about solving equations. In this traditional way secondary mathematics teachers usually show, using students' visual-graphical intelligence, that the solutions of the equation considered are the x-coordinates of the points where the graphs of the functions intersect each other. My study went a step further, making a musical analogy: if we consider the two functions on the left and right sides of the equation as two melodies, then the roots of the equation correspond to the time-points when the two melodies use the same musical note or sound frequency. Figure 2.1 on page 52 provides an example of this analogy.

3.3.5 Analogy 5

Vertical Translation of Piano-Key Function ≈ Music Transposition

The vertical translation of a function has the musical analogy of transposition of a melody from one scale to another scale. If the piano key number function $n = f(t)$ is vertically translated by a constant a, which means $n = f(t)+a$ then the same melody is played, but a half-tones higher.

As an example, if we consider the piano key number difference of 47 – 40 = 7 between the *C* sound (piano key number 40) and the *G* sound (piano key number 47) on the piano, then the piano key number function $f(t)$ and its vertical translation by $a = 7$, the vertically translated piano key number function $g(t) = f(t)+7$ are the function representations of the same melody played in *C-major*, respective in *G-major*. We see visually and we understand mathematically that we have a vertical function translation. We can hear musically that we have a melody transposition. If we understand this powerful analogy between the mathematical concept of vertical translation of function and the musical concept of transposition of melody, then we acquire a deep understanding of both of the concepts. An example is illustrated in Figure 3.3 as follows.

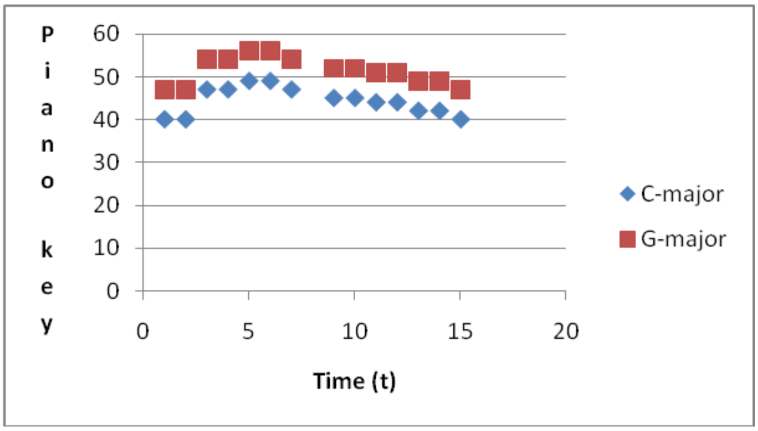

Figure 3.3: Vertical translation of piano-key function ≈ Transposition of music.

3.3.6 Analogy 6

Horizontal Translation of Frequency Function ≈ Repetition of Music

The horizontal translation of a function by a constant b, $g(t) = f(t+b)$, has an analogy in music: the musical repetition of the corresponding melody. Repetition patterns are frequently used in music. For example, both the frequency function $f(t)$ of the considered melody in C-major and its horizontal translation by $b = -20$, the frequency function $g(t) = f(t - 20)$, representing the repetition of the same melody 20 beats later, are displayed in Figure 3.4. In this case the horizontal translation constant is $b = -20$.

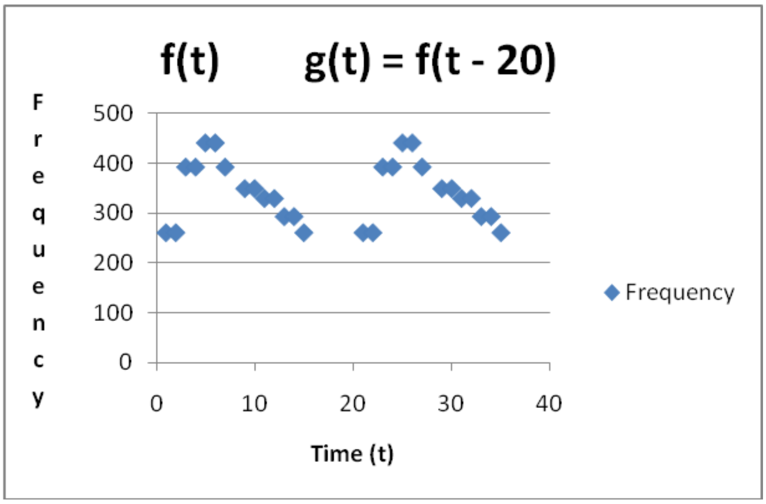

Figure 3.4: Horizontal translation of function ≈ Repetition of music.

The vertical dilation of a function by a real constant c > 0, g(x) = c × f(x), has two musical analogies, depending if the function y = f(x) is the amplitude function of the melody, as it is presented in section 3.3.7 (Analogy 7), or the function y = f(x) is the frequency function of the melody, as it is presented in section 3.3.8 (Analogy 8) below.

3.3.7 Analogy 7

Vertical Dilation of Amplitude Function ≈ Loudness of Music

If $y = A(t)$ is the amplitude function of a melody, then the vertical dilation $c \times A(t)$ represents the increase (if $c > 1$) or the decrease (if $0 < c < 1$) of the loudness of the melody.

It is good to know the relevant Physics concepts, that the loudness of the sound is different to the intensity of the sound. The loudness of sound depends from the intensity of sound, and is measured in decibels (dB) on a scale of 0 to 100 dB, where 0 dB is no sound, and 100 dB is the level of loudness that human ears usually found hardly supportable. The intensity (I) of sound is the measure of energy of the sound. Loy (2006) describes the intensity as follows:

> Energy from the motion of sound waves flows through the eardrums and into the inner ear, where it registers as sound. Intensity *I* is the energy *E* per unit of time *t* that is flowing across a surface of unit area a: $I = E/t/a^2$.
>
> (Loy, 2006, p. 118)

Loy (2006) also explains that:

> Just as the range of frequencies we can hear is limited, so is our perception of sound intensity. The threshold of hearing is the minimum amount of sound intensity required for a sinusoid to be detected by an average listener in a noiseless environment. The limit of hearing (also called the threshold of pain) is the intensity above which sound is registered as (possibly damaging) pain by most of us. Perception of loudness is not as straightforward as perception of pitch.
>
> (Loy, 2006, p. 119)

The relation between the loudness (D) of sound expressed in decibels, and intensity (I) of sound, expressed in watts/m^2 with respect to a reference intensity I_r, usually $I_r = 10^{-12}$ watts/m^2 is given by the formula:

$$\mathbf{D = 10 \times \log(I / I_r)}$$

Obviously, if we have a softer sound than the considered reference sound, which means that $I < I_r$, then using the above decibel calculation formula we obtain a negative decibel value. This is the reason why the intensity level meters on sound recording devices display negative values **(**Loy, 2006).

3.3.8 Analogy 8

Vertical Dilation of Frequency Function ≈ Transposition of Music

If $y = f(t)$ is the frequency function of a melody, then the vertical dilation of the function $f(t)$ by a real constant $k > 0$, the frequency function $y = k \times f(t)$ represents a musical transposition of the melody. For example, if $f_C(t)$ is the frequency function of a melody in C-major then the function $f_G(t)$, obtained by the vertical dilation of $f_C(t)$ by $k = 2^{7/12}$, $f_G(t)= 2^{7/12} \times f_C(t)$ represents the same melody transposed in G-major.

For example, the tabular form of the frequency function $\mathbf{F_C(t)}$, representing the considered melody in C-major, is given in Table 3.7 below.

Table 3.7

Tabular Form of Frequency Function of the Melody in C-major

Time (**t**)	1	2	3	4	5	6	7	8
Sound	C_4	C_4	G_4	G_4	A_4	A_4	G_4	Rest
$\mathbf{F_C(t)}$	261.63	261.63	394	394	440	440	394	0

Time (**t**)	9	10	11	12	13	14	15	16
Sound	F_4	F_4	E_4	E_4	D_4	D_4	C_4	Rest
$\mathbf{F_C(t)}$	349.23	349.23	329.63	329.63	293.66	293.66	261.63	0

Because the G-key is the 7th key from the C-key on the piano and the ratio of every two consecutive piano keys is $2^{1/12} = 1.05946...$, we can observe that the melody can be transposed from C-major into G-major by multiplying every frequency $\mathbf{F_C(t)}$ in Table 3.7 by $2^{1/12}$ x $2^{1/12}$ x $2^{1/12}$ x $2^{1/12}$ x $2^{1/12}$ x $2^{1/12}$ x $2^{1/12}$, which is the real constant

k = $(2^{1/12})^7 = 2^{7/12}$. The obtained vertically dilated frequency function is $\mathbf{F_G(t)= k \times F_C(t)}$ for $\mathbf{t}\ \varepsilon\ \{1, \dots, 16\}$, and this function represents the melody in G-major. The tabular form of this vertically dilated frequency function is given in Table 3.8 below.

Table 3.8

Tabular Form of Frequency Function of the Melody in G-major

Time (**t**)	1	2	3	4	5	6	7	8
Sound	G_4	G_4	D_5	D_5	E_5	E_5	D_5	Rest
$\mathbf{F_G(t)}$	392	392	587.33	587.33	659.26	659.26	587.33	0

Time (**t**)	9	10	11	12	13	14	15	16
Sound	C_5	C_5	B_4	B_4	A_4	A_4	G_4	Rest
$\mathbf{F_G(t)}$	523.25	523.25	493.88	493.88	440	440	392	0

$\mathbf{F_G(t)}$, the vertical dilation of the frequency function $\mathbf{F_C(t)}$ by the constant k = $2^{7/12}$ represents the melody in G-major. The graphs of these two frequency functions, $\mathbf{y_c}$ = $\mathbf{F_C(t)}$, the frequencies of sounds of the considered melody in C-major, and $\mathbf{y_g}$ = $\mathbf{F_G(t)}$, the frequencies of the sounds of the same melody in G-major, are displayed in Figure 3.5 below. We can musically hear the vertical function dilation $\mathbf{F_G(t) = 2^{7/12} \times F_C(t)}$. Students in the experimental group have observed through these examples the following four embodiments of vertical dilation of function: algebraic, tabular, graphical, and musical.

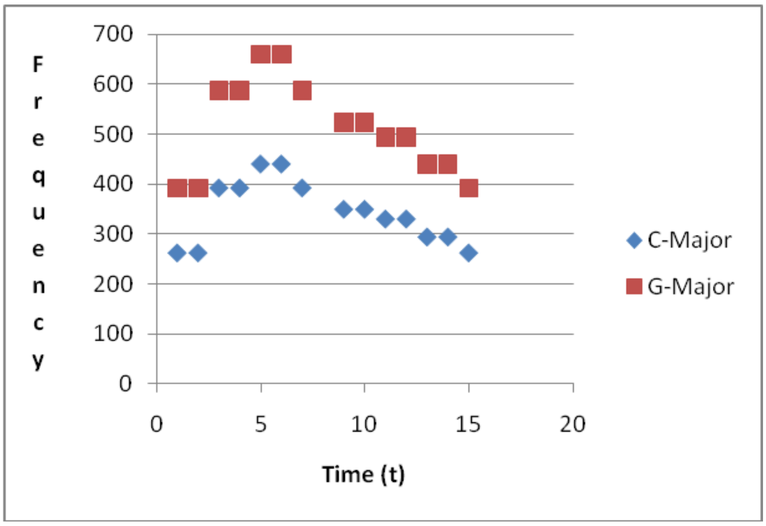

Figure 3.5: Vertical dilation of frequency function ≈ Transposition of music.

3.3.9 Analogy 9

Horizontal Dilation of Frequency ≈ Change of Tempo of Music

A horizontal dilation of a function $f(t)$ by a real constant $d > 0$, namely, $g(t) = f(d \times t)$ has an analogy in music: change of the tempo (speed) of music. Let $f(t)$ be the frequency function of a melody. If $0 < d < 1$, then $t > d \times t$ and as a result the sound frequency $g(t) = f(d \times t)$ will occur at a later time than the sound frequency $f(t)$. In this case the tempo (speed) of the melody represented by $g(t)$ is slower than the tempo of the melody represented by $f(t)$. For example, both the frequency function $f(t)$ of the considered melody and its horizontal dilation by $d = 0.5$, the frequency function of the 2 times slower melody $g(t) = f(0.5 \times t)$ are given in Figure 3.6 below.

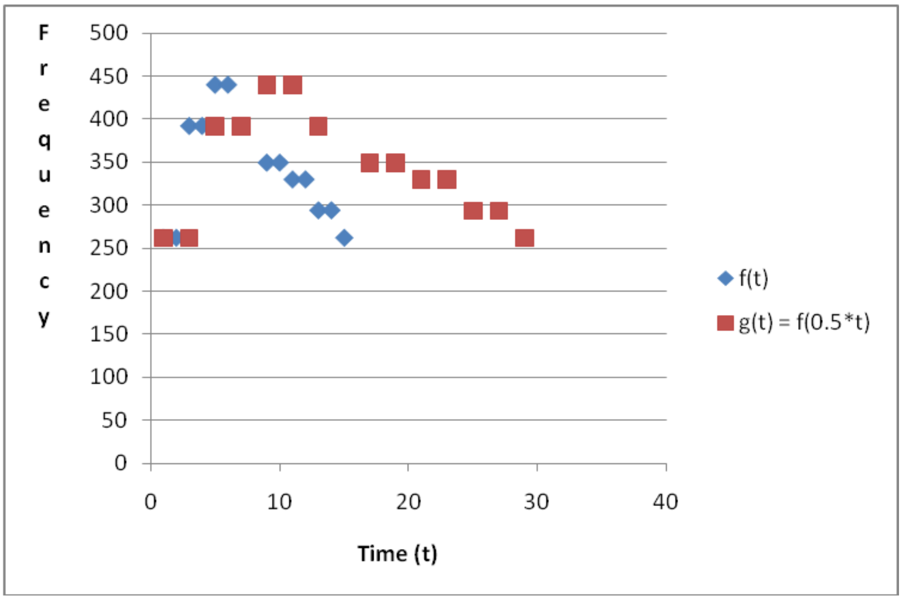

Figure 3.6: Horizontal dilation of frequency function ≈ Change of tempo of music.

I have presented to the students how the Music Speed Changer (2012) software allows to change the speed (tempo) of a melody simply by changing the horizontal dilation function parameter, the positive real constant d. If $d > 1$ then $t < d \times t$ and as a result the sound frequency $g(t) = f(d \times t)$ will occur at an earlier time than the original sound frequency $f(t)$. In this case the tempo (speed) of the melody represented by $g(t)$ is faster than the tempo (speed) of the melody represented by $f(t)$.

3.4 The Treatment Lesson

The treatment lesson, taught by me in a standard 90 minute double-lesson as a revision of the function topic, was a revision of all six targeted aspects of function in a non-traditional way by the active integration of mathematics and music, through the presentation of the nine self-designed musical analogies. The treatment lesson has been presented with the help of an author-designed Microsoft PowerPoint presentation, projected on a large screen in the classroom. Microsoft PowerPoint made possible to play the selected music examples of my study. The Microsoft PowerPoint presentation also made possible the performance of all activities within a predefined time-frame, which assured the coverage of all the planned material within the 90 minutes of the intervention time. During the treatment lesson there were different activities, where students' active contribution was encouraged – they had the opportunity to ask questions, work individually, and work in small groups. The students interacted with and observed the effects of the Original Sine (2014), Virtual Piano (2015) and Music Speed Changer (2012) software tools.

Introductory information regarding the musical concept of pitch of sound, loudness of sound and the corresponding physical properties of sound, namely sound frequency and amplitude, were presented verbally and in printed form at the beginning of the treatment lesson (see Appendix I). The concept of frequency and amplitude was already known by students from science and/or physics lessons. I have mentioned that next to the main properties of sound, frequency and amplitude, there are also other mathematical/physical descriptors or properties of sound, such as rhythm and timbre, which together make possible to describe or represent every musical sound by numbers. This fact implies the mathematical description of music, which is the key to create digital audio files, and audio CDs, and to use analogies between mathematics and music during mathematics lessons.

3.4.1 Activity One

Students were asked to listen to the sound produced by a tuning fork and to the same sound produced by a digital mp3 file sample. This sound is the A_4 musical sound, having the frequency of 440 Hz, which can be written mathematically as $f(A_4) = 440$. This sound has an important role in music as it is used as a reference sound when tuning instruments. This was a very short activity, but by itself it was a demonstration that a physical music sound can be described mathematically and can be represented and produced by a digital file, showing a correspondence between mathematics and music.

3.4.2 Activity Two

Students have had the opportunity to explore the effects of changing frequencies and amplitudes of sound waves using the provided Original Sine (2014) software. They were asked 'what do they hear when they double the frequency and/or the amplitude?'. They observed that the change of the frequency parameter has the effect of changing the pitch of the sound (higher and lower sounds) and the change of the amplitude parameter changes the loudness (the volume) of the sound. They concluded that the two important properties of sound, pitch and loudness, are determined by the frequency and amplitude of the sound wave. The students also examined a provided information sheet (see Appendix I), which presents the loudness and pitch of sound as mathematical graphs. I have obtained this information sheet from the Science Faculty of the College where the research took place.

3.4.3 Activity Three

Students explored the piano key frequencies using the Virtual Piano (2015) application and the provided information sheet, titled "*Piano Key Numbers, Names, and Frequencies (Hz)*" (see Appendix N). The students were asked to find the A_4 (the normal A, having 440 Hz frequency) note on the piano, which is the yellow key in Figure 3.7. The students observed that the other keys of the piano produce a musical sound with lower (e.g. the C_4 key, the middle C note, the blue key in Figure 3.7) or higher frequencies. The students acknowledged that the piano key frequencies form a geometric progression having the first term 27.50 and the common ratio $2^{1/12}$ because f(1st key) = 27.50 Hz, f(2nd key) = $27.50 \times 2^{1/12}$, f(3nd key) = $27.50 \times 2^{2/12}$ and so on for all 88 keys of a standard piano. The 49th key's frequency is f(49th key) = $27.50 \times 2^{48/12} = 440$ Hz (see Appendix N).

Students observed that the provided information sheet, the "*Piano Key Numbers, Names, and Frequencies (Hz)*" table (see Appendix N) constitutes three examples of bijective functions, given in tabular form. These three functions are defined between the sets of: Piano-key numbers and Piano-key names, Piano-key numbers and Frequencies, and Piano-key names and Frequencies.

3.4.4 Activity Four

Students freely interacted with the *Virtual Piano (2015)* application to observe how to produce musical melodies and harmonies, which are audible forms of mathematical functions and relations. They compared the audible difference between pressing in a sequence one and only one key at a time and pressing in a sequence simultaneously two or more keys at the same time. In this way students experimented audible (musical) embodiments of mathematical functions and relations.

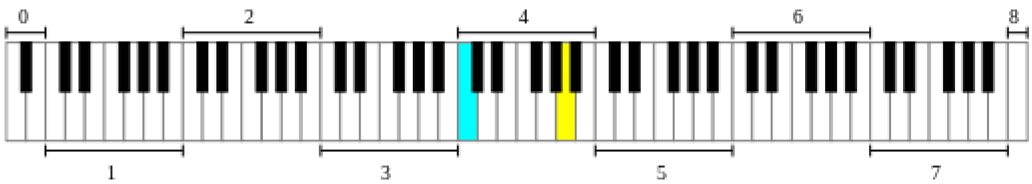

Figure 3.7: Experimenting Audible Functions and Relations on a Piano.

For example, students were asked to press the C_4 note (the blue key on the Figure 3.7) four times, then the A_4 note (the yellow key on the Figure 3.7) four times, then the C_4 note (the blue key on the Figure 3.7) four times again to produce a simple melody. This simple melody is different to pressing both keys at the same time, which produces a harmony. In the case of the melody we hear one and only one sound at any given time, and in the case of the harmony we hear two sounds of two different frequencies or pitches at the same time.

The students in the experimental group have observed and enjoyed both the activities and the nine musical analogies described above. During the treatment the students made positive gestures together with spontaneous comments such as “Awesome!”, “Unbelievable!”, “I've got it!”. “Very interesting!”, and “OK, this makes sense!”.

On the same day I also taught the control group in a standard 90-minute double-lesson reviewing all the six required aspects of the function topic in a traditional way by refreshing the definitions and solving relevant exercises, without presenting any musical analogy to the group. During this revision double-lesson, students' active contribution was encouraged, they had the opportunity to ask questions, work individually, and work in small groups.

Both groups' post-tests were administered three days after the treatment lesson.

This concludes the description of the methodology of my study. The following chapter presents the results of the study, including students' mathematical achievements and students' perceptions of their own beliefs and attitudes regarding mathematics and their learning environment. The following Chapter 4 also presents the statistical analyses of the collected data and after the discussion the two research questions of my study are answered.

Chapter 4

RESULTS, ANALYSIS, AND DISCUSSION

4.1 Introduction

The results, analysis, and discussion of the collected data – the scores of the mathematics tests and of the three questionnaires – are presented in this chapter. Three statistical analysis tests were performed: GLM, MANOVA and the independent t-test for two samples, using the IBM SPSS (Statistical Package for the Social Sciences) software package. The SPSS was the best option to consider because this statistical analysis package performs both Leverne's Test for equality variances and Welch's Test – without naming it explicitly – with adjusted degrees of freedom for unequal variances. When comparing the two groups' pre-test and post-test outcomes on mathematics tests and students' perceptions on the three questionnaires, the variance probabilities of the Leverne's Tests for equality of variances showed that for the collected eight pairs of samples data, equal variances can be assumed.

The performed GLM (General Linear Model) and MANOVA (Multivariate Analysis of Variance) tests, which offer multivariate comparisons of two groups, indicate that there were no significant group (E, C), gender (F, M) and music study (Yes, No) differences between the experimental and control groups ($p > .05$ on all pairs of these independent variables). Box's Test of Equality of Covariance Matrices, performed by the GLM process, also indicates that the correlations between the dependent variables, and the standard deviations of the two groups are similar, as Box's M = 13.06, $F(9, 3781.94) = 1.29$, $p = .235$. Levene's Test of Equality of Error Variances shows for the mathematics pre-tests of the two groups $F(3, 40) = 0.785$, $p = .510$; and for the mathematics post-tests $F(3, 40) = 1.291$, $p = .291$, meaning that there are no significant differences between the two groups error variances for both mathematics pre-test and post-test. The multivariate test results also show that regarding the independent variables the difference between the experimental and control groups is not significant as we have for the group variable: Wilks' Lambda = .001, $F(36, 1) = 36.67$, $p = .130$, for the gender variable: Pillai's Trace = .982, $F(36, 1) = 1.52$, $p = .578$, and for music study variable: Hotelling's Trace = 6906.43, $F(36, 1) = 191.85$, $p = .057$.

4.2 Null Hypotheses, Significance Levels

This study had two research questions as described earlier, the first research question being considered the main while the second one, the auxiliary research question. I formulated two Null Hypotheses accordingly.

The Null Hypothesis for the first research question ($H1_0$) was the following:
$H1_0$: The active integration of mathematics and music by using these nine analogies between different aspects of function and the corresponding aspects of music does not make a statistically significant improvement in my senior secondary mathematics students' understanding of function.

The Null Hypothesis of the second research question ($H2_0$) was the following:
$H2_0$: This type of integration of mathematics and music does not make a statistically significant improvement in my senior secondary mathematics students' attitudes and beliefs regarding mathematics.

In order to avoid committing any type of error in retaining or rejecting a Null Hypothesis, the statistical significance levels were set a priori at 5% for both questions, hence the confidence intervals can be stated with a 95% confidence level.

4.3 Results of the Mathematics Tests

The results of the mathematics tests of the students in the experimental group and of the students in the control group are summarised in Table 4.1E, respective in Table 4.1C below. In these tables the six mathematical dependent variables are coded as M22, M23, M24, M25, M26, and M27; the students in the experimental group are coded as E_1, E_2, …, E_{22}; and the students in the control group are coded as C_1, C_2, …, C_{22}. The achievable maximum marks of the respective mathematical dependent variables, as described earlier (see Table 3.1 on page 98), are displayed in the header of each table.

Table 4.1E

Experimental Group's Mathematics Test Results

Variable Test	M22 Pre	M22 Post	M22 Diff.	M23 Pre	M23 Post	M23 Diff.	M24 Pre	M24 Post	M24 Diff.	M25 Pre	M25 Post	M25 Diff.	M26 Pre	M26 Post	M26 Diff.	M27 Pre	M27 Post	M27 Diff.
Max.	**11**	**11**	**11**	**9**	**9**	**9**	**10**	**10**	**10**	**12**	**12**	**12**	**6**	**6**	**6**	**12**	**12**	**12**
E1	10	11	**1**	5	6	**1**	8	8	**0**	0	3	**3**	0	2	**2**	12	12	**0**
E2	9	11	**2**	8	9	**1**	8	8	**0**	9	10	**1**	6	4	**-2**	12	12	**0**
E3	2	7	**5**	2	8	**6**	7	8	**1**	9	12	**3**	0	2	**2**	11	11	**0**
E4	8	7	**-1**	5	3	**-2**	5	4	**-1**	0	0	**0**	0	0	**0**	11	12	**1**
E5	6	6	**0**	7	7	**0**	6	5	**-1**	6	6	**0**	2	3	**1**	12	12	**0**
E6	10	11	**1**	7	9	**2**	8	8	**0**	0	11	**11**	0	0	**0**	11	12	**1**
E7	6	6	**0**	8	4	**-4**	5	6	**1**	2	8	**6**	0	0	**0**	12	12	**0**
E8	11	11	**0**	9	9	**0**	10	8	**-2**	4	12	**8**	4	4	**0**	12	12	**0**
E9	7	10	**3**	8	8	**0**	7	8	**1**	10	11	**1**	0	0	**0**	12	12	**0**
E10	5	8	**3**	8	6	**-2**	8	5	**-3**	5	10	**5**	0	0	**0**	12	12	**0**
E11	5	10	**5**	6	9	**3**	8	5	**-3**	9	9	**0**	0	0	**0**	12	12	**0**
E12	11	11	**0**	8	8	**0**	8	10	**2**	7	11	**4**	3	3	**0**	12	12	**0**
E13	9	10	**1**	5	3	**-2**	3	4	**1**	6	10	**4**	0	0	**0**	12	12	**0**
E14	11	11	**0**	9	7	**-2**	10	8	**-2**	12	11	**-1**	1	1	**0**	12	11	**-1**
E15	7	11	**4**	9	9	**0**	8	10	**2**	7	12	**5**	0	6	**6**	12	12	**0**
E16	8	8	**0**	9	9	**0**	7	7	**0**	11	3	**-8**	0	2	**2**	12	12	**0**
E17	10	9	**-1**	2	0	**-2**	10	5	**-5**	0	11	**11**	0	0	**0**	12	11	**-1**
E18	6	8	**2**	9	8	**-1**	10	6	**-4**	10	12	**2**	0	0	**0**	12	12	**0**
E19	1	9	**8**	2	4	**2**	1	5	**4**	0	0	**0**	0	0	**0**	12	12	**0**
E20	9	6	**-3**	2	1	**-1**	10	5	**-5**	0	0	**0**	0	0	**0**	12	12	**0**
E21	7	9	**2**	9	9	**0**	7	8	**1**	6	7	**1**	0	0	**0**	12	12	**0**
E22	7	11	**4**	6	8	**2**	4	5	**1**	11	12	**1**	0	0	**0**	12	12	**0**

Table 4.1C

Control Group's Mathematics Test Results

Variable Test	M22 Pre	M22 Post	M22 Diff.	M23 Pre	M23 Post	M23 Diff.	M24 Pre	M24 Post	M24 Diff.	M25 Pre	M25 Post	M25 Diff.	M26 Pre	M26 Post	M26 Diff.	M27 Pre	M27 Post	M27 Diff.
Max.	**11**	**11**	**11**	**9**	**9**	**9**	**10**	**10**	**10**	**12**	**12**	**12**	**6**	**6**	**6**	**12**	**12**	**12**
C_1	7	8	**1**	7	9	**2**	8	8	**0**	0	0	**0**	0	0	**0**	12	12	**0**
C_2	8	10	**2**	5	9	**4**	8	5	**-3**	12	10	**-2**	0	4	**4**	12	12	**0**
C_3	3	3	**0**	5	5	**0**	6	7	**1**	0	3	**3**	0	0	**0**	10	11	**1**
C_4	2	9	**7**	1	7	**6**	3	4	**1**	0	0	**0**	0	0	**0**	12	8	**-4**
C_5	5	4	**-1**	2	3	**1**	5	4	**-1**	0	0	**0**	0	0	**0**	12	12	**0**
C_6	9	8	**-1**	8	8	**0**	8	8	**0**	0	0	**0**	0	0	**0**	12	12	**0**
C_7	4	4	**0**	0	1	**1**	0	1	**1**	0	2	**2**	0	0	**0**	12	12	**0**
C_8	9	9	**0**	7	9	**2**	6	6	**0**	0	0	**0**	0	0	**0**	12	12	**0**
C_9	6	7	**1**	3	5	**2**	3	3	**0**	0	0	**0**	0	0	**0**	12	12	**0**
C_{10}	6	6	**0**	7	7	**0**	8	6	**-2**	7	2	**-5**	0	0	**0**	12	12	**0**
C_{11}	5	7	**2**	7	7	**0**	8	6	**-2**	0	2	**2**	0	0	**0**	12	12	**0**
C_{12}	8	8	**0**	9	9	**0**	6	7	**1**	0	9	**9**	0	0	**0**	11	12	**1**
C_{13}	3	10	**7**	6	8	**2**	6	6	**0**	0	3	**3**	0	0	**0**	11	12	**1**
C_{14}	11	11	**0**	6	8	**2**	10	10	**0**	0	9	**9**	0	0	**0**	12	12	**0**
C_{15}	4	6	**2**	4	5	**1**	6	6	**0**	1	0	**-1**	0	0	**0**	12	12	**0**
C_{16}	6	3	**-3**	3	0	**-3**	0	0	**0**	0	0	**0**	0	0	**0**	12	12	**0**
C_{17}	7	3	**-4**	1	0	**-1**	7	0	**-7**	0	0	**0**	0	0	**0**	6	0	**-6**
C_{18}	6	6	**0**	1	3	**2**	6	0	**-6**	0	0	**0**	0	0	**0**	7	12	**5**
C_{19}	2	0	**-2**	3	3	**0**	6	3	**-3**	0	0	**0**	0	0	**0**	9	11	**2**
C_{20}	10	8	**-2**	8	5	**-3**	9	8	**-1**	3	8	**5**	0	0	**0**	11	11	**0**
C_{21}	9	7	**-2**	4	5	**1**	8	7	**-1**	9	8	**-1**	0	0	**0**	12	12	**0**
C_{22}	4	9	**5**	7	7	**0**	7	8	**1**	0	9	**9**	0	0	**0**	12	12	**0**

The function related concepts covered in the mathematics tests were not new to the students. The main goal of this study was to research the improvement of students' understanding of the concept of function compared to the pre-existing understanding of the concept showed in the pre-test. The mathematics tests were marked by me. In order to avoid any type of bias a strict marking scheme and the achievable percentages of variables were set a priori. I strictly applied the pre-defined mathematics test marking scheme as described in Chapter 3. The details of the marking scheme are summarised in Table 3.3 on page 99.

Based on students' mathematics scores, listed in Table 4.1E and Table 4.1C above, the pre-test and post-test achievements have been cumulated for both experimental and control group for the six targeted mathematical dependent variables.

The increase of understanding of function is indicated by the difference between the post-test and pre-test achievements. For the six mathematical dependent variables regarding the aspects of function, the mathematics tests' achievements of both groups have been considered and graphed. The graph of the experimental group's mathematics results – expressed in cumulated marks – is given in Figure 4.1 below.

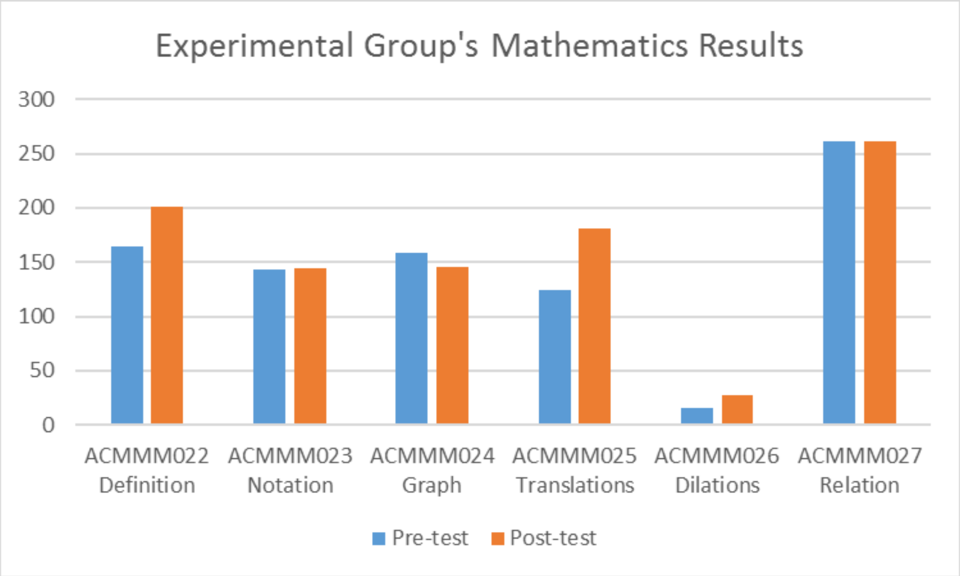

Figure 4.1: Experimental Group's Mathematics Results in Marks.

The graph of the control group's mathematics results is given in Figure 4.2 below.

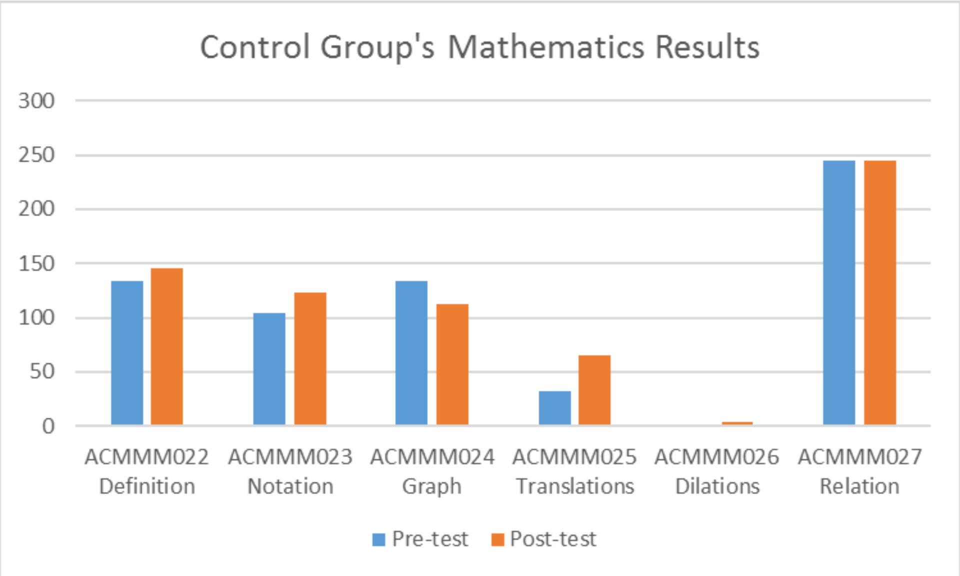

Figure 4.2: Control Group's Mathematics Results in Marks.

The independent t-test for two samples SPSS statistical analysis test was performed in order to state statistically correct statements regarding the achievement differences between the mathematics pre-tests and post-tests of both groups. The mathematical achievements of pre-tests and post-tests of both groups, expressed in marks and percentages, together with the statistical significance p-values at $\alpha = 0.05$ alpha level are given in Table 4.1A, respective Table 4.1R. The Table 4.1A gives the absolute group achievements of both groups for the six mathematical dependent variables, expressed in cumulated mark scores.

Table 4.1A

Absolute Mathematics Achievements and Test Differences in Marks

Math. Dependent Variable	**Exp. Gr. Pre-test (marks)**	**Exp. Gr. Post-test (marks)**	**Exp. Gr. Difference (marks)**	**Exp. Gr. Significance (p-value)**	**Ctrl. Gr. Pre-test (marks)**	**Ctrl. Gr. Post-test (marks)**	**Ctrl. Gr. Difference (marks)**	**Ctrl. Gr. Significance (p-value)**
ACMMM022 Definition	165	201	36	***p* = .006**	134	146	12	*p* = .378
ACMMM023 Notation	143	144	1	*p* = .922	104	123	19	*p* = .054
ACMMM024 Graph	158	146	-12	*p* = .292	134	113	-21	*p* = .052
ACMMM025 Translation	124	181	57	***p* = .008**	32	65	33	*p* = .066
ACMMM026 Dilation	16	27	11	*p* = .134	0	4	4	*p* = .329
ACMMM027 Relation	261	261	0	*p* = 1.000	245	245	0	*p* = 1.000
Total	**867**	**960**	**93**	***p* = .011**	**649**	**696**	**47**	***p* = .192**

The Table 4.1R gives the relative group achievements of both groups for the six mathematical dependent variables, expressed in percentages relative to the maximum marks achievable by the group on each of the six mathematical dependent variables. The maximum achievable marks by student of the six mathematical dependent variables are given in Table 3.1 on page 98. The maximum achievable marks by the group is the product between the maximum achievable marks by student and the number of students in the group. Because in both groups there were 22 participant students, the maximum achievable marks per group were calculated by multiplying the values of the maximum achievable marks per student (given in Table 3.1) by 22. For example, the experimental group's mathematics post-test achievement percentage of 83.06% on the ACMMM022 mathematical dependent variable (Understand the concept of function as a mapping between sets) has been calculated as follows: The maximum achievable marks per student on this variable is 11 marks (see Table 3.1).

As a result, the maximum achievable marks per group on this variable is 11 × 22 = 242 marks. The experimental group has achieved a total of 201 marks on this mathematical dependent variable (Table 4.1A). As a result, the relative percentage achievement of the experimental group on this mathematical dependent variable ACMMM022 is 201 / 242 × 100% = 83.057% = 83.06% as figured – together with the other relative percentages of both groups on all variables – in Table 4.1R.

Table 4.1R

Relative Mathematics Achievements and Test Differences in Percentages

Math. Dependent Variable	Exp. Gr Pre-test (%)	Exp. Gr Post-test (%)	Exp. Gr. Difference (%)	Exp. Gr. Significance (p-value)	Ctrl. Gr. Pre-test (%)	Ctrl. Gr. Post-test (%)	Ctrl. Gr. Difference (%)	Ctrl. Gr. Significance (p-value)
ACMMM022 Definition	68.18%	83.06%	14.88%	**p = .006**	55.37%	60.33%	4.96%	p = .378
ACMMM023 Notation	72.22%	72.73%	0.51%	p = .922	52.53%	62.12%	9.59%	p = .054
ACMMM024 Graph	71.82%	66.36%	-5.46%	p = .292	60.91%	51.36%	-9.55%	p = .052
ACMMM025 Translation	46.97%	68.56%	21.59%	**p = .008**	12.12%	24.62%	12.50%	p = .066
ACMMM026 Dilation	12.12%	20.45%	8.33%	p = .134	0.00%	3.03%	3.03%	p = .329
ACMMM027 Relation	98.86%	98.86%	0.00%	p = 1.000	92.80%	92.80%	0.00%	p = 1.000
Overall	**65.68%**	**72.73%**	**7.05%**	**p = .011**	**49.17%**	**52.73%**	**3.56%**	**p = .192**

In this Table 4.1R the group difference values are additive percentages, and are calculated as post-test minus pre-test values. For example, the experimental group's achievement increase regarding the first mathematical variable ACMMM022 is 14.88% = 83.06% - 68.18%. In both tables the indicator "Overall" shows students' overall understanding of function. In the "Overall"-values all six function aspects (ACMMM022, …, ACMMM027) are combined based on the pre-allocated achievable percentages of the six mathematical dependent variables and on the total of 60 marks of the mathematics test (see Table 3.1 on page 98). As a result, the formula to calculate the values of the "Overall" indicator is the following:

$$\textbf{Overall} = (11 \times M022 + 9 \times M023 + 10 \times M024 + 12 \times M025 + 6 \times M026 + 12 \times M027) / 60$$

We observe that the direct calculation mode of the experimental group's overall mathematics achievement percentage difference, 72.72% - 65.68% = 7.05%, gives the same result as calculating the experimental group's mathematics achievement percentage difference by indirect calculation mode, using the overall percentage difference calculation formula given above, as follows:

$$\textbf{Overall} = (11\times14.88\%+9\times0.51\%+10\times(-5.46\%)+12\times21.59\%+6\times8.33\%+12\times0.00\%) / 60 = 422.73\% / 60 = 7.0455\% = 7.05\%$$

As a result, we have obtained both by direct and indirect calculation modes the same result of 7.05% of experimental group's overall increase in the understanding of the concept of function. I also make a general observation regarding the distribution of the six mathematical dependent variables (ACMMM022, …, ACMMM027) in terms of how the experimental group's improvement of understanding of function compares to the control group's improvement of understanding of function. Compared to the control group, the experimental group has achieved:

- **better results** (larger increase or smaller decrease) on four mathematical dependent variables: ACMMM022, ACMMM024, ACMMM025, and ACMMM026;
- **equal result** on one mathematical dependent variable: ACMMM027; and
- **weaker result** (smaller increase than the control group) on one mathematical dependent variable: ACMMM023.

Regarding the mathematical dependent variable ACMMM027 (Recognise the distinction between functions and relations, and the vertical line test) it is to be observed that both groups have achieved sound, high achievement percentage results on the pre-tests (98.86% for the experimental group and 92.80% for the control group) and their overall results on this variable did not changed on the post-tests. For this mathematical dependent variable, the p-value for both experimental group and control group is $p = 1.0$, which indicates an exact equality of the two groups' improvements of understanding. The improvement per group in this case is zero, expressing the fact – the result obtained also in terms of cumulated marks and in terms of achievement percentages – that both groups have not increased or decreased their achievement on the mathematical dependent variable ACMMM027. Regarding this result, I also observed that there were differences between individual students' increases or decreases of understanding regarding this variable. For example, in the experimental group the students E_4 and E_6 increased their achievements by one mark each, but the students E_{14} and E_{17} have decreased their achievement by one mark each. All the other students in the experimental group achieved on the post-test the same result as on the pre-test, and as a result the general achievement of the experimental group on this mathematical dependent variable ACMMM027 has not changed. Because on this mathematical dependent variable ACMMM027 (Recognise the distinction between functions and relations, and the Vertical Line Test for Function) both groups have achieved the above mentioned good results (98.86%, respective 92.80% achievements) on both tests, the probability of an increase or a decrease was very low and neither group has showed an increase or decrease on this mathematical dependent variable.

It is also to be observed that on the mathematical dependent variable ACMMM024 (Understand the concept of the graph of a function) both groups showed a decrease in their achievements, but these decreases are statistically not significant at the 0.05 alpha level, showing $p = .292$ for the experimental group and $p = .052$ for the control group.

In terms of the statistical significance of the improvements of the experimental group on the six mathematical dependent variables, and based on the statistical analysis test of the independent samples t-test, we can state that the experimental group has achieved:

(a) **Statistically significant increases** on two variables:

- $p = .006$ on ACMMM022 (Understand the concept of function as a mapping between sets), an achievement increase of 14.88% from 68.18% to 83.06%.
- $p = .008$ on ACMMM025 (Examine translations and the graphs of $y = f(x) + a$ and $y=f(x+b)$), an achievement increase of 21.59% from 46.97% to 68.56%.

(b) **Statistically not significant increases** on two variables:

- $p = .922$ on ACMMM023 (Use function notation, domain and range, independent and dependent variables), a small achievement improvement from 72.22% to 72.73%.
- $p = .134$ on ACMMM026 (Examine dilations and the graphs of $y = c \times f(x)$ and $y = f(k \times x)$), an achievement improvement from 12.12% to 20.45%.

(c) **Statistically not significant decrease** on one variable:

- $p = .292$ on ACMMM024 (Understand the concept of the graph of a function), an achievement decrease (-5.46%) from 71.82% to 66.36%.

(d) **No change** on one variable:

- p = 1.000 on ACMMM027 (Recognise the distinction between functions and relations, and the vertical line test), sound achievements of 98.86% on both tests.

In terms of statistical significance of the improvements of the control group on the six mathematical dependent variables, and based on the performed statistical analysis test of independent samples t-test, we can state that the control group has achieved:

(a) **No statistically significant increase** on any mathematical dependent variable.

(b) **Statistically not significant increase** on four mathematical dependent variables:

- p = .378 on ACMMM022 (Understand the concept of function as a mapping between sets).
- p = .054 on ACMMM023 (Use function notation, domain and range, independent and dependent variables).
- p = .066 on ACMMM025 (Examine translations and the graphs of y = f(x) + a and y = f(x + b)).
- p = .329 on ACMMM026 (Examine dilations and the graphs of y = cf(x) and y = f(kx)).

(c) **Statistically not significant decrease** on one variable:

- p = .052 on ACMMM024 (Understand the concept of the graph of a function).

(d) **No change** on one variable:

- p = 1.000 on ACMMM027 (Recognise the distinction between functions and relations, and the Vertical line test).

The reason of the decrease of the achievements of both groups on the variable ACMMM024 (Understand the concept of the graph of a function) is not known. Considering the overall increase of 7.05% of the experimental group compared to the 3.56% overall increase of the control group can be observed that the general increase of the understanding of the concept of function in the case of the experimental group was almost double that of the general increase of the understanding of the concept of function of the control group.

4.4 Statistical Analysis of the Mathematics Achievements

Independent t-tests for the two samples statistical analyses were performed in order to enable making correct and valid statistical statements regarding the first research question. The first research question, RQ-1, was the main focus of the study: “Does the active integration of mathematics and music by using these nine analogies between different aspects of function and the corresponding aspects of music make a statistically significant improvement in my senior secondary mathematics students’ understanding of function?”

I present the relevant results of the independent t-tests and will state how they responded to the main research question regarding the improvement of my senior secondary (Year 11) mathematics students' understanding of function.

4.4.1 Analysis of the Group Statistics

The group statistics (N, Mean, Standard Deviation and Standard Error Mean) of the Mathematics pre-tests and post-tests are shown in the Table 4.2 below.

Table 4.2

Group Statistics of Mathematics Tests

Test	N	Mean	Std. Deviation	Std. Error Mean
Experimental Group, Mathematics Pre-test	22	39.41	8.86	1.89
Experimental Group, Mathematics Post-test	22	43.64	9.45	2.01
Control Group, Mathematics Pre-test	22	29.5	8.73	1.86
Control Group, Mathematics Post-test	22	31.64	11.89	2.53

I interpreted the Table 4.2 above of Group Statistics as follows: There were 22 valid observations in both experimental and control group for both mathematics tests. The means, the central tendencies of achievements of the experimental group are 39.41 marks for the mathematics pre-test and 43.64 marks for the mathematics post-test. The means, the central tendencies of achievements of the control group are 29.50 marks for the mathematics pre-test and 31.64 marks for the mathematics post-test. The standard deviations, expressing the spread of the data are 8.86 and 9.45 marks for the experimental group, and 8.73 and 11.89 marks for the control group. Howell (2010) defines the standard error as “the standard deviation of any sampling distribution is called the standard error of that distribution” (p. 205). The standard errors of the means are calculated relative to N, in our case as the corresponding ratios of the standard deviations and the square root of N = 22.

It can be observed that in the previous observation, based on percentage differences between the pre-test and post-test achievements, the increase of the understanding of the concept of function of the participant students in the experimental group was nearly the double of the increase of the understanding of the participant students in the control group which is reflected in the increase of the mean values as well. The increase of the mean of the experimental group is 43.64 - 39.41 = 4.23 marks, and the increase of the mean of the control group is 31.64 – 29.50 = 2.14 marks, which gives the ratio of mean increases Experimental Group Mean Increase / Control Group Mean Increase = 4.23 / 2.14 ≈ 1.98, this being in accordance with the percentage result presented above and is expressing a nearly double improvement of the understanding of the function concept of the students in the experimental group compared to that of the students in the control group. I believe that this result can be attributed to the treatment intervention, showing that this type of active integration of mathematics and music has improved my senior secondary mathematics students' understanding of function.

The standard deviations, expressing the spread of the measure of understanding of the function concept, have increased in different modes. The increase of the standard deviation of the experimental group is 9.45 - 8.86 = 0.59 marks, or (9.45 – 8.86) / 8.86 × 100% = 6.62% increase. The increase of the standard deviation of the control group is 11.89 – 8.73 = 3.16 marks or (11.89 – 8.73) / 8.73 × 100% = 36.20% increase. If we compare the two increases of standard deviations, then we obtain the control group / experimental group ratio of 36.20% / 6.62% = 5.47, which means that the increase of the standard deviation of the control group is nearly 5.5 times bigger than the increase of the standard deviation of the experimental group. These figures indicate that the distribution of the measure of understanding of the two groups became very different. While the pre-tests standard deviations were fairly close to each other, 8.86 marks for the experimental group and 8.73 marks for the control group, showing the equivalence of the two groups also in this way, the post-test standard deviations are different. The experimental group has kept the original distribution of the measures of understanding of the concept of function in a much more accentuated way than the control group. This means that the students in the experimental group have improved their understanding of function proportionally, but the students in the control group gained improvement of their understanding of function differently, they have spread their improvement of understanding differently than their colleagues in the experimental group, which means that some students improved very well, other students perhaps didn't improve at all, or some students perhaps have decreased their understanding, showing a negative improvement.

The following Table 4.3 presents the statistics regarding the two groups' mathematics pre-tests and post-tests.

Table 4.3

Mathematics Independent Samples Test

	Levene's Test for Equality of Variances		t-test for Equality of Means						
								95% CI of the diff.	
	F	**Sig.**	**t**	**df**	**Sig. (2-tailed)**	**Mean Diff.**	**Std. Err. Diff.**	**Lower**	**Upper**
Pre	0.32	0.58	3.74	42	.001	9.91	2.65	4.56	15.26
Post	1.14	0.29	3.71	42	.001	12	3.24	5.47	18.53

We interpret the above Table 4.3 of Mathematics Independent Samples Test as follows.

<u>Levene's Test for Equality of Variances</u>

The first column, "**F**", provides the F-statistic values (for the mathematics pre-tests and for the mathematics post-tests) of Levene's Test for Equality of Variances, the ratios which underline the analysis of the equality of variances tests.

The second column, "**Sig.**", provides the two-tailed p-values (significance value probabilities), regarding the null hypotheses that there are no differences in the variances of the two groups. Because both p-values are greater than the a priori set $\alpha = 0.05$ alpha level, we don't reject these hypotheses, and we state that Levene's statistics for equality of variances show that there is no statistically significant difference in the variance for both mathematics pre-test ($p = .58$) and mathematics post-test ($p = .29$). These results indicated that the variances of both mathematical tests were equal.

t-Test for Equality of Means

In the "**t**" and "**df**" columns provide the *t*-statistic, respective the *df*-statistic values. In Table 4.3 the *df*-statistic values, the degrees of freedom, are calculated as df = N – 2. In my study N = 44, so df = 44 - 2 = 42 in both cases. We can see that the t and df statistics are: t = 3.736, df = 42 for the mathematics pre-tests, and t = 3.707, df = 42 for the mathematics post-tests. The equality of the degrees of freedom is a possible sign of normality.

The "**Sig. (2-tailed)**" column provides the p-values (significance values, two-tailed probabilities) – calculated by using the corresponding t-distribution, and expressing the probability of observing a t-value of equal or greater value under the null hypothesis. If this p-value is less than the pre-set alpha significance level of $\alpha = 0.05$, then we conclude that the difference in means is significantly different from zero. In our case $p = .001 < .05 = \alpha$ for both mathematics tests, and so we can state that the difference in means is statistically significantly different to zero.

The "**Mean Diff.**" column provides the mathematics achievement mean differences between the experimental and control groups for both pre-test and post-test, which in this case are 9.91 marks for the mathematics pre-tests, and 12.00 marks for the mathematics post-tests.

The "**Std. Error Diff.**" column provides the statistics expressing the estimated standard deviations of the differences between the sample means. These are 2.65 marks for the mathematics pre-test and 3.24 for the mathematics post-test. These statistics provide a measure of the variability of the sample means.

The "**Lower**" and "**Upper**" columns provide the 95% confidence intervals of the differences of means, which in our case are [4.56, 15.26] for the mathematics pre-test and [5.47, 18.54] for the mathematics post-test.

4.4.2 Analysis of Mathematics Scores of the Experimental Group

The t-test statistical analysis tables regarding the experimental group's mathematics achievements are as follows. The Experimental group's paired samples statistics are displayed in Table 4.4 following.

Table 4.4

Experimental Group's Mathematics Paired Samples Statistics

	Mean	N	Std. Deviation	Std. Error Mean
Experimental Group's Mathematics Pre-test	39.41	22	8.86	1.89
Experimental Group's Mathematics Post-test	43.64	22	9.45	2.01

The following paired samples correlations table of the t-test of the experimental group, Table 4.5 shows that for the experimental group the mathematics pre-test and the mathematics post-test are statistically significantly positively correlated ($p < 0.001$, $r = 0.700$).

Table 4.5

Experimental Group's Mathematics Paired Samples Correlations

	N	Correlation	Significance
Experimental Group's Mathematics Pre-test and Post-test	22	r = 0.700	$p = .000$

There is statistical evidence that the increase of the mathematics achievement of the experimental group is statistically significant ($t = -2.788$, $p = .011$) as described in table of the experimental group's mathematics paired samples test, Table 4.6 below.

Table 4.6

Experimental Group's Mathematics Paired Samples Test

Exp. Group Mathematics	Mean	Std. Deviation	Std. Error Mean	95% C. I. Lower	95% C. I. Upper	t	df	Sig. (2-tailed)
Pre - Post	-4.23	7.11	1.52	-7.38	-1.07	t = -2.788	21	$p = .011$

On observing that the general improvement of the experimental group's understanding of function is statistically significant ($p = .011$) at alpha = 0.05 level, I realised that I would have to calculate the grade of the effect by using an effect size measure. Howell (2010) states that “we come to the issue of finding ways to present information to our readers that conveys the magnitude of the difference” (p. 209). I used an effect size measure based on *Cohen’s d.*

Cohen (1988, p. 20) defines the formula of the effect size index *d*, later named as *Cohen's d*, in terms of the means m_A and m_B of the two independent samples and – if the standard deviations are equal – the common standard deviation σ of two samples as follows:

$$\mathbf{d = |m_A – m_B| / \sigma}$$

This formula is expressing the mean difference between the two samples in terms of standard deviation units. Cohen (1988) considers the effect size "small: *d* = 0.20, medium: *d* = 0.50, and large: *d* = 0.80" (p. 40). It was not possible to use the above formula in my study because the standard deviations of the two samples were not equal.

Cohen (1988, p. 44) gives the formula for the effect size index *d* also in the case when the two standard deviations are not similar. In this case, if we have the two different standard deviations σ_A and σ_B then we calculate σ', the root mean value of the two standard deviations as follows:

$$\mathbf{\sigma' = [(\sigma_A^2 + \sigma_B^2) / 2]^{0.5}}$$

The effect size index can be calculated by using the similar formula:

$$\mathbf{d = |m_A – m_B| / \sigma'}$$

In the case of my study, we have the experimental group's mathematics pre-test mean m_A = 39.41 marks and standard deviation σ_A = 8.86 marks, and the experimental group's mathematics post-test mean m_B = 43.64 marks and standard deviation σ_B = 9.45 marks (see Table 4.4 above). The effect size index was calculated as follows.

Step 1: The root mean value of the two standard deviations was calculated:

$$\mathbf{\sigma' = [(\sigma_A^2 + \sigma_B^2) / 2]^{0.5} = [(8.86^2 + 9.45^2) / 2]^{0.5} = 9.16}$$

Step 2: The effect size index was calculated by substituting the values of m_A = 39.41, m_B = 43.64, and σ' = 9.16 into Cohen's formula:

$$\mathbf{d = (|m_A – m_B|) / \sigma' = (|39.41 - 43.64|) / 9.16 = 0.46}.$$

This result is also confirmed by the online effect size calculator: http://www.polyu.edu.hk/mm/effectsizefaqs/calculator/calculator.html.

Another index to measure the magnitude of the improvement is the *Glass' delta* effect size. Becker (2000) gives the definition of the *Glass' delta* effect size as "the mean difference between the experimental and control group divided by the standard deviation of the control group” (p. 4).

Stangroom (2018) states regarding the effect sizes:

> *Cohen's d* is the appropriate effect size measure if two groups have similar standard deviations and are of similar size. *Glass' delta*, which uses only the standard deviation of the control group, is an alternative measure if each group has a different standard deviation. *Hedges' g*, which provides a measure of effect size weighted according to the relative size of each sample, is an alternative where there are different sample sizes.

(Stangroom,“EffectSize”2018,http://www.socscistatistics.com/effectsize/Default3.aspx)

Using the N, Mean, and Std. Deviation parameters above, I calculated, with the help of the Social Science Statistics' online effect size calculator for t-test (Stangroom, 2018), the following.

The effects sizes between the experimental group's mathematics pre-test and post-test are:

Cohen's d = (43.64 - 39.41) ⁄ 9.159752 = 0.461803,

Glass' $delta$ = (43.64 - 39.41) ⁄ 8.86 = 0.477427, and

Hedges' g = (43.64 - 39.41) ⁄ 9.159752 = 0.461803.

In order to calculate the Hedges' g effect size index, the relevant formula where the equal standard deviation is not a condition was used. Using the t-statistic values t = 3.736 for the pre-test and t = 3.707 for the post-test, and the sizes of the two groups, n_1 = 22 and n_2 = 22, the *Hedges' g* effect size indices were used to determine the magnitudes of the differences. Substituting these values into the following formula given in Becker (2000, p. 4):

$$\mathbf{g = t \times [(n_1 + n_2) / (n_1 \times n_2)]^{0.5}}$$

produces the results $g_{pre} = 3.736 \times (44 / 484)^{0.5} = 1.13$ for the mathematics pre-test, and $g_{post} = 3.707 \times (44 / 484)^{0.5} = 1.12$ for the mathematics post-test.

The two effect size measures, *Cohen's d* and *Hedges' g* are in relation to each other. Given *N* and *df*, then *Hedges's g* effect size can be computed from *Cohen's d* effect size based on the formula given in Becker (2000, p. 4) as follows:

$$\mathbf{g = d / (N / df)^{0.5}}$$

In my study having the p-value p = .011, and the effect size indexes of *Cohen's d* = 0.46, *Glass' delta* = 0.48, and *Hedges' g* = 0.46, calculated above, we can state that the treatment's effect was a statistically significant, medium effect size improvement of experimental group's understanding of function.

Regarding the mathematics achievements of the experimental group we can state that the paired samples t-test indicated that marks were significantly higher for the experimental group's mathematics post-test (M = 43.64, SD = 9.45) than for the experimental group's mathematics pre-test (M = 39.41, SD = 8.86), $t(21) = -2.79$, p = .011, d = 0.46.

4.4.3 Analysis of Mathematics Scores of the Control Group

The relevant t-test statistical analysis tables regarding the control group's mathematics achievements are as follows.

The following Table 4.7 shows the control groups' mathematics related paired samples statistics. It presents statistical information regarding the control group's mathematics pre-tests and post-tests, such as the means, sample size, standard deviations and standard error means of these mathematics tests. These statistics reveal important information if we compare these two mathematics tests of the control group.

Table 4.7

Control Group's Mathematics Paired Samples Statistics

	Mean	N	Std. Deviation	Std. Error Mean
Control Group's Mathematics Pre-test	29.5	22	8.73	1.86
Control Group's Mathematics Post-test	31.64	22	11.89	2.53

The Table 4.7 above reveals that the control group's mean result has improved from 29.50 marks to 31.64 marks, which increase is an important statistic, a valuable outcome of my action research, informing my traditional teaching practice of function. I have concluded that my traditional method of revision of the function topic also has improved my Year 11 Mathematical Methods students' understanding of function, which fact underlines the observation that both methods work fairly well. I also have observed from Table 4.7 that the spread of the marks among the students in the control group also has increased, as the standard deviation values demonstrate that fact. This information suggests that some students gained a higher increase in understanding of function compared to others.

The following Table 4.8, the Paired Samples Correlations Table of the control group's mathematics tests, shows that for this group the mathematics pre-test and mathematics post-test are significantly positively correlated ($p < 0.001$, $r = 0.781$).

Table 4.8

Control Group's Mathematics Paired Samples Correlations

	N	Correlation	Significance
Control Group's Mathematics Pre-test and Post-test	22	r = 0.781	*p* = .000

The correlation index tells how these two samples of mathematics tests of the control group relate to each other.

Table 4.9 below presents the mean difference of the two mathematics samples of the control group, the t-statistic and the p-value. There is statistical evidence that the increase of the achievement of the control group is statistically not significant (t = -1.347, p = 0.192) at alpha = 0.05 level as presented in Table 4.9 below.

Table 4.9

Control Group's Mathematics Paired Samples Test

Control Group Mathematics	Mean	Std. Deviation	Std. Error Mean	95% C. I. Lower	95% C. I. Upper	t	df	Sig. (2-tailed)
Pre - Post	-2.14	7.44	1.59	-5.44	1.16	*t* = -1.347	21	*p* = .192

Using the *N*, Mean, and Std. Deviation statistics above, I calculated with the help of the Social Science Statistics' online effect size calculator for t-test (Stangroom, 2018), the effect sizes between the control group's mathematics pre-test and post-test as follows:

Cohen's d = (31.64 - 29.5) ⁄ 10.430364 = 0.20517,

Glass' delta = (31.64 - 29.5) ⁄ 8.73 = 0.245132, and

Hedges' g = (31.64 - 29.5) ⁄ 10.430364 = 0.20517.

As a result, regarding the control group's mathematics achievements, the following effect size indices have been considered *Cohen's d* = 0.21, *Glass' delta* = 0.25, and *Hedges' g* = 0.21. These effect size indices indicate that my teaching practice using the traditional way of reviewing functions had a small effect size.

Regarding the mathematics achievements of the control group the paired samples t-test indicated that marks were not significantly higher for the control group's mathematics post-test (M = 31.64, SD = 11.89) than for the control group's mathematics pre-test (M = 29.5, SD = 8.73), $t(21) = -1.35$, $p = .192$, $d = 0.21$.

The t-test result of the control group ($t = -1.347$, $p = 0.192$) together with the t-test result of the experimental group ($t = -2.788$, $p = .011$) contributed towards answering the first research question by rejecting the first Null Hypothesis, $H1_0$, and stating that this type of active integration of mathematics and music by using these nine analogies between different aspects of function and the corresponding aspects of music does make a statistically significant improvement in my senior secondary mathematics students’ understanding of function. Being able to provide the effect size for both groups, which was not asked for explicitly in the research question represents a very important piece of information for a practising teacher. *Cohen's d* for the experimental group is d = 0.46 representing a medium effect size, and *Cohen's d* for the control group is d = 0.21, representing a small effect size. These effect size indices informed my teaching practice that the non-traditional method of this study had a medium effect size and the traditional method a small effect size in improving my Year 11 mathematics students' understanding of function.

4.5 Statistical Analysis of the Collected Non-Cognitive Data

Non-cognitive data, that is students' perception regarding their mathematics-related beliefs and attitudes as well as their learning environment perceptions, were collected from all participants (N = 44) in pre-test and post-test form by using three questionnaires with proven validity and reliability: Mathematics Related Beliefs

Questionnaire (MRBQ) (Op't Eynde & De Corte, 2003), Test Of Mathematics Related Attitudes (TOMRA) (Fraser, 1981; Taylor, 2004) and What Is Happening In This Class (WIHIC) (Fraser, 1998). These three questionnaires use Likert-type measuring scales. Warmbrod (2014) describes the Likert-type measuring scales in the following terms:

> Psychologist Rensis Likert published a monograph, *A Technique for the Measurement of Attitudes*, describing the concepts, principles, and substantiative research basic to an instrument to quantify constructs describing psychological and social phenomena (Likert, 1932). A Likert-type scale consists of a series of statements that define and describe the content and meaning of the construct measured. The statements comprising the scale express a belief, preference, judgment, or opinion. The statements are composed to define collectively an unidimensional construct. Alternatively, clusters of statements within a scale may define one or more subscales that quantify more specific unidimensional subconstructs within the major scale.
>
> (Warmbrod, 2014, p. 31)

The following four scales regarding students' mathematics related beliefs were collected through the MRBQ questionnaire (Op't Eynde & De Corte, 2003), see Appendix C.

- Teacher as facilitator (11 items, questions 1 – 11),
- Student self-efficacy (12 items, questions 12 – 23),
- Functional necessity (13 items, questions 24 – 36),
- Real world relevance (11 items, questions 37 – 47).

The following two scales regarding students' mathematics related attitudes were also collected through the TOMRA questionnaire (Fraser, 1981; Taylor, 2004), see Appendix D.

- Enjoyment of mathematics lessons (10 items, questions 1, 3, 5, 7, 9, 11, 13, 15, 17, 19),
- Attitude to mathematics inquiry (10 items, questions 2, 4, 6, 8, 10, 12, 14, 16, 18, 20).

The following seven scales regarding students' learning environment were collected through the WIHIC questionnaire (Fraser, 1998), see Appendix B.

- Student cohesiveness (8 items, questions 1 – 8): extent to which students know, help and are supportive of one another;
- Teacher support (8 items, questions 9 – 16): extent to which teacher helps, trusts, and shows interest in students;
- Involvement (8 items, questions 17 – 24): extent to which students have attentive interest, participate in discussions, perform additional work, and enjoy the class;
- Investigation (8 items, questions 25 – 32): extent to which students use their skills and knowledge in processes of inquiry, problem solving and investigation;
- Task orientation (8 items, questions 33 – 40): extent to which it is important to complete activities planned and to stay on the subject matter;
- Cooperation (8 items, questions 41 – 48): extent to which students cooperate rather than compete with one another on learning tasks;
- Equity (8 items, questions 49 – 56): extent to which students are treated equally by the teacher.

Data was collected from all forty four participating students in pre-test and post-test forms. There were no missing or invalid responses. The descriptive statistics of the findings – the frequencies of the five Likert scores and the corresponding weighted means of the collected data – are presented below. The weighted mean of a scale, m_w is calculated by considering the values of students' Likert-type answers 5, 4, 3, 2, and 1, together with their corresponding frequencies F(5), F(4), F(3), F(2), and F(1) as follows:

$$\mathbf{m_w = [(5 \times F(5) + 4 \times F(4) + 3 \times F(3) + 2 \times F(2) + 1 \times F(1)] \,/\, [F(5) + F(4) + F(3) + F(2) + F(1)]}$$

This weighted mean m_w is a relative value because it is independent from N, the number of participant students. The weighted mean is expressing the group's perception within the range of the Likert-type scores 1 – 5. This type of mean indicator is different to the absolute mean indicator calculated by the t-test, where the mean is calculated as the sum of the scores divided by *N*, the number of participants. Obviously, both approaches are correct and they together offer a better understanding of the results of the study.

The following Table 4.10 and Table 4.11 present the summary of the experimental group's, respective control group's answers on the thirteen scales of the three questionnaires showing the frequencies F(5), F(4), F(3), F(2), and F(1) of the Likert-type answers 5, 4, 3, 2, respective 1, the weighted means for both pre-test and post-test, and the post – pre weighted mean differences.

Table 4.10

Experimental Group's Scores on Non-Cognitive Scales

Experimental group's perceptions	F(5) Pre	F(4) Pre	F(3) Pre	F(2) Pre	F(1) Pre	m_w Pre	F(5) Post	F(4) Post	F(3) Post	F(2) Post	F(2) Post	m_w Post	m_w Diff
Teacher as facilitator	142	93	7	0	0	**4.56**	144	82	12	2	0	**4.53**	**-0.03**
Student self-efficacy in mathematics	91	105	52	12	4	**4.01**	85	102	57	17	3	**3.94**	**-0.07**
Functional necessity of mathematics	52	114	66	45	9	**3.54**	64	106	59	45	12	**3.58**	**0.04**
Real world relevance of mathematics	134	82	21	5	0	**4.43**	131	83	25	3	0	**4.41**	**-0.02**
Enjoyment of mathematics lessons	62	83	53	20	2	**3.83**	74	75	52	16	3	**3.91**	**0.08**
Attitude to mathematics inquiry	32	110	60	18	0	**3.71**	60	91	46	19	4	**3.84**	**0.13**
Student cohesiveness	69	68	36	3	0	**4.15**	77	64	33	1	1	**4.22**	**0.07**
Teacher support	47	69	48	11	1	**3.85**	78	50	36	6	6	**4.07**	**0.22**
Involvement	21	53	63	25	14	**3.24**	23	58	45	32	18	**3.2**	**-0.04**
Investigation	12	55	68	27	14	**3.14**	30	55	50	22	19	**3.31**	**0.17**
Task orientation	91	63	21	0	1	**4.38**	88	69	15	4	0	**4.37**	**-0.01**
Cooperation	63	75	34	4	0	**4.12**	71	74	27	3	1	**4.2**	**0.08**
Equity	100	53	21	2	0	**4.43**	87	64	22	3	0	**4.34**	**-0.09**

Can be observed that the weighted mean score differences between the post-tests and pre-tests $m_w diff$ on all of the considered thirteen non-intellective variables is minimal, being in the range of $-0.10 < m_w\ diff < 0.23$. These data show that there were no major changes in experimental group students' perceptions of their mathematics related attitudes and beliefs nor in students' learning environment perceptions. This result is confirmed by the performed t-test as well.

Table 4.11

Control Group's Scores on Non-Cognitive Scales

Control group's perceptions	F(5) Pre	F(4) Pre	F(3) Pre	F(2) Pre	F(1) Pre	m_w Pre	F(5) Post	F(4) Post	F(3) Post	F(2) Post	F(1) Post	m_w Post	m_w Diff
Teacher as facilitator	107	96	34	5	0	**4.26**	107	101	27	4	1	**4.28**	**0.04**
Student self-efficacy in mathematics	39	104	104	17	0	**3.63**	63	103	84	10	4	**3.79**	**0.16**
Functional necessity of mathematics	69	94	76	38	9	**3.62**	57	106	84	21	18	**3.57**	**-0.05**
Real world relevance of mathematics	105	105	28	3	1	**4.28**	93	104	45	0	0	**4.19**	**-0.09**
Enjoyment of mathematics lessons	30	75	92	16	7	**3.48**	31	92	89	5	3	**3.65**	**0.17**
Attitude to mathematics inquiry	32	101	79	8	**0**	**3.71**	19	110	80	10	1	**3.62**	**-0.09**
Student cohesiveness	58	77	38	2	1	**4.07**	63	63	38	9	3	**3.99**	**-0.08**
Teacher support	69	51	44	10	2	**3.99**	57	59	53	4	3	**3.93**	**-0.06**
Involvement	15	37	83	26	15	**3.06**	31	33	76	26	10	**3.28**	**0.22**
Investigation	4	57	69	25	21	**2.99**	36	47	65	23	5	**3.49**	**0.5**
Task orientation	76	69	29	2	0	**4.24**	80	51	36	7	2	**3.12**	**-1.12**
Cooperation	45	75	43	11	2	**3.85**	43	60	67	5	1	**3.79**	**-0.06**
Equity	89	51	32	2	2	**4.27**	89	53	30	4	0	**4.29**	**0.02**

Can be observed that the weighted mean score differences between the post-tests and pre-tests, $m_w diff$ of the control group is in the range of $-1.13 < m_w\ diff < 0.51$. These data show that there were some changes in control group's perceptions of mathematics related attitudes and beliefs in students' learning environment perceptions. This result is also confirmed by the performed t-test.

The relevant independent t-test statistical analysis results regarding the students' mathematics related beliefs, attitudes and learning environment perceptions, expressed in the form of indicators as described above, in other words, the statistical analysis of the WIHIC (Fraser, 1998), MRBQ (Op't Eynde & De Corte, 2003) and TOMRA (Fraser, 1981; Taylor, 2004) questionnaires, show the findings described below.

4.5.1 Experimental Group's Statistics of Non-Cognitive Scales

The Paired Samples Correlations table of the independent t-test for the two samples of the experimental group shows that students' pre-test and post-test perceptions on all of the collected 13 non-cognitive indicators are significantly positively correlated ($p_i < 0.019$ and $0.500 < r_i < 0.858$, for all $i = 1, 2, \ldots, 13$). The following Table 4.12

shows the experimental group's main relevant statistic values such as the mean results, pre-test mean – post-test mean differences, sample sizes (*N*), t-statistics (*t*), degrees of freedom (*df*), and 2-tailed significance probabilities (*p*-values) regarding to the mean differences to be zero on all 13 considered non-cognitive dependent variables. The mean regarding a variable is the mean of the scores, and is calculated in t-test as the total of the scores regarding that variable divided by $N = 22$, the number of participant students in the group. For example, if we consider the variable "Teacher as facilitator" then we have the total of 142 x 5 + 93 x 4 + 6 x3 + 0 x 2 + 0 x 1 = 1103 scores, as a result the mean scores regarding this variable is 1103 / 22 = 50.1363 = 50.14 scores as figured in Table 4.12. This is the absolute mean score which is in correlation with the weighted or relative mean score given above.

Table 4.12

Experimental Group's Main Statistics of Non-Cognitive Scales

Experimental group's students' perceptions of:	Mean Pre	Mean Post	Mean Diff.	N	t	df	Sig. (2-tailed)
Teacher as facilitator	50.14	49.82	0.32	22	$t = 0.414$	21	$p = .683$
Student self-efficacy in math.	48.14	47.32	0.82	22	$t = 0.867$	21	$p = .396$
Functional necessity of math.	46.05	46.5	-0.46	22	$t = -0.309$	21	$p = .760$
Real world relevance of math.	48.68	48.55	0.14	22	$t = 0.163$	21	$p = .872$
Enjoyment of mathematics lessons	38.32	39.14	-0.82	22	$t= -0.923$	21	$p = .367$
Attitude to mathematics inquiry	37.09	38.36	-1.27	22	$t = -1.267$	21	$p = .219$
Student cohesiveness	33.23	33.77	-0.54	22	$t = -0.704$	21	$p = .489$
Teacher support	30.82	32.55	-1.73	22	$t = -1.378$	21	$p = .183$
Involvement	25.91	25.64	0.27	22	$t = 0.249$	21	$p = .806$
Investigation	25.09	26.5	-1.41	22	$t = -1.187$	21	$p = .249$
Task orientation	35.05	34.95	0.09	22	$t = 0.132$	21	$p = .896$
Cooperation	32.95	33.59	-0.64	22	$t = -0.747$	21	$p = .463$
Equity	35.41	34.68	0.73	22	$t = 0.911$	21	$p = .373$

It can be observed that the mean differences are minimal, they vary between -1.41 and 0.82 marks which fact is reflected in the p-values as well, every p-value being greater than 0.05, indicating that in terms of the 13 non-cognitive variables, there is no statistically significant difference between the experimental group's pre-test and the post-test perceptions on any of these 13 variables. Based on these facts it may be concluded that there is no statistical evidence to reject the second Null Hypothesis, $H2_0$. The answer to the second research question, RQ-2, is that this type of integration of mathematics and music did not make a statistically significant improvement in my senior secondary mathematics students' attitudes and beliefs regarding mathematics. This result confirms my observations I have acquired during my teaching practice, namely, that changing senior secondary students' attitudes and beliefs regarding mathematics cannot be achieved in a short period of time, through one intervention only. A significant positive change could only be the result of a much longer process. Based on this observation I agree with the statistical evidence and consider that this type of intervention is a step only in the long process of changing my senior secondary students' attitudes and beliefs regarding mathematics.

4.5.2 Control Group's Statistics of Non-Cognitive Scales

The Paired Samples Correlations table of the independent t-test for the two samples of the control group, shows that students' pre-test and post-test perceptions on eight variables out of the collected thirteen non-cognitive variables are significantly positively correlated and on five variables are not significantly positively correlated at 0.05 alpha level. The following Table 4.13 shows the control group's main statistics regarding the considered thirteen non-cognitive variables. $p > 0.05$ on twelve variables indicating the mean differences to be zero on these twelve variables.

Table 4.13

Control Group's Main Statistics of Non-Cognitive Scales

Control group's students' perceptions of:	**Mean Pre**	**Mean Post**	**Mean Diff.**	**N**	**t**	**df**	**Sig. (2-tailed)**
Teacher as facilitator	46.86	47.05	-0.18	22	t = -0.134	21	p = .895
Student self-efficacy in math.	43.5	45.59	-2.09	22	t = -1.780	21	p = 0.090
Functional necessity of math.	47.0	46.41	0.59	22	t = 0.418	21	p = 0.680
Real world relevance of math.	47.09	46.18	0.91	22	t = 0.734	21	p = 0.471
Enjoyment of mathematics lessons	34.77	36.5	-1.73	22	t = -2.055	21	p = 0.053
Attitude to mathematics inquiry	37.14	36.18	0.96	22	t = 1.380	21	p = 0.182
Student cohesiveness	32.59	31.91	0.68	22	t = 0.703	21	p = 0.490
Teacher support	31.95	31.41	0.55	22	t = 0.627	21	p = 0.537
Involvement	24.5	26.23	-1.73	22	t = -1.157	21	p = 0.260
Investigation	23.91	27.91	-4	22	t = -2.533	21	**p = 0.019**
Task orientation	33.95	33.09	0.86	22	t = 0.708	21	p = 0.486
Cooperation	30.82	30.32	0.5	22	t = 0.434	21	p = 0.669
Equity	34.14	34.32	-0.18	22	t = -0.222	21	p = 0.826

We can observe that the p-values are greater than 0.05 on twelve out of thirteen non-cognitive dependent variables, meaning that there is no statistically significant difference between the control students' pre-test and post-test perceptions on twelve out of these thirteen non-cognitive dependent variables. There is a statistically significant difference between the means of the control groups' pre-test and post -test scores regarding one out of 13 non-cognitive dependent variables. The statistically significant difference (p = .019) has been observed on the variable “Investigation”, expressing the extent of students' self-estimated level of using their mathematical skills and knowledge in mathematical inquiries, problem solving and investigation. This is a positive result and useful information for myself as a practicing secondary mathematics teacher, that my traditional method – which incorporates encouraging students to investigate different mathematical questions or problems – has statistically improved my students' perception of investigation.

4.6 Discussion

The independent t-test was used for statistical analysis of the data collected by using the IBM SPSS Statistic (SPSS) software package. I also have used the Microsoft Excel spreadsheet software to calculate and double-check different statistics and to graph relevant data.

The main descriptor statistics of the experimental group and control group were compared. The following statistics were calculated: N, Mean, Standard Deviation and Standard Error Mean. Both groups did show an improvement in the understanding of the concept of function. These statistics and the fact of the improvement of both groups demonstrates the equivalence of the two groups; also in this way and they indicate the internal validity of this research. Very important is the fact that, regarding the mode of increase of the understanding of the concept of function, there is a major difference between the two groups. The difference is that the improvement of the control group is statistically not significant (p = .192), but the improvement of the experimental group is statistically significant (p = .011). This statistical fact made possible to answer the first research question adequately.

Regarding the experimental group, the statistical analysis tests of the group show that the mathematics pre-test and the mathematics post-test are significantly positively correlated ($p < 0.001$, $r = 0.700$). Also, an important result of this study is presented here, namely that there is statistical evidence that the improvement of the experimental group's understanding of function is statistically significant at $\alpha = 0.05$ significance level ($p = .011$).

Regarding the control group, the statistical analysis tests of the group show that the mathematics pre-test and the mathematics post-test are significantly positively correlated ($p < 0.001$, $r = 0.781$). An interesting fact, a deciding result for this research is given here, namely that there is statistical evidence that the improvement of the control group's understanding of function statistically is not significant at $\alpha = 0.05$ significance level ($p = .192$).

The result of statistically significant increase of the understanding of the function concept of the experimental group is attributed to the treatment intervention. It is possible to answer the first research question because the first Null Hypothesis, $H1_0$ can be rejected and as a result the answer to the main research question is positive.

Yes, the active integration of mathematics and music by using these nine analogies between different aspects of function and the corresponding aspects of music does make a statistically significant improvement in my senior secondary mathematics students' understanding of function.

This fact confirms and also refines the results of researches stating that music can enhance students' outcomes in mathematics. This result is a refinement of the existing ones because in this case a secondary mathematics concept has been targeted through a particular method.

As mentioned before, based on the statistical evidence I have concluded and state regarding the second research question, that the second Null Hypothesis, $H2_0$, cannot be rejected. The answer to the second research question is the following:

No, this type of integration of mathematics and music does not make a significant improvement in my senior secondary mathematics students' attitudes and beliefs regarding mathematics.

This concludes the discussion of the results of the study. The following chapter presents a summary of the advantages of the nine author-designed analogies between function and music, the links to previous research, limitations of the study, and the conclusions.

Chapter 5

CONCLUSIONS

5.1 Introduction

In this chapter I conclude the main points of the study, giving the summary of the advantages of the nine author-designed analogies, the main links of the study to previous research, the limitations of the study, recommendations for further research, and final conclusions.

My effort here was to demonstrate to my students who were inclined towards mathematics how to “think Music” and to those inclined towards music how to “feel Mathematics”, using these expressions of the mathematician James J. Sylvester (1814-1897), as cited in Bell (1953, p. 445-446). This was possible, I believe, by presentation of these nine analogies, showing the similarities between the two structures. In this way, students inclined towards mathematics heard and observed how to think music, and students inclined towards music observed that mathematics is much more than numbers to make calculations and theorems to learn. I believe that the others, both the non-mathematical and non-musical students have also benefited observing these non-conventional mathematical-musical analogies. They also have observed that mathematics is not just about numbers and measurements, not just about quantity, but much more about quality and structure. As Solomon Marcus (1925-2016), mathematician and philosopher, who made analogies between mathematics and poetry, stated in Marcus (1998):

> Math conceived as the science of numbers and spatial forms is for a long time no longer acceptable. Math moved from quantity to quality and from numbers to structure; any attempt to capture in a single statement its great diversity fails.
>
> (Marcus, 1998, p. 175)

As an example of action research, the study aimed to improve the quality of my personal secondary mathematics teaching practice of function, indicated by the improvement of my students' understanding of this topic. The study has created new theoretical and practical knowledge regarding my mathematics teaching practice of function, and of how the active integration of mathematics and music through relevant mathematical-musical analogies can improve my senior secondary (Year 11) mathematics students' understanding of functions.

Comparing, for example, the effect size indices – *Cohen's d* values – of the mathematics achievement improvement of the experimental group (d = 0.46) and the mathematics achievement improvement of the control group (d = 0.21) it can be observed that the effect size of the non-traditional method on the experimental group was more than the double that of the effect size of the traditional method on the control group. These results inform my teaching practice as well. I concluded on one side, that my teaching practice using the traditional way of reviewing functions induced a statistically not significant improvement of my students' understanding of function, producing a small effect size. On other side, my teaching practice using the described non-traditional way of reviewing functions induced a statistically significant improvement of my students' understanding of function and produced a medium effect size.

The effects of the active integration of mathematics and music on senior secondary students' understanding of function has not been widely researched. This study represents my effort – that of a practicing secondary mathematics teacher – to fill this gap in the practice and literature by actively integrating mathematics and music through using a self-developed approach, targeting to improve my students' understanding of function by using nine relevant author-designed musical analogies of different aspects of a very important mathematical concept. The study has been designed in order to enhance my students' understanding of function in practice and to answer the following two research questions, the first question being the main focus of the study:

(1) Does the active integration of mathematics and music by using these nine analogies between different aspects of function and the corresponding aspects of music make a statistically significant improvement in my senior secondary mathematics students’ understanding of function?

(2) Does this type of integration of mathematics and music make a statistically significant improvement in my senior secondary mathematics students' attitudes and beliefs regarding mathematics?

I concluded that this non-traditional method of active integration of mathematics and music was an effective way to improve my Year 11 Mathematical Methods students' general understanding of function, especially in the case of two mathematical dependent variables: ACMMM022 (The concept of function as a mapping between sets) and ACMMM025 (Examine translations and the graphs of $y = f(x) + a$ and $y = f(x + b)$) where the improvement of students' understanding showed statistically significant increases.

I also concluded that the traditional method worked better in the case of the mathematical dependent variable MCMMM023 (Use function notation, domain and range, independent and dependent variables) as the control group achieved a greater increase on this variable compared to the experimental group.

These conclusions, together with the other outcomes of my study have influenced my mathematics teaching practice in a positive mode.

5.2 Advantages of The Nine Author-Designed Analogies

The following Table 5.1 provides a summary of the self-designed and actively integrated nine relevant analogies between different aspects of function and the corresponding aspects of music, along with the musical examples used in my research.

Table 5.1

Summary of the Analogies and Musical Examples

Analogy	Mathematical concept	Musical concept	Musical example
1	Function	Melody	Ancient French folk song
2	Relation	Harmony	Harmonised ancient French folk song
3	Vertical Line Test for function	Test of two aspects of music	*J.S. Bach: Cello Suite 2* *F. Chopin: Prelude in C*
4	Graphs of two or more functions on the same axes	Musical counterpoint	*J.S. Bach: Arioso from Cantata, BWV 156*
5	Vertical translation of piano-key function $g(x) = f(x) + c$	Musical transposition	Ancient French folk song in C-major and G-major
6	Horizontal translation of frequency function $g(x) = f(x + c)$	Musical repetition or restatement	Ancient French folk song repeated *J. Pachelbel: Gigue in D*
7	Vertical dilation of frequency function $g(x) = c \times f(x)$	Musical transposition	Ancient French folk song in C-major and G-major
8	Vertical dilation of amplitude function $g(x) = c \times A(x)$	Loudness of music	Ancient French folk song
9	Horizontal dilation of frequency function $g(x) = f(c \times x)$	Speed of music	Ancient French folk song

The main advantage of these analogies and musical examples above regarding my study was that they have been used effectively to improve my Year 11 mathematics students' knowledge and understanding of the function concept. Another advantage is that these analogies can be used in a natural, easy, and effective mode at any secondary school by the interested mathematics teachers. Another advantage is that the author-designed methodology of this study, including the nine analogies between the concept of function and music can be considered as a general guide for a future study which offers the freedom to expand or reduce the use of research instruments presented and described in this thesis, for example by selecting and using other musical examples such as different choral-, instrumental- or even rock music pieces. Another advantage is that the methodology is addressing the Australian Curriculum’s function related topics in a well presented, attractive mode, encourages both individual and group work, and is mathematically meaningful at Year 11 level.

5.3 Links to Previous Research

Despite the fact that the effects of active integration of mathematics and music on senior secondary students' understanding of function is not widely researched, this study has links to previous research.

My study was inspired generally by the works of Zoltan P. Dienes, Howard Gardner, and other authors as mentioned in this thesis.

The studies of Nisbet and Bain (1998, 2000), researching how primary students are matching melodies with visual graphs were an inspiration for this study. These authors, after performing their research with a sample of 51 primary students aged 10-11 years, concluded that children can match musical melodies with their corresponding visual graphs. In my eyes (and ears), matching a melody with a visual bar graph constitutes an analogy between music and mathematics. The outcomes of Nisbet and Bain study inspired me to think about analogies between mathematics and music which can be used at Year 11 Mathematical Methods level in order to improve my senior students’ understanding of function.

Benson's (2002) article, which is a “much expanded and augmented version of the course notes for the undergraduate course 'Mathematics and Music', given at the University of Georgia” (p. 1), inspired me how to present music concepts in a mathematics lesson. Benson (2008) describes that amplitude and loudness, frequency and pitch, spectrum and timbre, and duration and length are the physical and perceptual attributes of sound. In this study I have presented these connections in the form of self-designed analogies between function and music.

5.4 Limitations of the Study

There are several limitations of the study, concerning the following aspects: (i) sample size; (ii) results confined to sample only; (iii) duration of the treatment; and (iv) lack of a cyclic component within my action research.

The sample size (N = 44) is relative small compared to other studies, but this number was 93.62% of all 47 senior secondary Year 11 Mathematical Methods students at the college where the action research has been performed and where I have had a mathematics teaching contract during the year of the study.

My results are confined to my sample only because being performed an action research I wanted to find the answers to my research questions regarding my students and my teaching practice only. This is the reason that I do not generalise my findings to the larger population of Year 11 Mathematical Methods students. Generalised statements regarding senior secondary Year 11 Mathematical Methods students' improvement of understanding of the concept of function through this non-conventional method of active integration of mathematics and music by using these nine analogies may be formulated based on studies performed with larger sample size.

Regarding the duration of the treatment, I am aware of the fact that a longer treatment would have been better. The duration of the treatment lesson was 90 minutes, performed during students' normal timetable double-lesson. There was no more time allocated by the administration for the intervention. This 90 minute time-frame also determined the maximum number of musical analogies (nine) to be presented. This short time-frame was filled with quality, and for the students, accessible information regarding different aspects of function described in Australian Curriculum 2014. I designed the treatment lesson carefully, to cover the topic in the time available, and to avoid too much music detail or irrelevant information.

Regarding the lack of cyclic component of my action research I am aware of the fact that action research usually involves more than one design cycle, as English (2007) states:

> Such a process usually involves a series of 'iterative design cycles', in which trial outcomes are iteratively tested and revised in progressing towards the improvement of mathematics teaching and learning. Such developmental cycles leave auditable trails of documentation that reveal significant information about how and why the desired outcomes evolved.
>
> (English, 2007, p. 223)

Due to college logistics, available time and my teaching contract it was not possible to perform additional iterative cycles of plan-act-observe-reflect actions of my action research, but I feel compelled to say that I believe that the performed planning, acting, observing, and reflecting actions worked well and, as a result, I have found the answers to my research questions.

5.5 Recommendations for Further Research

I would recommend for interested secondary mathematics teachers of advanced students to use the intervention described and presented in this study as a one-off extension topic.

Further longitudinal research with larger samples is needed to confirm the outcome of this study regarding the cognitive skill transfer effect in the improvement of students' understanding of function. I would recommend to perform this study with larger sample sizes and compare both *p*-value and effect size indices with my result. Subsequent research should be performed to address the effects of this type of active integration of mathematics and music on non-cognitive skills, students' beliefs and attitudes. A further research question would be to address the ratio of resultant cognitive and non-cognitive skills.

This study has targeted 16-17 years old, Year 11 senior secondary mathematics students' improvement of understanding of the concept of function. I believe that there is a need to target also other important senior secondary level mathematics concepts in future research studies.

Out of the five main aspects of music (melody, harmony, tempo, rhythm, and timbre) this study has considered the first three aspects: the mathematical expression modes of melody, harmony, and tempo of music. Using also the rhythm and timbre in improving students' understanding of a mathematical concept would be another recommended topic for future research.

5.6 Conclusions

The first, and perhaps the most important conclusion of the study is that this author-designed methodology, including these nine analogies can be used effectively to significantly improve my senior secondary mathematics students' understanding and knowledge of function. Also, this methodology and these mathematical-musical analogies can be used effectively by the interested secondary mathematics teachers. An important observation is that in order to use this methodology, these nine analogies, and the musical examples during a senior secondary mathematics lesson, as a revision of the function topic, neither the teacher nor the students need to be musically educated.

The second important conclusion is that the methodology of this study is “user friendly”, because it is adaptable and expandable. A possible approach to adapt the research methodology is to keep these nine analogies, but to consider other music pieces as examples. This also can be a homework for the students. The considered musical examples in this study were classical music pieces, but – as mentioned above – the defined analogies offer the freedom to select and use accordingly other classical music examples, for example vocal, choral, instrumental, or musical examples from other music genres, for example jazz, blues, rock, pop, country, and rap music pieces. A possible form to expand the study is to design a similar one, but to target another mathematical concept rather than function and perhaps other contexts.

In this study I have described and demonstrated an author-designed methodology of active integration of mathematics and music for improving Year 11 senior secondary mathematics students' understanding of the six aspects of the concept of function required by the Australian Curriculum 2014. The main conclusion of the study is that this type of active integration of mathematics and music significantly improved my students' understanding of function. The main implication of the study is that after further affirmative research results with larger samples, this type of integrated approach may be considered to be employed with success by teachers addressing the Curriculum.

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Mathematics Test

SACE Number: __________

Directions

This test is not for grading purposes. Use a blue or black pen. Answers given on the answer sheets should be labelled accordingly. Time allowed: 40 minutes.

Please complete

You are: Male Female

Did you study music: Yes No

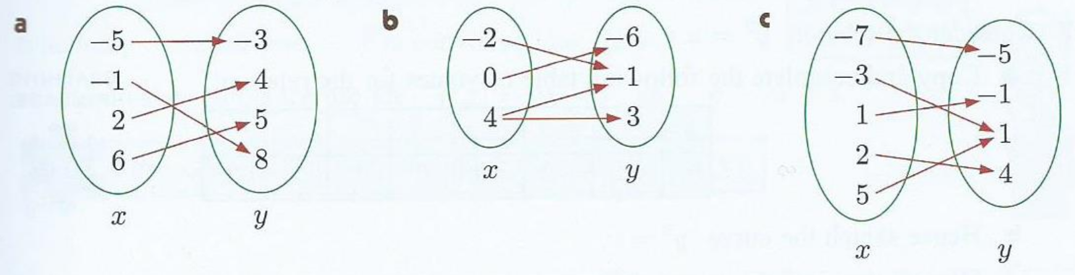
If Yes, then: Number of years:

Instrument:

Music theory: Yes No

Question 1

Which of the following mappings are functions? Give reasons for your answers.



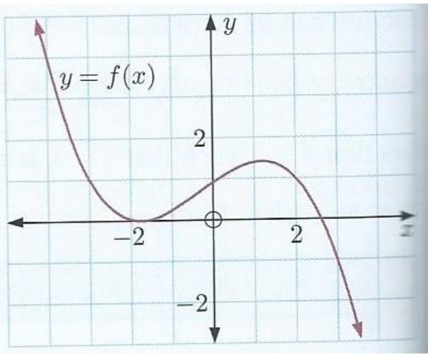
Question 2

The graph of $y = f(x)$ is shown alongside.

a Find:

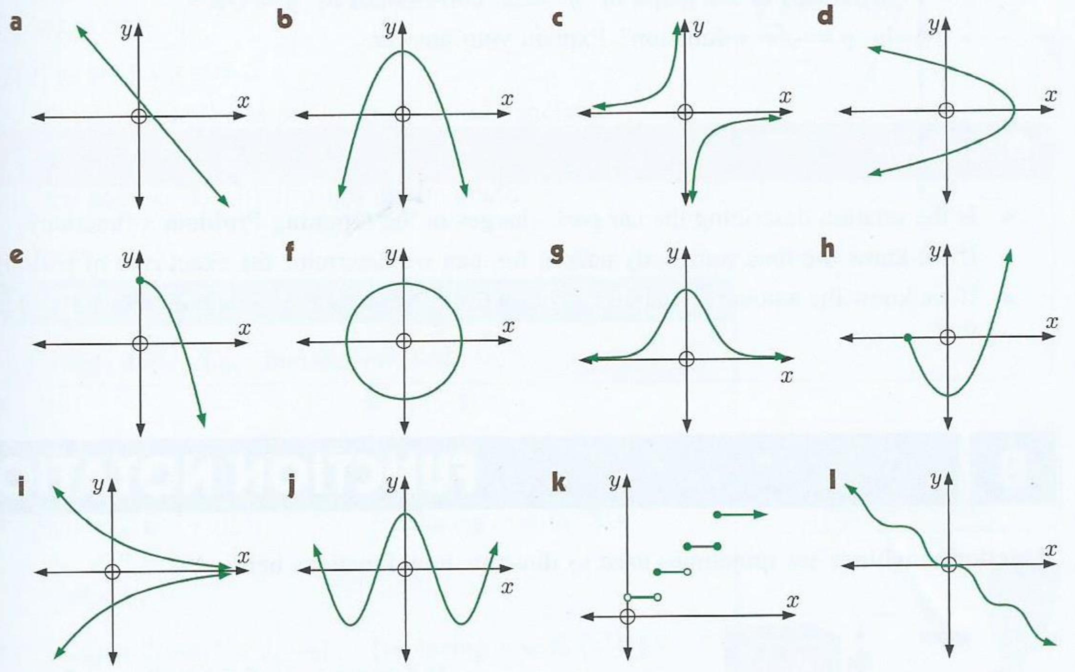
i $f(2)$ **ii** $f(3)$

b Find the value of x such that $f(x) = 4$.



Question 3

Use the vertical line test to determine which of the following relations are functions:



Question 4

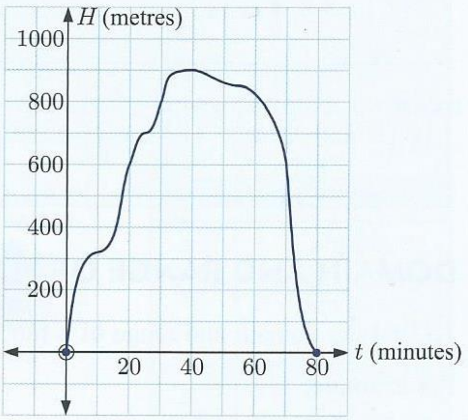
For a hot air balloon ride, the function $H(t)$ gives the height of the balloon after t minutes. Its graph is shown alongside.

a Find $H(30)$, and explain what your answer means.

b Find the values of t such that $H(t) = 600$. Interpret your answer.

c For what values of t was the height of the balloon recorded?

d What range of heights was recorded for the balloon?

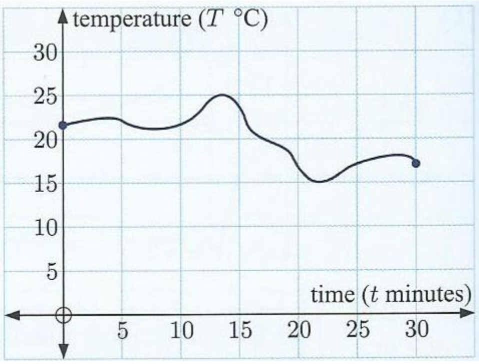


Question 5

This graph shows the temperature in Perth over a 30 minute period as the wind shifts.

a Explain why a temperature graph like this must be a function.

b Find the domain and range of the function.



Question 6

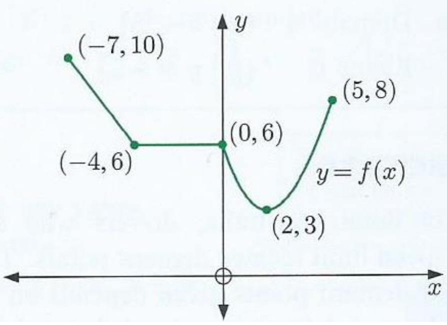
Consider the graph of $y = f(x)$ alongside. Decide whether each statement is true or false:

a -5 is in the domain of f.

b 2 is in the range of f.

c 9 is in the range of f.

d $\sqrt{2}$ is in the domain of f.



Question 7

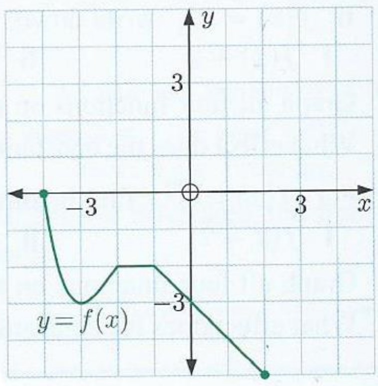
Is it possible for a function to have more than one y-intercept? Explain your answer.

Question 8

Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

a $f(x) + 5$ **b** $f(x - 3)$

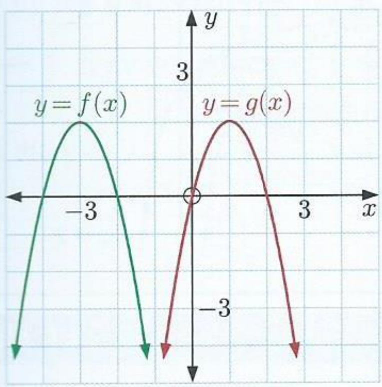
c $f(x - 3) + 5$



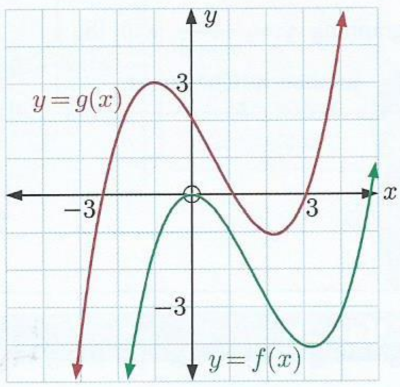
Question 9

Write $g(x)$ in terms of $f(x)$:

a



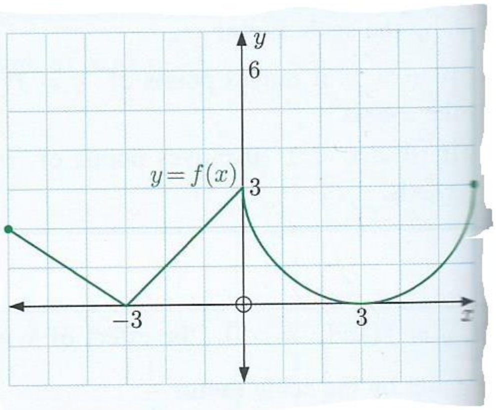
b



Question 10

Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

a $y = 2f(x)$ **b** $f(3x)$



-End of Test-

Appendix B **The WIHIC Questionnaire**

Curtin University

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WIHIC

What Is Happening In This Class - Questionnaire

SACE Number: __________

Directions:

This questionnaire is not for grading purposes. You will be asked to identify how you feel regarding different statements. There are no "right" or "wrong" answers. Your opinion is what is wanted. Use a blue or black pen. All answers should be given on this sheet. Identify how you feel today regarding the following statements using the scale below:

AN - if you **ALMOST NEVER** feel this way in this class,
SD - if you **SELDOM** feel this way in this class,
ST - if you **SOMETIMES** feel this way in this class,
OFT - if you **OFTEN** feel this way in this class,
AA - if you **ALMOST ALWAYS** feel this way in this class.

Only one response should be circled for each statement. If you change your mind about an answer, then cross it out and circle another answer. You are asked to indicate your opinion about all statements.

WIHIC – What Is Happening In This Class Questionnaire

AN = Almost Never **SD** = Seldom **ST** = Sometimes **OFT** = Often **AA** = Almost Always

1.	**I make friends amongst students in this class.**	AN	SD	ST	OFT	AA
2.	**I know other students in this class.**	AN	SD	ST	OFT	AA
3.	**I am friendly to members of this class.**	AN	SD	ST	OFT	AA
4.	**Members of the class are my friends.**	AN	SD	ST	OFT	AA
5.	**I work well with other class members.**	AN	SD	ST	OFT	AA
6.	**I help other class members who are having trouble with their work.**	AN	SD	ST	OFT	AA
7.	**Students in this class like me.**	AN	SD	ST	OFT	AA
8.	**In this class, I get help from other students.**	AN	SD	ST	OFT	AA
9.	**The teacher takes a personal interest in me.**	AN	SD	ST	OFT	AA
10.	**The teacher goes out of his/her way to help me.**	AN	SD	ST	OFT	AA
11.	**The teacher considers my feelings.**	AN	SD	ST	OFT	AA
12.	**The teacher helps me when I have trouble with the work.**	AN	SD	ST	OFT	AA
13.	**The teacher talks with me.**	AN	SD	ST	OFT	AA
14.	**The teacher is interested in my problems.**	AN	SD	ST	OFT	AA
15.	**The teacher moves about the class to talk with me.**	AN	SD	ST	OFT	AA
16.	**The teacher's questions help me to understand.**	AN	SD	ST	OFT	AA
17.	**I discuss ideas in this class.**	AN	SD	ST	OFT	AA
18.	**I give my opinions during class discussions.**	AN	SD	ST	OFT	AA
19.	**The teacher asks me questions.**	AN	SD	ST	OFT	AA
20.	**My ideas and suggestions are using during classroom discussions.**	AN	SD	ST	OFT	AA
21.	**I ask the teacher questions.**	AN	SD	ST	OFT	AA
22.	**I explain my ideas to other students.**	AN	SD	ST	OFT	AA
23.	**Students discuss with me how to go about solving problems.**	AN	SD	ST	OFT	AA
24.	**I am asked to explain how I solve problems.**	AN	SD	ST	OFT	AA
25.	**I carry out investigations to test my ideas.**	AN	SD	ST	OFT	AA
26.	**I am asked to think about the evidence for my statements.**	AN	SD	ST	OFT	AA
27.	**I carry out investigations to answer questions coming from discussions.**	AN	SD	ST	OFT	AA
28.	**I explain the meaning of statements, diagrams, and graphs.**	AN	SD	ST	OFT	AA
29.	**I carry out investigations to answer questions that puzzle me.**	AN	SD	ST	OFT	AA
30.	**I carry out investigations to answer teacher's questions.**	AN	SD	ST	OFT	AA
31.	**I find out answers to questions by doing investigations.**	AN	SD	ST	OFT	AA
32.	**I solve problems by using information obtained from my own investigations.**	AN	SD	ST	OFT	AA

33.	**Getting a certain amount of work done is important to me.**	AN	SD	ST	OFT	AA
34.	**I do as much as I set out to do.**	AN	SD	ST	OFT	AA
35.	**I know the goals for this class.**	AN	SD	ST	OFT	AA
36.	**I am ready to start class on time.**	AN	SD	ST	OFT	AA
37.	**I know what I am trying to accomplish in this class.**	AN	SD	ST	OFT	AA
38.	**I pay attention during this class.**	AN	SD	ST	OFT	AA
39.	**I try to understand the work in this class.**	AN	SD	ST	OFT	AA
40.	**I know how much work I have to do.**	AN	SD	ST	OFT	AA
41.	**I cooperate with other students when doing assignment.**	AN	SD	ST	OFT	AA
42.	**I share my books and resources with other students when doing assignments.**	AN	SD	ST	OFT	AA
43.	**When I work in groups in this class, there is teamwork.**	AN	SD	ST	OFT	AA
44.	**I work with other students on projects in this class.**	AN	SD	ST	OFT	AA
45.	**I learn from other students in this class.**	AN	SD	ST	OFT	AA
46.	**I work with other students in this class.**	AN	SD	ST	OFT	AA
47.	**I cooperate with other students on class activities.**	AN	SD	ST	OFT	AA
48.	**Students work with me to achieve class goals.**	AN	SD	ST	OFT	AA
49.	**The teacher gives as much attention to my questions as to other students' questions.**	AN	SD	ST	OFT	AA
50.	**I get the same amount of help from the teacher as do other students.**	AN	SD	ST	OFT	AA
51.	**I have the same amount of say in this class as other students do.**	AN	SD	ST	OFT	AA
52.	**I am treated the same as other students in this class.**	AN	SD	ST	OFT	AA
53.	**I receive the same encouragement from the teacher as other students do.**	AN	SD	ST	OFT	AA
54.	**I get the same opportunity to contribute to class discussions as other students.**	AN	SD	ST	OFT	AA
55.	**My work receives as much praise as other students' work.**	AN	SD	ST	OFT	AA
56.	**I get the same opportunity to answer questions as other students.**	AN	SD	ST	OFT	AA

For evaluator's use only: SC __________ TS __________

INVMT ___________ INVST __________

TO ___________ C __________

E ___________

Appendix C **The MRBQ Questionnaire**

Curtin University of Technology, Science and Mathematics Education Centre
GPO Box U1987 Perth WA 6845. Protocol Approval SMEC-55-14

MRBQ

Mathematics-Related Beliefs Questionnaire

SACE Number: __________

Directions:

This questionnaire is not for grading purposes. This questionnaire contains a number of statements about mathematics, mathematics lessons and learning mathematics. You will be asked what you yourself think about these statements. There are no "right" or "wrong" answers. Your opinion is what is wanted. Use a blue or black pen. All answers should be given on this test sheet. For each statement draw a circle around one of answers that best describes your opinion:

SA - if you **STRONGLY AGREE** with the statement,

A - if you **AGREE** with the statement,

N - if you are **NOT SURE** about the statement,

D - if you **DISAGREE** with the statement,

SD - if you **STRONGLY DISAGREE** with the statement.

Only one response should be circled for each statement. If you change your mind about an answer, then cross it out and circle another answer. Although some sentences in this test are fairly similar to other statements, you are asked to indicate your opinion about all statements.

Practice item statement: "It would be interesting to learn about boats." Suppose that you AGREE with this statement, then you would circle A, like this:

SA $\boxed{\text{A}}$ N D SD.

MRBQ – Mathematics-Related Beliefs Questionnaire

SA = Strongly Agree **A** = Agree **N** = Neutral/Not sure **D** = Disagree **SD** = Strongly Disagree

1.	**The teacher really wants us to enjoy learning new things.**	SA	A	N	D	SD
2.	**The teacher understands our problems and difficulties with mathematics.**	SA	A	N	D	SD
3.	**The teacher tries to make the mathematics lessons interesting.**	SA	A	N	D	SD
4.	**The teacher listens carefully to what we say.**	SA	A	N	D	SD
5.	**The teacher is friendly to us.**	SA	A	N	D	SD
6.	**The teacher appreciates it when we try hard, even if our results are not so good.**	SA	A	N	D	SD
7.	**The teacher always shows us, step by step, how to solve a mathematical problem, before giving us exercises.**	SA	A	N	D	SD
8.	**The teacher always gives us time to really explore new problems and try out different solution strategies.**	SA	A	N	D	SD
9.	**The teacher wants us to understand the content of this mathematics course**	SA	A	N	D	SD
10.	**The teacher explains why mathematics is important.**	SA	A	N	D	SD
11.	**The teacher thinks mistakes are okay as long as we are learning from them.**	SA	A	N	D	SD
12.	**I can understand even the most difficult topics taught me in mathematics.**	SA	A	N	D	SD
13.	**I understand everything we have done in mathematics this year.**	SA	A	N	D	SD
14.	**I think I will do well in mathematics this year.**	SA	A	N	D	SD
15.	**I like doing mathematics.**	SA	A	N	D	SD
16.	**Compared with others in the class, I think I'm good at mathematics.**	SA	A	N	D	SD
17.	**I'm very interested in mathematics.**	SA	A	N	D	SD
18.	**I expect to do well on the mathematics tests and assessments we do.**	SA	A	N	D	SD
19.	**I don't have to try too hard to understand mathematics.**	SA	A	N	D	SD
20.	**I can usually do mathematics problems that take a long time to complete.**	SA	A	N	D	SD
21.	**I think mathematics is an important subject.**	SA	A	N	D	SD
22.	**I find I can do hard mathematics problems with patience.**	SA	A	N	D	SD
23.	**I prefer class work that is challenging so I can learn new things.**	SA	A	N	D	SD
24.	**If I cannot solve a mathematics problem quickly, I quit trying.**	SA	A	N	D	SD
25.	**Only the mathematics to be tested is worth learning.**	SA	A	N	D	SD
26.	**Only very intelligent students can understand mathematics.**	SA	A	N	D	SD
27.	**If I cannot do a mathematics problem in a few minutes, I probably cannot do it at all.**	SA	A	N	D	SD

28.	**Ordinary students cannot understand mathematics, but only memorise the rules they learn.**	SA	A	N	D	SD
29.	**It's a waste of time when our teacher makes us think on our own.**	SA	A	N	D	SD
30.	**By doing the best I can in mathematics I try to show my teacher that I'm better than other students.**	SA	A	N	D	SD
31.	**The teacher wants us just to memorise the content of this mathematics course.**	SA	A	N	D	SD
32.	**Everybody has to think hard to solve a mathematics problem.**	SA	A	N	D	SD
33.	**Mathematics learning is mainly about having a good memory.**	SA	A	N	D	SD
34.	**There is only one way to find the correct solution to a mathematics problem.**	SA	A	N	D	SD
35.	**Getting the right answer in mathematics is more important than understanding why the answer works.**	SA	A	N	D	SD
36.	**My only interest in mathematics is getting a good grade.**	SA	A	N	D	SD
37.	**Mathematics is used all the time in people's daily life.**	SA	A	N	D	SD
38.	**I think mathematics is an important subject.**	SA	A	N	D	SD
39.	**Mathematics enables us to better understand the world we live in.**	SA	A	N	D	SD
40.	**Knowing mathematics will help me earn a living.**	SA	A	N	D	SD
41.	**I study mathematics because I know how useful it is.**	SA	A	N	D	SD
42.	**Mathematics is a worthwhile and necessary subject.**	SA	A	N	D	SD
43.	**I can use what I learn in mathematics in other subjects.**	SA	A	N	D	SD
44.	**Discussing different solutions to a mathematics problem is a good way of learning mathematics.**	SA	A	N	D	SD
45.	**Time used to understand why a solution works is time well spent.**	SA	A	N	D	SD
46.	**I think it is important to learn different strategies for solving the same problem.**	SA	A	N	D	SD
47.	**Routine exercises are very important in the learning of mathematics.**	SA	A	N	D	SD

For evaluator's use only:

T __________ S __________ F __________ R __________

Appendix D The TOMRA Questionnaire

Curtin University of Technology, Science and Mathematics Education Centre
GPO Box U1987 Perth WA 6845. Protocol Approval SMEC-55-14

TOMRA

Test of Mathematics-Related Attitudes

SACE Number: __________

Directions:

This test is not for grading purposes. This test contains a number of statements about mathematics, mathematics lessons and learning mathematics. You will be asked what you yourself think about these statements. There are no "right" or "wrong" answers. Your opinion is what is wanted. Use a blue or black pen. All answers should be given on this test sheet. For each statement draw a circle around one of answers that best describes your opinion:

SA - if you **STRONGLY AGREE** with the statement,

A - if you **AGREE** with the statement,

N - if you are **NOT SURE** about the statement,

D - if you **DISAGREE** with the statement,

SD - if you **STRONGLY DISAGREE** with the statement.

Only one response should be circled for each statement. If you change your mind about an answer, then cross it out and circle another answer. Although some sentences in this test are fairly similar to other statements, you are asked to indicate your opinion about all statements.

Practice item statement: "It would be interesting to learn about boats." Suppose that you AGREE with this statement, then you would circle A, like this:

SA $\boxed{\textbf{A}}$ N D SD.

TOMRA – Test Of Mathematics-Related Attitudes

SA = Strongly Agree **A** = Agree **N** = Not sure **D** = Disagree **SD** = Strongly Disagree

1.	**Mathematics lessons are fun.**	SA	A	N	D	SD
2.	**I would prefer to know a solution by solving a mathematics problem than by being told.**	SA	A	N	D	SD
3.	**I dislike mathematics lessons.**	SA	A	N	D	SD
4.	**Solving mathematics problems is not as good as finding out solutions from teachers.**	SA	A	N	D	SD
5.	**Schools should have more mathematics lessons each week.**	SA	A	N	D	SD
6.	**I would prefer to solve mathematics problems than to read about them.**	SA	A	N	D	SD
7.	**Mathematics lessons bore me.**	SA	A	N	D	SD
8.	**I would rather agree with other people than to solve a mathematics problem to find out for myself.**	SA	A	N	D	SD
9.	**Mathematics is one of the most interesting school subjects.**	SA	A	N	D	SD
10.	**I would prefer to solve my mathematics problems than to find out information from a teacher.**	SA	A	N	D	SD
11.	**Mathematics lessons are a waste of time.**	SA	A	N	D	SD
12.	**I would rather find out about a solution by asking an expert than by solving a mathematics problem.**	SA	A	N	D	SD
13.	**I really enjoy going to mathematics lessons.**	SA	A	N	D	SD
14.	**I would rather find a solution by solving a mathematics problem than be told the answer.**	SA	A	N	D	SD
15.	**The material covered in mathematics lessons is uninteresting.**	SA	A	N	D	SD
16.	**It is better to ask the teacher the answer than to find it out by solving a mathematics problem.**	SA	A	N	D	SD
17.	**I look forward to mathematics lessons.**	SA	A	N	D	SD
18.	**I would prefer to solve a mathematics problem than to read about it in a mathematics book.**	SA	A	N	D	SD
19.	**I would enjoy school more if there were no mathematics lessons.**	SA	A	N	D	SD
20.	**It is better to be told mathematics resultsS than to find them out from solving a mathematics problem.**	SA	A	N	D	SD

For evaluator's use only: E ____________ I ____________

Appendix E Written permission from Haese Mathematics

<u>Email received from Haese Mathematics Pty Ltd, the publisher of the textbook</u>

Dear Istvan,

Thank you for your enquiry.

I have checked with management and they are more than happy for you to photocopy exercises for your educational research.

Acknowledgement of Haese Mathematics in your thesis would be greatly appreciated.

Regards,
Elaine

Elaine Beveridge
Haese Mathematics

Email: elaine@haesemathematics.com.au
Web: www.haesemathematics.com.au
Phone: +61 8 8210 4666
Fax: +61 8 8354 1238

<u>Email sent to Haese Mathematics Pty Ltd, the publisher of the mathematics textbook</u>

Dear Sir/Madam,

I am a mathematics teacher and currently a Mathematics Education postgraduate research student at Curtin University of Technology.

I would like to ask for Haese Mathematics' permission to photocopy and use 10-15 exercises from the Haese Mathematics: Mathematics for Australia 11 Mathematical Methods (First edition, 2015) textbook in my planned educational research.

These 10-15 Haese Mathematics' textbook exercises, subject to your permission, will form both the mathematics pre-test and mathematics post-test questions of my research, as standard Year 11, paper based mathematics tests. The research will have about 46 participant students. The research integrates mathematics and music and it is aiming to measure Year 11 Mathematical Methods students' understanding of the concept of function.

In the resultant thesis and in any further publications I will acknowledge Hease Mathematics' contribution to the research.

Thank you,
Yours sincerely,
Istvan Nagy
Adelaide

Appendix F Written permission from the Virtual Piano software

Email from Crystal Magic Studio Ltd, the developer of the Virtual Piano software

Dear Istvan,

Thanks for your email and for your interest in Virtual Piano.

Yes, please do go ahead with the use of VP in your educational program.

We haven't built the feature yet but we're in the middle of building the next version which will allow various key combinations.

Do send us a link to your project if you publish your course.

All the best

Habib

Habib Amir

Director, Digital Innovation

CMAGICS

Cross-Platform Digital Engagement

Web: http://www.cmagics.com

Email: habib@cmagics.com

Social: @CMAGICS | Google+ | LinkedIn

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Telephone: +44203 397 4988

14 Basil Street, Knightsbridge, London SW3 1AJ

CMAGICS helps brands:

- Establish a strategic digital presence
- Build and improve online brand awareness
- Increase online revenue and consumer advocacy

CMAGICS is a trading name for Crystal Magic Studio Ltd, registered in England and Wales under company # 05476350.

Email sent to Crystal Magic Studio Ltd, the developer of the Virtual Piano software

To:

Mr. Habib Amir
Managing Director
Crystal Magic Studio
14 Basil Street
Knightsbridge
London SW3 1AJ
UK

Dear Mr Amir,

Congratulations to the Virtual Piano, which is a great application.

I would like to ask for your permission to use the Virtual Piano in an educational research project, involving about 25 secondary students, which integrates mathematics and music.

I also would like to ask you if the Virtual Piano has the feature to produce the same sound by using two different combinations of keys, for example to produce the sound G-sharp (Shift+O) and the same sound A-flat by using a different key combination, e.g. Alt+P?

Thank you very much for your attention and for your answer.

Yours sincerely,
Istvan Nagy
Mathematics teacher
Adelaide, Australia

Appendix G Written permission from the Music Speed Changer software

Email received from the developer of the Music Speed Changer software

[My translation and the original email in German:]

Dear Mr Nagy,

You can install the software on both devices.
Best regards,
Harald Meyer

Sehr geehrter Herr Nagy,

Sie können die Software auf beiden Geräten installieren.

Mit freundlichen Grüßen
Harald Meyer

Email sent to Harald Meyer, the developer of the Music Speed Changer software

Dear Mr. Meyer,

I have downloaded the setup file but didn't install it yet.

I would like to ask you if it is possible to install and run the software on both my Windows 10 PC and Windows 10 Tablet?

Thank you,
István Nagy

Sehr geehrter Herr Meyer,

Ich möchete Sie fragen die folgende Frage: ist es möglich die Software auf mein Windows 10 PC und Windows 10 Tablet zu installieren?

Schönen Dank,
István Nagy

Appendix H Written permission from the Original Sine software

Email received from Dr Braden Phillips, the developer of the Original Sine software

Hi Istvan,

I would be very pleased to distribute the software. I'd just like to spend a little more time to tidy it up and package it with a proper installation program.

Hopefully you will be able to find the software at https://bitbucket.org/bjphillips/original-sine-trigonometry-and-sound.
It would be great to try it on your School computers well ahead of the practical session. Do go ahead and use it for your research and thesis.

I'm interested to hear how you go with it.

Braden

Email sent to Dr Braden Phillips, the developer of the Original Sine software

To:

Dr Braden Phillips

Senior Lecturer

The University of Adelaide

Dear Dr Phillips,

I am a secondary mathematics teacher and I have participated on your presentation of the Original Sine software during the STEM Equipped workshop on 16/5/14.

I would like to ask you if it would be possible to use the Original Sine software with my students? As a Doctor of Mathematics Education postgraduate student I am working on my thesis involving the relations between mathematics and music and I think that I could integrate the Original Sine software into the research very well.

Thank you for your answer in anticipation,

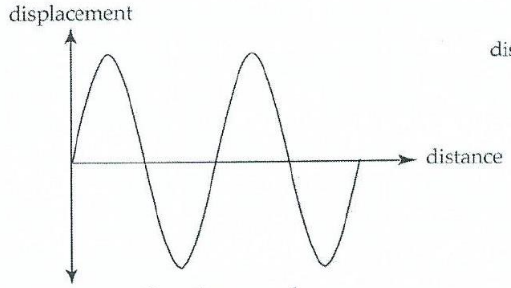
With best wishes,

Istvan Nagy

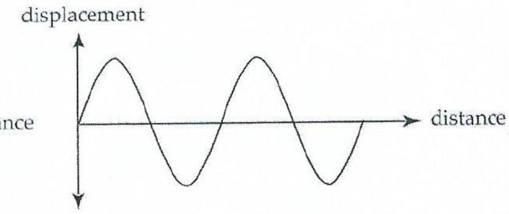
Appendix I Loudness (Amplitude) and Pitch (Frequency) of Sound

Loudness

Amplitude is a measure of how loud a sound is. A wave that we hear as a loud sound will have a higher amplitude than a softer sound.



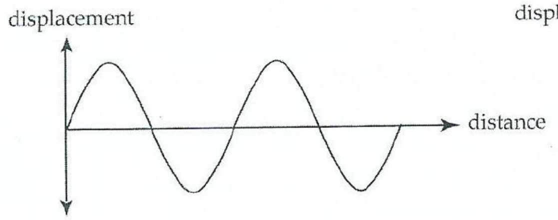
Louder sound



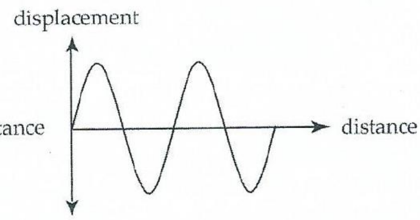
Softer sound

Pitch

A musical note may be high or low. This is called the pitch of the note. Sounds with a high pitch have a high frequency. Sounds with a low pitch have a low frequency. For example, middle C on the piano has a frequency of 256 Hz. The C which is an octave (eight notes) above it on the piano has a frequency of 512 Hz.



Lower pitch



Higher pitch

(Fact sheet, Source not identified, n.d., n.p.)

Appendix J Ethics Approval Memorandum from Curtin University

Curtin University

Memorandum

To	Istvan Nagy, SMEC
From	Mun Yin Cheong, Form C Ethics Co-ordinator Faculty of Science and Engineering
Subject	Protocol Approval **SMEC-55-14**
Date	11 August 2014
Copy	David Treagust & John Malone, SMEC

Office of Research and Development

Human Research Ethics Committee

Telephone 9266 2784
Facsimile 9266 3793
Email hrec@curtin.edu.au

Thank you for your "Form C Application for Approval of Research with Low Risk (Ethical Requirements)" for the project titled *"Melody of functions and graphs: Improving senior secondary mathematics students' understanding of the concept of function by the integration of mathematics and music"*. On behalf of the Human Research Ethics Committee, I am authorised to inform you that the project is approved.

Approval of this project is for a period of 4 years **7th August 2014** to **6th August 2018**.

Your approval has the following conditions:

(i) Annual progress reports on the project must be submitted to the Ethics Office.

(ii) It is your responsibility, as the researcher, to meet the conditions outlined above and to retain the necessary records demonstrating that these have been completed.

The approval number for your project is **SMEC-55-14**. Please quote this number in any future correspondence. If at any time during the approval term changes/amendments occur, or if a serious or unexpected adverse event occurs, please advise me immediately.

Regards,

Mun Yin

MUN YIN CHEONG
Form C Ethics Co-ordinator
Faculty of Science and Engineering

Please Note: The following standard statement must be included in the information sheet to participants:
This study has been approved under Curtin University's process for lower-risk Studies (Approval Number xxxx). This process complies with the National Statement on Ethical Conduct in Human Research (Chapter 5.1.7 and Chapters 5.1.18-5.1.21). For further information on this study contact the researchers named above or the Curtin University Human Research Ethics Committee. c/- Office of Research and Development, Curtin University, GPO Box U1987, Perth 6845 or by telephoning 9266 9223 or by emailing hrec@curtin.edu.au.

Appendix K Research Approval from Catholic Education South Australia

catholic
education
south australia

Adelaide Catholic Education Centre
116 George Street, Thebarton SA 5031
PO Box 179, Torrensville Plaza SA 5031
T +61 8 8301 6600 **F** +61 8 8301 6611
E director@cesa.catholic.edu.au
W www.cesa.catholic.edu.au

Istvan Nagy
PO Box 81
CAMPBELLTOWN SA 5074

Dear Istvan

RE Melody of functions and graphs: Improving Senior Secondary Mathematics Students' Understanding of the Concept of Function by the Integration of Mathematics and Music

Thank you for your email of 22 July 2015 in which you seek permission to conduct research on the Melody of Functions and Graphs: Improving Senior Secondary Mathematics Students' Understanding of the Concept of Function by the Integration of Mathematics and Music. I understand that Stage 1 mathematics students will be targeted on a voluntary basis and will participate by using a text-book based mathematics test and three standard 5 – 10 minute questionnaires.

In the normal course, permission of the principal of the school is required. Research in Catholic schools is granted on the basis that individual students, schools and the Catholic sector itself is not specifically identified in published research data and conclusions.

Approval is also contingent upon the following conditions, i.e. that:

- the permission of parents of each child involved in the study and the participating teachers has been obtained
- the research complies with the ethics proposal of Curtin University
- the research complies with any provisions under the Privacy Act that may require adherence by you as researcher in gathering and reporting data
- the presentation in the school is carried out within view of the classroom teacher or authorised school observer
- no comparison between schooling sectors is made
- sector requirements relating to child protection and police checks are met by researchers:
    - where researchers obtain information in relation to a student which suggests or indicates abuse, this information must be immediately conveyed to the Director of Catholic Education SA
    - all researchers and assistants, who in the course of the research interact in any way with students, are required to provide evidence of an acceptable police clearance direct to the school

Please accept my very best wishes for the research process.

Yours sincerely

M H Conway.

MONICA CONWAY
ASSISTANT DIRECTOR

23 July 2015

Appendix L **Informed Consent Form**

Curtin University of Technology, Science and Mathematics Education Centre
GPO Box U1987 Perth WA 6845

Informed Consent Form

Research Study Title:

Melody of functions and graphs: Improving senior secondary mathematics students' understanding of the concept of function by the integration of mathematics and music.

Research Study Summary:

The aim of this research study is to investigate senior secondary students' understanding of the mathematical concept of function and their attitudes toward mathematics by the integration of mathematics and music in normal senior secondary mathematics lessons. The study will involve completing three 5-10 minutes questionnaires and a maths test both at the beginning and end of the research. The used questionnaires and tests will not contain participants' names or personal details.

By signing below, you are agreeing that:

(1) you have read and understood the Participant Information Sheet of this research study,

(2) you are a full time senior secondary mathematics student, less than 18 years of age, enrolled in a South Australian school,

(3) questions about your participation in this study (if any) have been answered satisfactorily,

(4) you are aware of the potential risks (if any), and

(5) you are taking part in this research study voluntarily (without coercion).

____

Participant's Name (Printed)

____ ____________________________________

Participant's signature Date

____

Parent's/Guardian's Name (Printed)

____ ____________________________________

Parent's/Guardian's signature Date

ISTVAN NAGY
Name of study coordinator (Printed)

____

Signature of study coordinator

Appendix M **Participant Information Sheet**

Curtin University of Technology, Science and Mathematics Education Centre

GPO Box U1987 Perth WA 6845

Participant Information Sheet

Invitation: You are invited to participate in a research study on effects of music on mathematics learning and senior secondary students' attitudes toward mathematics.

Study Title: Melody of functions and graphs: Improving senior secondary mathematics students' understanding of the concept of function by the integration of mathematics and music.

Postgraduate student researcher: My name is Istvan Nagy. I am a secondary mathematics teacher in South Australia, conducting this research. The research will form part of the requirements for a Doctor of Mathematics Education degree at Curtin University, Perth, Western Australia.

About the research: The aim of this research project is to investigate senior secondary students' understanding of the mathematical concept of function and their attitudes toward mathematics by the integration of mathematics and music in normal mathematics lessons. The study will involve completing three questionnaires and a maths test both at the beginning and end of the research in a school term. The questionnaires and tests will contain participants' codes only in order to match the questionnaires and tests. The questionnaires record your answers (rankings as e.g. Strongly Agree, Agree, Not sure, Disagree or Strongly Disagree) to different questions about maths, maths lessons and learning maths. Each questionnaire is anticipated to take 5-10 minutes to complete. The two maths tests, similar to normal school maths tests, will be a set of relevant maths exercises regarding the concept of function, an already covered topic from participants' mathematics textbook. Participants will be attending their normal mathematics lessons and will learn maths with their teacher, not music, but some mathematical examples will be presented using relevant musical examples which may improve the understanding of the studied mathematical concept of function and/or may improve the attitudes toward maths. The procedures used in the research are considered part of normal school activities.

Criteria of participation: To participate in the research you must be a full time senior secondary mathematics student, less than 18 years of age, enrolled in a South Australian school. Musical pre-training is not a prerequisite for participation.

Consent to participate: Your involvement in this research is entirely voluntary. If you agree to participate in this research study then you and your parent/guardian will be asked to sign a separate Informed Consent Form which has to be returned to the research coordinator. Participation or non-participation will not affect your normal school activities. You have the right to withdraw from the research project at any time without prejudice or negative consequences and doing so will not affect your normal school activities. Notice of withdrawing should be made in writing to the study coordinator.

Confidentiality: Only my university supervisors and I will have access to the data you provide. Your consent form, completed questionnaires and maths tests will be filed and will be kept in a locked filing cabinet at Curtin University. All data recorded and stored in the study database will be in a coded form without your name or other personal details and held in a secure electronic environment within Curtin University. Password protection will be used for the computer containing study data. All data will be kept for seven years then will be destroyed. It is intended that this research will be published in specialist national and international journals. Only combined data will be published and participants will not be personally identifiable in any publication.

Risks and Benefits: There are no anticipated risks to you as a result of participating. The convenience and comfort of participants have been given the highest priority in the design of this research. While there are no financial benefits involved in this research project, your participation may help you to gain valuable new knowledge, skill and experience.

Participants' rights: You may decide to stop being a part of the research study at any time without explanation. You have the right to ask that any data you have supplied to that point be withdrawn/destroyed. You have the right to have your questions about the procedures answered. If you have any questions as a result of reading this information sheet, you should ask the study coordinator before the study begins.

Further information: This study has been approved under Curtin University's process for lower-risk Studies (Approval Number SMEC-55-14). This process complies with the National Statement on Ethical Conduct in Human Research (Chapter 5.1.7 and Chapters 5.1.18-5.1.21). For further information on this study contact the researchers named above or the Curtin University Human Research Ethics Committee. c/- Office of Research and Development, Curtin University, GPO Box U1987, Perth 6845 or by telephoning 9266 9223 or by emailing hrec@curtin.edu.au.

For further information about the research, please feel free to contact me.

Thank you for your attention,

Istvan Nagy
Curtin University HDR student, study coordinator
Ph. 0450 032 230
Email: istvan.nagy@postgrad.curtin.edu.au

Curtin University research supervisors:

Dr John Malone
Emeritus Professor
Email: J.Malone@curtin.edu.au

Dr David Treagust
John Curtin Distinguished Professor
Email: D.Treagust@curtin.edu.au

Appendix N Piano Key Numbers, Names, and Frequencies (Hz)

Piano key Numbers, Names and Frequencies (Hz)

1	2	3	4	5	6	7	8	9	10	11	12
A_0	$A_0^\#/B_0^b$	B_0	C_1	$C_1^\#/D_1^b$	D_1	$D_1^\#/E_1^b$	E_1	F_1	$F_1^\#/G_1^b$	G_1	$G_1^\#/A_1^b$
27.50	29.14	30.86	32.70	34.65	36.70	38.89	41.20	43.65	46.24	48.99	51.91

13	14	15	16	17	18	19	20	21	22	23	24
A_1	$A_1^\#/B_1^b$	B_1	C_2	$C_2^\#/D_2^b$	D_2	$D_2^\#/E_2^b$	E_2	F_2	$F_2^\#/G_2^b$	G_2	$G_2^\#/A_2^b$
55.00	58.27	61.73	65.40	69.29	73.41	77.78	82.40	87.30	92.49	97.99	103.82

25	26	27	28	29	30	31	32	33	34	35	36
A_2	$A_2^\#/B_2^b$	B_2	C_3	$C_3^\#/D_3^b$	D_3	$D_3^\#/E_3^b$	E_3	F_3	$F_3^\#/G_3^b$	G_3	$G_3^\#/A_3^b$
110.00	116.54	123.46	130.81	138.59	146.83	155.56	164.81	174.61	184.99	195.99	207.65

37	38	39	40	41	42	43	44	45	46	47	48
A_3	$A_3^\#/B_3^b$	B_3	C_4	$C_4^\#/D_4^b$	D_4	$D_4^\#/E_4^b$	E_4	F_4	$F_4^\#/G_4^b$	G_4	$G_4^\#/A_4^b$
220.00	233.08	246.93	261.62	277.18	293.66	311.12	329.62	349.22	369.99	391.99	415.30

49	50	51	52	53	54	55	56	57	58	59	60
A_4	$A_4^\#/B_4^b$	B_4	C_5	$C_5^\#/D_5^b$	D_5	$D_5^\#/E_5^b$	E_5	F_5	$F_5^\#/G_5^b$	G_5	$G_5^\#/A_5^b$
440.00	466.16	493.88	523.25	554.36	587.32	622.25	659.25	698.45	739.98	783.99	830.60

61	62	63	64	65	66	67	68	69	70	71	72
A_5	$A_5^\#/B_5^b$	B_5	C_6	$C_6^\#/D_6^b$	D_6	$D_6^\#/E_6^b$	E_6	F_6	$F_6^\#/G_6^b$	G_6	$G_6^\#/A_6^b$
880.00	932.32	987.76	1046.50	1108.73	1174.65	1244.50	1318.51	1396.91	1479.97	1567.98	1661.21

73	74	75	76	77	78	79	80	81	82	83	84
A_6	$A_6^\#/B_6^b$	B_6	C_7	$C_7^\#/D_7^b$	D_7	$D_7^\#/E_7^b$	E_7	F_7	$F_7^\#/G_7^b$	G_7	$G_7^\#/A_7^b$
1760.00	1864.65	1975.53	2093.00	2217.46	2349.31	2489.01	2637.02	2793.82	2959.95	3135.96	3322.43

85	86	87	88
A_7	$A_7^\#/B_7^b$	B_7	C_8
3520.00	3729.31	3951.06	4186.00

Useful information:

These frequencies form a geometric progression: every frequency is obtained by multiplying the previous one by $2^{1/12}$. E.g. $f(2) = f(1) * 2^{1/12}$. The ratio $2^{1/12}$ assures that after 12 keys the frequency is doubled. E.g. $f(49+12) = 2 * f(49)$. The tuning reference note is A_4 (440 Hz), key number 49. We can see that the frequency of the n^{th} key can be calculated as $\mathbf{f(n) = 440 * (2^{1/12})^{(n-49)}}$, $n = 1,\ldots,88$.

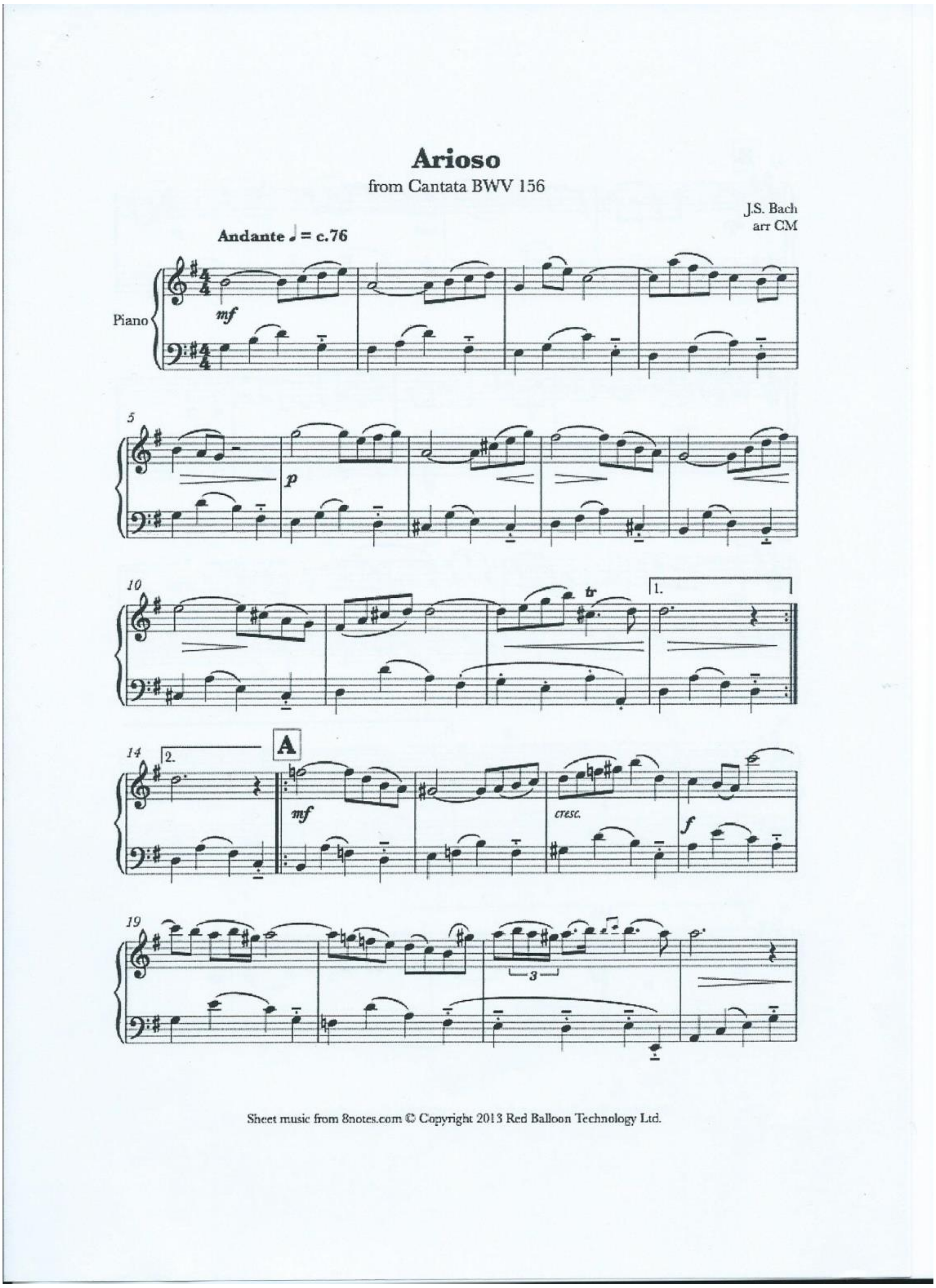
Lowest notes: bass: E_1 (41.20 Hz), guitar: E_2 (82.40 Hz), violin: G_3 (195.99 Hz)
Speed of sound: 345 m/s. Wave equation: **speed = wavelength * frequency**

Curtin University, Research project: Melody of Functions and Graphs, HDR: Istvan Nagy, 2016

Appendix O Sheet Music of F. Chopin: Prelude in C-minor Op. 28 No. 20

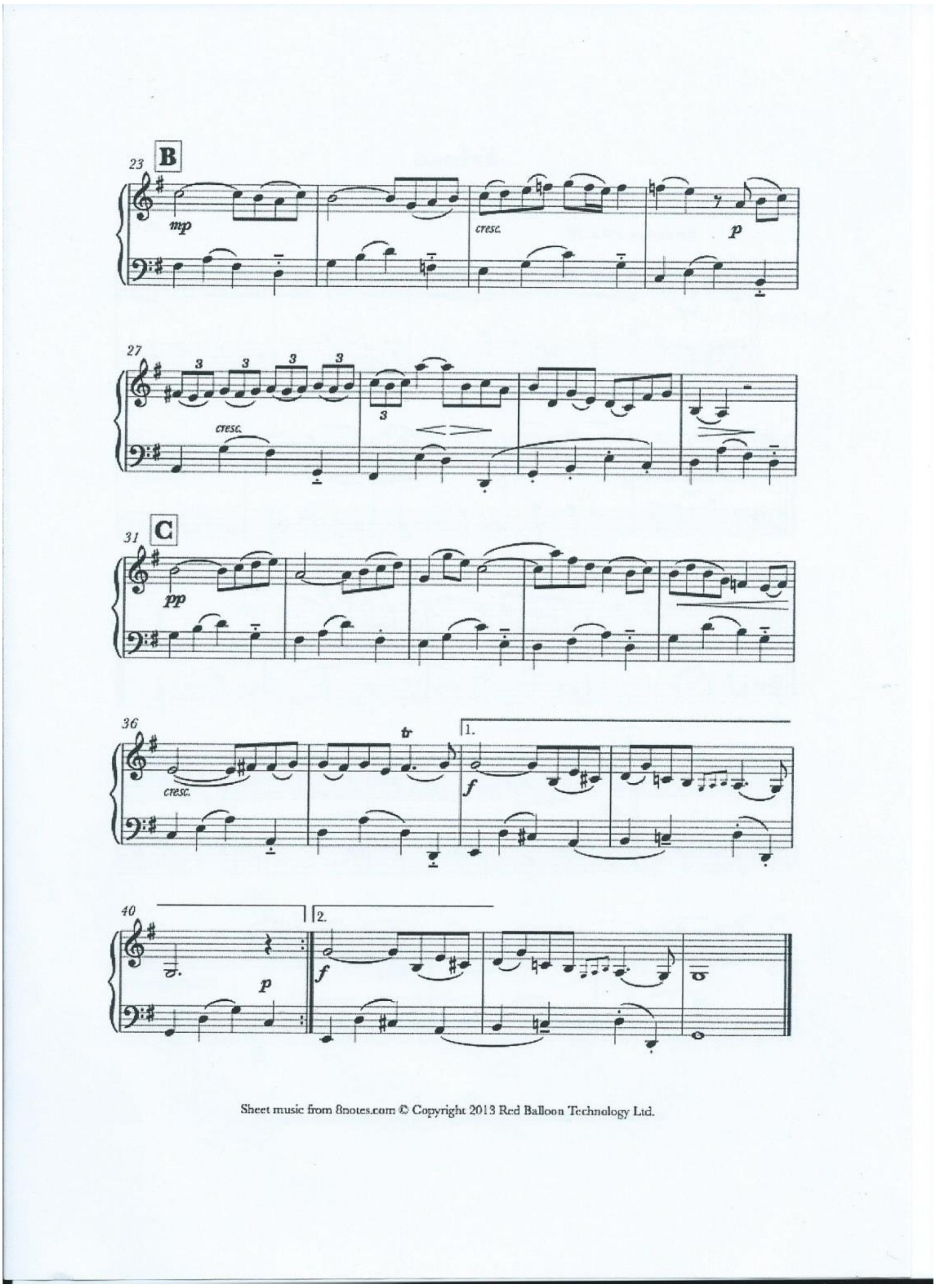
(Source: www.8notes.com, Red Balloon Technology Ltd.)

Appendix P Sheet Music of J.S. Bach: Arioso from Cantata, BWV 156



(Source: www.8notes.com, Red Balloon Technology Ltd.)

Appendix P (cont.) **Sheet Music of J.S. Bach: Arioso from Cantata, BWV 156**



(Source: www.8notes.com, Red Balloon Technology Ltd.)

Appendix Q Sheet Music of J.S. Bach: Cello Suite No. 2 in D-Minor, BWV 1008