Department of Mathematics and Statistics

Existence and Estimates of Solutions for Various Elliptic Equation Models

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Declaration

This thesis contains no meterial which has been accepted for the award of any other degree or diploma in any univeristy.

To the best of my Knowledge and belief this thesis contains non material previously published by any other person except where due acknowledge has been made.

Signature:
Jiang Yongsheng

Date: July 2017

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Abstract

The thesis is devoted to studying the existence and estimates of the solutions of various elliptic equations or degenerate elliptic equations, including the Kohn Laplace equation in the Heisenberg group, the nonlinear elliptic equations with negative exponent. These differential equation models arise from the study of quantum mechanics and differential geometry and have been extensively applied to solve classical physics or geometry problems, such as Brunn-Minkoski's theory in Geometry.

The research includes three key components. The first component is concerned with the partial Schauder estimates to a degenerate elliptic equation in the Heisenberg group. Without the commutativity of the differential operator in the Heisenberg group, smooth estimations cannot be obtained by only using the Maximum principle and a priori estimates for a harmonic function as the classical argument in Euclidean spaces. To overcome this difficulty, we develop a new estimate to the Newton potential for getting the partial derivatives of the solution which in some directions are Hölder continuous but may be fail in others.

For the second component, we study the existence of periodic solutions to a system of nonlinear ordinary differential equations. By using the truncated technique we study the solvability of the dual Minkowski problems in two dimension by using the periodic solution of a nonlinear ordinary differential equation.

For the third component, we give a new variational functional to the nonlinear differential equation related to the 2-dimensional L_p Minkowski problem. Then a constrained variational method is used to study the solvability of the L_p Minkowski problems in two dimensions.

Keywords: Schauder estimate, elliptic equation, variational methods, periodic solutions, the Minkowski problem.

List of Publications Related to This Thesis

1. N. Wei, Y. Jiang and Y. Wu, Partial Schauder estimates for a sub-elliptic equation in the Heisenberg group. *Acta Mathematica Scientia*, 36B(3), pp945-956, 2016.

2. Y. Jiang and Y. Wu, On the 2-dimensional dual Minkowski problem, *Journal of Differential Equations*, 263(6), pp3230-3243, 2017.

3. Y. Jiang, An application of the variational method to the solvability of the 2-dimensional L_p type of Minkowski problem. *To be submitted*.

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Chapter 1

Introduction

1.1 General

In Geometry or Physics, many classical problems have been studied via the existence and estimates of solutions to various elliptic equation models, resulting in a series of profound mathematical theories. For example, Strauss [67] studied the existence of radial symmetric solutions to the scalar field equation by the compact embedding property of function in the Sobolev space. To prove the existence of the ground state solution to a general scalar field equation, Lions [48] established the principle of concentration and compactness. Caffarelli [9–11] established various estimates for studying fully nonlinear equations, and Wang et al. [16, 75] developed a variational framework to study the Hessian equations. The previous research and outstanding achievements in this field shows that the study of existence and estimates of solutions to nonlinear differential equation models is important and challenging in Mathematics.

The Schauder estimates for elliptic equations play an important role in the theory of partial differential equations. One could derive the existence of solutions for elliptic equations by using fixed points argument in the Hölder space under the help of Schauder estimates [10, 31, 61]. One also could get a weak solution for elliptic equations firstly, then prove its smoothness by Schauder estimates [31]. Many mathematicians have studied the Schauder estimates for elliptic equations by using various different methods [10,11,19,22,32,58,61,64,76]. However, there are still many open problems such as the partial Schauder estimates for sub-elliptic equations, which require further study.

The existence of solutions for nonlinear elliptic equations is challenging. Many methods in nonlinear analysis have been used to study the existence of solutions for nonlinear elliptic equations [3,21,39,48,60,67–69,77]. Among them, the fixed points argument and variational methods have been used to get fruitful results [12,46,47,49,70,71,71,78]. The key work of applying the fixed points argument is to get a priori estimates of the solutions to nonlinear equations. However, it is difficult to get a priori estimates for some nonlinear elliptic equations with negative exponent, which causes essential difficulty when the fixed points argument is used to study these problems [43]. By the variational method, the problem of establishing the conditions for the existence of solutions to a nonlinear elliptic equation is transformed to the problem of finding the critical points of a related functional [60, 77]. The key work in using the variational method is to get a compact sequence of approximate solutions to the original equation. Although the principle of concentration and compactness [48, 83] could be used directly to deal with many of those problems, there are further work to do for deriving a compact sequence of approximate solutions for various equations with complicated or unconventional functionals.

In this thesis, we focus on the partial Schauder estimates to a degenerate elliptic equation, and the existence of solutions to nonlinear elliptic equations with complicated and unconventional functionals, such as the Minkowski problem in two dimensions.

1.2 Objectives

The main objective of this research is to study the existence and estimates of solutions to various elliptic equation models, which includes the Kohn-Laplace equation in the Heisenberg group, the dual Minkowski problems and the L_p Minkowski problem in two dimensions. The specific objectives are detailed as follows.

(1) Establish the partial Schauder estimates for a degenerate equation in the Heisenberg group.

The phenomenon that the partial derivatives of the solution in some directions are Hölder continuous but fail in others are called partial Schauder estimates. One of our objectives is to study this phenomenon for sub-elliptic equations such as

$$\sum_{i,j=1}^{m} a_{ij}(\xi) Z_i Z_j u(\xi) = f(\xi)$$

in the Heisenberg group $\mathbb{H}^n$, where $\{Z_j's, j = 1, \cdots, m\}$ span the first layer of the Lie algebra of Heisenberg group. We choose the Kohn Laplace equation in the Heisenberg group as a specific subject of investigation model. The Kohn Laplace equation on a unit ball is as follows

$$\Delta_{\mathbb{H}^n} u(\xi) = f(\xi) \ \text{ in } B_1(0), \tag{1.1}$$

where $B_1(0) = \{\eta \in \mathbb{H}^n, d(\eta, 0) < 1\}$ is the unit ball in the Heisenberg group $\mathbb{H}^n$, and f is the given data. Under some assumptions on the smoothness of f, we aim to establish the estimates of internal Hölder norms for some second derivatives of the solution u to (1.1). For each $i = 1, 2 \cdots, 2n$, we should calculate the derivatives of u in both directions z_i and t if we consider $Z_i u$. In the light of this observation, it is natural to consider the partial Schauder estimates for the solution of (1.1) when f is smooth on those planes related to the variables z_i and t. In this sense, the partial Schauder estimates reveal the natural effect of the smoothness of the given data on the Hölder norm of partial derivatives for the solution u.

(2) Study the dual Minkowski problems in two dimension.

Analytically, the study of the dual Minkowski problem is equivalent to studying the following fully nonlinear elliptic equation

$$u(u^2 + |\nabla u|^2)^{\frac{k-n}{2}} det(u_{ij} + e_{ij} u) = g(v), \ v \in \mathbb{S}^{n-1}, \tag{1.2}$$

where $k \in \mathbb{R}$ and ∇u denotes the gradient vector of u respect to a frame on $\mathbb{S}^{n-1}$. In two dimensions, (1.2) is the following nonlinear problem

$$u''(\theta) + u(\theta) = g(\theta) u^{-1} (u^2 + u'^2)^{(2-k)/2}, \ \theta \in \mathbb{S} \tag{1.3}$$

where $g(\theta)$ is the given data. It is difficult to study the solvability of (1.3) directly. By using the truncated technique, it is possible to use the solution of a truncated problem to construct a solution of (1.3). The key research problem here is how to choose an appropriate truncated problem, for which it is convenient to study the solvability and to derive good estimates of the solution.

(3) Study a new variational functional related to the L_p Minkowski problem in two dimensions.

By using a new variational functional, we consider an application of variational method to the solvability of the L_p Minkowski problem in two dimensions:

$$u'' + u = g(x)u^{-(1+p)},\ x \in \mathbb{S}. \tag{1.4}$$

where $p \geq 0$ and $g(x)$ is a $2\pi-$periodic positive function. From the point of view of differential equations, (1.4) is a nonlinear problem with negative exponent. Its solution should be positive for all $x \in \mathbb{S}$. When applying the variational method to study the solvability of (1.4), the critical point theory has to be applied in a positive cone which consists of positive functions. It leads to additional difficulties and makes the application of variational method more complex. To simplify the problem, we apply the exponential mapping to transform problem (1.4) into

$$-w'' + w'^2 + 1 = \frac{g(\theta)e^{(p+2)w}}{\int_{\mathbb{S}} g(\theta)e^{pw}d\theta}, \tag{1.5}$$

which is a nonlinear elliptic problem with a gradient term. Let $H^1(\mathbb{S})$ be the Hilbert space. For a $L^1(\mathbb{S})$ function $g(\theta)$, the solution of (1.4) belongs to $H^1(\mathbb{S})$ without additional constrained condition. Hence, we have a well defined functional on $H^1(\mathbb{S})$:

$$I(w) = \frac{1}{2}\int e^{-2w(\theta)}d\theta - \frac{1}{2}\int e^{-2w(\theta)}w'^2(\theta)d\theta \\ + \frac{1}{p}\ln\left(\int g(\theta)e^{pw(\theta)}d\theta\right), w(\theta) \in H^1(\mathbb{S}).$$

We aim to study the geometry property of functional I, to study the existence of solution of (1.5) by the critical points of functional I, and then use the solution of (1.5) to construct the solution of (1.4).

1.3 Outline of the thesis

To achieve the aims of this thesis, we further develop the classical techniques for partial differential equations and the variational method to overcome the difficulties appeared in the process of deriving the partial Schauder estimates for a sub-elliptic equation, the solvability of the dual Minkowski problems in two dimensions, and the solvability of the L_p Minkowski problems by using new variational functional. To make a clear statement of the research and the results achieved from this research work, the thesis is divided to five chapters.

Chapter 1 introduces the general of the research and presents the objectives of this Ph. D project.

Chapter 2 discusses the partial Schauder estimates to the solution of a sub-elliptic equation.

Chapter 3 is concerned with the solvability of a dual Minkowski problem in two dimensions.

Chapter 4 gives a new variational functional to the L_p Minkowski problem in two dimensions. The solvability of the L_p Minkowski problem is obtained by using the variational method.

In Chapter 5, we summarize the main results and conclusions achieved from this project, and discuss some problems for further research.

Chapter 2

Partial Schauder estimates for a sub-elliptic equation

2.1 General

In the Euclidean space, Schauder estimates for elliptic and parabolic equations have been well studied in [10], [11], [61], [76], etc., which play an important role in the theory of partial differential equations. In brief, if $u \in C^2$ is a solution of $\Delta u = f$, then one can have the estimates for the modulus of $D^2 u$ when f is Hölder continuous. The partial Schauder estimates for the solutions of elliptic equations in Euclidean spaces can be derived under incomplete Hölder continuity assumptions, see [19], [22], [72]. Here, the partial Schauder estimates means that the partial derivatives of the solution in some directions are Hölder continuous but fail in others. One of the motivations of this chapter is to study the phenomenon about the partial Schauder estimates for sub-elliptic equations.

Some research has been done for the Schauder estimates of the operators structured on the non-abelian vector field. Capogna and Han [58] showed the Schauder estimates for the operator $L = \sum_{i,j=1}^{m} a_{ij}(x) X_i X_j$ in the Carnot group, where $\{X_j's, j = 1, \cdots, m\}$ span the first layer of the Lie algebra of a Carnot group in $\mathbb{R}^N$. Then, Gutiérrez and Lanconelli [32] considered the Schauder estimates for a class of more general sub-elliptic equations $L = \sum_{i,j=1}^{m} a_{ij}(x) X_i X_j + X_0$ by using the Taylor formula. Bramanti and Brandolini [8] gave the Schauder estimates for the operators $a_{ij} X_i X_j - \partial_t$ with X_i satisfying Hörmander's rank condition, by making use of the properties of the fundamental solution of the frozen operator. Capogna showed the C^α regularity of Quasi-linear equations in the Heisenberg group (denoted by $\mathbb{H}^n$) [13], which is the simplest non-abelian nilpotent Lie group. The Kohn Laplace equation in the Heisenberg group is a classical sub-elliptic equaion. We have derived the Schauder estimates to the Kohn Laplace equation

$$\Delta_{\mathbb{H}^n} u = f \ \text{ in } B_1(0), \tag{2.1}$$

where $B_1(0) = \{\eta \in \mathbb{H}^n, d(\eta, 0) < 1\}$ is the unit ball in the Heisenberg group $\mathbb{H}^n$ and f is Dini continuous. However, only very few results have been obtained on the partial Schauder estimates for sub-elliptic equations. One of the purposes of this chapter is to extend the results in [44] to the case of partial Schauder estimates under incomplete Dini continuity assumptions.

Let $\xi := (\xi_1, \xi_2, \cdots, \xi_{2n}, t) \in \mathbb{H}^n$ be a given point, and η_m be a given unit vector in the plane

$$P_m := L\{\xi_m, t\} \text{ which is generated by} \{\xi_m, t\}. \tag{2.2}$$

Analogously to the definition of Dini continuous in one direction [72], we say that f is Dini continuous in P_m $(m = 1, 2, \cdots, 2n)$ if

$$\int_0^1 \frac{\omega_{f,m}(r)}{r} dr < \infty,$$

where

$$\omega_{f,m}(r) = \sup\{|f(\xi) - f(\xi \circ t\eta_m)| \text{ where } \xi, \xi \circ t\eta_m \in B_1, \eta_m \in P_m, |t| < r\}. \tag{2.3}$$

This definition of Dini continuous for a function in a plane can be regarded as an extension from the classical concept of Dini continuous. Indeed, the space $\mathbb{H}^n$ can be spanned by a collection of planes such as $\{P_1, P_2, \cdots, P_{2n}\}$, and there exists a point $\eta_m \in P_m$ for each $m(= 1, 2, \cdots, 2n)$ such that $\xi \circ \eta^{-1} = \xi \circ \eta_1^{-1} \circ (\eta_1 \circ \eta_2^{-1}) \circ \cdots (\eta_i \circ \eta_{i+1}^{-1}) \circ \cdots (\eta_{2n-1} \circ \eta_{2n}^{-1}) \circ \eta_{2n} \circ \eta^{-1}$. Hence,

$$\omega(r) = \sup_{\xi,\eta \in B_1 \ d(\xi,\eta)<r} |f(\xi) - f(\eta)| \leq \sum_{m=1}^{2n} \omega_{f,m}(r), \tag{2.4}$$

which means that $\int_0^1 \frac{\omega(t)}{t} dt < \infty$($f$ is Dini continuous) when f is Dini continuous in the plane P_m for all $m = 1, 2, \cdots, 2n$.

The rest of this chapter is organized as follows: Section 2.2 presents the notions of the Heisenberg group and some preliminary lemmas. Section 2.3 presents two theorems developed for Schauder estimates of a sub-elliptic equation. Section 2.4 is devoted to the proof of Theorems 2.3.1 and 2.3.4.

2.2 Preliminaries

In this section, we introduce some notations. We denote the points of the Heisenberg group $\mathbb{H}^n$ by $\xi = (z,t) = (x,y,t)$ where $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ and $z_i = x_i, z_{n+i} = y_i$ for $i = 1, 2, \cdots, n$. The group law on $\mathbb{H}^n$ is given by

$$\xi \circ \xi' = (z + z', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle)),$$

where $\xi' = (z', t') = (x', y', t')$ and $\langle x', y\rangle = \sum_{i=1}^n x_i' y_i$. The Kohn Laplacian on $\mathbb{H}^n$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^n (Z_j^2 + Z_{n+j}^2),$$

where

$$Z_j = \frac{\partial}{\partial z_j} + 2z_{n+j}\frac{\partial}{\partial t},\ Z_{n+j} = \frac{\partial}{\partial z_{n+j}} - 2z_j\frac{\partial}{\partial t},\ \ j \in \{1, \cdots, n\}.$$

We call $\nabla_{\mathbb{H}^n} = (Z_1, \cdots, Z_n, Z_{n+1}, \cdots, Z_{2n})$ the horizontal gradient operator on $\mathbb{H}^n$, and $T = \frac{\partial}{\partial t}$ the vertical derivative. Let $u(\xi)$ be a smooth function on $\mathbb{H}^n$. For each $m = 1, 2 \cdots, 2n$, we should calculate the derivatives of u in both directions z_m and t if we consider $Z_m u$. That's the motivation on the definition of the plane P_m in (2.2). However, the case for the derivative of a function on the Euclidean space is in one direction. The partial regularity for elliptic equations is obtained under the assumptions of Dini continuity in one direction, see [72]. It is also important to note that $\{Z_1, \cdots, Z_n, Z_{n+1}, \cdots, Z_{2n}, T\}$ is a basis for the left-invariant vector fields on $\mathbb{H}^n$, whose commutation relations are as follows:

$$\begin{aligned} &[Z_{n+s}, Z_s] = 4T, [Z_{n+s}, Z_t] = 0,\ s, t = 1, 2, \cdots, n \text{ and } s \neq t. \\ &[Z_j, Z_i] = [Z_{n+j}, Z_{n+i}] = [T, Z_i] = [Z_{n+j}, T] = 0,\ i, j = 1, 2, \cdots, n. \end{aligned} \tag{2.5}$$

A natural group of dilations on $\mathbb{H}^n$ is given by $\delta_\lambda(\xi) = (\lambda z, \lambda^2 t)$ with $\lambda > 0$. The Jacobian determinant of δ_λ is λ^Q, where $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$. The operators $\nabla_{\mathbb{H}^n}$ and $\Delta_{\mathbb{H}^n}$ are invariant with respect to the left translations τ_ξ of $\mathbb{H}^n$ and homogeneous with respect to the dilations δ_λ of degree one and of degree two, respectively. A remarkable analogy between the Kohn Laplacian and the classical Laplace operator, given in [24], is that a fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is given by

$$\Gamma(\xi) = \frac{c_Q}{d(\xi)^{Q-2}},$$

where c_Q is a suitable positive constant and

$$d(\xi) = (|z|^4 + t^2)^{1/4}. \tag{2.6}$$

Moreover, if we define $d(\xi, \xi') = d(\xi'^{-1} \circ \xi)$, then d is a distance on the Heisenberg group $\mathbb{H}^n$. More details on the Heisenberg group and the Kohn Laplacian can be found, for example, in [26] and [29].

We first present a priori estimate of the derivatives of the $\Delta_{\mathbb{H}^n}$-harmonic function u, which plays an important role in the follows.

Lemma 2.2.1. *([73],Proposition 2.1) Let Ω be an open subset of $\mathbb{H}^n$, and let u solve $\Delta_{\mathbb{H}^n} u = 0$ and $\overline{B_r}(\xi) \subseteq \Omega$. Then for every*

$$Z_1, \cdots, Z_k \in \{Z_1, \cdots, Z_n, Z_{n+1}, \cdots, Z_{2n}\}$$

we have

$$|Z_1 ... Z_k u(\xi)| \le c r^{-k} \sup_{B_r} |u|, \tag{2.7}$$

where $c = c(k, Q)$.

Lemma 2.2.1 gives a classical estimate to the horizontal gradient of $\Delta_{\mathbb{H}^n}$-harmonic functions. In the follows, we also need the estimates to the vertical derivatives of $\Delta_{\mathbb{H}^n}$-harmonic functions as follows.

Lemma 2.2.2. *Under the same assumption of Lemma 2.2.1, we have*

$$|Tu(\xi)| \leq cr^{-2} \sup_{B_r} |u|, \tag{2.8}$$

and

$$|TZ_k u(\xi)| \leq cr^{-3} \sup_{B_r} |u|, k = 1, 2, \cdots, 2n. \tag{2.9}$$

Proof. By using (2.5), we have

$$|Tu(\xi)| \leq 4\left(|Z_{n+s}Z_t u(\xi)| + |Z_t Z_{n+s} u(\xi)|\right). \tag{2.10}$$

Then by applying Lemma 2.2.1 in (2.10), we get the estimate (2.8). Using a similar argument for (2.8) by replacing u with $Z_k u$, we derive (2.9). □

To prove our conclusion, we also need the following maximum principle for the solution of $\Delta_{\mathbb{H}^n} u = f$, see [30].

Lemma 2.2.3. *Let Ω be a bounded open subset of $\mathbb{H}^n$ and $f \in L^p(\Omega)$ for some $p > \frac{Q}{2}$, then u belongs to $L^\infty(\Omega)$, and one has the estimate*

$$\sup_{\Omega} |u| \leq C \sup_{\partial\Omega} |u| + C(\mathbb{H}^n, p) diam(\Omega)^{2-\frac{Q}{p}} \|f\|_{L^p(\Omega)}. \tag{2.11}$$

We apply the mean value theorem in the homogeneous Carnot group [6] to our case to obtian the following result.

Lemma 2.2.4. *There exists a constant $c_1 > 0$, depending only on $\mathbb{H}^n$ and on the homogeneous norm d, such that for all $f \in C^1(\mathbb{H}^n, \mathbb{R})$ and every $\xi, h \in \mathbb{H}^n$,*

$$|f(\xi \circ h) - f(\xi)| \leq d(h, 0) \sup_{\eta : d(\xi,\eta) \leq d(h,0)} |Zf(\eta)|. \tag{2.12}$$

where $Zf = (Z_1 f, \cdots, Z_n f, Z_{n+1} f, \cdots, Z_{2n} f)$.

In order to prove our main result, we also need to use the Taylor formula in [5], which is established in $\mathbb{H}^n$ and is given by the Lemma below

Lemma 2.2.5. *Assume $u \in C^{m+1}(\mathbb{H}^n)$ $(m \geq 0)$. Then, for any $\xi = (z,t) \in \mathbb{H}^n$, $s = 0, 1, \cdots$, and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2n})$ where $\alpha_i = 0, 1, ..., (i = 1, 2, \cdots, 2n)$, we have*

$$u(\xi) = \sum_{k=0}^{m} \sum_{|\alpha|+2s=k} \frac{1}{\alpha! s!} \left(Z^\alpha T^s u\right)(0) z^\alpha t^s + O(d^{m+1}(\xi)),$$

where $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_{2n}!$, $Z^0 = id$, $T = \frac{\partial}{\partial t}$ and $Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} \cdots Z_{2n}^{\alpha_{2n}}$.

2.3 Partial Schauder estimates

We assume that the solution is smooth, for example $u \in C^3(B_1)$. By approximation, the following estimates hold for weak solutions.

Theorem 2.3.1. *Let u be a solution of (2.1). If f is Dini continuous in a plane P_m, then $\forall \xi$ and $\eta \in B_{\frac{1}{4}}(0)$, we have*

$$\begin{aligned} |Z_i Z_m u(\xi) - Z_i Z_m u(\eta)| \leq C_Q \Bigg[& d \sup_{B_1} |u| + d \sup_{B_1} |f| \\ & + \int_0^d \frac{w_{f,m}(t)}{t} dt + d \int_d^1 \frac{w_{f,m}(t)}{t^2} dt \Bigg], \end{aligned} \tag{2.13}$$

$$\begin{aligned} |Z_m Z_i u(\xi) - Z_m Z_i u(\eta)| \leq C_Q \Bigg[& d \sup_{B_1} |u| + d \sup_{B_1} |f| \\ & + \int_0^d \frac{w_{f,m}(t)}{t} dt + d \int_d^1 \frac{w_{f,m}(t)}{t^2} dt \Bigg], \end{aligned} \tag{2.14}$$

where $Z_i \in \{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ are the horizontal gradient operators on $\mathbb{H}^n$, and $d = d(\xi, \eta) = \|\eta^{-1} \circ \xi\|$, $C_Q > 0$ depends on Q.

Here, we assume that f is Dini contiouous in the plane P_m only. By (2.4), we see that the conclusions about Schauder estimates in [44] is a corollary of Theorem 2.3.1. The proof of Theorem 2.3.1 follows the same ideas as in [72]. In brief, we decompose the difference $u(\xi) - u(\eta)$ as the sum of a Newton potential and a sequence of $\Delta_{\mathbb{H}^n}$-harmonic functions. Since the $\Delta_{\mathbb{H}^n}$-harmonic function is sufficiently smooth, the main difficulties of this chapter are to establish a reasonable decomposition of $u(\xi) - u(\eta)$ and to study the smoothness of the Newton potential (see Lemma 2.4.1 below). The above estimates imply the partial Schauder estimates for the Kohn Laplace equation. To precisely describe the partial Schauder estimates for the Kohn Laplace operator, we need a precise definition as follows.

Definition 2.3.2. *(Partial Hölder space) Let P_m be a plane in $\mathbb{H}^n$ by (2.2), $\alpha \in (0,1]$, and $v : \Omega \subset \mathbb{H}^n \to \mathbb{R}$ be a continuous function. We say that v is Hölder continuous in the plane P_m with Hölder exponent α if*

$$P_{m,\alpha}(v) = \sup\left\{\frac{|v(\xi) - v(\xi \circ \eta)|}{d(\xi, \xi \circ \eta)^\alpha} \text{ where } \xi, \xi \circ \eta \in \Omega, \eta \in P_m\right\} < +\infty,$$

and denote $v \in C^\alpha_{P_m}(\Omega)$ with the norm

$$\|v\|_{C^\alpha_{P_m}(\Omega)} = \sup_{\xi \in \Omega} |v(\xi)| + P_{m,\alpha}(v). \tag{2.15}$$

Remark 2.3.3. *For $\alpha \in (0,1]$, if $v : \Omega \to \mathbb{R}$ is Hölder continuous in all planes $P_m(m = 1, 2, \cdots, 2n)$ with Hölder exponent α. As (2.4), we have*

$$\|v\|_{C^\alpha(\Omega)} = \sup_{\xi \in B_1(0)} |v(\xi)| + \sup\left\{\frac{|v(\xi) - v(\eta)|}{d(\xi, \eta)^\alpha} : \xi, \eta \in \Omega, \xi \neq \eta\right\} < +\infty,$$

which means that v is Hölder continuous in Ω with exponent α, i.e.

$$v \in C^\alpha(\Omega) = \{v : \Omega \to \mathbb{R} : \|v\|_{C^\alpha} < \infty\}.$$

By Definition 2.3.2 and a simple calculation, we see that the terms

$$\int_0^d \frac{w_{f,m}(t)}{t} dt + d \int_d^1 \frac{w_{f,m}(t)}{t^2} dt$$

in (2.13) and (2.14) can be controlled by the Hölder norm (2.15) with $v = f$ (see (2.44) and (2.45) below). Then we have the following partial Schauder estimate for the solution of (2.1).

Theorem 2.3.4. *Let $f \in C^{\alpha}_{P_m}(B_1)$, then, for any $\alpha \in (0,1)$,*

$$\begin{aligned} &\|Z_i Z_m u\|_{C^{\alpha}(B_{\frac{1}{4}})} + \|Z_m Z_i u\|_{C^{\alpha}(B_{\frac{1}{4}})} \\ &\le C_Q \left[\sup_{B_1} |u| + \left(1 + \frac{1}{\alpha(1-\alpha)}\right) \|f\|_{C^{\alpha}_{P_m}(B_1)} \right]. \end{aligned} \tag{2.16}$$

In this theorem, f is assumed to be Hölder continuous in a plane P_m ($m = 1, 2, \cdots, 2n$) only. But $Z_i Z_m u$ and $Z_m Z_i u$ are Hölder continuous in all variables for all $i = 1, 2, \cdots, 2n$. If f is Hölder continuous in $\mathbb{H}^n$, then it is Hölder continuous in the plane P_m for all $m = 1, 2, \cdots, 2n$. By using Theorem 2.3.4 we get that $Z_i Z_j u$ are Hölder continuous in all variables for $i, j = 1, 2, \cdots, 2n$, which is the classical regularity for sub-elliptic equations and hence extends the results in [44]. Moreover, if f is Hölder continuous in planes $P_k, P_{k+1}, \cdots, P_{2n}$, then $Z_i Z_j u$ is Hölder continuous for all $i, j \geqslant k$. Similar results for elliptic and parobolic equations in Euclidean space have been derived in [19] by using the maximum principle and the Krylov-Safonov theory. We should address that the partial regularity for elliptic operators and parobolic operators have been proved by Wang and Tian in [72], where the commutativity of the gradient operators in Euclidean spaces helps getting the smooth estimations by mainly using the Maximum principle and a priori estimates for the derivatives of a harmonic function. The gradient operators in the Heisenberg group are non commutative, see (2.5) below. It seems impossible to get a harmonic function by differentiating an auxiliary equation as it is in [72]. So, it is difficult to get the smooth estimates by only using the Maximum principle and the property of harmonic functions. In this chapter, we need more estimates to the Newton potential as it is in Lemma 2.4.1 below to overcome this difficulty.

2.4 Proof of the main results

In this section, we prove our main results (Therem 2.3.1) of this chapter by using the perturbation argument established in [76]. To get the smooth estimate of the difference $u(\xi) - u(\eta)$, we decompose it into the sum of a Newton potential and a sequence of $\Delta_{\mathbb{H}^n}$-harmonic functions. Firstly, we derive the estimates for the Newton potential by using the ideas in [72,76].

Lemma 2.4.1. *Let $f(\xi)$ be an integrable function on B_1 and*

$$h(\xi) = \int_{B_1(0)} \frac{f(\xi')}{d^{Q-2}(\xi \circ \xi'^{-1})} d\xi', \quad \text{where } \xi \in B_{1/4}(0).$$

If $f(\xi) = f(\xi_1, \ldots, \xi_{2n}, t)$ does not depend on t and ξ_m ($m = 1, 2, \ldots, 2n$), then we have

$$|Z_m h(\xi)| \leq C \int_{B_1(0)} f(\xi') d\xi', \tag{2.17}$$

and

$$|Z_m Z_j h(\xi)| + |Z_j Z_m h(\xi)| \leq C \int_{B_1(0)} f(\xi') d\xi', \tag{2.18}$$

where $j = 1, 2, \ldots, 2n$.

Proof. Without loss of generality, we prove (2.18) with $m = n$. let $Z'_n = \frac{\partial}{\partial x'_n} + 2y'_n \frac{\partial}{\partial t}$ and $(\xi_1, \cdots, \xi_{2n}, t) = (x_1, \cdots, x_n, y_1, \cdots, y_n, t)$. By the definition of d in (2.6), a simple calculating implies that $Z_n d(\xi \circ \xi'^{-1}) = -Z'_n d(\xi \circ$

ξ'^{-1}), and hence,

$$\begin{aligned} Z_n h(\xi) &= \int_{B_1(0)} -Z_n \frac{f(\xi')}{d^{Q-2}(\xi\circ\xi'^{-1})} d\xi' \\ &= -\int_{\beta\le 1} f(\xi') \int_{-\alpha}^{\alpha} Z_n' \frac{1}{d^{Q-2}(\xi\circ\xi'^{-1})} d\xi', \end{aligned} \tag{2.19}$$

where $\alpha = \sqrt{\sqrt{1-t'^2} - (\sum_{i=1}^{n-1}(x'^2_i + y'^2_i) + y'^2_n)}$, $\beta = t'^2 + (\sum_{i=1}^{n-1}(x'^2_i + y'^2_i) + y'^2_n)^2$. Based on this observation, we prove (2.17) by the following two cases.

Case I: $\beta < \frac{1}{2}$.

We see that

$$\begin{aligned} \frac{1}{2} &\le \left(\sqrt{1-t'^2}\right)^2 - \left(\sum_{i=1}^{n-1}(x'^2_i + y'^2_i) + y'^2_n\right)^2 \\ &= \alpha^2 \left[\sqrt{1-t'^2} + \sum_{i=1}^{n-1}(x'^2_i + y'^2_i) + y'^2_n\right] \le 2\alpha^2. \end{aligned}$$

Thus $\alpha \ge \frac{1}{2}$. If $x_n' = \pm\alpha$, then $|\xi'| \ge \frac{1}{2}$. Since $f(\xi)$ does not depend on $\xi_m = x_n$, it follows from $|\xi| \le \frac{1}{4}$ that

$$\begin{aligned} & \left| \int_{\beta\le 1/2} \int_{-\alpha}^{\alpha} f(\xi') \frac{\partial}{\partial x_n'} \frac{1}{d^{Q-2}(\xi\circ\xi'^{-1})} d\xi' \right| \\ \le & \int_{\alpha\ge 1/2} \left| \frac{1}{d^{Q-2}(\xi\circ\xi'^{-1})} \Big|_{x_n'=-\alpha}^{x_n'=\alpha} \right| \left|f(\xi')\right| dx_1' \cdots dx_{n-1}' dy_1' \cdots dy_n' dt' \\ \le & \; 4^Q \int_{\alpha\ge 1/2} \alpha |f(\xi')| dx_1' \cdots dx_{n-1}' dy_1' \cdots dy_n' dt' \\ \le & \; 4^{Q-1} \int_{\alpha\ge 1/2} |f(\xi')| dx_1' \cdots dx_{n-1}' dy_1' \cdots dy_n' dt' \int_{-\alpha}^{\alpha} dx_n' \\ < & \; 4^{Q-1} \int_{B_1(0)} |f(\xi')| d\xi'. \end{aligned} \tag{2.20}$$

Case II: $\beta \ge \frac{1}{2}$.

In this case, we have $|\xi'| > 1/2$ and $d(\xi\circ\xi'^{-1}) > 1/4$ for any $\xi \in B_{1/4}(0)$, and

$$\begin{aligned}
&\left|\int_{\beta\geq 1/2,|\xi'|<1} f(\xi')\int_{-\alpha}^{\alpha}\frac{\partial}{\partial x_n'}\frac{1}{d^{Q-2}(\xi\circ\xi'^{-1})}d\xi'\right| \\
\leq\ & \int_{d(\xi\circ\xi'^{-1})\geq 1/4,|\xi'|<1}|f(\xi')|\int_{-\alpha}^{\alpha}\frac{C}{d^{Q-1}(\xi\circ\xi'^{-1})}d\xi' \\
\leq\ & C\int_{|\xi'|<1}|f(\xi')|\int_{-\alpha}^{\alpha}d\xi' = C\int_{B_1(0)}|f(\xi')|d\xi'. \qquad (2.21)
\end{aligned}$$

It follows from (2.20) and (2.21) that

$$\begin{aligned}
&\int_{B_1(0)}\frac{\partial}{\partial x_n'}\frac{f(\xi')}{d^{Q-2}(\xi\circ\xi'^{-1})}d\xi' \qquad (2.22)\\
=\ & \int_{\beta\leq 1} f(\xi')\int_{-\alpha}^{\alpha}\frac{\partial}{\partial x_n'}\frac{1}{d^{Q-2}(\xi\circ\xi'^{-1})}d\xi' \\
\leq\ & \left|\int_{\beta\geq 1/2,|\xi'|<1}+\int_{\beta\leq 1/2,|\xi'|<1} f(\xi')\int_{-\alpha}^{\alpha}\frac{\partial}{\partial x_n'}\frac{1}{d^{Q-2}(\xi\circ\xi'^{-1})}d\xi'\right| \\
\leq\ & C\int_{B_1(0)}|f(\xi')|d\xi'. \qquad (2.23)
\end{aligned}$$

Similarly we have

$$Th(\xi)=\int_{B_1(0)}\frac{\partial}{\partial t}\frac{f(\xi')}{d^{Q-2}(\xi\circ\xi'^{-1})}d\xi'\leq C\int_{B_1(0)}|f(\xi')|d\xi'. \qquad (2.24)$$

By applying (2.22), (2.24) in (2.19), we get (2.17). Let

$$d_j(\xi,\xi')=Z_j'\frac{1}{d^{Q-2}(\xi\circ\xi'^{-1})}.$$

Then proceeding as for (2.17), we have

$$\left|\int_{B_1(0)} f(\xi')\left(\frac{\partial}{\partial x_n'}+2y_n'\frac{\partial}{\partial t}\right)d_j(\xi,\xi')d\xi'\right|\leq C\int_{B_1(0)}|f(\xi')|d\xi'. \qquad (2.25)$$

As for derivating (2.19), the estimate (2.25) helps us getting that

$$\begin{aligned}
|Z_nZ_jh(\xi)| &= \left|\int_{B_1(0)}\left(\frac{\partial}{\partial x_n'}+2y_n'\frac{\partial}{\partial t}\right)d_j(\xi,\xi')f(\xi')d\xi'\right| \\
&\leq C\int_{B_1(0)}|f(\xi')|d\xi'. \qquad (2.26)
\end{aligned}$$

By using (2.24), (2.26) and the commutation relations of the left-invariant vector fields (2.5), we have

$$|Z_j Z_n h(\xi)| \leq |Z_n Z_j h(\xi)| + |Th(\xi)| \leq C \int_{B_1(0)} |f(\xi')| d\xi'. \tag{2.27}$$

Then (2.18) is obtained by combining (2.26) and (2.27). $\square$

Remark 2.4.2. *Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2n})$ where $\alpha_i = 0, 1, 2, \cdots$ and $\alpha_m \geq 1$, $Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} \cdots Z_{2n}^{\alpha_{2n}}$, $f(\xi)$ and $h(\xi)$ be given by Lemma 2.4.1. By a similar argument for Lemma 2.4.1, we see that $\sup\limits_{B_{1/4}(\xi)} |Z^\alpha h(\xi)| \leq C \int_{B_1(0)} f(\xi') d\xi'$.*

Now, we give the proof of Theorem 2.3.1. The method is similar to [76], but we have to use Lemma 2.4.1 instead of harmonic function to estimate the Newton potential due to the non-commutativity of the horizontal gradient operators on $\mathbb{H}^n$. For convenience of the reader, we present it entirely here.

Proof of Theorem 2.3.1: Without loss of generality, let $m = n$. For any given point η near the origin, we have

$$\begin{aligned}
|Z_i Z_n u(\eta) - Z_i Z_n u(0)| &\leq |Z_i Z_n u_k(\eta) - Z_i Z_n u_k(0)| \\
&\quad + |Z_i Z_n u_k(0) - Z_i Z_n u(0)| + |Z_i Z_n u(\eta) - Z_i Z_n u_k(\eta)| \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

where u_k is the solution of

$$\Delta_{\mathbb{H}^n} u_k = f(x', y, 0) \text{ in } B_{\rho^k}, u_k = u \text{ on } \partial B_{\rho^k},\ \rho = 1/2,$$

with $x' = (x_1, \cdots, x_{n-1}, 0)$. For simplicity, we use B_k to denote B_{ρ^k} in the following. By the fact that $\Delta_{\mathbb{H}^n}(u_k - u) = f(x', y, 0) - f(x, y, t)$ and using

Lemma 2.2.3, we have

$$\begin{aligned}\|u_k - u\|_{L^\infty(B_k)} &\le C(2\rho^k)^{2-\frac{Q}{p}}\left(\int_{B_k}|f(x',y,0)-f(x,y,t)|^p d\xi\right)^{1/p}\\ &\le C(\rho^k)^{2-\frac{Q}{p}}\omega_{f,n}(\rho^k)|B_k|^{1/p}\\ &\le C(\rho^k)^{2-\frac{Q}{p}}\omega_{f,n}(\rho^k)(\rho^k)^{Q/p}\\ &= C\rho^{2k}\omega_{f,n}(\rho^k). \qquad (2.28)\end{aligned}$$

Hence,

$$\begin{aligned}\|u_k - u_{k+1}\|_{L^\infty(B_k)} &\le \|u_k - u\|_{L^\infty(B_k)} + \|u_{k+1} - u\|_{L^\infty(B_{k+1})}\\ &\le C\rho^{2k}\omega_{f,n}(\rho^k). \qquad (2.29)\end{aligned}$$

It is clear to see that $u_{k+1} - u_k$ is $\Delta_{\mathbb{H}^n}$-harmonic, and thus by using (2.7) and (2.29), we have

$$\begin{aligned}\|Z_j(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} &\le C\rho^{-k}\sup_{B_k}|u_k - u_{k+1}|\\ &\le C\rho^{-k}\rho^{2k}\omega_{f,n}(\rho^k) = C\rho^k\omega_{f,n}(\rho^k),\end{aligned}$$

and

$$\begin{aligned}\|Z_iZ_j(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} &\le C\rho^{-2k}\sup_{B_k}|u_k - u_{k+1}|\\ &\le C\rho^{-2k}\rho^{2k}\omega_{f,n}(\rho^k) = C\omega_{f,n}(\rho^k).\end{aligned} \qquad (2.30)$$

By using Lemma 2.2.5 and (2.28), we have

$$\lim_{k\to+\infty} Z_iZ_ju_k(0) = Z_iZ_ju(0) \text{ for any } i,j = 1,2,\cdots,2n. \qquad (2.31)$$

Let $k \ge 1$ such that $\rho^{k+4} \le d(\eta) \le \rho^{k+3}$, then by (2.30) and (2.31), we have

$$\begin{aligned} I_2 &= |Z_iZ_nu_k(0) - Z_iZ_nu(0)| \le \sum_{l=k}^{\infty}|Z_iZ_nu_l(0) - Z_iZ_nu_{l+1}(0)|\\ &\le C\sum_{j=k}^{\infty}\omega_{f,n}(\rho^j) \le C\int_0^{d(\eta)}\frac{\omega_{f,n}(r)}{r}dr. \qquad (2.32)\end{aligned}$$

Now we can estimate I_3 in a similar way. Let v_l be the solution of

$$\Delta_{\mathbb{H}^n} v_l = f(\hat{\eta}_n) \text{ in } B_l(\eta), v_l = u \text{ on } \partial B_l(\eta),$$

where $l = k, k+1, \cdots$, $\hat{\eta}_n = (x_1, \cdots, x_{n-1}, \hat{x}_n, y, \hat{t})$, $\hat{x}_n$ is equal to the value of the n-th component of η and $\hat{t}$ is equal to the value of the $(2n+1)$-th component of η. Then

$$\begin{aligned} I_3 &= |Z_i Z_n u(\eta) - Z_i Z_n u_k(\eta)| \\ &\leq |Z_i Z_n u(\eta) - Z_i Z_n v_k(\eta)| + |Z_i Z_n v_k(\eta) - Z_i Z_n u_k(\eta)|. \end{aligned} \tag{2.33}$$

Similar to (2.32), we have

$$\begin{aligned} |Z_i Z_n u(\eta) - Z_i Z_n v_k(\eta)| &\leq \sum_{l=k}^{\infty} |Z_i Z_n v_l(\eta) - Z_i Z_n v_{l+1}(\eta)| \\ &\leq C \sum_{j=k}^{\infty} \omega_{f,n}(\rho^j) \leq C \int_0^{d(\eta)} \frac{\omega_{f,n}(r)}{r} dr. \end{aligned} \tag{2.34}$$

Let $w_k(\xi) = v_k(\xi) - u_k(\xi)$, then

$$\Delta_{\mathbb{H}^n} w_k(\xi) = f(\hat{\eta}_n) - f(x', y, 0), \ \ \xi \in B_{k+1}(\eta).$$

Let $\hat{w}_k(\xi) = w_k(\delta_{\rho^{k+1}}\xi \circ \eta)$ with $\delta_{\rho^{k+1}}\xi = (\rho^{k+1}x, \rho^{k+1}y, \rho^{2(k+1)}t)$, then

$$\Delta_{\mathbb{H}^n} \hat{w}_k(\xi) = \rho^{2(k+1)}[f(\delta_{\rho^{k+1}}\hat{\eta}_n \circ \eta) - f(\rho^{k+1}(x', y, 0) \circ \eta)], \ \ \xi \in B_1(0).$$

By using Theorem 2.1 and Corollary 2.8 in [25] with $M = B_1(0)$ and $\mathscr{L} = \Delta_{\mathbb{H}^n}$, we get

$$\hat{w}_k(\xi) = \rho^{2(k+1)} \int_{B_1(0)} \frac{f(\hat{\xi}') - f(\bar{\xi}')}{d^{Q-2}(\xi \circ \xi'^{-1})} d\xi' + g(\xi), \tag{2.35}$$

where

$$\begin{aligned} \hat{\xi}' &= (\rho^{k+1}\xi_1', \cdots, \rho^{k+1}\xi_{n-1}', \rho^{k+1}\hat{x}_n, \rho^{k+1}\xi_{n+1}', \cdots, \rho^{k+1}\xi_{2n}', \rho^{2(k+1)}\hat{t}), \\ \bar{\xi}' &= (\rho^{k+1}\xi_1', \cdots, \rho^{k+1}\xi_{n-1}', 0, \rho^{k+1}\xi_{n+1}', \cdots, \rho^{k+1}\xi_{2n}', 0), \end{aligned} \tag{2.36}$$

and $\Delta_{\mathbb{H}^n} g(\xi) = 0$ in $B_{1/2}(0)$,

$$g(\xi) = \hat{w}_k(\xi) - \rho^{2(k+1)} \int_{B_1(0)} \frac{f(\hat{\xi}') - f(\bar{\xi}')}{d^{Q-2}(\xi \circ \xi'^{-1})} d\xi' \text{ on } \partial B_{1/2}(0).$$

It follows that

$$\sup_{|\xi| \leq 1/2} |g(\xi)| \leq \sup_{|\xi|=1/2} |\hat{w}_k(\xi)| + C\rho^{2(k+1)} \omega_{f,n}(\rho^{k+1}),$$

and that

$$\sup_{|\xi|=\frac{1}{1}} |\hat{w}_k(\xi)| \leq \sup_{|\xi \circ \eta^{-1}|=\rho^{k+2}} (|u_k(\xi) - u(\xi)| + |u(\xi) - v_k(\xi)|) \leq \rho^{2(k+1)} \omega_{f,n}(\rho^{k+1})$$

can be obtianed similarly as (2.28). Therefore,

$$\sup_{|\xi| \leq 1/2} |g(\xi)| \leq C\rho^{2k+2} \omega_{f,n}(\rho^{k+1}) + C\rho^{2(k+1)} \omega_{f,n}(\rho^{k+1}).$$

This together with Lemma 2.2.1 implies that

$$|Z_i Z_j g(0)| \leq C \sup_{B_{1/2}} |g(\xi)| \leq C\rho^{2k+2} \omega_{f,n}(\rho^k). \tag{2.37}$$

Let $h(\xi) = \int_{B_1(0)} \frac{f(\hat{\xi}') - f(\bar{\xi}')}{d^{Q-2}(\xi \circ \xi'^{-1})} d\xi'$, from (2.36) we see that the value of $F(\xi) = f(\hat{\xi}') - f(\bar{\xi}')$ does not depend on both the n-th component and the $(2n+1)$-th component of ξ. Thus by using Lemma 2.4.1, we have that

$$|Z_i Z_n h(0)| \leq C \int_{B_1} f(\hat{\xi}') - f(\bar{\xi}') d\xi \leq C\omega_{f,n}(\rho^k). \tag{2.38}$$

Combining (2.35)-(2.38), we have

$$|Z_i Z_n \hat{w}_k(0)| \leq C\rho^{2(k+1)} \omega_{f,n}(\rho^k),$$

therefore,

$$|Z_i Z_n w_k(\eta)| = \rho^{-2(k+1)} |Z_i Z_n \hat{w}_k(0)| \leq C\omega_{f,n}(\rho^k). \tag{2.39}$$

Then, by (2.33), (2.34) and (2.39), we have

$$I_3 \le C \int_0^{d(\eta)} \frac{\omega_{f,n}(r)}{r} dr + C\omega_{f,n}(\rho^k) \le C \int_0^{d(\eta)} \frac{\omega_{f,n}(r)}{r} dr. \tag{2.40}$$

Next we estimate I_1. Let $h_k = u_k - u_{k-1}$, by (2.29), Lemmas 2.2.1 and 2.2.4, we see that

$$\begin{aligned}
|Z_iZ_nh_k(\eta) - Z_iZ_nh_k(0)| &\le d(\eta) \sup_{\theta: d(\theta)\le d(\eta)} |Z_lZ_iZ_nh_k(\theta)| \\
&\le Cd(\eta)\rho^{-3k} \sup_{B_k(\theta)} |h_k(\theta)| \\
&\le Cd(\eta)\rho^{-3k}\rho^{2k}\omega_{f,n}(\rho^k) \\
&= C\rho^{-k} d(\eta)\omega_{f,n}(\rho^k).
\end{aligned}$$

Similarly to (2.39), we have

$$|Z_iZ_nu_0(\eta) - Z_iZ_nu_0(0)| \le Cd(\eta)\left(\sup_{B_1}|u| + \sup_{B_1}|f|\right).$$

Then,

$$\begin{aligned}
I_1 &= |Z_iZ_nu_k(\eta) - Z_iZ_nu_k(0)| \\
&\le |Z_iZ_nu_{k-1}(\eta) - Z_iZ_nu_{k-1}(0)| + |Z_iZ_nh_k(\eta) - Z_iZ_nh_k(0)| \\
&\le |Z_iZ_nu_{k-2}(\eta) - Z_iZ_nu_{k-2}(0)| + |Z_iZ_nh_{k-1}(\eta) - Z_iZ_nh_{k-1}(0)| \\
&+ |Z_iZ_nh_k(\eta) - Z_iZ_nh_k(0)| \le \cdots \\
&\le |Z_iZ_nu_0(\eta) - Z_iZ_nu_0(0)| + \sum_{l=1}^{k} |Z_iZ_nh_l(\eta) - Z_iZ_nh_l(0)| \\
&\le Cd(\eta)\left(\sup_{B_1}|u| + \sup_{B_1}|f|\right) + Cd(\eta)\sum_{l=1}^{k}\rho^{-l}\omega_{f,n}(\rho^l) \\
&\le Cd(\eta)\left(\sup_{B_1}|u| + \sup_{B_1}|f| + C\int_{d(\eta)}^1 \frac{\omega_{f,n}(r)}{r^2}\right)
\end{aligned} \tag{2.41}$$

Finally, (2.13) can be obtained by combining (2.32), (2.40) and (2.41). And (2.14) can be obtained by using (2.13) and the commutation relations of the left-invariant vector fields. Indeed, from (2.5) we have

$$
\begin{aligned}
&|Z_m Z_i u(\xi) - Z_m Z_i u(\eta)| \\
&\leqslant |Z_i Z_m u(\xi) - Z_i Z_m u(\eta)| + |Tu(\xi) - Tu(\eta)|.
\end{aligned}
\tag{2.42}
$$

The estimate for $|Z_i Z_m u(\xi) - Z_i Z_m u(\eta)|$ can be obtianed by using (2.13). Then we have to give the estimate to $|Tu(\xi) - Tu(\eta)|$ as follows:

$$
\begin{aligned}
&|Tu(\xi) - Tu(\eta)| \leqslant |Tu(\xi) - Tu(0)| + |Tu(0) - Tu(\eta)| \\
&\leqslant |Tu(\xi) - Tu(0)| + |Tu(0) - Tu(\eta)| \\
&\leqslant C\left[d \sup_{B_1}|u| + d \sup_{B_1}|f| + \int_0^d \frac{\omega_{f,m}}{t}dt + d\int_d^1 \frac{\omega_{f,m}(t)}{t^2}dt\right],
\end{aligned}
\tag{2.43}
$$

whose last inequality is obtianed by applying the same argument for (2.13), since we have estimates (2.8), (2.9) and (2.24). □

Proof of Theorem 2.3.4 . Let $r \in (0,1)$, by the defintion (2.3) we have a sequence $\{(\xi_n, \eta_n)\}$ in the set

$$
S_{m,r} = \{(\xi,\eta) \in B_1 \times P_m : |\eta| = 1, \xi \circ t\eta \in B_1, |t| < r\},
$$

such that

$$
\begin{aligned}
w_{f,m}(r) &= \sup_{(\xi,\eta)\in S_{m,r}} |f(\xi) - f(\xi \circ t\eta)| = \lim_{n\to\infty} \sup_{|t|<r} |f(\xi_n) - f(\xi_n \circ t\eta_n)| \\
&\leq r^\alpha \sup_{\xi,\eta\in S_{m,r},|t|<r} \frac{|f(\xi) - f(\xi \circ t\eta)|}{d(\xi, \xi \circ t\eta)^\alpha}
\end{aligned}
$$

This together with the definition of the partial Hölder norm (2.15) shows that

$$
w_{f,m}(r) \leqslant r^\alpha \|f\|_{C^\alpha_{P_m}(B_1)}.
$$

Hence,

$$
\int_0^d \frac{w_{f,m}(r)}{r}dr = \int_0^d r^{\alpha-1}dr\|f\|_{C^\alpha_{P_m}(B_1)} = \frac{1}{\alpha}d^\alpha \|f\|_{C^\alpha_{P_m}(B_1)},
\tag{2.44}
$$

$$
\begin{aligned}
d\int_d^1 \frac{w_{f,m}(r)}{r^2} &= d\int_d^1 r^{\alpha-2}dr\|f\|_{C^{\alpha}_{P_m}(B_1)} \\
&= \frac{1}{1-\alpha}(d^{\alpha}-d)\|f\|_{C^{\alpha}_{P_m}(B_1)}.
\end{aligned}
\tag{2.45}
$$

Then (2.16) follows from (2.13), (2.14), (2.44) and (2.45). □

Chapter 3

The dual Minkowski problems in two dimensions

3.1 General

A Minkowski problem is to establish the necessary and sufficient conditions for a given finite Borel measure to arise as a measure generated by a convex body, which includes, for example, the surface area measure of a convex body in the classical Minkowski problem [2,53], the L_p surface area measure of a convex body in the L_p Minkowski problem [51], and the dual curvature measure in the dual Minkowski problem introduced recently by Huang et al. [37]. Analytically, the studying of a Minkowski problem is equivalent to studying a degenerate Monge-Ampère equation [54, 57, 63]. Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, and $u : \mathbb{S}^{n-1} \to (0, +\infty)$ be the unknown supporting function of a convex body. Denote by $e_{i,j}$ the standard Riemannian metric on $\mathbb{S}^{n-1}$, and g the density function of a finite Borel measure on $\mathbb{S}^{n-1}$, then the classical Minkowski problem is equivalent to the Monge-

Ampère equation

$$det(u_{ij} + e_{i,j}u) = g(v),\ v \in \mathbb{S}^{n-1},$$

which has led to a series of important works [9, 11, 15, 54, 57]. The L_p Minkowski problem given by Lutwak [51] is a natural L_p extension of the classical Minkowski problem. For a fixed $p \in \mathbb{R}$, if the given Borel measure on $\mathbb{S}^{n-1}$ has a density function g, then the L_p Minkowski problem can be formulated by a fully nonlinear partial differential equation on sphere $\mathbb{S}^{n-1}$, namely

$$u^{1-p} det(u_{ij} + e_{ij}u) = g(v), v \in \mathbb{S}^{n-1}. \tag{3.1}$$

This problem has attracted extensive attention, and many important results have been obtained, see [1,7,14,17,20,34–36,40–43,45,50–52,80–82] and their references.

Recently, Huang et al. [37] introduced the dual curvature measure and studied the existence for the dual Minkowski problem under the assumptions that $1 \leq k \leq n$ for the k-th dual curvature measure and that the given Borel measure is even. Moreover, if the given Borel measure has a density function $g : \mathbb{S}^{n-1} \to \mathbb{R}$, then the dual Minkowski problem relates to the following fully nonlinear elliptic equation

$$u(u^2 + |\nabla u|^2)^{\frac{k-n}{2}} det(u_{ij} + e_{ij}u) = g(v),\ v \in \mathbb{S}^{n-1}, \tag{3.2}$$

where $k \in \mathbb{R}$ and ∇u denotes the gradient vector of u respect to a frame on $\mathbb{S}^{n-1}$. It is clear that equation (3.2) with $k = n$ is the same as (3.1) with $p = 0$, which is related to the logarithmic Minkowski problem [7, 65, 80]. Li et al. [59] obtained the regularities of the dual Minkowski problem by a flow method. From [37] (see also [59]), equation (3.2) may be seen as the Euler-Lagrange equation with respect to the functional

$$\mathbb{F}(K) = \begin{cases} \int_{\mathbb{S}^{n-1}} g(v) \log u_K dv - \int_{\mathbb{S}^{n-1}} \log r_K(\xi) d\xi, \\ \int_{\mathbb{S}^{n-1}} g(v) \log u_K dv - \frac{1}{k} \int_{\mathbb{S}^{n-1}} r_K^k(\xi) d\xi, \ k \neq 0, \end{cases}$$

where (u_K, r_K) is the support function and radial function of convex body K.

In this paper, we study the existence of solutions to the 2-dimensional dual Minkowski problem for all $k > 1$ via studying the existence of positive solutions to (3.2) with $n = 2$. We deal with a more general case, by introducing a parameter $l \in [0, 1]$ as the coefficient of the term $|\nabla u|^2$ in equation (3.2) and then derive the following nonlinear problem

$$u''(\theta) + u(\theta) = g(\theta) u^{-1} (u^2 + l u'^2)^{(2-k)/2}, \ \theta \in \mathbb{S}, \tag{3.3}$$

where $g(\theta)$ is a continuous $2\pi/m$-periodic function with $m \in \mathbb{N}$, $k \in \mathbb{R}$, and $l \in [0, 1]$ is a parameter. The special form of (3.3) with $l = 0$ is equivalent to the 2-dimensional L_p Minkowski problem with $p = 2 - k$; while (3.3) with $l = 1$ relates to the 2-dimensional dual Minkowski problem. The nonlinear model (3.3) with $l = 0$ also appears in the generalized curve shortening problem, see [1,4,42] and the references therein.

Many authors have derived fruitful results for (3.3) with $l = 0$, that is, for the following equation

$$u''(\theta) + u(\theta) = \frac{g(\theta)}{u^{k-1}}, \ \theta \in \mathbb{S}. \tag{3.4}$$

For example, Andrews studied (3.4) with $g(\theta) \equiv 1$ for all real k in [4]. For

$g(\theta) \not\equiv constant$, many authors study (3.4) under some assumptions on $g(\theta)$ for different ranges of k, see [1,14,18,20,27,28,38,41,42,65,66,74,79]. Among those references, a priori estimate to the solution or the Blaschke-Santaló inequality plays essential roles for proving the existence of solutions for (3.4). Actually, one can get a priori estimate to the solution of (3.4) for all $k > 1$ when $g(\theta)$ is a positive $\pi/2$-periodic function [18]. A priori estimate to the π-periodic solution of (3.4) with $k = 4$ can also be obtained when the π-periodic C^2 function $g(\theta)$ is positive and *B-nondegenerate* in the sense that

$$\int_0^{\pi} \frac{g(\theta+t) - g(\theta) - 2^{-1}g'(\theta)\sin 2t}{\sin^2 t} dt \neq 0 \text{ at any critical points of } g(\theta),$$

which can be refered to [1] or [42,43] for the case of 2π-periodic solutions. To study (3.4) for a large range of k and a more general form of $g(\theta)$, the variational method was introduced [14,20,74,79]. Chen [14] obtained the solvability of (3.4) for $k \in [2,4]$ without imposing the traditional convexity condition by using the variational method and a generalized Blaschke-Santaló inequality such as

$$\left\{\int_{\mathbb{S}} u^2 - u'^2 d\theta\right\}\left\{\int_{\mathbb{S}} \frac{1}{u^2} d\theta\right\} \leq 4\pi^2$$

when $u \in H^1(\mathbb{S})$ and

$$\int_{\mathbb{S}} \frac{1}{u^3(\theta)} \sin\theta d\theta = 0 = \int_{\mathbb{S}} \frac{1}{u^3(\theta)} \cos\theta d\theta.$$

Based on the idea of Chen in [14], Dou and Zhu in [20] proved that (3.4) with $k \geq 4$ is solvable when the continuous fucntion g is positive at one point and is π/m-periodic for $m > 1$; Sun and Long [79] studied (3.4) with $k \geq 2$ when the $L^1(\mathbb{S})$ function g is nonnegative and is $2\pi/m$-periodic for $m > 2$.

The nonlinearity of (3.3) with $l \neq 0$ is more complex comparing to the one in (3.4). Existing methods in the literature are not applicable directly for finding a priori estimate to the solution of (3.3) when $l \neq 0$, and little about the variational functional of equation (3.3) is known for $l \neq 0$. These facts make it difficulty to study the solvablility of (3.3) when $l \neq 0$. Motivated by [43], we use Poincaré's map and the truncated technique to study the solvability of (3.3). Firstly, using Poincaré's map we prove the existence of periodic solutions to a general equation

$$u''(\theta) + u(\theta) = g(\theta) f(u, u') - \epsilon, \ \theta \in \mathbb{R}, \tag{3.5}$$

where $\epsilon > 0$, and $f(s,t)$ satisfies the global Lipschitz condition

$$|f(s_1,t_1) - f(s_2,t_2)| < L\sqrt{(s_1 - s_2)^2 + (t_1 - t_2)^2}, \text{ where } L \text{ is constant.} \tag{3.6}$$

We see that (3.5) with some initial value condition has a global solution under the Lipschitz assumption (3.6). Using Brouwer's fixed point theorem, with some further assumptions such as $g(\theta)$ is a T-periodic continuous function for $T < 2\pi$, we prove that problem (3.5) has a T-periodic solution. To construct a solution of (3.3) by using the solution of (3.5), we introduce a truncated function

$$\gamma_\epsilon(t) = \begin{cases} t + \epsilon, & t > -\epsilon/2, \\ \epsilon/2, & t \leq -\epsilon/2, \end{cases} \tag{3.7}$$

where $\epsilon > 0$ is a parameter. Suppose that the nonlinearity in (3.5) has the form of

$$f(u, u') = \frac{1}{\gamma_\epsilon(u)[\gamma_\epsilon^2(u) + l u'^2]^{\frac{k-2}{2}}},$$

which satisfies the assumption (3.6) for all $k > 1$ and $l \in [0, 1]$. So (3.5) with this special nonlinearity has a T-periodic solution u when g is a T-periodic continuous function for $T < 2\pi$. Let $\epsilon \to 0^+$ and $T = 2\pi/m$ with $m > 1$ in (3.5), if the solution u of (3.5) is great than $-\epsilon/2$ for all $\theta \in [0, 2\pi/m]$, then $u + \epsilon$ is a solution of (3.3). So, another difficulty in this paper is to get the estimation of solution such that $u(\theta) \geq -\epsilon/2$ for $\theta \in [0, 2\pi/m]$. In Section 3, by using the Green function and the eigenfunction of a differential operator we prove that the solution u of (3.5) with $f = f_\epsilon$ is great than $-\epsilon/2$ under some assumptions on k and $g(\theta)$.

The rest of the chapter is organized as follows. In Section 2, we establish the existence of T-periodic solutions of (3.5) when $T < 2\pi$. In Section 3, we give a sufficient condition for the solvability of (3.3).

3.2 T-periodic solutions of the truncated problem.

In this section, we study the periodic solution of problem (3.5) by using fixed point argument. We firstly introduce some notations. We denote by $\|\cdot\|_2$ the 2-norm of a vector in $\mathbb{R}^n$ or the 2-norm of a matrix. The 2-norm of a vector is the usual Euclidean norm in $\mathbb{R}^n$, and the 2-norm of a matrix B is the square root of the largest eigenvalue of the positive-semidefinite matrix B^*B, where B^* is the adjioint matrix of B.

Let $\epsilon > 0$ be given by (3.5), we rewrite (3.5) as the following system of ordinary differential equations

$$\frac{d}{d\theta}U = AU + G(\theta, U),\ \theta \in \mathbb{R}, \tag{3.8}$$

where

$$U := U(\theta) = \begin{pmatrix} u(\theta) \\ u'(\theta) \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } G(\theta, U) = \begin{pmatrix} 0 \\ g(\theta)f(u, u') - \epsilon \end{pmatrix}.$$

Since (3.5) is equivalent to system (3.8), in what follows, we study the existence of periodic solutions to (3.8). The main conclusion of this section is as follows.

Theorem 3.2.1. *Assume $g(\theta)$ is a T-periodic continuous function, $f(u, u')$ is the nonlinearity of (3.8) satisfying (3.6), and there exists a positive constant f_0 such that*

$$|f(s, t)| < f_0, \text{ for all } (s, t) \in \mathbb{R}^2. \tag{3.9}$$

If $T < 2\pi$, then problem (3.8) has at least one T-periodic solution

$$U(\theta) = \begin{pmatrix} u(\theta) \\ u'(\theta) \end{pmatrix}.$$

To prove this theorem, we firstly study the existence and properties of the global solution to an Cauchy's problem of system (3.8) by using the following two classical results from [33].

Theorem 3.2.2. *If $F(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz with respect to x. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, there exists an interval I containing t_0 and $\varphi : I \to \mathbb{R}^n$ satisfies the Cauchy's problem*

$$\begin{cases} \frac{dx}{dt} = F(t, x), \\ x(t_0) = x_0. \end{cases} \tag{3.10}$$

Moreover, φ is the unique solution on the interval I.

Theorem 3.2.3. *Assume that $F(t,x):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous and locally Lipschitz with respect to $x\in\mathbb{R}^n$. If there exist $R>0$ and $c:\mathbb{R}\to[0,+\infty)$ such that*

$$\|F(t,x)\|_2 \le c(t)\|x\|_2,$$

for $\|x\|_2 \ge R$. Then there exists a global solution $\varphi(t;x_0)$ of (3.10). Moreover, for any initial data $x_0\in\mathbb{R}^n$ and compact interval $[a,b]\subset\mathbb{R}$, the mapping $x_0\to\varphi(\cdot;x_0)\in C([a,b],\mathbb{R}^n)$ is continuous.

When one gets a global solution $U(\theta)$ to the Cauchy problem related to (3.8), one can use it to get a mapping from an initial value $U(0)$ to $U(T)$ by using the uniqueness. By the help of the continuous property and the estimates to the solution of the Cauchy problem, one get that the mapping from $U(0)$ to $U(T)$ is continuous and contracted. A fixed point argument can be used to establish the existence of periodic solution to system (3.8). This idea is due to Poincaré, and the argument above is somehow standard. For the convenience of the reader, we reproduce it entirely here.

The proof of Theorem 3.2.1. Under the assumption (3.6), by the definition of G we see that $H(\theta,U)=AU+G(\theta,U)$ is continuous with respect to $\theta\in\mathbb{R}$ and $U\in\mathbb{R}^2$. Let $\theta\in\mathbb{R}$, and

$$U_1=\begin{pmatrix} u_1(\theta)\\ u_1'(\theta)\end{pmatrix},\ U_2=\begin{pmatrix} u_2(\theta)\\ u_2'(\theta)\end{pmatrix}.$$

By(3.6), (3.9) and the definition of 2-norm we have

$$\begin{aligned}\|H(\theta,U_1)-H(\theta,U_2)\|_2 &\le \|A\|_2\|U_1-U_2\|_2+\|G(\theta,U_1)-G(\theta,U_2)\|_2\\ &= \|U_1-U_2\|_2+\left|g(\theta)[f(u_1,u_1')-f(u_2,u_2')]\right|\\ &\le \|U_1-U_2\|_2+L\sup_{\theta\in\mathbb{R}}|g(\theta)|\|U_1-U_2\|_2.\end{aligned}$$

It follows that H is also global Lipschitz with respect to $U \in \mathbb{R}^2$. Based on these facts, we can apply Theorems 3.2.2 and 3.2.3 to get the unique global solution $U(\theta) := U(\theta, V)$ of (3.8) with an initial data $U(0) = V \in \mathbb{R}^2$. Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be the unit matrix, for fixed $T \in (0, 2\pi)$ the solution $U(T, V)$ can be formulated as

$$U(T,V) = (E - B(T))^{-1} B(T) \int_0^T B^{-1}(\theta) G(\theta, U(\theta, V)) d\theta,$$

where

$$B(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is a matrix satisfying the Cauchy problem

$$\begin{cases} \frac{d}{d\theta} B(\theta) = AB(\theta), \\ B(0) = E. \end{cases}$$

By a simple calculation, we get that $\|B(\theta)\|_2 = \|B^{-1}(\theta)\|_2 = 1$ for all $\theta \in \mathbb{R}$,

$$E - B(\theta) = \begin{pmatrix} 1 - \cos\theta & -\sin\theta \\ \sin\theta & 1 - \cos\theta \end{pmatrix},$$

and $\|(E - B(\theta))^{-1}\|_2 = 1$ for all $\theta \in (0, 2\pi)$. Let $W(\theta) = U(\theta + T)$. If $U(T, V) = V$ we have $W(0) = U(T) = U(0) = V$. A direct calculation leads to

$$W'(\theta) = U'(\theta+T) = AU(\theta+T) + G(\theta+T, U(\theta+T)) = AW(\theta) + G(\theta, W(\theta)),$$

which means that $W(\theta)$ is also a solution of (3.8) with initial data $W(0) = V$. By the uniqueness of the solution to (3.8) via Theorem 3.2.2, we have

$W(\theta) = U(\theta)$ for all $\theta \in \mathbb{R}$, which means that $U(\theta, V)$ is a T-periodic solution of (3.8). Hence, we need to prove the existence of a special initial data $V_0 \in \mathbb{R}^2$ such that $U(T, V_0) = V_0$ in the following. For this, we define a mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\Phi(V) := U(T, V) = (E - B(T))^{-1} B(T) \int_0^T B^{-1}(\theta) G(\theta, U(\theta, V)) d\theta. \quad (3.11)$$

By Theorem 3.2.3 we know that the solution $U(\theta, V)$ of (3.8) is continuous with respect to the initial data $V \in \mathbb{R}^2$, that is

$$\max_{\theta \in [0,T]} \|U(\theta, V_i) - U(\theta, V)\|_2 \to 0 \text{ as } \|V_i - V\|_2 \to 0.$$

This together with the definitions of the matrix $B(\theta)$ and the mapping Φ in (3.11) show that the mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous. So (3.11) shows that

$$\|\Phi(V)\|_2 \le \left\|(E - B(T))^{-1} B(T)\right\|_2 \int_0^T \left\|B^{-1}(\theta) G(\theta, U(\theta, V))\right\|_2 d\theta. \quad (3.12)$$

By using the properties of the matrix $B(\theta)$ above and the assumption (3.9) in (3.12) we get

$$\|\Phi(V)\|_2 \le T \|G(\theta, U(\theta, V))\|_2 \le T(f_0 + \epsilon). \quad (3.13)$$

Let $R = T(f_0 + \epsilon)$ and $\bar{K}_R$ be the closure of a ball $K_R(0)$ in $\mathbb{R}^2$ with center at origin, it follows from (3.13) that the continuous mapping Φ is contracted on $\bar{K}_R$, that is $\Phi(\bar{K}_R) \subset \bar{K}_R$. According to Brouwer's fixed point theorem, we get that Φ has at least one fixed point $V_0 \in \bar{K}_R$. That is $\Phi(V_0) = V_0$. $\square$

3.3 Solvability of the Minkowski problem in two dimension

In this section, we study the solvability of (3.3) by using Theorem 3.2.1. In the following, we denote by ϵ a positive parameter, and frequently denote by C, C_1 fixed positive constants. The values of C and C_1 may vary, but always are independent of the parameter ϵ.

From (3.7), we see that $\gamma_\epsilon(t)$ is Lipschitz continuous, namly

$$|\gamma_\epsilon(t_1) - \gamma_\epsilon(t_2)| \leq |t_1 - t_2| \text{ for all } t_1, t_2 \in \mathbb{R}. \tag{3.14}$$

Let $l \in [0, 1]$, $k \in \mathbb{R}$, $\epsilon > 0$ be fixed, and

$$f_\epsilon(s,t) = \frac{1}{\gamma_\epsilon(s)[\gamma_\epsilon^2(s) + lt^2]^{(k-2)/2}}, \ (s,t) \in \mathbb{R}^2. \tag{3.15}$$

By (3.15) and the definition of γ_ϵ in (3.7) we see that $|f_\epsilon(s,t)| < 2^{k-1}/\epsilon^{k-1}$ for $k > 1$, hence f_ϵ satisfies (3.9) when $k > 1$. For any $s_1, s_2, t_1, t_2 \in \mathbb{R}$, a simple calculation shows that

$$\|(\gamma_\epsilon(s_1), t_1) - (\gamma_\epsilon(s_2), t_2)\|_2 \leq \sqrt{2}\|(s_1, t_1) - (s_2, t_2)\|_2. \tag{3.16}$$

Let $(s_1, t_1) \neq (s_2, t_2)$. If $k > 1$, by (3.15), (3.16) and through simple calculation, we get that

$$\begin{aligned}
&\frac{|f_\epsilon(s_1,t_1) - f_\epsilon(s_2,t_2)|}{\|(s_1,t_1) - (s_2,t_2)\|_2} \\
&= \frac{|f_\epsilon(s_1,t_1) - f_\epsilon(s_2,t_2)|}{\|(\gamma_\epsilon(s_1),t_1) - (\gamma_\epsilon(s_2),t_2)\|_2} \frac{\|(\gamma_\epsilon(s_1),t_1) - (\gamma_\epsilon(s_2),t_2)\|_2}{\|(s_1,t_1) - (s_2,t_2)\|_2} \\
&\leq \sqrt{2}\left(\frac{|f_\epsilon(s_1,t_1) - f_\epsilon(s_2,t_1)|}{|\gamma_\epsilon(s_1) - \gamma_\epsilon(s_2)|} + \frac{|f_\epsilon(s_2,t_1) - f_\epsilon(s_2,t_2)|}{|t_2 - t_1|}\right) \\
&\leq C/\epsilon^k,
\end{aligned}$$

which implies that f_ϵ satisfies (3.6) for $\epsilon > 0$. Hence we can apply Theorem 3.2.1 to get the following result.

Theorem 3.3.1. *For $\epsilon > 0$, let f_ϵ be given by (3.15) with $k > 1$, and let $g(\theta)$ be a continuous, T-periodic nonnegative function. If $T < 2\pi$, the following equation*

$$u''(\theta) + u(\theta) = g(\theta) f_\epsilon(u, u') - \epsilon, \;\; \theta \in \mathbb{R}. \tag{3.17}$$

has a T-periodic solution for all $l \in [0, 1]$.

In the following, let u be the T-periodic solution of (3.17) given by Theorem 3.3.1. If ε is small enough, we show that the T-periodic function $u + \varepsilon$ is a solution of (3.3) when $T = 2\pi/m$ for $m > 1$. For this aim, we need the following lemmas giving the properties of the solution to (3.17), and we also need two kinds of assumptions about $g(\theta)$ and $l \in [0, 1]$:

(I) $l = 0$ and the nontrivial function g is nonnegative.

(II) $l \in (0, 1]$ and g is positive.

Lemma 3.3.2. *Assume (I) or (II) holds. If $k > 1$ and ϵ is small enough, then u is positive at some point $\theta_0 \in [0, T]$.*

Proof. Otherwise $u(\theta) \leq 0$ for all $\theta \in \mathbb{R}$, then we have $\epsilon/2 \leq \gamma_\epsilon(u(\theta)) \leq \epsilon$ and hence

$$f_\epsilon(u, u') \geq \frac{1}{\epsilon(\epsilon^2 + l u'^2)^{(k-2)/2}} \geq \frac{1}{\epsilon^{k-1}}. \tag{3.18}$$

Under assumption (II) we have $\min\limits_{\theta \in [0,T]} g(\theta) > 0$. Let $u(\theta_1) = \max\limits_{\theta_1 \in [0,T]} u(\theta)$, then $u(\theta_1) \leq 0$, $u'(\theta_1) = 0$ and $u''(\theta_1) \leq 0$. If $k > 1$, from (3.17) and (3.18) we derive

$$0 \geq u(\theta_1) \geq g(\theta_1) f_\epsilon(u, u') - \epsilon \geq \min_{\theta \in [0,T]} g(\theta)/\epsilon^{k-1} - \epsilon \stackrel{\epsilon \to 0}{\longrightarrow} +\infty,$$

which leads to a contradiction when ϵ is small. Thus the conclusion is proved under assumption (II). Under assumption (I) we have $\int_0^T g(\theta)d\theta > 0$ and $l = 0$, if $u(\theta) \leq 0$ for all $\theta \in \mathbb{R}$, then we have

$$u''(\theta) + u(\theta) \geq \frac{g(\theta)}{\epsilon^{k-1}} - \epsilon. \tag{3.19}$$

By integrating both sides of the inequality (3.19) over the interval $[0, T]$ we get

$$0 \geq \int_0^T u(\theta)d\theta \geq \frac{1}{\epsilon^{k-1}} \int_0^T g(\theta)d\theta - T\epsilon.$$

If $k > 1$ and ϵ is small enough, we get a contradiction. $\square$

Lemma 3.3.3. *Assume (I) or (II) holds. Let $\alpha_0 \in (2\pi/3, 2\sqrt{2})$ be a constant. If $k > 1$and ϵ is small enough, then there exists an interval (a, b) with $b - a > \alpha_0$ such that $u(\theta) > 0$ for $\theta \in (a, b)$.*

Proof. If $u(\theta) > 0$ for all $\theta \in \mathbb{R}$, we get the conclusion. Otherwise, there exists a point $\theta_0 \in \mathbb{R}$ such that $u(\theta_0) \leq 0$. If the parameter ϵ is small, by applying Lemma 3.3.2 we get that $u(\theta)$ is positive at some points. Since u is a continuous periodic function, the set $\{\theta \in \mathbb{R} : u(\theta) > 0\}$ may consist of many subintervals (a_i, b_i) with $u(\theta) > 0$ for $\theta \in (a_i, b_i)$ and $u(a_i) = u(b_i) = 0$, $(i = 1, 2, 3, \ldots)$. On these intervals, we consider the following Dirichlet problem on a typical subintervals (a, b):

$$\begin{cases} u'' = h(\theta), \theta \in (a, b), \\ u(a) = u(b) = 0. \end{cases} \tag{3.20}$$

where $h(\theta) = g(\theta)f_\epsilon(u, u') - u - \epsilon$. By using the Green function we get a formula for the solution u of (3.20):

$$u(\theta) = \frac{-1}{b-a}\left[(b-\theta)\int_a^\theta (s-a)h(s)ds + (\theta - a)\int_\theta^b (b-s)h(s)ds\right]. \tag{3.21}$$

Since $g(\theta) \geq 0$, it is clear that $h(\theta) \geq -\max_{\theta\in[a,b]} u(\theta) - \varepsilon$ for $\theta \in [a,b]$. By applying this inequality in (3.21) we get

$$u(\theta) \leq \frac{(b-a)^2}{8}(\max_{\theta\in[a,b]} u(\theta) + \varepsilon) \text{ for all } \theta \in [a,b]. \tag{3.22}$$

If each branch of the set $\{\theta \in \mathbb{R} : u(\theta) > 0\}$ has length less than α_0, that is $b - a \leq \alpha_0 < 2\sqrt{2}$. From (3.22) we derive that $\max_{\theta\in[a,b]} u(\theta) < 8\varepsilon/(8-\alpha_0^2)$, which is an estimate to the positive part of u. Hence we get an estimate of u on $[0,T]$ as

$$\max_{\theta\in[0,T]} u(\theta) < \frac{8\varepsilon}{8-\alpha_0^2}. \tag{3.23}$$

Under the assumption (I), we can estimate the lower bound of $f_\epsilon(u,u')$ by using (3.23), that is

$$f_\epsilon(u(\theta), u'(\theta)) > \frac{C}{\epsilon^{k-1}},\ \theta \in [0,T].$$

By using this inequality in (3.17) we have

$$u''(\theta) + u(\theta) > \frac{Cg(\theta)}{\epsilon^{k-1}} - \epsilon,\ \theta \in [0,T]. \tag{3.24}$$

Then by integrating both sides of (3.24) over the interval $[0,T]$ and applying the estimation (3.23), we get

$$C_1\epsilon \geq \int_0^T u(\theta)d\theta > \frac{C}{\epsilon^{k-1}}\int_0^T g(\theta)d\theta - T\epsilon,$$

which is a contradiction when $k > 1$ and ϵ is small enough. Hence there exists an interval (a,b) such that $u(\theta) > 0$ for $\theta \in (a,b)$ and $b - a \geq \alpha_0$.

Under the assumption (II), we get $\min_{\theta\in[0,T]} g(\theta) > 0$. Let $u(\theta_0) = \max_{\theta\in[0,T]} u(\theta)$, then $u''(\theta_0) \leq 0$ and $u'(\theta_0) = 0$. We can still estimate the lower bound of $f_\epsilon(u(\theta_0), u'(\theta_0))$ by applying (3.23). Similarly, as for derivating (3.24), we get

$$u(\theta_0) > \frac{Cg(\theta_0)}{\epsilon^{k-1}} - \epsilon > \frac{C}{\epsilon^{k-1}}\min_{\theta\in[0,T]} g(\theta) - \epsilon,$$

which contradicts (3.23) when $k > 1$ and ϵ is small enough. $\square$

Now we give the conditions for the solvability of (3.3).

Theorem 3.3.4. *Assume (I) or (II) holds. Let $k > 1$ and $g(\theta)$ be a $2\pi/m$-periodic continuous function. If $m > 2$, then problem (3.3) with $l \in [0, 1]$ has a $2\pi/m$-periodic positive solution.*

Proof. Let $T = 2\pi/m$ in Theorem 3.3.1 with $m \in \mathbb{N}$, and let u be the periodic solution of (3.17) given by Theorem 3.3.1. If $m > 2$, then $2\pi/m \le 2\pi/3 < 2\sqrt{2}$. Let ϵ be small enough, it follows by Lemmas 3.3.2 and 3.3.3 that $u(\theta) > 0$ for all $\theta \in [0, 2\pi/m]$ when $m > 2$. Hence $u(\theta) > 0$ for all $\theta \in \mathbb{R}$ since u is $2\pi/m-$periodic. So, $\gamma_\epsilon(u) = u + \epsilon$ and

$$f_\epsilon(u, u') = \frac{1}{(u+\epsilon)[(u+\epsilon)^2 + lu'^2]^{\frac{k-2}{2}}}.$$

It follows that $u + \epsilon$ is a $2\pi/m$-periodic solution of (3.3) when $m > 2$ and ϵ is small enough. $\square$

To study the case $m = 2$ (i.e.$g(\theta)$ is π-periodic), we need more estimates as follows

Lemma 3.3.5. *Let $l = 0$, and let g be a positive continuous function. Assume $u(\theta) > 0$ for all $\theta \in (a, b)$ and $u(a) = u(b) = 0$. If $k > 1$, $b - a < \pi$ and ϵ is small enough, then*

$$\pi - (b - a) \le C\epsilon^{\frac{k}{k-1}}. \tag{3.25}$$

Proof. Let $\varphi(\theta) = \sin(\frac{\theta-a}{b-a}\pi)$. Multipling (3.17) with φ and then integrating both sides of the equation over interval $[a, b]$, we have

$$\int_a^b \left(\frac{g(\theta)}{(\epsilon+u)^{k-1}} - \epsilon + \left(\frac{\pi}{b-a}\right)^2 u - u \right) \varphi d\theta = 0. \tag{3.26}$$

By simple calculation we have

$$\begin{aligned}F(u+\epsilon) &:= \frac{g(\theta)}{(\epsilon+u)^{k-1}} - \epsilon + \left(\frac{\pi}{b-a}\right)^2 u - u \\ &= \frac{g(\theta)}{(\epsilon+u)^{k-1}} + \left[\left(\frac{\pi}{b-a}\right)^2 - 1\right](\epsilon+u) - \left(\frac{\pi}{b-a}\right)^2 \epsilon \\ &\geq \frac{\min_{\theta\in[0,T]} g(\theta)}{(\epsilon+u)^{k-1}} + \left[\left(\frac{\pi}{b-a}\right)^2 - 1\right](\epsilon+u) - \left(\frac{\pi}{b-a}\right)^2 \epsilon.\end{aligned}$$

To estimate F, we consider function $h(t) = Dt^{-(k-1)} + Mt$ for $t > 0$, where $D = \min_{\theta\in[0,T]} g(\theta) > 0$ and $M = \pi^2/(b-a)^2 - 1 > 0$ since $b - a < \pi$. Then $h(t) \geq h(t_0) = C_0 D^{1/k} M^{(k-1)/k}$ for all $t \geq 0$, where $t_0 = (k-1)^{1/k} D^{1/k}/M^{1/k}$ and $C_0 = k(k-1)^{1/k-1}$. Hence,

$$F(u+\epsilon) \geq k(k-1)^{\frac{1}{k}-1} D^{\frac{1}{k}} M^{\frac{k-1}{k}} - \left(\frac{\pi}{b-a}\right)^2 \epsilon.$$

This and (3.26) imply that

$$k(k-1)^{\frac{1}{k}-1} \min_{\theta\in[0,T]} g^{\frac{1}{k}} \left(\frac{\pi^2}{(b-a)^2} - 1\right)^{\frac{k-1}{k}} - \left(\frac{\pi}{b-a}\right)^2 \epsilon \leq 0.$$

By simplifying this inequality we get (3.25). □

Theorem 3.3.6. *Let $l = 0$, and $g(\theta)$ be a π-periodic positive continuous function. If $k \in (1,3)$, then problem (3.3) has a π-periodic positive solution.*

Proof. Let $T = \pi$ in Theorem 3.3.1, and u be the periodic solution of (3.17) given by Theorem 3.3.1. If $u(\theta) \geq -\epsilon/2$ for all $[0,\pi]$, then $\gamma_\epsilon(u(\theta)) = u(\theta) + \epsilon$ for all $\theta \in \mathbb{R}$ since u is π-periodic. We get that $u + \epsilon$ is a positive solution of problem (3.3). Otherwise $u(\theta_0) < -\epsilon/2$ for some $\theta_0 \in [0,\pi]$, and we can derive a contradiction for small ϵ as follows. Lemma 3.3.2 shows that $\max_{\theta\in[0,\pi]} u(\theta) > 0$ for small ϵ. Hence we have two intervals (a,b) and (b,c) such that $c - a \leq \pi$, $u(a) = u(b) = u(c) = 0$, $u(\theta) > 0$ for $\theta \in (a,b)$ and

$u(\theta) < 0$ for $\theta \in (b, c)$. Without loss of generality, we choose interval (b, c) such that $\theta_0 \in (b, c)$. Then we have

$$\max_{\theta \in (a,b)} |u(\theta)| \geq |u(\theta_0)| > \frac{\epsilon}{2}. \tag{3.27}$$

Let ϵ be small enough, by using Lemma 3.3.5 we have $\pi - (b-a) < C\epsilon^{k/(k-1)}$. Hence,

$$c - b \leq \pi - (b - a) < C\epsilon^{k/(k-1)}. \tag{3.28}$$

This inequality implies that $c - b \to 0$ when $\epsilon \to 0^+$. If ϵ is small, then the interval (b, c) can be estimated by the small quantity $\epsilon^{k/(k-1)}$. To estimate the bound of $c - b$ from below, we consider the Dirichlet problem

$$\begin{cases} u''(\theta) = g(\theta) f_\epsilon(u, u') - u - \epsilon, \theta \in (b, c), \\ u(b) = u(c) = 0. \end{cases} \tag{3.29}$$

Since $u(\theta) < 0$ for all $\theta \in (b, c)$, for small ϵ we have

$$|g(\theta) f_\epsilon(u, u') - u - \epsilon| < C/\epsilon^{k-1} + \max_{\theta \in [b,c]} |u(\theta)|. \tag{3.30}$$

Using (3.30) and the Green function we can estimate the solution u of (3.29) by

$$\begin{aligned} |u(\theta)| &\leq \frac{C/\epsilon^{k-1} + \max_{\theta \in [b,c]} |u(\theta)|}{c - b} \left| (c - \theta) \int_b^\theta (s - b) ds + (\theta - b) \int_\theta^c (c - s) ds \right| \\ &= \frac{(c - \theta)(\theta - b)}{2} \left(C/\epsilon^{k-1} + \max_{\theta \in [b,c]} |u(\theta)| \right) \\ &\leq \frac{(c - b)^2}{8} \left(C/\epsilon^{k-1} + \max_{\theta \in [b,c]} |u(\theta)| \right), \quad \theta \in (b, c). \end{aligned}$$

If ϵ is small enough, we get $c - b < 2$ by (3.28), hence $(c - b)^2/[8 - (c - b)^2] > 0$. It follows that

$$\max_{\theta \in (b,c)} |u(\theta)| \leq \frac{(c - b)^2}{8 - (c - b)^2} \frac{C}{\epsilon^{k-1}}.$$

This and (3.27) show that $c - b > C_1 \epsilon^{\frac{k}{2}}$. If $k < 3$, then it contradicts (3.28) when the parameter ϵ is small. $\square$

From Theorems 3.3.4 and 3.3.6, we get the main conclusion of this chapter as following:

Theorem 3.3.7. *Let $k \in \mathbb{R}$ and $l \in [0, 1]$. Assume that $g(\theta)$ is a $2\pi/m$-periodic continuous function with $m \in \mathbb{N}$. If $g(\theta)$ is positive and $m > 2$, then problem (3.3) with $l \in [0, 1]$ is solvable for all $k > 1$. If $g(\theta)$ is positive and $m = 2$, then problem (3.3) with $l = 0$ is solvable for all $k \in (1, 3)$. If the nontrivial function $g(\theta)$ is nonnegative and $m > 2$, then problem (3.3) with $l = 0$ is solvable for all $k > 1$.*

By Theorem 3.3.7, we see that (3.2) with $n = 2$ is solvable for all $k > 1$ when the positive continuous function $g(\theta)$ is $2\pi/m$-periodic for $m > 2$. Obviously, Theorem 3.3.7 gives a sufficient condition on the existence of solutions to equation (3.3) and consequently the solution to the dual Minkowski problem for all $k > 1$ in 2-dimensions. For the case $m = 3$, the density function g of a given measure is not π-periodic, and hence not even. In this sense, we get a more general sufficient condition in Theorem 3.3.7 for the existence of solution to the 2-dimensional dual Minkowski problem than the one in [37] which solves the problem for the case of even measure and $k \in [1, 2]$. Moreover, we give a new method to study the solvability of (3.4) for all $k > 1$, which needs neither a priori estimate to the solution of (3.4) required in [1, 18, 42, 42], nor the Blaschke-Santaló inequality.

Chapter 4

An application of variational method to a L_p Minkowski problem

4.1 General

Lutwak showed in [51] that for each Firey's p-sum ($p \leq -1$) there is an associated Brunn-Minkowski theory. The existence and uniqueness of the generalized Minkowski problem was also obtained in [23, 51] in the case of even positive functions on $\mathcal{S}^n$. Later, Lutwak and Oliker in [52] established the regularity of the solutions for this case. The corresponding Monge-Ampère equation for the L_p Minkowski problem related to Firey's p-sum is:

$$\det(u_{ij} + u\delta_{ij}) = u^{-(p+1)} f, \tag{4.1}$$

where $f(x)$ is a positive function on the unit sphere $\mathcal{S}^n$, u is the supporting function of a convex body, u_{ij} is the convariant differentiation of u on $\mathcal{S}^n$.

By using the theories of partial differential equations, the existence and regularity of the L_p Minkowski problem was studied by Guan and Lin [55] and Chou and Wang [17]. Guan-Lin [55] derived a solution for the case of $p > n$. Chou-Wang [17] solved the problem for $p > n$ as well as for polytopes when $p < -1$. For the case $p = -1$, the problem becomes the well-known classical Minkowski problem dealing with existence, uniqueness, regularity and stability of closed convex hyper-surfaces. The major contributions to this problem are due to Minkowski, Aleksandrov, Fenchel, Jessen and Lewy (see [9,11,13,15,54,56,57,62]).

In this chapter, we consider an application of variational method to the solvability of the L_p Minkowski problem in two dimensions, that is the following equation [14,74].

$$u'' + u = g(x)u^{-(1+p)},\ x \in \mathbb{R}. \tag{4.2}$$

where $p \geq 0$ and $g(x)$ is a $2\pi-$periodic positive function. Equation (4.2) is also used to study the generalized curve shortening problem [17,42]. For $g(x) \equiv 1$, Andrews solved the problem for all real p in [4]. For $g(x) \not\equiv 1$, the existence of solutions of problem (4.2) has been studied under different assumptions on g and the ranges of p. For the case $p = 0$, Gage and Li [27,28] studied this problem when g is smooth data; Stancu solved this case when $g(x)$ is a discrete measure. Ai et al. got a $\pi-$ periodic solution of problem (4.2) under the assumption that $g(x)$ is a $\pi-$periodic and $B-$nondegenerate function [1]. Jiang obtained a $2\pi-$ periodic solution of problem (4.2) under $B-$nondegenerate assumption on $2\pi-$ periodic data $g(x)$ [42].

The variational method has been applied to study (4.2) for more general assumptions on $g(x)$ and larger range of p. For example, Umanskiy studied (4.2) when $g(x)$ is a periodic function of period $T < 1$ in [74] for all $p \notin \{0, 2\}$. For $p \in [0, 2)$, Chen showed that (4.2) has a periodic solution when the periodic function $g(x)$ is positive at some points. Dou and Zhu in [20] showed that (4.2) has a $\pi/2$ periodic solution if $p \geq 2$ and the $\pi/2-$ periodic continuous function g is positive at one point; more recently, Sun and Long [79] studied (4.2) when the nonnegative function g is integrable over $\mathbb{S}$. The key point of solving (4.2) mentioned-above is to deal with the problem caused by the nonlinearity $u^{-(1+p)}$ which is singular when $u \to 0^+$.

By the exponential mapping transformation, problem (4.2) becomes

$$-w'' + w'^2 + 1 = \frac{g(\theta)e^{(p+2)w}}{\int_{\mathbb{S}} g(\theta)e^{pw}d\theta}. \tag{4.3}$$

Problem (4.3) is a nonlinear elliptic problem with a gradient term. Let $\mathbb{S}$ be the unit circle parameterized by the angle θ. Let $H^1(\mathbb{S})$ be the Hilbert space equipped with the usual norm

$$\|w\| = \left\{ \int_{\mathbb{S}} w'^2(\theta) + w^2(\theta)d\theta \right\}^{1/2}.$$

In the following, we replace $\int_{\mathbb{S}}$ by $\int$ for simplicity. If $g(\theta)$ is a nonnegative

$L^1(\mathbb{S})$ function, the functional of (4.3) is well defined in $H^1(\mathbb{S})$ as

$$I(w) = \frac{1}{2}\int e^{-2w(\theta)}d\theta - \frac{1}{2}\int e^{-2w(\theta)}w'^2(\theta)d\theta + \frac{1}{p}\ln\left(\int g(\theta)e^{pw(\theta)}d\theta\right), w(\theta) \in H^1(\mathbb{S}). \tag{4.4}$$

To the authors' knowledge, it seems that (4.4) is a new functional. In this chapter, we firstly study the solution of (4.3) by the critical points of function I. Then using the solution of (4.3) to construct the solution of (4.2).

4.2 Preliminaries

In this section, we give two lemmas, which are the basis in the following proofs for our theorems. Let

$$G(w) = \int e^{-2w(\theta)}d\theta - \int e^{-2w(\theta)}w'^2(\theta)d\theta - 1,\ w(\theta) \in H^1(\mathbb{S}). \tag{4.5}$$

Then I in (4.4) and G in (4.5) are Fréchet differentiable which are listed as follows.

Lemma 4.2.1. *Let $g(\theta)$ be a nonnegative $L^1(\mathbb{S})$ function, then the functionals $I(w)$ and $G(w)$ are C^1 in $H^1(\mathbb{S})$, and*

$$I'(w)\varphi = \int e^{-2w}w'^2\varphi d\theta - \int e^{-2w}w'\varphi' d\theta - \int e^{-2w}\varphi d\theta + \left(\int g(\theta)e^{pw}d\theta\right)^{-1}\int g(\theta)e^{pw}\varphi d\theta,\ \textit{for all } w, \varphi \in H^1(\mathbb{S}). \tag{4.6}$$

Remark 4.2.2. *Moreover, if $w \in C^2(\mathbb{S})$ we have*

$$I'(w)\varphi = \int e^{-2w}(w'' - w'^2 - 1)\varphi d\theta + \left(\int g(\theta)e^{pw}d\theta\right)^{-1}\int g(\theta)e^{pw}\varphi d\theta,\ \textit{for all } \varphi \in H^1(\mathbb{S}). \tag{4.7}$$

Proof. We first show that I is Fréchet differentiable in $H^1(\mathbb{S})$ and then prove that $I'(w)$ is continuous. For any $w, \varphi \in H^1(\mathbb{S})$, let

$$J(w,\varphi) = \int e^{-2w} w'^2 \varphi d\theta - \int e^{-2w} w'\varphi' d\theta - \int e^{-2w}\varphi d\theta + \frac{\int g(\theta) e^{pw}\varphi d\theta}{\int g(\theta) e^{pw} d\theta}.$$

We claim: for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, w)$ such that

$$|I(w+\varphi) - I(w) - J(w,\varphi)| \leq \varepsilon \|\varphi\|, \tag{4.8}$$

provided that $\|\varphi\| < \delta$. Therefore I is Fréchet differentiable and $I'(w)\varphi = J(w,\varphi)$, which shows that (4.6) is true. By calculating, we have

$$\begin{aligned} I(w+\varphi) - I(w) - J(w,\varphi) &= \underbrace{\frac{1}{2}\int (e^{-2(w+\varphi)} - e^{-2w}) d\theta + \int e^{-2w}\varphi d\theta}_{(I)} \\ &+ \underbrace{\frac{1}{2}\int (-e^{-2(w+\varphi)}(w'+\varphi')^2 + e^{-2w}w'^2) d\theta - \int e^{-2w}(w'^2\varphi - w'\varphi') d\theta}_{(II)} \\ &+ \underbrace{\frac{1}{p}\ln\left(\int g(\theta) e^{p(w+\varphi)} d\theta\right) - \frac{1}{p}\ln\left(\int g(\theta) e^{pw} d\theta\right) - \frac{\int g(\theta) e^{pw}\varphi d\theta}{\int g(\theta) e^{pw} d\theta}}_{(III)}. \end{aligned} \tag{4.9}$$

To estimate the left term of (4.8), we also need the Sobolev embedding, i.e.

$$|w|_\infty \leq c_1 \|w\| \text{ for any } w \in H^1(\mathbb{S}). \tag{4.10}$$

Now, using (4.10) we estimate (I), (II) and (III) respectively as follows,

$$\begin{aligned} |(I)| &= \left| \frac{1}{2}\int (e^{-2(w+\varphi)} - e^{-2w}) d\theta + \int e^{-2w}\varphi d\theta \right| \\ &= 2\left| \int (e^{-2(w+\eta_1(\theta)\eta_2(\theta)\varphi)} \eta_1(\theta)\varphi^2 d\theta \right|, \text{where } 0 < \eta_1, \eta_2 < 1 \\ &\leq 2e^{2(|w|_\infty + |\varphi|_\infty)} |\varphi|_2^2 \leq 2e^{2c_1(\|w\| + \|\varphi\|)} \|\varphi\|^2. \end{aligned} \tag{4.11}$$

$$
\begin{aligned}
(II) &= \frac{1}{2}\int(-e^{-2(w+\varphi)}(w'+\varphi')^2+e^{-2w}w'^2)d\theta-\int e^{-2w}(w'^2\varphi-w'\varphi')d\theta\\
&= \int(e^{-2(w+\eta_3(\theta)\varphi)}(w'+\varphi')^2\varphi d\theta-\int e^{-2w}(w'+\eta_4(\theta)\varphi')\varphi'd\theta\\
&+\int w^{-2w}(w'\varphi'-w'^2\varphi)d\theta\\
&= 2\int(e^{-2(w+\eta_3(\theta)\varphi)}(w'+\eta_5(\theta)\varphi')\varphi'\varphi d\theta-\int e^{-2w}\eta_4(\theta)\varphi'^2d\theta\\
&-2\int(e^{-2(w+\eta_3(\theta)\eta_6(\theta)\varphi)}w'^2\eta_3(\theta)\varphi\varphi'd\theta,
\end{aligned}
$$

$$
|(II)|\le\left(e^{2c_1(\|w\|+\|\varphi\|)}(1+c_1\|w\|^2)+e^{2c_1\|w\|}\right)\|\varphi\|^2. \tag{4.12}
$$

$$
\begin{aligned}
(III) &= \frac{1}{p}\ln\left[\frac{\int ge^{p(w+\varphi)}d\theta-\int ge^{pw}d\theta}{\int ge^{pw}d\theta}+1\right]\\
&-\left\{\int ge^{pw}d\theta\right\}^{-1}\int ge^{pw}\varphi d\theta\\
&= \frac{1}{p}\ln\left[\frac{p\int ge^{p(w+\eta_7(\theta)\varphi)}\varphi d\theta}{\int ge^{pw}d\theta}+1\right]-\left\{\int ge^{pw}d\theta\right\}^{-1}\int ge^{pw}\varphi d\theta.
\end{aligned}
$$

Note that $\frac{\ln(1+x)}{x}=1-x/2+o(x)$ as $x\to 0$ and

$$
\left|\int ge^{p(w+\eta_7(\theta)\varphi)}\varphi d\theta\right|\le c_1\|\varphi\|e^{pc_1(\|w\|+\|\varphi\|)}\int g(\theta)d\theta.
$$

As $\|\varphi\| \to 0$, we have

$$
\begin{aligned}
(III) &= \frac{\int g e^{p(w+\eta_7(\theta)\varphi)}\varphi d\theta}{\int g e^{pw} d\theta} - (\frac{1}{2}+o(1))\left\{\frac{\int g e^{p(w+\eta_7(\theta)\varphi)}\varphi d\theta}{\int g e^{pw} d\theta}\right\}^2 \\
&- \left\{\int g e^{pw} d\theta\right\}^{-1}\int g e^{pw}\varphi d\theta \\
&= \frac{p\int g e^{p(w+\eta_7(\theta)\eta_8(\theta)\varphi)}\eta_7(\theta)\varphi^2 d\theta}{\int g e^{pw} d\theta} \\
&- (\frac{1}{2}+o(1))\left\{\frac{\int g e^{p(w+\eta_7(\theta)\varphi)}\varphi d\theta}{\int g e^{pw} d\theta}\right\}^2,
\end{aligned}
$$

hence

$$
|(III)| \le c_1^2 p e^{pc_1(\|w\|+\|\varphi\|)}\frac{\int g e^{pw} d\theta}{\int g d\theta}\left(1 + c_1^2 p e^{pc_1(\|w\|+\|\varphi\|)}\frac{\int g e^{pw} d\theta}{\int g d\theta}\right)\|\varphi\|^2. \tag{4.13}
$$

Let

$$
\begin{aligned}
c(g,w) &= 2e^{2c_1(\|w\|+1)} + e^{2c_1(\|w\|+1)}(1+c_1\|w\|^2) + e^{2c_1\|w\|} \\
&+ c_1^2 p e^{pc_1(\|w\|+1)}\frac{\int g e^{pw} d\theta}{\int g d\theta}\left(1 + c_1^2 p e^{pc_1(\|w\|+1)}\frac{\int g e^{pw} d\theta}{\int g d\theta}\right)
\end{aligned}
$$

and $\delta(\varepsilon, w) = \min(1, \frac{\varepsilon}{3c(g,w)}, \varepsilon)$. Combing (4.10)-(4.13) we have (4.8).

To prove that $I'(w)$ is continuous, we let $w_n \to w$ in $H^1(\mathbb{S})$. Then $w_n \to w$ in $C^\alpha(\mathbb{S})$ for $\alpha < 1/2$, and $|e^{-2w_n} - e^{-2w}|_\infty \to 0$. Hence, it is easy to see that

$$
\sup_{\|\varphi\|\le 1}\left|\int e^{-2w_n}\varphi - e^{-2w}\varphi d\theta\right| \le c|e^{-2w_n} - e^{-2w}|_\infty = o(1),
$$

$$
\sup_{\|\varphi\|\le 1}\left|\frac{\int g(\theta)e^{pw_n}\varphi d\theta}{\int g(\theta)e^{pw_n} d\theta} - \frac{\int g(\theta)e^{pw}\varphi d\theta}{\int g(\theta)e^{pw} d\theta}\right| = o(1).
$$

Moreover, we have

$$
\begin{aligned}
&\left|\int e^{-2w_n}w_n'^2\varphi - e^{-2w}w'^2\varphi d\theta\right| + \left|\int e^{-2w_n}w_n'\varphi' - e^{-2w}w'\varphi' d\theta\right| \\
&\le c_1\|\varphi\|e^{c_1\|w_n\|}\|w_n - w\|^2 + c_1\|\varphi\|\|w\|^2\left|e^{-2w_n} - e^{-2w}\right|_\infty \\
&+ \|\varphi\|e^{c_1\|w\|}\|w_n - w\| + \|\varphi\|\|w_n\|\left|e^{-2w_n} - e^{-2w}\right|_\infty
\end{aligned}
$$

Combing those inequalities, if $w_n \to w$ in $H^1(\mathbb{S})$, we have

$$\begin{aligned}
&\|I'(w_n) - I'(w)\| = \sup_{\|\varphi\|\le 1} \left|I'(w_n)\varphi - I'(w)\varphi\right| \\
&\le \sup_{\|\varphi\|\le 1} \left|\int e^{-2w_n} w_n'^2 \varphi - e^{-2w} w'^2 \varphi d\theta\right| \\
&+ \sup_{\|\varphi\|\le 1} \left|\int e^{-2w_n} w_n' \varphi' - e^{-2w} w' \varphi' d\theta\right| + \sup_{\|\varphi\|\le 1} \left|\int e^{-2w_n} \varphi - e^{-2w} \varphi d\theta\right| \\
&+ \sup_{\|\varphi\|\le 1} \left|\frac{\int g(\theta) e^{pw_n} \varphi d\theta}{\int g(\theta) e^{pw_n} d\theta} - \frac{\int g(\theta) e^{pw} \varphi d\theta}{\int g(\theta) e^{pw} d\theta}\right| \\
&= o(1).
\end{aligned}$$

The Fréchet differentiable of functional $G(w)$ is similar as $I(w)$. □

Theorem 4.2.3. *The Lagerange Theorem: Let X be a Banach space, X^* be the dual space of X and $J_1, J_2 : X \to \mathbb{R}$ be functionals of the class $C^1(X)$. Let $w \in X$ be a point of a minimum of the function J_1 under the condition $J_2 = c_0$, where c_0 is a constant, that is*

$$J_1(w) = \min_{v \in X, J_1(v) = c_0} J_1(v).$$

Then there exist constants λ_1, λ_2 which are not simultaneously equal to zero, and

$$\lambda_1 J_1'(w) + \lambda_2 J_2'(w) = 0, \text{ in } X^*.$$

4.3 The variational frame and existence of solution

The following are main results of this chapter.

Theorem 4.3.1. *Assume that $g(x)$ is $\frac{\pi}{k}$ periodic($k > 1$), continuous nonnegative function. Then problem (4.2) has solution for all $p > 0$.*

To prove Theorem 4.3.1 by using variational method, we let $k \in \mathbb{N}$, γ be a constant and

$$M_{\gamma,k} = \left\{ w \in H^1(\mathbb{S}) | \int_{\mathbb{S}} e^{-2w} - e^{-2w} w'^2(\theta) d\theta = \gamma, w(\theta + \frac{\pi}{k}) = w(\theta) \right\}.$$

It is clear to see that $M_{\gamma,k} \neq \emptyset$ for $\gamma > 0$. In this section we find the minimum of the functional $-I$ on $M_{1,k}$. Let $\{u_n\} \subset M_{1,k}$ satisfy that

$$\begin{aligned} c &= \lim_{n\to+\infty} -I(u_n) = \inf_{u\in M_{1,k}} -I(u) \\ &= -\frac{1}{2} - \frac{1}{p} \sup_{u\in M_{1,k}} \ln\left(\int g e^{pw(\theta)} d\theta\right) < +\infty, \end{aligned} \tag{4.14}$$

where c could be $-\infty$. In the following we show that $\{u_n\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{S})$ and $c > -\infty$ when $k > 1$. We firstly give a priori estimates of the minimum and maximum of those u_n on $\mathbb{S}$ as

Lemma 4.3.2. *Let $k \in \mathbb{N}$. There exists a constant c_0 such that*

$$\max_{\theta\in\mathbb{S}} u_n(\theta) \geq c_0 \text{ and } \min_{\theta\in\mathbb{S}} u_n(\theta) \leq \frac{\ln 2\pi}{2}, \text{ for all } n \in \mathbb{N}. \tag{4.15}$$

Proof. If $\max_{\theta\in\mathbb{S}} u_n(\theta) \to -\infty$, we have $\int g(\theta)e^{pw_n(\theta)}d\theta \to 0$, hence $c = +\infty$ which contradicts with the definition of c in (4.14). This shows that the first part of (4.15) is true. For any $u \in M_{1,k}$ with $k \in \mathbb{N}$ we have

$$2\pi e^{-2\min_{\theta\in\mathbb{S}} u(\theta)} \geq \int_{\mathbb{S}} e^{-2w(\theta)}d\theta > \int_{\mathbb{S}} e^{-2w(\theta)} - e^{-2w(\theta)}w'^2(\theta)d\theta = 1.$$

It follows that $\min_{\theta\in\mathbb{S}} u(\theta) < \frac{\ln 2\pi}{2}$, which shows that (4.15) is true for $u_n \in M_{1,k}$. □

Furthermore we can get the lower bound for $\{\min_{\theta\in\mathbb{S}} u_n(\theta)\}_{n=1}^{\infty}$.

Lemma 4.3.3. *If $k > 1$, there exists $c_1 > -\infty$ such that*

$$\min_{\theta\in\mathbb{S}} u_n(\theta) \geq c_1, \text{ for all } n \in \mathbb{N}. \tag{4.16}$$

Proof. We apply an approximate process to prove this lemma. Since $C^1(\mathbb{S})$ is dense in $H^1(\mathbb{S})$ and $G(u)$ in Lemma 4.2.1 is $C^1(H^1(\mathbb{S}))$ functional, there exist a sequence of $C^1(\mathbb{S})$ functions $w_n \in M_{\gamma_n,k}$ satisfying

$$\lim_{n\to+\infty} |\gamma_n - 1| + \|u_n - w_n\| = 0 \tag{4.17}$$

Let $w_n(\alpha_n) = \min_{\theta\in\mathbb{S}} w_n(\theta)$ and $w_n(\beta_n) = \max_{\theta\in\mathbb{S}} w_n(\theta)$. Then

$$\int_{\alpha_n}^{\alpha_n+\frac{\pi}{k}} e^{-2w_n(\theta)} - e^{-2w_n(\theta)} w_n'^2(\theta) d\theta = \frac{\gamma_n}{2k}. \tag{4.18}$$

Without loss of generality, we assume that $\alpha_n < \beta_n < \alpha_n + \frac{\pi}{k}$. By Hölder inequality and simple calculation we get

$$|e^{-w_n(\alpha_n)} - e^{-w_n(\beta_n)}|^2 = \left|\int_{\alpha_n}^{\beta_n} (e^{-w_n})' d\theta\right|^2 \leq |\alpha_n - \beta_n| \int_{\alpha_n}^{\beta_n} e^{-2w_n} w_n'^2 d\theta, \tag{4.19}$$

and

$$\begin{aligned} |e^{-w_n(\alpha_n)} - e^{-w_n(\beta_n)}|^2 &= |e^{-w_n(\alpha_n+\frac{\pi}{k})} - e^{-w_n(\beta_n)}|^2 \\ &\leq \left|\alpha_n + \frac{\pi}{k} - \beta_n\right| \int_{\beta_n}^{\alpha_n+\frac{\pi}{k}} e^{-2w_n} w_n'^2 d\theta. \end{aligned} \tag{4.20}$$

Without loss of generality, let $\beta_n - \alpha_n = \frac{\tau_n \pi}{k}$ where $\tau_n \in (0, \frac{1}{2}]$ are parameters, then $\alpha_n + \frac{\pi}{k} - \beta_n = \frac{(1-\tau_n)\pi}{k}$. From (4.18)-(4.20) we get the following two estimates

$$|e^{-w_n(\alpha_n)} - e^{-w_n(\beta_n)}|^2 \leq \tau_n \frac{\pi^2}{k^2} \left(e^{-2w_n(\alpha_n)} - \frac{\gamma_n}{2\pi}\right),$$

$$\begin{aligned} |e^{-w_n(\alpha_n)} - e^{-w_n(\beta_n)}|^2 &\leq \max\{\frac{\tau_n\pi}{2k}, \frac{(1-\tau_n)\pi}{2k}\} \int_{\alpha_n}^{\alpha_n+\frac{\pi}{k}} e^{-2w_n} w_n'^2 d\theta \\ &= \frac{(1-\tau_n)\pi}{2k} \left(\int_{\alpha_n}^{\alpha_n+\frac{\pi}{k}} e^{-2w_n} d\theta - \frac{\gamma_n}{2k}\right) \\ &\leq \frac{1-\tau_n}{2} \frac{\pi^2}{k^2} \left(e^{-2w_n(\alpha_n)} - \frac{\gamma_n}{2\pi}\right). \end{aligned}$$

We see that $\tau_n < \frac{1-\tau_n}{2}$ for $\tau_n \in (0, \frac{1}{3})$ and $\tau_n \geq \frac{1-\tau_n}{2}$ for $\tau_n \in [\frac{1}{3}, \frac{1}{2}]$. Hence ,

$$|e^{-w_n(\alpha_n)} - e^{-w_n(\beta_n)}|^2 \leq \frac{\pi^2}{3k^2} \left(e^{-2w_n(\alpha_n)} - \frac{\gamma_n}{2\pi}\right). \tag{4.21}$$

By using the Sobolev inequality together with (4.15) and (4.17) we see that the set $\{w_n(\beta_n)\}_{n=1}^\infty$ is bounded from below and $\{\gamma_n\}$ is a bounded sequence. If $\{w_n(\alpha_n)\}$ is unbounded, by multiplying inequality (4.21) with

$e^{2w_n(\alpha_n)}$ then setting $n \to +\infty$, we get $e^{w_n(\alpha_n)-w(\beta_n)} = o(1)$, hence

$$|1-o(1)|^2 \leq \frac{\pi^2}{3k^2}\left(1-o(1)\right),$$

which contradicts with $k > 1$. So $\{w_n(\alpha_n)\}$ is unbounded from below. This together with (4.17) and the Sobolev inequality gives (4.16) □

Now we prove that $\{u_n\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{S})$.

Lemma 4.3.4. *If $k > 1$, then $\{u_n\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{S})$.*

Proof. Since $\{u_n\}_{n=1}^{\infty} \subset M_{1,k}$, by applying the reverse Hölder inequality we derive that

$$\begin{aligned}\int_{\mathbb{S}} e^{-2u_n(\theta)}d\theta &= 1 + \int_{\mathbb{S}} e^{-2u_n(\theta)}u_n'(\theta)d\theta \\ &\geq 1 + \left\{\int_{\mathbb{S}} e^{2u_n(\theta)}d\theta\right\}^{-1}\left\{\int_{\mathbb{S}} |u_n'(\theta)|d\theta\right\}^2.\end{aligned}$$

It follows that

$$\int_{\mathbb{S}} e^{2u_n(\theta)}d\theta \int_{\mathbb{S}} e^{-2u_n(\theta)}d\theta \geq \int_{\mathbb{S}} e^{-u_n(\theta)}d\theta + \left\{\int_{\mathbb{S}} |u_n'(\theta)|d\theta\right\}^2. \tag{4.22}$$

We see that $u_n \in M_{1,k}$ is $\pi-$periodic function for all $n \in \mathbb{N}$. Then by using the Blaschke-Santaló inequality we have

$$\int_{\mathbb{S}} e^{2u_n(\theta)}d\theta = \int_{\mathbb{S}} e^{2u_n(\theta)}d\theta \int_{\mathbb{S}} e^{-2u_n(\theta)}\left(1-u'^2\right)d\theta \leq 4\pi^2. \tag{4.23}$$

Let c_1 be a constant given by (4.16). By combing (4.22), (4.23) and (4.16) we have

$$\left\{\int_{\mathbb{S}} |u_n'(\theta)|d\theta\right\}^2 \leq 4\pi^2 \int_{\mathbb{S}} e^{-2u_n(\theta)}d\theta \leq 8\pi^3 e^{-2\min\limits_{\theta\in\mathbb{S}} u_n(\theta)} \leq 8\pi^3 e^{-2c_1}. \tag{4.24}$$

By (4.15) and (4.16) we see that $\{\min\limits_{\theta\in\mathbb{S}} u_n(\theta)\}_{n=1}^{\infty}$ is bounded. This together with (4.24) implies that $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^{\infty}(\mathbb{S})$, that is, there is a constant $C_1 > 0$ such that

$$|u_n|_{L^{\infty}(\mathbb{S})} < C_1. \tag{4.25}$$

Using the Blaschke-Santaló inequality together with (4.16) and (4.25), we have

$$1 + e^{-2C_1} \int_{\mathbb{S}} u_n'^2(\theta) d\theta \leq 1 + \int_{\mathbb{S}} e^{-2u_n} u_n'(\theta) d\theta = \int_{\mathbb{S}} e^{-2u_n(\theta)} d\theta \leq 2\pi e^{-2c_1}.$$

Then we have

$$\int_{\mathbb{S}} u_n'^2(\theta) d\theta < e^{2C_1} \left(2\pi e^{-2c_1} - 1\right). \tag{4.26}$$

It follows from (4.25) and (4.26) that $\{u_n\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{S})$. $\square$

If $k > 1$, by Lemma 4.3.4 we see that there exists a subsequence of $\{u_n\}$(still denote it by $\{u_n\}$ in the following) and $u_0 \in H^1(\mathbb{S})$ such that

$$u_n \overset{n\to+\infty}{\longrightarrow} u_0, \text{weakly in } H^1(\mathbb{S}). \tag{4.27}$$

Moreover, u_0 is a minimum of $-I$, which is proved in the following.

Lemma 4.3.5. *Let u_0 be given by (4.27). If $k > 1$, then u_0 is a minimum of functional $-I$ on $M_{1,k}$, that is, $-I(u_0) = \inf_{u \in M_{1,k}} -I(u)$.*

Proof. By using (4.27) and the compact imbedding from $H^1(\mathbb{S})$ to $C^\alpha(\mathbb{S})$ with $\alpha \in [0, 1/2]$, we see that

$$u_n \overset{n\to+\infty}{\longrightarrow} u_0, \text{strongly in } C^0(\mathbb{S}).$$

Hence we have

$$\lim_{n\to+\infty} \int_{\mathbb{S}} e^{-2u_n(\theta)} d\theta = \int_{\mathbb{S}} e^{-2u_0(\theta)} d\theta, \tag{4.28}$$

$$\lim_{n\to+\infty} \int_{\mathbb{S}} g(\theta) e^{pu_n(\theta)} d\theta = \int_{\mathbb{S}} g(\theta) e^{pu_0(\theta)} d\theta, \tag{4.29}$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{S}} u_n'^2(\theta) e^{-2u_n} d\theta \geq \int_{\mathbb{S}} u_0'^2(\theta) e^{-2u_0} d\theta. \tag{4.30}$$

We thus get that

$$\lim_{n\to+\infty}\int_{\mathbb{S}} u_n'^2(\theta)e^{-2u_n}d\theta = \int_{\mathbb{S}} u_0'^2(\theta)e^{-2u_0}d\theta. \tag{4.31}$$

Otherwise, by (4.30) we have $\lim\limits_{n\to+\infty}\int_{\mathbb{S}} u_n'^2(\theta)e^{-2u_n}d\theta > \int_{\mathbb{S}} u_0'^2(\theta)e^{-2u_0}d\theta$, which together with (4.28) and $u_n \in M_{1,k}$ implies that

$$\int_{\mathbb{S}} e^{-2u_0(\theta)} - e^{-2u_0(\theta)}u_0'^2(\theta)d\theta > 1.$$

Then there exists $t_0 > 0$ such that

$$\int_{\mathbb{S}} e^{-2u_0(\theta)} - e^{-2u_0(\theta)}u_0'^2(\theta)d\theta = 1 + t_0. \tag{4.32}$$

Combing (4.14), (4.28), (4.29) and (4.32), we get that

$$\begin{aligned} -I(u_0) &= \frac{-1-t_0}{2} - \lim_{n\to+\infty}\frac{1}{p}\ln\left(\int_{\mathbb{S}} g(\theta)e^{pu_n}d\theta\right) \\ &= \lim_{n\to+\infty} -I(u_n) - \frac{t_0}{2} = c - \frac{t_0}{2}, \end{aligned}$$

where c is given by (4.14). And there exists $\alpha_0 > 0$ such that $\int_{\mathbb{S}} e^{-2(u_n+\alpha_0)} - e^{-2(u_n+\alpha_0)}u'^2(\theta)d\theta = 1$, that is, $u_n + \alpha_0 \in M_{1,k}$. Obviously we see that $1 + t_0 = e^{2\alpha_0}$. It follows that

$$\begin{aligned} -I(u_0+\alpha_0) &= \frac{1}{2}\int_{\mathbb{S}} e^{-2(u_0+\alpha_0)}u'^2(\theta) - e^{-2(u_0+\alpha_0)}d\theta \\ &\quad - \frac{1}{p}\ln\left(\int_{\mathbb{S}} g(\theta)e^{p(u_0+\alpha_0)}d\theta\right) \\ &= -I(u_0) + \frac{1-e^{-2\alpha_0}}{2}\int_{\mathbb{S}} e^{-2u_0} - e^{-2u_0}u'^2(\theta)d\theta - \alpha_0 \\ &= -I(u_0) + \frac{e^{2\alpha_0}-1}{2} - \alpha_0 \\ &= c - \alpha_0 < c, \end{aligned}$$

which contradicts with the definition of c in (4.14). By using (4.28)-(4.29) and (4.31), we have

$$-I(u_0) = \lim_{n\to+\infty} -I(u_n) = \inf_{u\in M_{1,k}} -I(u) = c,$$

which is the end of the proof. $\square$

Now we apply Theorem 4.2.3 to get the main conclusion of this chapter.

Theorem 4.3.6. *Assume that $p > 0$, and $g(\theta)$ is a $\frac{\pi}{k}-$period function in $L^1(\mathbb{S})$ with $k > 1$. Then there exists a solution for (4.3) in the weak sense of*

$$\begin{aligned} 0 = \int e^{-2u}u'^2\varphi d\theta - \int e^{-2u}u'\varphi' d\theta - \int e^{-2u}\varphi d\theta \\ + \left(\int g(\theta)e^{pu}d\theta\right)^{-1}\int g(\theta)e^{pu}\varphi d\theta, \end{aligned} \tag{4.33}$$

for all $\varphi \in H^1(\mathbb{S})$.

Proof. Since u_0 is a minimum of $-I$ on $M_{1,k}$ in Lemma 4.3.5. We apply Theorem 4.2.3 with $X = H^1(\mathbb{S})$ and get

$$-\lambda_1 I'(u_0) + \lambda_2 G'(u_0) = 0,$$

where λ_1 and λ_2 are not simultaneously equal to zero. For any $\varphi(\theta) \in H^1(\mathbb{S})$, we have

$$\begin{aligned} -\lambda_1\left(\int e^{-2u_0}u_0'^2\varphi d\theta - \int e^{-2u_0}u_0'\varphi' d\theta - \int e^{-2u_0}\varphi d\theta\right) \\ = \lambda_2\left(\int g(\theta)e^{pu_0}d\theta\right)^{-1}\int g(\theta)e^{pu_0}\varphi d\theta, \end{aligned} \tag{4.34}$$

Let $\varphi = 1$ in (4.34), we derive that

$$\lambda_2 = -\lambda_1\left(\int e^{-2u_0}u_0'^2 d\theta - \int e^{-2u_0}d\theta\right) = \lambda_1 \neq 0. \tag{4.35}$$

Hence (4.33) follows from (4.34) and (4.35). $\square$

Proof of Theorem 4.3.1. Let u_0 be the solution of (4.3). If $g \in C^0(\mathbb{S})$ then $u_0 \in C^2(\mathbb{S})$. Let $v = \left(\int g(\theta)e^{pu_0}d\theta\right)^{-1/(p+2)} e^{-u_0}$, then $v \in C^2(\mathbb{S})$ and

$$v'' + v = \frac{g(\theta)}{v^{p+1}}.$$

$\square$

Chapter 5

Summary and Further Research

5.1 Summary

This thesis focuses on the existence and estimates of solutions to various elliptic equation models, including a sub-elliptic equation and a nonlinear elliptic equation with negative exponent. The main results achieved in this thesis are summarized as follows.

1. To study the partial Schauder estimates of solution to the sub-elliptic equation, we firstly introduce some new concepts about Dini contiouous and Hölder continuous in a plane for functions, which can be regarded as an extension from the classical concept of Dini continuous and Hölder continuous. Then, by using the perturbation argument we decompose the difference of the second partial derivative of solution at any two different points as the sum of a Newton potential and a sequence of $\Delta_{\mathbb{H}^n}$-harmonic functions. Since the $\Delta_{\mathbb{H}^n}$-harmonic function is sufficiently smooth, the partial Schauder estimates are de-

rived by using the smoothness of the Newton potential and the sum of a sequence of $\Delta_{\mathbb{H}^n}$-harmonic functions.

2. To study the solvability of the dual Minkowski problem in two dimensions, we establish a nonlinear differential equation with a small parameter by using the truncated technique. Then we derive the existence of periodic solutions to the truncated equation by using Poincaré's map and fixed point theorem. A prior estimates to the periodic solution of the truncated equation is also obtained by using the Green function and the eigenfunction of a differential operator. Base on these results, we establish that the dual Minkowski problem is solvable in two dimensions when the data function is positive and $2\pi/3-$ periodic.

3. The model related to the L^p Minkowski problem is a nonlinear differential equation with negative exponent, and its solution should be a positive function. The critical point theory has to be applied in a positive cone which consists of positive functions when applying the variational method. To overcome this kind of difficulties, we firstly transform it into a nonlinear elliptic problem with a gradient term, then give the variational functional defined on the whole space instead of a positive cone. This treatment helps us to apply the variational method directly to study the solvability of the L^p Minkowski problem.

5.2 Further research

In this thesis, we develop the truncated technique and the variational method to establish the existence of solution to the nonlinear elliptic equation with negative exponent. we also derive the partial Schauder estimate to the solution of a special sub-elliptic equation by using the perturbation argument. Based on the framework established in this thesis, the following areas are proposed for further study.

One of the areas for further research is to study the partial Schauder estimate of the solution to the sub-elliptic equation in general form. The key point is to establish some estimates to the related Green function.

Another area for further research is to study the solvability of the dual Minkowski problem in two dimensions for general data functions instead of the data function with some symmetry conditions. The key step is to develop an appropriate truncated equation model to derive good estimates for the solution.

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Appendix 1. Statement of Candidate's Contributions to Joint-Authored Paper #1

To Whom It May Concern

I, *Yongsheng JIANG,* made major contributions (*in the design of the research work, development of theories, analysis of research data/results and writing of the research work)* to the paper entitled *Partial Schauder estimates for a subelliptic equation. Acta Mathematica Scientia, 36(3), pp.945-956, 2016.*

Jiang Yongsheng

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Na Wei

Wei Na

Yong Hong Wu Wu YongHong

Appendix 2. Statement of Candidate's Contributions to Joint-Authored Paper #2

To Whom It May Concern

I, *Yongsheng JIANG,* made major contributions (*in the design of the research work, development of theories, analysis of research data/results and writing of the research work*) to the paper entitled *On the 2-dimensional dual Minkowski problem, Journal of Differential Equations, 263(6), pp.3230-3243, 2017.*

Jiang Yongsheng

I, Yong Hong Wu, as the Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Wu YongHong

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