School of Electrical Engineering, Computing and Mathematical Sciences

Multi-scale Volatility in Option Pricing

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Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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Abstract

In this thesis, we investigate a generalized hybrid model to simulate various financial derivatives including time-independent European options, time-dependent variance swaps as well as portfolio selection of credit risk. Different from the existing literature, our modelling framework consists of the equity driven by the dynamics of stochastic interest rate, stochastic volatility, and jump-diffusion processes. In particular, time-scale property of stochastic processes has been taken into consideration in our thesis with the application of various numerical approaches.

The contribution of our study consists of main two aspects. The first aspect in extending the classical Black-Sholes model to a generalized model for various applications. The multi-factor stochastic volatilities, stochastic interest rate and the jump-diffusion process are all taken into account in our model to simulate the behaviour of real financial market. The model is then applied to study both derivative pricing and portfolio selection. For the derivative pricing, we focus on the time-independent European option and the time-dependent variance swaps. The portfolio optimization technique is applied to the critical problem of credit risk in the bank system. The effect of stochastic volatility, stochastic interest rate and jump diffusion process have also been studied in this thesis, and we find that the effects of jump process and stochastic volatility process are significant.

The second aspect of contribution is the development of numerical and analytical solutions for the underlying mathematics problems. The option pricing problem under our hybrid model generates a high-dimensional partial integral differential equation, while the credit risk portfolio optimization problem generates a high-dimensional non-linear partial differential equation. In this research, we apply the generalized Fourier transformation, asymptotic approximation, finite element method and Monte Carlo simulation to solve the associated partial differential equations. Through numerical examples, we discover that the incorporation of multi-scale volatility process and jump-diffusion term have a significant impact on both option pricing and credit risk measuring process. For the variance swap pricing, the semi-closed solution is derived via generalized Fourier transforma-

tion, and the integral term arising from jump-diffusion process is solve by Fourier convolution.

List of publications during PhD candidature

- Shican Liu& Xiangyu Ge, Benchawan Wiwatanapataphee ,YongHong Wu ,The Study on the Pricing of Credit Risk under the Fast Stochastic volatility, International Conference in Identification, Information and knowledge in the internet of things, 2016.

- Shican Liu& Yanli Zhou, Yonghong Wu, Xiangyu Ge, The Simulation of Stochastic Variance Swap, proceeding of ICDSBA, 2017.

- Shican Liu& Yanli Zhou, Benchawan Wiwatanapataphee ,YongHong Wu, Xiangyu Ge, 'The Study of Utility Valuation of Single-Name Credit Derivatives with the Fast-Scale Stochastic Volatility Correction, Sustainability, 10(4):1027, DOI: 10.3390/su10041027

Acknowledgements

Contents

CHAPTER 1

Introduction

1.1 Background

As derivatives emerge and become the most important aspect of our daily life, it is important to study the pricing of derivatives. Financial derivatives originally served as an approach for companies to hedge against the risk exposure. Other than the risk management, derivatives also help in price speculation and improve the efficiency of the underlying asset.

Financial derivatives include future, forward, option and swap. The financial derivatives evolved in the 19th century. A market called 'the Chicago Board of Trade' was established to help farmers to conquer the difficulty of reaching the potential buyers. Forward is the simplest form of derivatives, it is an agreement to buy or sell an asset at a certain future time for a certain price, and it is always traded over the counter(OTC). Compared to the forward, the contract of the future is more standardized and specifies every detailed feature of the underlying asset. Unlike forward and future, the option provides the investor a right to buy or sell the underlying asset at a certain time for a certain price, and it is traded both OTC and on the exchange market. Swaps are newly derived financial derivatives which aim at exchanging cash flows between two parties.

The pricing techniques are very important in order to trade the derivatives in the real market. As suggested by its name, the price of derivatives is based on the underlying assets, including the stock price, the stock index, the commodity price, currencies, etc. Since 1950, many research has been conducted for pricing derivatives. Appell, Boussinesq and Poincare [1] proved that the price increment follows the normal distribution and it is independent of the present and past values, which was later developed as the Geometric Brownian motion. The binomial option pricing model is an option valuation method proposed by Cox, Ross, and Rubinstein in 1979 [2]. The benchmark of the derivative pricing

is the Black-Scholes model, in which the author assumed that the stock price is a geometric Brownian motion and evolves continuously [3]. Under the assumption of martingale, the backward stochastic differential equation can be solved analytically and by applying the Monte Carlo simulation, the derivative price can be calculated by taking the expectation of the underlying asset process [4] [5] [6]. Partial differential equations have also been used to price the derivatives. From the Feynman-Kac theorem, it has been proved that the solution of the backward stochastic differential equation can be solved by a partial differential equation under the assumption of risk-neutral [7].

The classical Black-Scholes model was established on the assumption of a complete market, which does not really exist in the real market. Thus research on the Black-sholes model under the incomplete market emerged to relax the rigid assumptions. The relax of the interest rate leads to the stochastic interest rate model, and the two most famous interest models are the Hull-White model and the CIR Model. The Hull-White model assumes that the interest rate follows a Gaussian process, while the CIR model assumes that the interest rate is driven by a non-central chi-square process [8] [9]. The relax of the constant volatility results in the development of local volatility models and stochastic volatility models, which are widely studied to capture the phenomenon of volatility skew. The local volatility model assumes that the local volatility of the stock is a function of stock price and time t, while the stochastic volatility model assumes that the volatility itself is a stochastic process correlated with the underlying process. In addition to the stochastic interest rate model and the stochastic volatility model [10] [11] [12] [13], the jump process has been used to sketch the unexpected abrupt change of stock price within a short period. The pioneering work of Merton assumes that the asset return process follows a Brownian motion plus a jump process [14], and the jump process is a compound Poisson process with constant jump intensity and normally distributed jump-size distribution. Two important applications of the jump-diffusion model are the Merton's model and the Kou's model. Different from Merton's Model, Kou [15] assumed that the distribution of the jump-size is a double exponential distribution instead of a normal distribution for the simplicity of computation. Transaction cost is also studied to make the derivative pricing more reliable. Though the aforementioned one-factor models play a significant role in derivative pricing, it is not realistic in some aspects, and it is suggested that all the extensions from the classified model could be considered together to reach a more realistic result. However, the combination of all those factors in the extended models will bring difficulties to

the study of the problem and further research is required.

In this thesis, a general stock model is established by taking account of the stochastic interest rate, the jump-diffusion process, and the multi-factor stochastic volatility processes. The general model is then applied to price financial derivatives such as European options and the variance swap. Both numerical and analytical approaches are studied and compared in our thesis.

1.2   Objectives of the thesis

This thesis focuses on the study of various financial derivative pricing problems with the underlying stock process driven by a generalized multi-scale volatility model. The multi-scale volatility model results in a high-dimensional partial differential equation under the risk-neutral assumption. The main objective of this research is to obtain analytical solutions and numerical solutions for the model, investigate the influence of the scale rate and other parameters on the prices of varies types of derivatives.

The specific objectives of this work are as follows:
(1) Establish the financial derivative pricing model with the underlying asset driven by the Geometric Brownian motion, and the volatility being assumed to be a function of two factors, which are driven by a fast-scale and a slow-scale stochastic volatility process respectively;
(2) Study the boundary value problem of the high-dimensional partial differential equation(PDE) derived from the corresponding financial derivative pricing models;
(3) Obtain analytical and numerical solutions for some special cases of the underlying high-dimensional financial based linear/non-linear PDEs;
(4) Investigate the influence of time-scale rate on the price of various types of options and show the significance of the work.

1.3   Outline of the thesis

The thesis is organized into five chapters.

Chapter 1 gives an overview of the research background and highlights the objectives of the research.

Chapter 2 reviews previous work relevant to the scope of this thesis. Some financial principles and mathematical methods closely related to the research are also proposed.

Chapter 3 studies the European option with multi-scale volatility correction and jump-diffusion process. The finite element method and dimension reduction technique are applied to obtain the approximate solution of the classical European option. The effects of time-scale and jump rate are studied in the Chapter.

Chapter 4 studies the credit risk pricing problem in the framework of the structural model and utility-based portfolio selection. The payoffs of financial derivatives are replicated by varying trading strategies of the underlying assets in a complete financial market. The asymptotic approach is applied to obtain the approximation of the value function.

Chapter 5 studies the pricing of the discrete sampling variance swap taking into account the effect of imposing multi-scale stochastic volatility into the stochastic process.

Chapter 6 investigates the variance swap pricing problem under a hybrid model. The effect of jump, stochastic volatility and stochastic interest rate on variance swap pricing is studied in this section. A semi-closed solution is derived via the generalized Fourier transformation. The integration term arising from the jump diffusion process is tackled by Fourier convolution.

In Chapter 7, the main results of this thesis are summarized, and discussion for further research is given.

CHAPTER 2

Mathematics and Finance Preliminaries

2.1 General

The research focuses on the pricing of financial derivatives, which involves the use of many finance principles and mathematical methods. Thus, in this chapter, we will first review the major types of financial derivatives, then present the finance and mathematical preliminaries required for the study of derivative pricing, including the risk-neutral pricing, the Feynman-Kac theorem, Monte Carlo simulation, analytical and numerical methods for option pricing. Then a brief review of previous work and models for the pricing of derivatives is given.

The rest of the chapter is organized as follows. Section 2.2 describes the major types of financial derivatives. Section 2.3 introduces the financial essentials for building up the foundations of option pricing. Section 2.4 is the mathematics foundation for solving the partial differential equations arising from option pricing problems, including the Fourier transform method, the finite difference method, the finite element method, and asymptotic approximation. Section 2.5 briefly reviews previous work on derivative pricing. A concluding remark based on the literature review is then given in Section 2.6.

2.2 Types of Financial Derivatives

Financial derivatives refer to securities/contracts which promise to make a payment at a specified time in the future and the amount of payment depend on the behaviour of the underlying security/securities up to and including the time of payment. There are various types of derivatives including forward, future, option, and swaps.

Forward is the simplest form of derivatives. It is an agreement to buy or sell an asset at a certain future time for a certain price. Compared to forward, the future

is more like an exchange, and the contract of the future is more standardized and specifies every detailed feature of the underlying asset.

Unlike forward and future, the option gives an investor the right to buy or sell the underlying asset at a certain time for a certain price. There are two basic types of options: call options and put options. Call options give the option holder the right but not obligation to buy the underlying asset at the specified price in the future. Put options give the holder the right but not the obligation to sell the asset at the specified price in the future trading day. The price written on the contract is named the strike price, and the specified date on the contract is the maturity date. The European options can only be traded at the maturity, while the American options can exercise at any time before the maturity day. Generally, the pricing of European options is easier than American options for the reason that the price of an European option is independent of the path, while the price of an American option is path dependent. The early exercise of the American call is unfavourable when there is no dividend paying, while the early exercise of American put is favourable when there is no dividend payment.

European options and American options are all categorized in the group of plain vanilla products. Other derivatives, such as Bermuda option, Asian option and Barrier option are termed as exotic options. A Bermuda option is an intermediate product of the European option and the American options, with the exercise date fixed at a certain date before the maturity date. The value of Barrier option, including knock-in and knock-out options, depends on whether the price of the underlying asset attains a predetermined level of price or not. The payoff function of Asian option depends on the average of the underlying asset price.

Another important exotic option is the variance and volatility swap. Variance swaps and volatility swaps are financial derivatives which enable us to exchange the realized volatility against the implied volatility(see Demeterfi et al. (1999) [16]). The first variance swap is traded in late 1998, and become increasingly popular with the development of the replication technique. There are two main reasons for investors to trade variance swap, namely to hedge against risk exposure of volatility, and to speculate on the difference in volatility across time and product. Long variance position will benefit when the realized volatility is higher than the strike price, while the short variance position benefits when the realized volatility is lower than the strike price. Figure 2.1 is generated by the VIX option monthly trading volume data downloaded from the Chicago Board Options Exchange(CBOE) website, and it is clear that the total trading volume

of the VIX option has been increasing every year. The realized volatility is approximated by the historical data of the underlying asset price, while the implied volatility is derived through the pricing formula. The distinction between the realized volatility and implied volatility is shown in figure 2.2. The realized variance is approximated by the SPX data downloaded from the CBOE website, while the proxy of implied volatility is the corresponding VIX index. Some properties of the variance can be observed from figure 2.2: the variance is a mean-reverted process and anti-correlated with the underlying assets, which is known as "skew effect". The main difference between the variance swap and the volatility swap
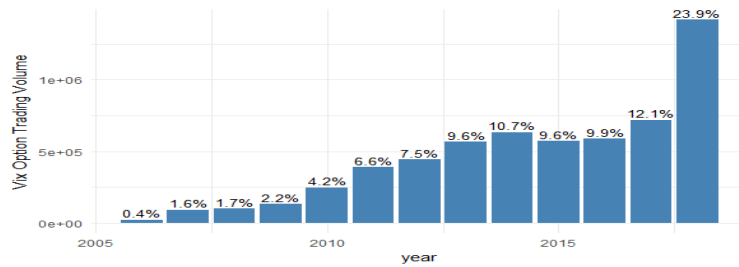
Figure 2.1: VIX Options Year Volume
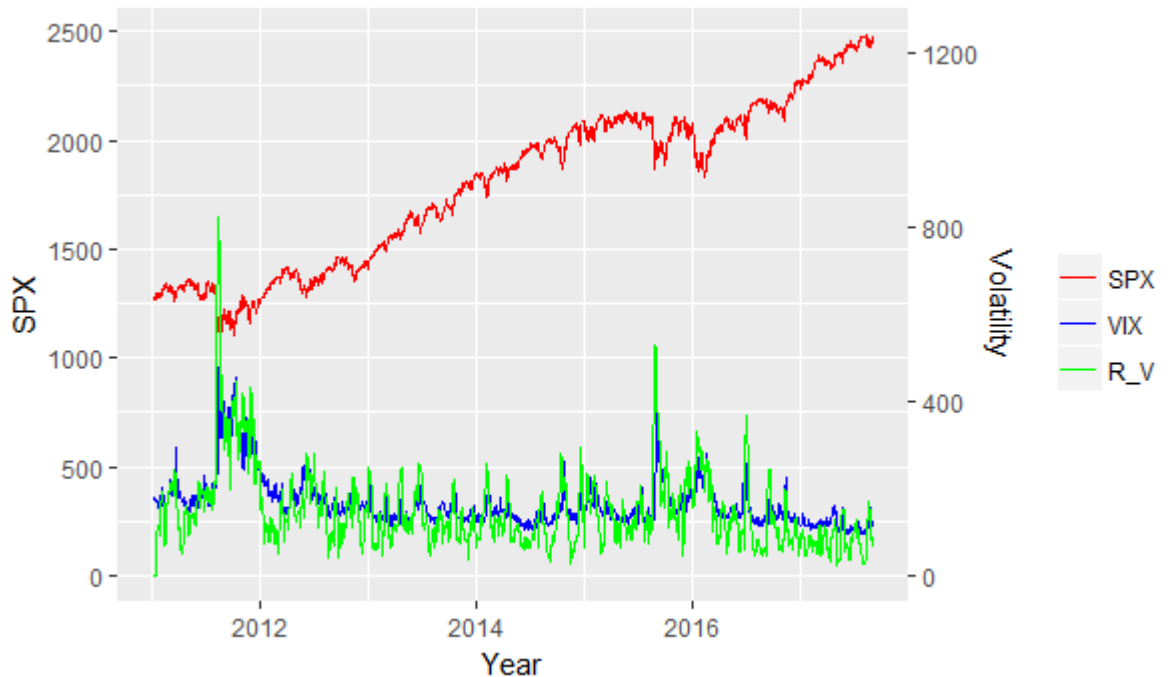
Figure 2.2: Realized volatility Versus Implied Volatility

is the payoff function. The payoff function of the volatility swap is the square root of the variance swap. Therefore, the payoff of the variance swap is convex

in comparison to the payoff of volatility swap, and more profitable as shown in figure 2.3.
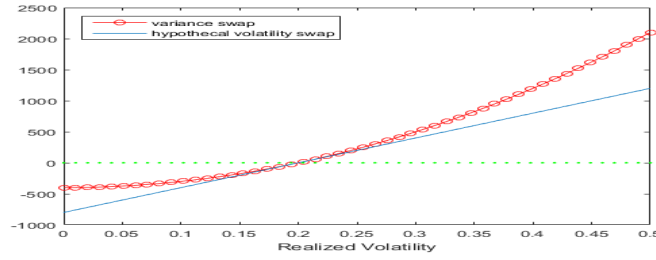
Figure 2.3: Convexity Property of Variance Swap

For the pricing of the variance swap, many researchers contribute to the literature. In Windcliff and Forsyth's work, the variance swap pricing problem is described by three different models: i) Geometric Brownian motion with constant volatility, ii) local volatility surface and iii) jump diffusion model [17]. They suggested that stochastic volatility should be incorporated into the model to obtain a better estimate of the fair strike price. Swishchuk et al. [18] used the probability approach to determine the variance swap price based on the CIR process with non-central χ square distribution ignoring the distribution of the payoff function. Lian and Zhu [19] applied the Fourier transformation to price variance swaps with discrete sampling times and obtained a closed-form solution of the Heston's two-factor stochastic volatility model [10]. Cao and Lian also obtained the semi-analytic solution of the variance swap pricing problem based on the Heston-CIR hybrid model via the generalized Fourier tramsformation [20]. However, Cao and Lian's approaches can be used only if the stochastic volatility process is a one factor CIR process [21].

2.3 Financial Preliminary

2.3.1 Risk-Neutral Pricing

In this subsection, we demonstrate the basic concept of derivative pricing. Risk-neutral measure, or equivalent probability measure, is a probability measure under which the derivative price can be seen as a discounted value of its final pay-off function. Under the risk-neutral measure, the stochastic process of the underlying asset becomes a martingale. The concept of martingale and change of measure are defined in Definition 2.3.1 and Definition 2.3.1 .

Definition 1.1. *[22] Equivalent Martingale Measure:the measure Q is an equivalent measure to P only if they have the same null sets.*

The equivalent martingale measure changes only the drift of the stochastic process, but the volatility of the process keeps the same.

Definition 1.2. *[23] Change of the probability measure: we change the old probability P to $\tilde{P}$ with the density Z*

$$Z = \frac{d\tilde{P}}{dP}.$$

We call the density Z Radon-Nikodym derivative, and we define the new probability measure by $\tilde{P} = ZdP$, and it is easy to prove that

$$\tilde{E}X = \int X d\tilde{P} = \int X Z dP = E(XZ).$$

There are two reasons of applying risk-neutral measure. First of all, only when the discounted stock process is a martingale, we can apply the Monte Carlo simulation to simulate expectation of the discounted payoff to price the stock or derivatives. Secondly, the risk-neutral measure is the fundamental of the Feynman-Kac theorem, which connects the stochastic process with the corresponding PDE. In all, the complicated financial problem will reduce to a solvable mathematical problem under the risk-neutral measure and proper assumptions.

Theorem 2.1. *[Girsanov Theorem] [24] Let $\mathbf{W}(t)(t \in [0,T])$ be a Brownian motion on a probability space$(\Omega, \mathcal{F}, \mathcal{P})$, and let $\{\mathcal{F}(t)|t \in [0,T]\}$ be its filtration. Let $\Gamma(t)$ be an adapted process. Define*

$$Z(t) = e^{-\int_0^t \gamma(d)d\mathbf{W}(s) - \frac{1}{2}\|\gamma(s)\|^2 ds}, \tag{2.1}$$

$$\tilde{W}(t) = W(t) + \int_0^t \gamma(s)ds, \tag{2.2}$$

and assume that

$$E^{\mathcal{P}}\left[\int_0^T \|\gamma(s)\|^2\right] < \infty$$

where $Z = Z(t)$ is the Radon-Nikodym. Then $E^P(Z) = 1$, and under the probability measure Q generated by Z, the process $\{\tilde{\mathbf{W}}(t), t \in [0.T]\}$ is an n-dimensional Brownian motion.

Girsanov's theorem provides us a way to change the measure to the risk-neutral measure, and can be easily proved by applying the Ito formula. To

demonstrate this, an example is specified based on the one dimensional Black-scholes model. It is assumed that the stock price S is driven by the following stochastic process

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{2.3}$$

The discounted price $\tilde{X}_t = e^{-rt} S_t$ can be rewritten as

$$d\tilde{X}_t = (\mu - r)\tilde{X}_t dt + \sigma \tilde{X}_t dW_t, \tag{2.4}$$

which is not a martingale since $\mu \neq r$ and the expectation of the process is not zero. If a process is a martingale with no arbitrage chances, we obtain $E^*(e^{-r(T-t)}V_T) = V_t$. Applying the Girsanov theorem, we can prove that under the risk-neutral probability Q, the discounted stock price process $\tilde{X}_t = e^{-rt} S_t$ is a martingale. Let $\gamma(s) = \frac{\mu - r}{\sigma}$ in (2.2). (2.4) can be transformed to

$$d\tilde{X}_t = \sigma \tilde{X}_t d\tilde{W}_t, \tag{2.5}$$

which is a martingale with no drift.

2.3.2 Feynman-Kac Theorem

The Feynman-Kac theorem is the key theorem of this part. As a preparation of the Feynman-Kac theorem proof, Markov representation of stock price will be displayed.

Theorem 2.2. (***Markov process***) *[25] Let $X(u), u \geq 0$, be a solution to the stochastic differential equation (2.4) with initial condition given at time 0. Then, for $0 \leq t \leq T$,*

$$E[h(X(T)) \mid \mathcal{F}(t)] = g(t, X(t)). \tag{2.6}$$

The Markov property guarantees that the price of derivative is a function of time and the state processes. The main reason we introduce the Markov process is that if we want to get the value of the derivative at time t, we should calculate the expectation of conditional payoff (from the risk neutral assumption). For the reason that the process is a Markov process, the filtration of $\mathcal{F}$ is only related to the state at time t and independent of the time effect before t, that is why we call it the state process. In order to obtain the value of the derivative at time t, we introduce the Markov process, from which the filtration is only related to the state at time t, so that the derivative value can easily be calculated by taking expectation of the conditional payoff function.

Theorem 2.3. *(**Feynman-Kac**) [26] Consider the stochastic differential equation*

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u). \qquad (2.7)$$

Define the function

$$g(t, x) = E^{t,x}h(X(T)), \qquad (2.8)$$

then $g(t, x)$ satisfies the partial differential equation

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2 g_{xx}(t, x) = 0, \qquad (2.9)$$

and the terminal condition

$$g(T, x) = h(x(T)) \quad for \ all \quad x. \qquad (2.10)$$

Proof. [7] By applying the Markov property, the Ito formula, and using (2.9), we have,

$$\begin{aligned} dg &= g_t dt + g_x dx + \frac{1}{2}g_{xx}[dx, dx] \\ &= \left(g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx} \right) dt + \gamma g_x dW. \end{aligned}$$

For the reason that $g(t, x)$ is a martingale, from the martingale representation theorem, the tendency of dg should be zero. Thereby, setting the coefficient of dt to zero leads to the PDE (2.9). □

Remark: Even though the proof part of the Feynman-Kac theorem is straightforward, there are still two facts we should notice:

- The notation of g represents the value of the derivative, which is denoted by V or U in our work.

- The terminal condition in Theorem 2.3 is denoted by $h(X(T))$. In the classical option pricing, it is always written in a discounted payoff style under the risk-neutral measure, which is $V(t) = \tilde{E}[e^{-r(T-t)}h(S(T)) \mid \mathcal{F}(t)]$.

2.3.3   Monte Carlo Simulation

Monte Carlo simulation can be used for derivative pricing. Monte Carlo Methods are based on the analogy between the probability and the volume. Monte Carlo calculates the volume by interpreting it by probability. The convergence of Monte

Carlo simulation is proved by the large number theorem. For instance, the volume or area is usually written in an integral form, while integral often relates to the expectation. The derivative pricing can be interpreted as the expected value of the final pay-off, and this is the reason why we apply the Monte Carlo method in derivative pricing problems. To apply the Monte Carlo simulation in finance, the discretization of the stochastic differential equation is vital. Two frequently used methods are the Euler-Maruyama(EM) method and the Milstein method.

Euler-Maruyama Method

Consider

$$dS = \mu(S_t)dt + \sigma(S_t)dW(t), \tag{2.11}$$

where when $\mu(S_t) = \mu S_t$, and $\sigma(S_t) = \sigma S_t$, (2.11) reduces to (2.3).

In Euler-Maruyama Method, we first discretize the stochastic model (2.11) , then generate random numbers, and scale them according to the definition of Brownian motion, and then generate thousands of paths in order to obtain an accurate result.

The stochastic process (2.11) is discretized as

$$S_j = S_{j-1} + \mu(S_{j-1})\Delta t + \sigma(S_{j-1})(w_j - w_{j-1}), \tag{2.12}$$

where $w_j - w_{j-1} = W(jRdt) - W((j-1)Rdt) = \sum_{k=jR-R+1}^{jR} dW_k$, in which $dW_k = \sqrt{dt}Z$, and Z is a random number with normal distribution.

Milstein Method

The idea of the Euler-Maruyama method is similar to the Euler method, but with an additional stochastic process. It is a modification of the Euler-Maruyama method and has a higher order of convergence. The strong convergence of the EM method is of order 1/2. By considering the expansion of μ_t and σ_t via the Ito Lemma, the order can be increased.

The Milstern discretization of the above equation is

$$S_j = S_{j-1} + \Delta t * \mu_{j-1} + \sigma_{j-1}(W(\tau_j) - W(\tau_{j-1}))$$
$$+ \frac{1}{2}\sigma_{j-1}\sigma'_{j-1}((W(\tau_j) - W(\tau_{j-1}))^2 - \Delta t), j = 1, 2, 3, \cdots, L. \tag{2.13}$$

In comparison with the Euler-Marugama scheme (2.12), the Milstern scheme (2.13) has an additional term, which can be proved by applying Ito's formula to both the drift $mu_t = \mu(S_t)$ and the volatility of volatility $\sigma_t = \sigma(S_t)$.
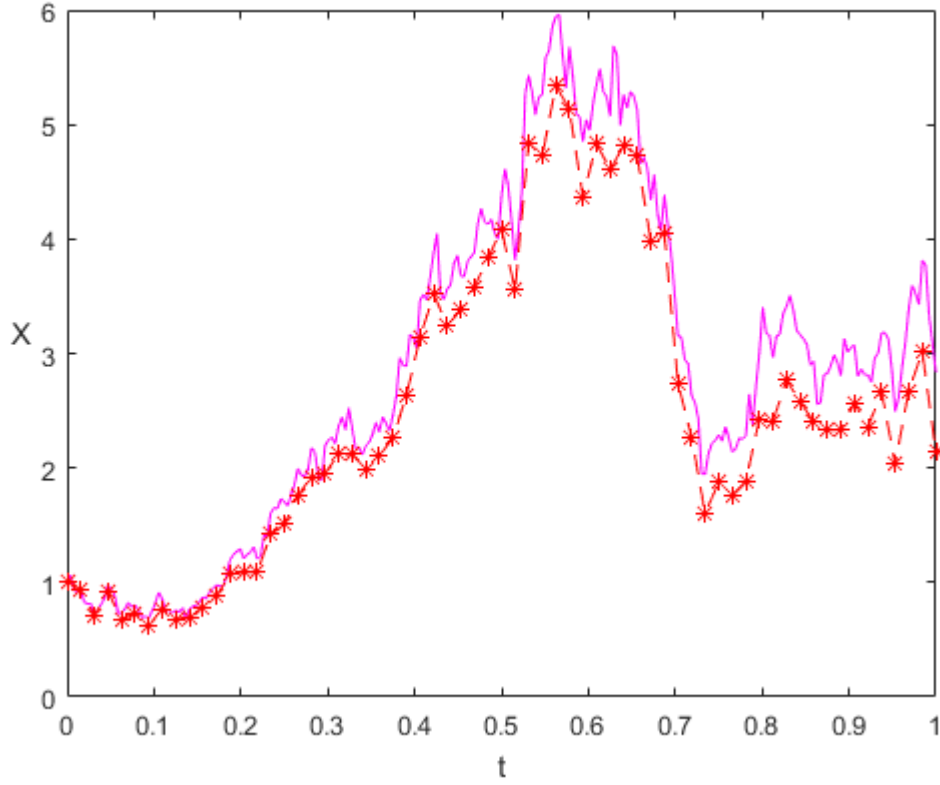
Figure 2.4: Euler-Maruyama Simulation of 1D

Firstly, the integral of (2.11) is

$$S_{t+dt} = S_t + \int_t^{t+dt} \mu_s ds + \int_t^{t+dt} \sigma_s dW(s). \qquad (2.14)$$

From the Ito formula, we obtain

$$d\mu_t = \mu_t' dS_t + \frac{1}{2}\mu_t''[dS_t, dS_t]$$

$$= [\mu_t'\mu_t + \frac{1}{2}\mu_t'\sigma_t^2]dt + \mu_t'\sigma_t dW(t).$$

$$\mu_s = \mu_t + \int_t^s [\mu_t'\mu_t + \frac{1}{2}\mu_t'\sigma_t^2]dt + \int_t^s \mu_t'\sigma_t dW(t), \qquad (2.15)$$

and

$$d\sigma_t = \sigma_t' dS_t + \frac{1}{2}\sigma_t''[dS(t), dS_t]$$

$$= [\sigma_t'\mu_t + \frac{1}{2}\sigma_t''\sigma_t^2]dt + \sigma_t'\sigma_t dW(t),$$

$$\sigma_s = \sigma_t + \int_t^s [\sigma_t'\mu_t + \frac{1}{2}\sigma_t''\sigma_t^2]dt + \int_t^s \sigma_t'\sigma_t dW(t), \qquad (2.16)$$

where $t < s < t + dt$.

Substituting (2.15),(2.16) into (2.14) and ignoring the high order term, we obtain

$$
\begin{aligned}
S_{t+dt} &= S_t + \int_t^{t+dt} \mu_s ds + \int_t^{t+dt} \sigma_s dW(s) \\
&= S_t + \int_t^{t+dt} \left(\mu_t + \int_t^s [\mu_t'\mu_t + \frac{1}{2}\mu_t'\sigma_t^2]dt + \int_t^s \mu_t'\sigma_t dW(t) \right) ds \\
&\quad + \int_t^{t+dt} \left(\sigma_t + \int_t^s [\sigma_t'\mu_t + \frac{1}{2}\sigma_t''\sigma_t^2]dt + \int_t^s \sigma_t'\sigma_t dW(t) \right) dW(s) \\
&= S_t + \int_t^{t+dt} \mu_t ds + \int_t^{t+dt} \sigma_t dW(s) + \int_t^{t+dt} \int_t^s \sigma_t'\sigma_t dW(t)dW(s). \quad (2.17)
\end{aligned}
$$

Now the next problem is to solve the following double integral,

$$
\begin{aligned}
\int_t^{t+dt} \int_t^s \sigma_t'\sigma_t dW(t)dW(s) &= \sigma_t'\sigma_t \int_t^{t+dt} W(s) - W(t)dW(s) \\
&= \sigma_t'\sigma_t[\int_t^{t+dt} W(s)dW(s) - W(t+dt)W(t) + W^2(t)] \\
&= \frac{1}{2}\sigma_t'\sigma_t[(W(t+dt) - W(t))^2 - \Delta t], \quad (2.18)
\end{aligned}
$$

where $\int_t^{t+dt} W(s)dW(s)$ can be evaluated by applying Ito's formula to the integral of $Y = W^2(t)$, that is

$$
\begin{aligned}
dY &= 2W\,dW + dt, \\
W\,dW &= \frac{1}{2}(dY - dt), \\
\int_t^{t+dt} W\,dW &= \frac{1}{2}[Y(t+dt) - Y(t) - dt]
\end{aligned}
$$

The difference between the Euler-Maruyama method and the Milstein method is that the Milstein method adds a correction term to the Euler-Maruyama method. The convergence rate of the Euler-Maruyama method is $1/2$, while the convergence rate of the Milstein method is 1.

2.3.4 Multiscale Stochastic Volatility

The assumption of the classical Black-Scholes model is too restrictive and has a lot of drawbacks. One of the drawbacks is the famous volatility 'smile' and 'smirk'. Option 'smile' denotes the relationship between the strike price and the implied volatility, which is visually a curve instead of the flatten one as Black-

schole's model assumes. Volatility 'smile' is symmetric, but the volatility 'smirk' is more realistic with non-symmetric skew. The option 'smile' indicates that the options are preferred in the money or out of the money than at the money. The empirical study of volatility curve can refer to the estimation by Rubinstein(1985) and Rubinstein et al. (1996) [27] [28]; the former applied data before the financial crisis of 1991, while the later research of Rubinstein applied the after crash data. In order to make a remedy of the unrealistic assumptions of Black-sholes formula, many attempts have been made to solve the problem by extending the classical Black-Scholes model to more general and realistic models. One of the most famous approaches is the volatility model, including the local volatility model and the stochastic volatility model, which is widely studied to capture the phenomenon of the volatility skew. The idea of local volatility and stochastic volatility is motivated by the Leptokurtic characteristic of the asset return distribution with higher peak and fatter tail, and these characters indicate that the distribution is not exactly a normal distribution, but a mixture distribution with different variances. The local volatility model assumes that the local volatility of the stock is a function of stock price and time t rather than a simple constant. For example, in Dupire's Model, the classical Black-Scholes model is modified to include a time-dependent local volatility rather than a constant volatility [29]. Constant elasticity of variance models(CEV model) attempts to capture the stochastic volatility and the leverage effect by assuming the volatility of the stock process is in the form of $\sigma(S_t) = S_t^\gamma$, which was firstly developed by Cox,et,al.(1976) [30], and then applied to calibrate and estimate the energy commodity market by Geman, H, and Shih, YF.(2009) [31].

Different from the local volatility model, the stochastic volatility model assumes that the volatility process is related to another stochastic process instead of the stock price process itself. For the early research of the stochastic volatility model, we refer the reader to Hull and White (1987) [32], Scott (1987) [33]. The most popular stochastic volatility model is the Heston model, from which Heston generalized the Black-Scholes model to a two-dimensional stochastic model by allowing the volatility to follow a Cox Ingersoll Ross model(CIR) process, and derived a semi-closed form solution by applying the method of characteristic function [10]. Besides the Heston model, Stein and Stein also promoted a stochastic volatility model driven by the Ornstein Uhlenbeck(OU) process, and established a closed-form solution in their paper "Stock Price Distributions with Stochastic Volatility: an analytic approach" [11]. Other stochastic volatility models are proposed for different forms of stochastic volatility. The traditional Heston model

assumes that the underlying volatility process is a CIR process with the power of 1/2; the 3/2 model assumes that the diffusion of the volatility process is a flipped CIR process, raising the power of 3/2 [12]. The 4/2 process is the combination of the CIR process and the flipped CIR process [13].

Most of the volatilities we applied are the one-factor volatility model, which means we take only one stochastic volatility into consideration. However, the structure of stochastic volatility is much more complicated than we expect. The idea that distinguishes the mean reversion rate of low-frequency data and high-frequency data has been noticed by French, Schwert, and Stambaugh(1987) [34], Schwert(1989) [35], and Campbell and Hentschel(1900) [36]. Thus, it is concluded by Chacko and Viceira [37] that at least two volatilities should be considered in the same model, and according to the authors, the volatilities should be classified into two groups, including the fast volatility and the slow volatility. The fast volatility is related to the short period high-frequency date, for instance, the intraday data, while the slow volatility is observed in the long run with low-frequency data. It is also proved by Heston that one-factor models are not accurate, and at least two factors should be taken into consideration. In Heston's multifactor models, it is assumed that all volatilities are mutually independent, and the two-factor models are approximated by principle analysis. The concept of time-scale is developed by Fouque, et, al. considering the volatility process as a combination of fast-scale and slow-scale process [38–40], from which they also studied the correlation of the volatility processes. In 2008, Fouque proposed a numerical algorithm based on asymptotic approximation and asymptotic homogenization to study the effect of the fast and the slow scale of the volatility Ornstein Uhlenbeck(OU) process on option pricing [40]. The definition of time-scale is distinguished by the fluctuation frequency of the observed volatility process. The fast-scale volatility relates to the highly frequent short period fluctuation, which is a singular perturbation, while the slow-scale volatility is a less frequent long-term variation, and relates to the regular perturbation. The phenomenon of time-scale can be observed by the stock prices generated by using the 27 years daily SPX data downloaded from the Chicago Board Options Exchange(CBOE) website, as shown in figure 2.5. The Slow scale volatility can be tracked from the long period variation, and it does not have to be mean-reverted, while the fast scale volatility is the smaller but drastic oscillations between the peak and the bottom.

An alternative approach to capture the leptokurtic features and implied volatility 'smile' is the jump-diffusion model. The jump process can be used to sketch the unexpected abrupt change of stock price within a short period. The pioneer-

Figure 2.5: Variation of SPX price with time

ing work of Merton assumes that the asset return process follows a Brownian motion plus a jump process [14], and the jump process is a compound Poisson process with constant jump intensity and normally distributed jump-size distribution. Different from Merton's Model, Kou assumed that the distribution of the jump-size is a double exponential distribution instead of a normal distribution for the simplicity of computation [15]. In 1987, Madan and Seneta studied the Australia stock market data and suggested that increments of log-prices follow a variance gamma(VG) distribution [41] [42]. The VG distribution is a special case of the generalized hyperbolic(GH) distribution, and for other cases, we refer to the GH distribution in Eberlein and Keller's model and the normal inverse Gaussian(NIG) distribution of Barndorff Nielson's model [43] [44]. More recent work proves that combination of the stochastic model and the traditional jump-diffusion model leads to more accurate models. Bates introduced the SVJ(stochastic volatility with jumps) model by allowing both jump diffusion and stochastic volatility in the return process. The SVJ model is then extended by Duffie et al. to incorporate the jump term not only in the return process but also in the stochastic process [45]. The SVJ model is also studied by Pillay and O'Hara [46], from which they assumed an affine structure of characteristic function, and applied the Fast Fourier transformation to solve the SVJ problem to obtain a semi-analytic solution.

Besides the relaxation of volatility, a lot of researchers show interests in modeling stochastic interest rate and its application. The stochastic interest rate was introduced by Hull& White, and the closed form solution of the Black-Scholes-Hull-White model was derived by Brigo and Mercurio for European Style options. However, even though the stochastic interest rate model can describe the fluctuation and enhance the long-term accuracy, it cannot sketch the skew effect or the 'option smile'. To overcome the drawbacks, the stochastic interest

model is always used together with the stochastic volatility model [47] [48] [49]. The Heston-Hull-White model and Heston-CIR model are studied by Grzelak and Oosterlee, from which they derived an affine structure solution by applying Fourier transformation. Different from the aforementioned literature, the correlation effects are also considered in their work [49]. Recently, Kim et.al. studied the multiscale volatility model and stochastic interest rate model by applying the technique of asymptotic approximation and derived the leading term and the first order correction term of European type option [49].

2.4 Mathematical Preliminary

From the previous section, the expected payoff can be transformed to the solution of a parabolic PDE by using the Feynman-kac theorem. In this section, we will discuss the analytic and the numerical methods commonly used to solve the PDE arising from financial problems.

2.4.1 Generalized Fourier Transform

The Fourier transform plays a significant role in Quantitative Finance, especially in solving the partial differential equation arising from the option pricing problem. In this section, we discuss the fundamental of the generalized Fourier transform.

Definition 1.1. *(Fourier Transform [50]) Let $U(x)$ be a payoff function of contingent claim(derivative), which is assumed to be a function of the underlying asset x. The generalized Fourier transform $V(w)$ is defined by*

$$V(w) = \mathcal{F}[U(x)] = \int_{-\infty}^{\infty} U(x)e^{-jwx}dx, \qquad (2.19)$$

with $j = \sqrt{-1}$ and w being the Fourier transform frequency.

- *The differentiation property of the generalized Fourier transform*

$$\mathcal{F}(\frac{\partial^n U(x)}{\partial x^n}) = (jw)^n V(w). \qquad (2.20)$$

- *The Generalized Fourier transform of delta function*

$$\mathcal{F}(\delta(x - x_0)) = e^{-jwx_0}. \qquad (2.21)$$

By applying Fourier transform, along with the affine structure assumption of the underlying asset processes, the complicated high dimensional partial differential equation reduces to a series of one-dimensional ordinary differential equations. The generalized partial differential equation is always showing in pairs with the inverse Fourier transform.

Definition 1.2. *The generalized inverse Fourier transform is given by*

$$U(x) = \mathcal{F}^{-1}[V(w)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(x)e^{iwx}dw. \tag{2.22}$$

2.4.2 Numerical Method in Option Pricing

Analytic solutions can be derived only for specific models. However, for most of the time-dependent models, especially American models with free boundary, and portfolio selection problems with non-linear terms, closed form solutions are not possible to obtain, thereby the study of alternative numerical methods is necessary. The commonly used numerical methods, including the finite difference method and the finite element method, are studied and compared in this subsection. The classic one-dimensional Black-Scholes PDE and two dimensional Heston models are selected as examples and the partial integral differential equation(PIDE) arising from the jump process will be discussed in this section. The vector form of the parabolic PDE is

$$\frac{\partial U}{\partial \tau} - \nabla \cdot A\nabla U - D \cdot \nabla U + rU = 0, \tag{2.23}$$

For the classic Black-Scholes model, we have the corresponding one dimensional PDE

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + \mu S \frac{\partial U}{\partial S} - rU = 0, \tag{2.24}$$

along with the initial boundary conditions

$$U(S,T) = [\phi(S-K)]^+, \tag{2.25}$$

$$U(S_{min},t) = \frac{1-\phi}{2}Ke^{-rT}, \tag{2.26}$$

$$U(S_{max},t) = \frac{1+\phi}{2}e^{-rT}. \tag{2.27}$$

Let $\tau = T - t$, and $x = ln(S)$, we obtain

$$\frac{\partial U}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} - \mu \frac{\partial U}{\partial x} + rU = 0, \tag{2.28}$$

The explicit solution of the above problem is

$$U(S,t) = \phi\left(SN(d_1) - Ke^{-r(T-t)}N(d_2)\right),$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,$$

$$d_{1,2} = \phi\frac{log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{2.29}$$

$$\begin{cases} dS_t = rS_t dt + \sqrt{Y_t}S_t dB_t^S \\ dY_t = k^*(\theta^* - Y_t)dt + \sigma_V\sqrt{Y_t}dB_t^V, \end{cases} \tag{2.30}$$

With the same assumptions, the corresponding two dimensional PDE can be rewritten as

$$U_t + rSU_S + k^*(\theta^* - Y)U_Y + \frac{1}{2}YS^2U_{SS} + \frac{1}{2}\sigma^2YU_{YY} + \frac{1}{2}\rho\sigma_V YSU_{SY} - rU = 0, \tag{2.31}$$

For European style options, the Boundary Condition is in the following form:

$$U(S,T) = [\Phi(S - K)]^+ \tag{2.32}$$

$$U(0,Y,t) = \frac{1-\Phi}{2}Ke^{-r(T-t)} \tag{2.33}$$

$$U_S(\infty,Y,t) = \frac{1+\Phi}{2} \tag{2.34}$$

$$U(S,\infty,t) = \frac{1+\Phi}{2}Se^{-r(T-t)} \tag{2.35}$$

$$U(S,0,t) = U_{BS}(S,t), \tag{2.36}$$

where $U_{BS}(S,t)$ denotes the Black-Sholes formula(2.29). For European call options, $\Phi = 1$; for European put options, $\Phi = 0$. Let $\tau = T - t, x = ln(S)$, we obtain

$$U_\tau - rU_x - k^*(\theta^* - Y)U_Y - \frac{1}{2}YU_{xx} - \frac{1}{2}\sigma^2YU_{YY} - \frac{1}{2}\rho\sigma_V YU_Y + rU = 0. \tag{2.37}$$

(A) Finite Difference Method

The application of the finite difference method(FDM) has more than 200 years history. The main idea behind the finite difference method for solving initial boundary value problems(IBVP) is to discretize the space and the time, and apply the boundary and initial conditions to retrieve the unknown function values at internal points. To the IBVP, the boundary type has a vital impact on the final solution. Generally, there are three different boundary types, including Dirichlet

boundary, Neunman boundary, and Robin boundary. Dirichlet boundary conditions specify the value of the unknown function on the boundary. The Neunman boundary conditions specify the value of the normal derivative of the unknown function on the boundary. The Robin type boundary condition is a combination of the Dirichlet boundary condition and the Neunman boundary condition. The discretizing scheme of finite difference method varies through the type of derivatives. Both explicit and implicit schemes will be investigated in our work. The scheme of explicit methods are conditional stable, and the implicit Euler and Crank-Nicolson are unconditional stable. Many financial engineering literature uses the Crank-Nicolson method because it is second order accurate. According to the FDM method, PDE (2.2) is discretized as follows

$$U_i^{k+1} + p*(AU_{i+1}^{k+1} + BU_i^{k+1} + CU_{i-1}^{k+1}) = U_i^k + (1-p)*(AU_{i+1}^k + BU_i^k + CU_{i-1}^k), \quad (2.38)$$

where

$$A = -(0.5\sigma^2\bar{h} + \mu Sl),$$
$$B = \sigma^2\bar{h} + \mu l + rk,$$
$$C = -0.5\sigma^2\bar{h}.$$

$\bar{h} = \frac{k}{h^2}, l = \frac{k}{h}$, with k denoting the time step size, and h denoting the space step size. If $p = 0$, (2.38) is fully explicit; if $p = 1$, (2.38) is a fully implicit; if $p = \frac{1}{2}$, (2.38) is a Crank-Nicholson method.

(B) Finite Element Method

The fundamental of the finite difference method(FDM) is based on the strong form of PDE. The fundamental of the finite element method(FEM) method is local Taylor expansion, from which the discretization of the differential equation is very intuitive and straightforward to apply, and the truncation error can be derived directly from the Taylor expansion. However, with respect to topology, FDM is fixed to rectangular shapes, and errors expand with dimensions. Different from the FDM method, the FEM method is based on the weak form of PDE and is more flexible and accurate.

The variational statement for (2.23) is : Find $U \in H_h^1(\Omega)$, such that $U = constant\,value$ on the boundary, and $\forall\, U \in H_0^1(w)$,

$$\int_\Omega \frac{\partial U}{\partial \tau} V d\Omega - \int_{\partial\Omega} (A\nabla U \cdot \vec{n}) V d\Gamma + \int_\Omega A\nabla U \cdot \nabla V d\Omega - \int_\Omega D \cdot \nabla U V d\Omega + \int_\Omega rUV d\Omega = 0,$$
$$(2.39)$$

where $H_0^1(w) = \{V|V \in H_0^h(\Omega) \quad \text{vanish on the boundary}\}$ By assuming

$$U_h(x, y, \tau) = \sum_{j=1}^{N} u_j(\tau)\Phi_j(x)$$

$$V_h(x, y) = \sum_{i=1}^{N} v_i\Phi_i(x). \tag{2.40}$$

We obtain

$$M\dot{U} + RU = 0, \tag{2.41}$$

where

$$M = \sum_{j=1}^{N} \int_w \Phi_i\Phi_j dw \tag{2.42}$$

$$R = -\sum_{j=1}^{N} \int_w (A\nabla\Phi_j) \cdot \nabla\Phi_i - D \cdot \nabla\Phi_j\Phi_i dw \tag{2.43}$$

This is a general form. In 1D case, $A = 0.5\sigma^2 S^2, D = \mu S - \sigma^2 S$.

Compared to the FDM method, the topology of the finite element method(FEM) is much more flexible, especially for dealing with complicated boundaries and multidimensional problems. The FEM is based on the weak form of the boundary value problem, which relaxes the smooth condition of PDE by applying Green and Gauss formula.

2.4.3 Asymptotic Approximation

In this subsection, the asymptotic approach is demonstrated for the derivative pricing arising from the multi-scale stochastic volatility model. To specify this approach, we firstly introduce the multi-scale stochastic volatility model. The difference between the multi-scale volatility model and the Heston stochastic model is in the stochastic processes. The Heston model assumes that the volatility of asset price is driven by a one-factor stochastic process, while the multi-scale volatility model is driven by a two factor stochastic process (2.44).

$$dS = rSdt + f(y, z)Sdw_t^{(0)} + SdJ^S,$$
$$dy = (\frac{1}{\xi}\alpha(y) - \frac{1}{\sqrt{\xi}}\Lambda(y)\beta(y))dt + \frac{1}{\sqrt{\xi}}\beta(y)dw_t^{(1)},$$
$$dz = (\sigma c(z) - \sqrt{\sigma}T(y, z)g(z))dt + \sqrt{\sigma}g(z)dw_t^{(2)}, \tag{2.44}$$

where r is the risk-free interest rate. Functions $T(y, z)$ and $\Lambda(y)$ denote market prices of volatility. According to the Feyman- Kac theorem, the option price is determined by the solution of the following partial differential equation

$$\mathcal{L}^{\xi,\sigma} P^{\xi,\sigma} = 0, \qquad (2.45)$$

where $t \in [0, T]$, and the operator $\mathcal{L}^{\xi,\sigma}$ is given by

$$\mathcal{L}^{\xi,\sigma} = \mathcal{L}_0 + \frac{1}{\sqrt{\xi}}\mathcal{L}_1 + \frac{1}{\xi}\mathcal{L}_2 + \sqrt{\sigma}\mathcal{M}_1 + \sigma\mathcal{M}_2 + \frac{\sqrt{\sigma}}{\sqrt{\xi}}\mathcal{M}_{12} = 0, \qquad (2.46)$$

with

$$\begin{cases} \mathcal{L}_0 = \partial_t + \frac{1}{2}f^2(y, z)S^2\partial_{SS} + rS\partial_S - (r + \lambda) \\ \mathcal{L}_1 = \rho_1\beta(y)f(y, z)S\partial^2_{Sy} \\ \mathcal{L}_2 = \frac{1}{2}\beta^2(y)\partial^2_{yy} + \alpha(y)\partial_y \\ \mathcal{M}_1 = \rho_2 g(z)f(y, z)S\partial^2_{Sz} \\ \mathcal{M}_2 = \frac{1}{2}g^2(z)\partial^2_{zz} + c(z)\partial_z \\ \mathcal{M}_{12} = \rho_{12}\beta(y)g(z)\partial^2_{yz}. \end{cases} \qquad (2.47)$$

According to the asymptotic approximation theory, the option price can be approximated by the addition of the leading term and the first order correction terms

$$P^{\xi,\sigma} = P_{0,0} + P^\xi_{1,0} + P^\sigma_{0,1}, \qquad (2.48)$$

where $P_{0,0}$ denotes the option price without the volatility correction, $P^\xi_{1,0} = \sqrt{\xi}P_{1,0}$ is the fast scale volatility correction, and $P^\sigma_{0,1} = \sqrt{\sigma}P_{1,0}$ is the slow scale volatility correction. We prove in this section that $P_{0,0}$, $P_{1,0}$, and $P_{0,1}$ are of the form of $f(t, S, z)$, which is independent of the fast-scale volatility y.

The asymptotic approximation is made up by the combination of a singular perturbation with respect to a fast scale volatility and a regular perturbation with respect to a slow scale volatility. To construct a singular perturbation expansion, we expand the asymptotic price $P^{\xi,\sigma}$ in the form of

$$P^{\xi,\sigma} = \sum_i^n \xi^{\frac{i}{2}} P^\sigma_i(t, S, y, z). \qquad (2.49)$$

To construct a regular perturbation expansion, the asymptotic price P^σ is ex-

panded in the form of

$$P^\sigma = \sum_j^n \sigma^{\frac{j}{2}} P_{i,j}(t, S, y, z). \tag{2.50}$$

Substituting (2.49) and (2.50) back into $\mathcal{L}^{\xi,\sigma} P = 0$ and collecting the like terms up to order $1/2$, one has

$$\mathcal{O}(1/\xi) : \mathcal{L}_2 P_{0,0} = 0 \tag{2.51}$$

$$\mathcal{O}(1/\sqrt{\xi}) : \mathcal{L}_2 P_{1,0} + \mathcal{L}_1 P_{0,0} = 0 \tag{2.52}$$

$$\mathcal{O}(1) : \mathcal{L}_2 P_{2,0} + \mathcal{L}_1 P_{1,0} + \mathcal{L}_0 P_{0,0} = 0 \tag{2.53}$$

$$\mathcal{O}(\sqrt{\xi}) : \mathcal{L}_2 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_0 P_{1,0} = 0 \tag{2.54}$$

$$\mathcal{O}(\sqrt{\sigma}/\xi) : \mathcal{L}_2 P_{0,1} = 0 \tag{2.55}$$

$$\mathcal{O}(\sqrt{\sigma}/\sqrt{\xi}) : \mathcal{L}_2 P_{1,1} + \mathcal{L}_1 P_{0,1} + \mathcal{M}_{12} P_{0,0} = 0 \tag{2.56}$$

$$\mathcal{O}(\sqrt{\sigma}) : \mathcal{L}_2 P_{2,1} + \mathcal{L}_1 P_{1,1} + \mathcal{L}_0 P_{0,1} + \mathcal{M}_1 P_{0,0} + \mathcal{M}_{12} P_{1,0} = 0. \tag{2.57}$$

As shown in (2.47), the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are in terms of y. From (2.51), (2.52), (2.55), and (2.56), we conclude that $P_{0,0}$, $P_{1,0}$, $P_{0,1}$ and $P_{1,1}$ are independent of y. As the fact that $\mathcal{L}_1 P_{1,0} = 0$, equations (2.53),(2.54) and (2.57) are Possion equations of the form

$$\mathcal{L}_2 P + <\mathcal{G}> = 0, \tag{2.58}$$

with $<\mathcal{G}> := \int g(y)\Pi(dy)$ according to the centring resolvability of the Possion equation, and Π is a invariant distribution with respect to y. Thus, one gets

$$<\mathcal{L}_0> P_{0,0} = 0 \tag{2.59}$$

$$<\mathcal{L}_1 P_{2,0}> + <\mathcal{L}_0> P_{1,0} = 0 \tag{2.60}$$

$$<\mathcal{M}_1> P_{0,0} + <\mathcal{L}_0> P_{0,1} = 0. \tag{2.61}$$

The terminal condition gives $P_{0,0}(T, S, z) = max(K - S, 0)$, and $P_{1,0}(T, S, z) = P_{0,1}(T, S, z) = 0$.

Definition 1.1. *The leading term in (2.59) is determined by*

$$<\mathcal{L}_0> = \partial_t + \frac{1}{2}\bar{\delta}^2 S^2 \partial_{SS} + rS\partial_S - r, \tag{2.62}$$

with $\bar{\delta}^2 = < f^2(y, z) > := \int f^2(y, z)\Pi(dy)$ denoting the mean historical volatility of stock, and

$$<\mathcal{L}_0> - \mathcal{L}_0 = \frac{1}{2}(\bar{\delta}^2 - f^2(y, z))S^2 \partial_{SS}. \tag{2.63}$$

Subtracting (2.63) by (2.60), and letting

$$\mathcal{L}_2 \phi = -\frac{1}{2}(\bar{\delta}^2 - f^2(y,z)), \tag{2.64}$$

one gets

$$< \mathcal{L}_1 P_{2,0} > = < \mathcal{L}_1 \mathcal{L}_2^{-1}(< \mathcal{L}_0 > -\mathcal{L}_0)P_{0,0} > = < \mathcal{L}_1 \phi S^2 \partial_{SS} P_{0,0} >, \tag{2.65}$$

with ϕ in the form of $\phi(t, S, y, z)$. $P_{0,0}$ is independent of y.

Definition 1.2. *The first order correcting term in (2.60) is determined by*

$$< \mathcal{L}_0 > P_{1,0} = - < \mathcal{L}_1 P_{2,0} > = V_0 S^3 \frac{\partial^3 P_{0,0}}{\partial S^3} + 2 * V_0 S^2 \frac{\partial^2 P_{0,0}}{\partial S^2}, \tag{2.66}$$

where $V_0 = \rho_1 < \beta(y) > \bar{\sigma} < \Phi_y >$.

We assume

$$P_{1,0} = -(T-t)\mathcal{V}P_{0,0}, \tag{2.67}$$

with $\mathcal{V} = V_0 S^3 \frac{\partial^3}{\partial S^3} + 2V_0 S^2 \frac{\partial^2}{\partial S^2}$. Substituting (2.69) into (2.66), we obtain

$$< \mathcal{L}_0 > P_{1,0} = -\mathcal{V}P_{0,0} + (T-t)\mathcal{V}(< \mathcal{L}_0 > P_{0,0} = -\mathcal{V}P_{0,0}$$

According to (2.61), we obtain

$$< \mathcal{L}_0 > P_{0,1} = - < \mathcal{M}_1 > P_{0,0} \tag{2.68}$$

Similarly,

$$P_{0,1} = -(T-t) < \mathcal{M}_1 > P_{0,0} = -(T-t)\mathcal{V}_\infty S\partial_{Sz}^2, \tag{2.69}$$

where $\mathcal{V}_\infty = \bar{\delta}g(z)$.

The accuracy of the asymptotic approximation is given precisely by theorem2.4:

Theorem 2.4. *In the case of option pricing problem with smooth payoff h, there exists a positive constant*

$$|P^{\xi,\sigma} - P^{\tilde{\xi},\sigma}| \leq C(\xi + \sigma + \sqrt{\xi\sigma}), \tag{2.70}$$

with $\xi \in [0,1]$ and $\sigma \in [0,1]$.

2.5   Pricing of the Financial Derivatives

The history of option pricing can date back to early 1950s. In P.Appell, J.Boussinesq and H.Poincare's thesis [51], the authors pointed out that the change of price over small time intervals is independent of present and past values, and they proved that the price increment follows the normal distribution by applying the central limit theorem. They also derived the Chapman Kolmogorov Equation by the Markov property and firstly proposed the concept of arbitrage. The concept of geometric Brownian motion applying in stock price process was first studied in Paul Cootner's paper(1964) [1]. Geometric Brownian motion, also called the Wiener process, is a continuous-time stochastic process and can be applied to sketch the stochastic movement of stock price along with the drift. The binomial option pricing model is an option valuation method proposed by Cox, Ross, and Rubinstein in 1979 [2]. The binomial pricing model traces the evolution of the option price by the means of binomial trees. The benefit of the method is that it can handle various situations, especially the time-dependent options, and it is very straightforward to understand. However, this method is not efficient compared to the Black-Scholes model. The Black-Scholes model is the corner stone of option pricing theory, in which the author assumed that the stock price follows a geometric Brownian motion and evolves continuously [3].

The above methods are all based on the assumption of an efficient market. The efficient market assumes that the interest rate is risk-free, and the volatility is a constant. Also, the stock does not pay any dividend and the market is frictionless with no transaction costs. Under all these strict assumptions, they derived the famous Black-Scholes formula from the corresponding Black-Scholes partial differential equation. Another option pricing method including the risk-neutral measure and martingale pricing theory is widely used, especially when we relax the assumptions of the efficient market. A risk-neutral measure is a probability measure, under which the option price can be calculated by taking the expectation of the discounted share price [30]. Martingale pricing works under the assumption of risk-neutrality and can be applied to a variety of derivatives. The idea of martingale pricing was firstly developed by Harrison and Kreps, et.al. [52].

By applying the martingale method, the expectation of martingale price can be calculated by Monte Carlo simulation. The Monte Carlo simulation method was first applied to price European option in 1977 [4]. However, European style options are time independent. In terms of the time-dependent model, Broadie

and Glasserman priced Asian option by Monte Carlo simulation in 2001, Longstaff and Schwartz developed a Monte Carlo method to price American-style option with early exercise [5] [6]. By applying the Monte Carlo approach to evaluate the option price, we have to discretize the stochastic differential equation and generate random numbers. There are two ways to discretize the stochastic differential equation, including the Euler-Maruyama and the Milstein Schemes. The difference between these two methods is the convergence rate; the Euler-Maruyama has a convergence rate of order 1/2, while Milstein has a strong convergence of order 1 by adding a correction to the Euler-Maruyama method. The accuracy of the Monte Carlo approach is proportional to $\delta/\sqrt{n}$, where n denotes the sample volume and δ denotes the sample variance. Two different ways can be applied to enhance the accuracy of Monte Carlo methods. The simplest way is to increase the sample number, and we usually set $n = 10,000$ in our work. Another way is to apply the variance reduction technique. The disadvantage of Monte Carlo Simulation is that compared to analytical methods, the execution time is too long and grows exponentially with dimension. Besides, the Monte Carlo method is not an ideal approach to simulate variance and volatility.

Partial Differential Equations(PDEs) play a significant role in option pricing. The Black-Sholes formula can be obtained by solving the underlying PDE. Most stochastic differential equations(SDE) have their corresponding PDEs under the risk-neutral assumption, and they can be derived from the Ito formula and the Feynman-Kac theorem. The Ito lemma is widely employed in option pricing and provides us a way to find differential of a function with stochastic variables. The Feynman-Kac formula connects the SDE with PDE, and it has been proved that the solution of the corresponding PDE is equivalent to the expectation of the payoff function under risk-neutral measurement, as detailed in [7]. To solve the corresponding PDE, there are four ways worth to mention about. Most works of literature apply Fourier transform and obtain an analytic or semi-analytic solution of the governing PDE. Fourier transform has been applied in the field of Finance by Merton in 1973 [14]. Stein and Stein(1991) applied the Fourier transform method to find the distribution of the stochastic volatility model [11]. Heston(1993) applied the inversed Fourier transform, along with the characteristic function, to find semi-analytic solutions for an European style option [10]. In 2000, Bakshi and Madan(2000) laid the foundation for characteristic functions and extended the valuation formula that could be applied in other more complex payoff functions [53]. A more comprehensive survey is made by Duffie, especially on the incorporation of exponential affine jump diffusions [45]. A pioneer work of

fast Fourier transform(FFT) was done by Carr and Madan(1999), from which they mapped the Fourier transform directly to call option prices via the characteristic function of the underlying price process [54]. The FFT is a fast algorithm of discretized Fourier transform(DFT). This method is then extended by Carr and Wu(2004) to a more generalized model with time changed Levy processes and generalized affine models [55].

In financial pricing problems, most problem of solving stochastic differential equations can be converted to problems of solving the associated partial differential equations under risk-neutral assumption, or partial integral differential equations(PIDE) arising from the jump-diffusion model. Consequently, numerical approaches are applied to approximate the solution of the boundary and initial value problems(BIVP). Three most widely used approaches are the finite difference method(FDM), the finite element method(FEM), and the finite volume method(FVM).

FDM is by far the most popular one with simplest discretization form. The classical Black-sholes formula is a convection-diffusion parabolic equation, and the finite difference scheme has been studied in detail by Duffy [56]. In Hull and White's paper [57], the authors suggested a modification to the explicit finite difference method for valuing derivatives, which leads to a more accurate approximation with small time steps, and the established approach was used to value bond options under two different interest rate processes [58]. Rama Cont and Ekaterina Voltchkova presented a finite difference method(FDM) to price the PIDE arising from a jump-diffusion model, and the authors also proposed an explicit-implicit(IMEX) FDM scheme for pricing European and Barrier options with Levy process. The IMEX splits the time step, and solves the stiff matrix implicitly and the nonstiff matrix explicitly. Convergence and stability are also considered in their work [59]. The FDM approach, together with a front fixing method, was applied by Wu and Kwok to price American option and generate the optimal boundary [60]. Ikonen and Toivanen proposed an operating splitting method for solving the linear complementarity problem arising from American option, and their approach is approved to be more efficient [61]. Another important paper of the application of FDM method is due to Leif Andersen and Jesper Andersen [62], from which the alternating direction implicit method(ADI) was applied to solve the PIDE arising from the Poisson jump. The ADI approach is proved to be unconditionally stable and efficient when it is combined with the FFT methods. For other exotic options, Little& Pant(2001) [63] applied the finite difference method(FDM) to solve the variance swaps problem based on the

assumption of constant volatility, in which a two-dimensional(2D) problem is reduced to a system of one-dimensional partial differential equations, and the price of variance swap is obtained as an average of all the solutions.

The Finite element method(FEM) ensures more flexibility and adaptivity of mesh compared to the FDM. The FEM is suitable for pricing almost all option types. Achdou illustrated three simple applications of the FEM approach in option pricing, including the standard BS equation, the stochastic volatility model, and the path-dependent Asian option [64]. The FEM was also applied in studying the multi-asset American type options by Pavlo KovalovVadim Lipetsk [65]. By adding the penalty term with continuous Jacobian and solving the final ordinary differential equations(ODEs) with an adaptive variable order and variable step size solver SUNDIALS, they proved that their approach is efficient even for multi-dimensional PDEs.

In finance, it is useful for pricing the Asian options when the PDE becomes hyperbolic near maturity. The study of the FVM for derivative price is more advanced and some novel results were obtained by Wang(2004) [66]. In 2007, Angermann and Wang extended the fitted finite volume spatial discretization to both European option and American option, and the convergence of the method is proved in the reference [67]. All these numerical methods result in ODE systems with respect to time.

For two scale volatility models, a specific approach named perturbation is applied. The multiscale model introduced above has two stochastic volatility factors, including the fast scale volatility, and the slow scale volatility. Under the risk-neutral assumption, the option price can be obtained as a solution of the corresponding partial differential equation. Thus, the multiscale model is changed to solving a high dimensional partial differential equation with small parameters, which can be viewed as a combination of singular and regular perturbations, and the asymptotic approach can be applied to derive an approximation of option price. The main idea behind this approach is to discretize the option price into a zero-order term and correction terms. The zero-order term, also called the leading term, is calculated by the underlying asset process with long-term constant volatility. The correction terms, including the fast-scale correction and slow scale correction, are calculated by expanding the operator of the PDE into different power orders. By doing so, the high dimensional problem reduces to lower dimensional linear problems. An advantage of the perturbation analysis is that we do not have to calibrate every parameter of the model, but only a few parameters regarding the volatility skew is needed, and thus, we simplify

the problem by a large extent. The perturbation technique was firstly adopted by Fouque in 2000 [68], from which the author derived the first-order approximation with fast-scale correction. The interest models such as the Vasicek or the CIR model with the fast mean-reverting stochastic volatility were also studied by Peter Cotton, Fouque and Papanicolaou(2001) [69]. They proved that small correction can affect the shape of the term structure of interest rate. The short correction of path-dependent American option was studied by Fouque, Papanicolaou and Sircar in 2000 [70]; by applying the asymptotic approximation, the governing two-dimensional free-boundary problem was reduced to two one-dimensional PDEs subject to free-boundary conditions . Regular perturbations can be found in Fournie et al.(1997) [71], Sircar and Papanicolaou(1999) [72], and also Hull and White(1988) [57]. Fouque included both fast-scale and slow-scale volatilities in the stochastic volatility model, and the techniques of singular perturbation and regular perturbation are combined to approximate the solution of the multiscale volatility model [73]. The multiscale model can also be applied to study default models and credit derivatives. In Fouque [74], et al. the authors studied the specific credit derivative contract CDO under the framework of singular-regular perturbation and the impact of volatility scales on the default distribution of the set of firms. The asymptotic technique can also be applied to price exotic options, Asian option driven by the stochastic volatility with different time scales. Incorporation of two scale processes in the Asian option will result in a four-dimensional PDE. Using singular-regular perturbation, together with the change of numeraire, will reduce the dimension of the problem [75]. The asymptotic techniques we mention above are all first-order correction techniques. For the second-order correction techniques, we refer the reader to a more recent work of Fouque et. al. [76]. Asymptotic analysis extending to second order approximation will bring the difficulty of terminal layer regarding the singular perturbation, which is solved by imposing the average terminal condition according to the ergodic theorem [77]. The multi-scale volatility can also fit in the portfolio optimization, and the corresponding PDE portfolio optimization problem is a non-linear Hamilton-Jacobi-Bellman PDE. By applying the singular-regular perturbation analysis together with the Taylor approximation, the high dimensional non-linear PDE can be reduced to a low-dimensional linear PDE problem, which is much easier to handle.

2.6 Concluding Remark

In this section, we review both the financial and mathematical essentials, which will be used later in the subsequent chapters. The relationship between the partial differential equation and the option pricing is detailed in this chapter. According to the Feynman-Kac theorem, the price of the derivatives can be determined by a PDE under the risk-neutral assumption. Only if the stochastic process is a martingale, the expectation of discounted payoff function can be solved from the PDE. Several mathematical approaches can be used to solve the underlying partial differential equation, including the Fourier transformation method, the finite difference method, the finite element method, and the asymptotic method. Literature of derivative pricing problems has been reviewed. Though the extension of the classical Black-Scholes model has been studied for years, further research is still worthwhile to make the model more realistic. Thus, in the upcoming chapters, more realistic models are developed and solved both analytically and numerically.

CHAPTER 3

Option pricing under the jump diffusion and multifactor stochastic processes

3.1 General

In this Chapter, we incorporate both multi-scale volatility processes and jump diffusion process to price European options and discretely-sampled variance swaps and solve the corresponding partial integral differential equation(PIDE) by absorbing the integral term into the test function of the FEM approach. Inclusion of both the two-factors and the jump diffusion in the model results in a high dimensional partial integral differential equation(PIDE), which is difficult to solve both numerically and analytically. In order to reduce the dimension, we embedded our variance swap problem into Little and Pant's framework with some modification. The payoff function of the variance swap is treated as a function of the current stock price and the previous stock price, with the former following a stochastic process, while the later being determined at the current time. In this case, four three-dimensional PIDEs are reduced to a three dimensional partial differential equation(PDE) in two different periods. For numerical solutions, we apply the finite element method(PDE) to solve the partial integral differential equation system. The chosen element is eight-nodal hexahedron, which can be seen as a tensor product of three one-dimensional iso-parametric elements. This largely simplifies the problem by absorbing the integral part in only one tensor(one dimensional problem). The rest of the chapter is organised as follows. Section (3.2) introduces the mathematical model and formulation. Section (3.3) presents the numerical algorithm we apply in this model. Numerical results are given in section (3.4), followed by a conclusion in section (3.5).

3.2 Mathematical Formulation

The price of stock is assumed to follow the following stochastic process,

$$dS = rSdt + f(y, z)Sdw_t^{(0)} + SdJ^S, \tag{3.1}$$

where $f(y, z)$ is a function of y and z which denote respectively the fast and slow scale volatilities. If $f(y, z) = \sqrt{y} + \sqrt{z}$, the volality process is formed by a CIR process; if $f(y, z) = \sqrt{y} + 1/\sqrt{z}$, the volatility is a 4/2 process, which can be viewed as a combination of the CIR process and the 3/2 process, and the assumption is in line with the consideration that the volatility should not be too close to zero [13]. It is assumed that y and z follow the stochastic processes

$$dy = \frac{1}{\xi}\alpha(y)dt + \frac{1}{\sqrt{\xi}}\beta(y)dw_t^{(1)}, \tag{3.2}$$

$$dz = \sigma c(z)dt + \sqrt{\sigma}g(z)dw_t^{(2)}. \tag{3.3}$$

The fast-scale and slow-scale volatilities are distinguished by the frequencies of the observed volatility data, and Chacko and Viceria(2005) [37] suggested to consider these volatilities simultaneously. Additionally, we assume that the Brownian motion $(w_t^{(0)}, w_t^{(1)}, w_t^{(2)})$ are correlated with the following correlation $Cov(w_t^{(0)}, w_t^{(1)}) = \rho_1, Cov(w_t^{(0)}, w_t^{(2)}) = \rho_2$ and $Cov(w_t^{(1)}, w_t^{(2)}) = 0$ for simplicity.

In this chapter, we consider both the European option and the variance swap. For the case of European put option, the payoff function at the maturity time is

$$U(T, S, y, z) = max\{K - S, 0\}. \tag{3.4}$$

Variance and volatility swaps are well known financial derivatives which allow investors to trade the realized volatility against the current implied volatility. Different from European options, variance swap and volatility swap are time-dependent. This phenomenon indicates that the variance swap will boost the gains and discount the losses, which explains why the variance swap is more attractive than the volatility swap. The difference between the realized volatility and the implied volatility is that the realized volatility σ_R^2 is calculated by applying the historical data of option prices, while the implied one is derived from the prices of options.

The realized volatility is commonly approximated by the following two for-

mulas:

$$\sigma_R^2 = \frac{AF}{N} \sum_{i=0}^{N-1} \left(\frac{S_{i+1} - S_i}{S_i}\right)^2, \tag{3.5}$$

or

$$\sigma_R^2 = \frac{AF}{N} \sum_{i=0}^{N-1} \left(ln(\frac{S_{i+1}}{S_i})\right)^2, \tag{3.6}$$

where S_{i+1} denotes the underlying stock price at the $(i+1)$th time step, AF is the annualized factor and $AF = 12$ if the sampling frequency is every month. In this chapter, we let $AF = \frac{N}{T}$ as a simplification. The payoff of the variance swap is

$$V(T, x, y, z) = L * E^Q(\sigma_R^2 - K), \tag{3.7}$$

which is equal to zero under the assumption of zero entry costs. Therefore, the fair strike price can be defined as $K = E^Q[\sigma_R^2]$. As a result, the variance swap pricing problem becomes calculating the expected value of the realized variance in the risk neutral world.

We apply the dimension reduction technique due to Little&Pant[2001] [63] by introducing a new variable I_t driven by the underlying process

$$I_t = \int_0^t \delta(t_{i-1} - \tau)S_\tau d\tau, \tag{3.8}$$

where δ is the Dirac delta function, which means $I_t = 0$ if $t < t_{i-1}$, and $I_t = S_{i-1}$ if $t \geq t_{i-1}$. The terminal condition becomes

$$U_i(T, S, Y, Z, I) = (\frac{S_i}{I_i} - 1)^2. \tag{3.9}$$

For the reason that we are more interested in the relationship between the maturity time and the strike price, we construct a new variable $X = ln(S/I)$, and then obtain

$$U_i(T, S, Y, Z, I) = (e^{X_i} - 1)^2. \tag{3.10}$$

According to the Ito formula and (3.1), we obtain a new process

$$dx = \mu dt + f(y, z)dw_t^{(0)} + dJ, i = 1, 2 \tag{3.11}$$

If the problem in question is an European put option,

$$\mu = \left(r - \frac{1}{2}f^2(y, z) + \lambda(1 - E(e^z))\right), \tag{3.12}$$

and the payoff function is

$$U(T, S, y, z) = max\{K(1 - e^x), 0\}. \tag{3.13}$$

If the investigated problem is a variance swap, we have two different situations,

$$\mu = \mu_1 = \left(r - \frac{1}{2}f^2(y, z) + \lambda(1 - E(e^z))\right), \quad t_{i-1} \le t \le t_i$$

$$\mu = \mu_2 = \left(r - e^x - \frac{1}{2}f^2(y, z) + \lambda(1 - E(e^z))\right), \quad 0 \le t \le t_{i-1}, \tag{3.14}$$

where $E(e^z) = \frac{p\eta_1}{1-\eta_1} + \frac{(1-p)\eta_2}{\eta_2+1}$ if the jump rate follows the double exponential distribution as in Kou's model with the density of

$$p(z) = p\eta_1 e^{-\eta_1 z} I_{z \ge 0} + (1 - p)\eta_2 e^{\eta_2 z} I_{z < 0}. \tag{3.15}$$

In contrast to the model (3.1), which absorbs the jump in the stock process only, the multidimensional jump process is more interesting. With this motivation, we include the jump process in both the stock price process and the multi-scale volatility process, namely

$$dy = \frac{1}{\xi}\alpha(y)dt + \frac{1}{\sqrt{\xi}}\beta(y)dw_t^{(1)} + dJ^Y, \tag{3.16}$$

$$dz = \sigma c(z)dt + \sqrt{\sigma}g(z)dw_t^{(2)} + dJ^Z. \tag{3.17}$$

However, incorporating more factors makes the model harder to tackle with, and thus development of an efficient numerical method for high dimensional PIDE is of great importance.

3.3   Algorithm of FEM

By using the Feynman-Kac theorem, we obtain the following partial differential equation,

$$u_t + \mathfrak{D}u + \mathfrak{C}u + \lambda \int_R [u(x + \eta) - u(x)]\Gamma(d\eta) - ru = 0 \tag{3.18}$$

with the infinitesimal generator of the three-dimensional Markov process (x_t, y_t, z_t). Let $\tau = T - t =$ time to expiry, we obtain

$$u_\tau - \mathfrak{D}u - \mathfrak{C}u - \lambda \int_R [u(x+\eta) - u(x)]\Gamma(d\eta) + ru = 0, \qquad (3.19)$$

with

$$\mathfrak{D}u(x) = \frac{1}{2}f^2(y,z)U_{xx} + \frac{1}{2}\frac{1}{\xi}\beta^2(y)U_{yy} + \frac{1}{2}\sigma g(z)U_{zz} +$$

$$\rho_1 \frac{1}{\sqrt{\xi}}\beta(y)f(y,z)U_{xy} + \rho_2\sqrt{\sigma}f(y,z)g(z)U_{xz} + \rho_{12}\frac{\sqrt{\sigma}}{\sqrt{\xi}}\beta(z)g(z)U_{yz}, \qquad (3.20)$$

$$\mathfrak{C}u(x) = \mu U_x + \frac{1}{\xi}\alpha(y)U_y + \sigma c(z)U_z + \lambda \int_R U(x+\eta)\Gamma(d\eta) - rU, \qquad (3.21)$$

which can be rewritten in vector form by

$$\frac{\partial u}{\partial \tau} - \nabla \cdot \bar{A}\nabla u - D \cdot \nabla u + (r+\lambda)u - \lambda \int_R u(x+\eta)\Gamma(d\eta) = 0; \qquad (3.22)$$

where

$$\bar{A} = \begin{bmatrix} \frac{1}{2}f^2(y+z) & \frac{1}{2}\frac{1}{\sqrt{\xi}}\rho_1\beta(y)f(y,z) & \frac{1}{2}\sqrt{\sigma}\rho_2g2(z)f(y,z) \\ \frac{1}{2}\frac{1}{\sqrt{\xi}}\rho_1\beta(y)f(y,z) & \frac{1}{2}\frac{1}{\xi}\rho_1\beta^2(y) & \frac{1}{2}\rho_{12}\frac{\sqrt{\sigma}}{\sqrt{\xi}}\beta(z)g(z) \\ \frac{1}{2}\sqrt{\sigma}\rho_2g(z)f(y,z) & \frac{1}{2}\rho_{12}\frac{\sqrt{\sigma}}{\sqrt{\xi}}\beta(z)g(z) & \frac{1}{2}\sigma\rho_2g^2(z) \end{bmatrix},$$

$$D = \begin{bmatrix} \mu_i \\ \frac{1}{\xi}\alpha(y) \\ \sigma c(z) \end{bmatrix}, \quad i = 1, 2.$$

In order to obtain option price, we have to solve the differential equation (3.22). However, different from μ_1, μ_2 is a dynamic process which is related to time. Let $n = \frac{T}{\nabla t}$, than (3.22) is divided into n different partial differential equations. We can then solve them one by one and then substitute the solutions back into (3.7) to obtain the σ_R^2.

The weak form of (3.22) can be written as,

$$\int_\Omega \left(\frac{\partial u}{\partial \tau} - \nabla \cdot \bar{A}\nabla u - D \cdot \nabla u + (r+\lambda)u - \lambda \int_R u(x+\eta) \right) v d\Omega = 0 \qquad (3.23)$$

Thus, by applying Green's Theorem, we obtain

$$(\frac{\partial u}{\partial \tau}, v) + (\bar{A}\nabla u, \nabla v) - (D \cdot \nabla u, v) - \lambda(\int_R u(x+\eta)\Gamma(d\eta), v) + (r+\gamma)(u, v) = 0 \quad (3.24)$$

which is derived by using the divergence theorem $\int_\Omega \bar{A}\nabla u \cdot \nabla v + \nabla \cdot \bar{A}\nabla uv d\Omega = \oiint \bar{A}\nabla uv \cdot \vec{n} dS$, and we assume that the test function vanishes on the boundary, and (a, b) denotes inner product.

Let $u = \sum_{i=1}^n u_i(\tau)\phi_i$, $v = \sum_{j=1}^n u_j\phi_j$, then we obtain the following ODE system,

$$M\dot{u} + Du - Cu - Bu = 0, \quad (3.25)$$

where the mass matrix $M = \sum_{i=1}^n (\phi_i, \phi_j)$, the matrix of the diffusion part $D = \sum_{i=1}^n (A\nabla\phi_i, \nabla\phi_j)$, the matrix of the convection part $C = \sum_{i=1}^n (D \cdot \nabla\phi_i, \phi_j)$, $A = r\sum_{i=1}^n (\phi_i, \phi_j), B = \sum_{i=1}^n (\mathfrak{B}\phi_i, \phi_j)$ denotes the matrix of the integral part,

$$\mathfrak{B}\phi_i = \lambda \int_R \phi_i(x+\eta, y, z)\Gamma(d\eta) = \lambda \int_R \phi_i(x+\eta, y, z)p(\eta)d\eta \quad (3.26)$$

By applying the 8 node hexahedral elements,

$$\phi_i^e = \frac{1}{8}(1 + \epsilon\epsilon_i)(1 + \eta\eta_i)(1 + \zeta\zeta_i), \quad (3.27)$$

with ϵ_i, η_i, ζ_i denoting the natural coordinates of the ith nodes. To be specific, $[x_i, x_{i+1}]$ is mapped to $[-1, 1]$.

Therefore, the basis function can be seen as the tensor product of three one-dimensional linear elements,

$$\phi_i^e(x, y, z) = \phi_i^e(x)\phi_i^e(y)\phi_i^e(z) \quad (3.28)$$

Substituting (3.28) into (3.26), the integral term can be rewritten as

$$\mathfrak{B}\phi_i = \lambda \int_R \phi_i(x+\eta, y, z)p(\eta)d\eta = \lambda \int_R \phi_i(x+\eta)p(\eta)d\eta\phi_i(y)\phi_i(z) = \Phi_i(x)\phi_i(y)\phi_i(z),$$
$$(3.29)$$

with the function $\Phi_i(x) = \lambda \int_R \phi_i(x+\eta)p(\eta)d\eta$ approximating by the finite element interpolation,

$$\Phi_i(x) \approx I_n\Phi_i(x) = \sum_l \Phi_i(x_l)\phi_l(x), \quad (3.30)$$

where

$$
\begin{aligned}
\Phi_i(x_l) &= \int_R \phi_i(x_l + \eta) p(\eta) d\eta \\
&= \int_{x_i}^{x_{i+1}} \phi_i(\bar{x}) p(\bar{x} - x_l) d\eta \\
&= \frac{h}{2} \int_{-1}^{1} \phi_i(\xi) p((\frac{\xi}{2} + i - l)h) d\xi \\
&= \frac{h}{4} \int_0^1 \xi p((\frac{\xi}{2} + i - l - \frac{1}{2})h) d\xi + \frac{h}{4} \int_0^1 (1 - \xi) p((\frac{\xi}{2} + i - l)h) d\xi, \quad (3.31)
\end{aligned}
$$

where $p(\cdot)$ is a double exponential density function and according to (3.28), $\Phi_i(x_l)$ is determined by the relationship between integers i and l. By simple calculation, we obtain

$$
\begin{cases}
\frac{p\lambda}{4\eta_1 h} e^{-\eta_1(i-l-1)h} \left(e^{-\frac{\eta_1 h}{2}} - 1\right)^2 & i - l \geq 1 \\
\frac{1}{4}\lambda + \frac{p\lambda}{4\eta_1 h}\left(e^{-frac\eta_1 h 2} - 1\right) + \frac{(1-p)\lambda}{4\eta_2 h}\left(e^{-\frac{\eta_2 h}{2}} - 1\right) & i = l \\
\frac{(1-p)\lambda}{4\eta_2 h} e^{-\eta_2(i-l-1)h} \left(e^{-\frac{\eta_2 h}{2}} - 1\right)^2 & i - l \leq -1
\end{cases}
$$

Therefore, B can be seen as a kronnecker product of inner products in three dimensions,

$$
B = B_x \otimes B_y \otimes B_z = \sum_{i=1}^{n} (\Phi_i(x), \phi_j(x))(\phi_i(y), \phi_j(y))(\phi_i(z), \phi_j(z)), \quad (3.32)
$$

By allowing the jump term in the volatility processes, as shown in model (3.16), we obtain B in the following form under the assumption of independence,

$$
B = B_x \otimes B_y \otimes B_z = \sum_{i=1}^{n} (\Phi_i(x), \phi_j(x))(\Phi_i(y), \phi_j(y))(\Phi_i(z), \phi_j(z)), \quad (3.33)
$$

Let $R = D - C + B$, then (3.25) can be written as

$$
M\dot{u} + Ru = 0. \quad (3.34)
$$

To solve the ODE system (3.34), we simply apply the backward Euler method, considering its unconditional stability property,

$$
(\frac{M}{\triangle t} + R)U_{n+1} = U_n. \quad (3.35)
$$

3.4 Numerical results and Discussion

In this section, we present our numerical results for European options and variance swaps taking into account both multiscale volatility and jump properties. Firstly, we start simulating both the stock price process and the multiscale volatiliy processes to show the motivation of our study. Then we apply the FEM algorithm to solve the three dimensional PIDE. The validity of our algorithm is verified by comparing our results with the results of the two factors Heston Model [78]. Pricing of variance swap is also studied in this chapter as an application.

3.4.1 Validility and Motivation of Our Model

To show the motivation of our model, we firstly apply Monte Carlo simulation to generate a sample path of the stock price. Figure 3.1 is the stock price generated by the model (3.1) and (3.2) by the classic Euler Maruyama Method [79]. As we can see from the figure, the asset process is a martingale process with upward sloping. Figure 3.2 shows the underlying trajectory of the fast scale volatility process, which is highly oscillated due to the small fast-scale rate $\xi = 0.01$. The slow scale volatility is simulated in Figure 3.3 with the slow scale rate $\sigma = 0.01$.
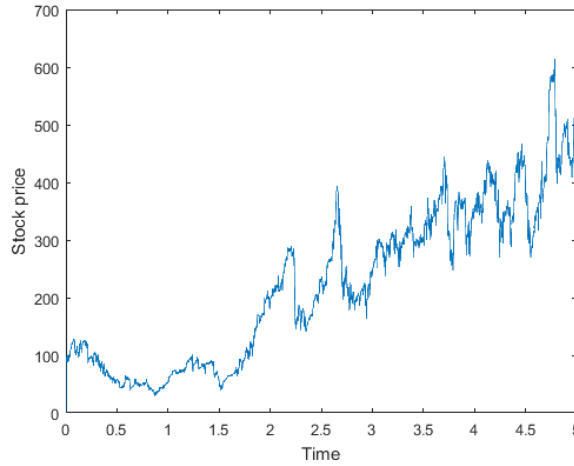
Figure 3.1: Simulation of Stock Price Process

The jump term is driven by the compound possion process, as shown in Figure 3.4 with the jump intensity $\lambda = 15$. The incorporation of the jump process in both stock price processes has practical significance, as shown in Figure 3.5.

In terms of the algorithm validity , we apply our FEM method to solve the model and compare the result with the semi-analytical result shown in [78]. It is
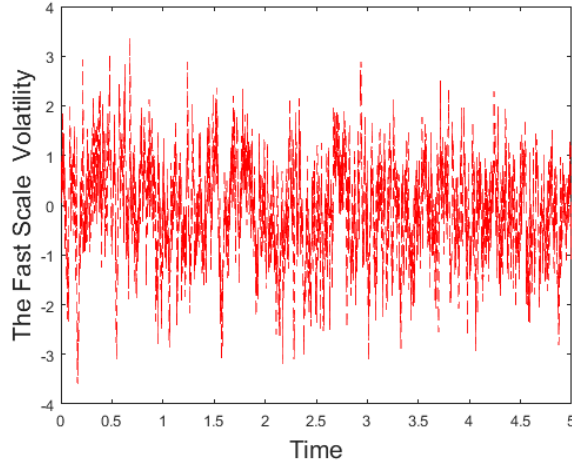
Figure 3.2: Simulation of Fast-scale Volatility Process
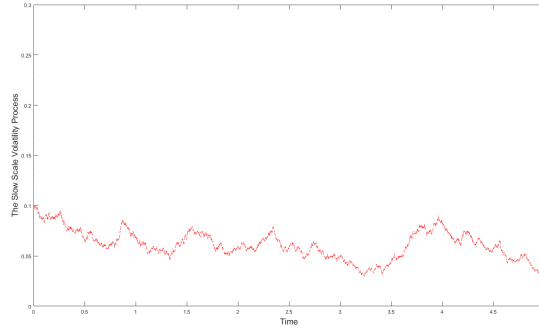
Figure 3.3: Simulation of Slow-scale Volatility Process
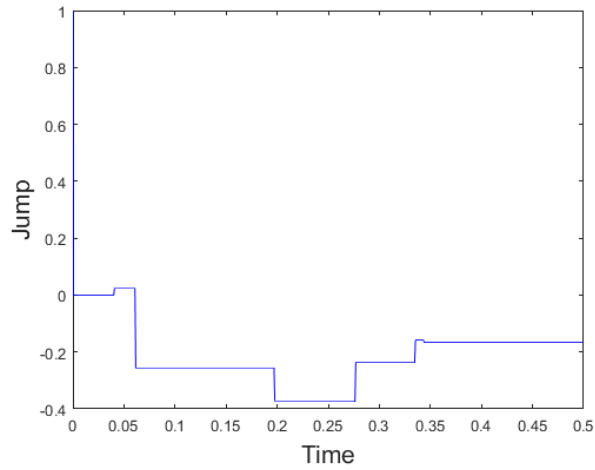
Figure 3.4: Simulation of Jump

seen from figure 3.6 that our result is well fitted.

$$dS = rSdt + \sqrt{V_1}Sdz_1 + \sqrt{V_2}Sdz_2, \qquad (3.36)$$

$$dV_1 = (m_1 - b_1V_1)dt + \delta_1\sqrt{V_1}dz_3, \qquad (3.37)$$

$$dV_2 = (m_2 - b_2V_2)dt + \delta_2\sqrt{V_2}dz_4, \qquad (3.38)$$
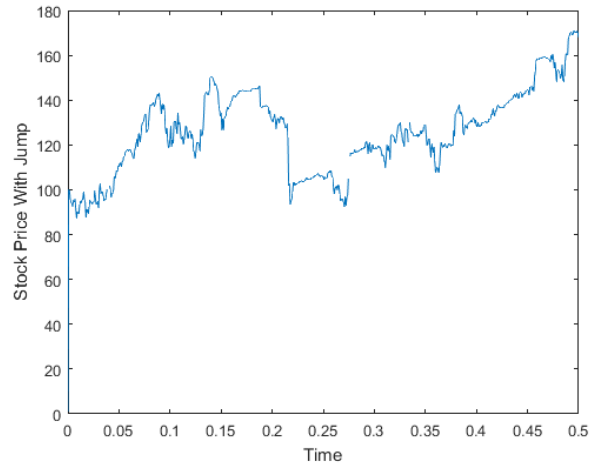
Figure 3.5: Simulation of Stock Price Process with Jump

However, analytic solution only exsited for some special cases if we can find the characterist function. For other models, it is not possible to obtain.
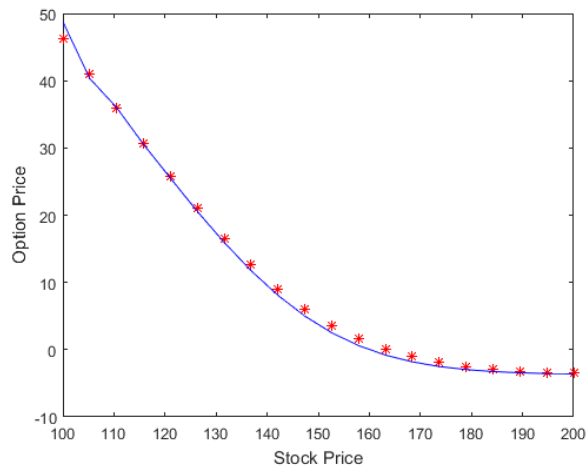
Figure 3.6: Comparison FEM with Semi-analytic solution

3.4.2 The Effects of Multiscale Volatility and Jump Term

Our method is applied to solve the option price of the classical European option model (3.13) as well as the strike price of variance swaps with the payoff function shown in (3.10). To be specific, let $\alpha(y) = k_1(a1 - b1 * y), \beta(y) = \delta_1\sqrt{y}, \alpha(y) = k_2(a2 - b2 * y), \beta(y) = \delta_2\sqrt{z}, f(y, z) = \sqrt{y} + \sqrt{z}$. Both the fast-scale process and the slow scale process are assumed to be mean reverted process. The parameters we selected are from the calibrated results of JP.Fouque et. al. [80]. The parameters are shown in table 3.1. The jump process here is assumed to be a double exponential process with $\eta_1 = 25, \eta_2 = 50, p = 0.3$.

Table 3.1: Parameters of model

k_1	$m1$	ρ_1	σ_1
17.38863	0.04480	-0.99000	3.70537
k_2	$m2$	ρ_2	σ_2
16.20866	0.04275	-0.82897	2.77650

For $\lambda = 0$, our model reduces to the original multiscale volatility model by JP.Fouque, et.al. [73]. Figure 3.7(a) and figure 3.7(c) are the surface plot of the option price and strike price of the variance swap when z is fixed at 0.0278. If both stock price and volatilities are all variables, we obtain the three dimensional plot shown in figure 3.7(b) and figure 3.7(d).

When $\lambda \neq 0$, it can be seen from figure 3.8a and figure 3.8b that jump intensity has significant effect onto option. Hence, our model is more general compared to multi-factor Heston model. The option price increases with the jump intensity λ, mainly because the growth of jump intensity leads to large uncertainty and risk exposure rate, which offers investors more possibilities to be in the money. Also, the jump terms can also be incorporated into both fast scale volatility and slow scale volatility processes. The change of the option price, though small. can be seen from figure 3.9. In figure 3.9 , MSJ denotes the multi-scale stochastic volatility model with jump in the stock price process, MS1J denotes the MSJ model with one jump term in the fast scale volatility, MSV2J denotes the MSJ model incorporating jump terms in each of the three processes.

Different from jump, the effects of stochastic volatility is a combination result. The effects of fast scale rates are displayed in figure 3.10a. while the effects of the slow scale rate are shown in figure3.10b. As we can see from figure 3.10a, the option price increases with the fast-scale rate. While in figure 3.10b, the option price decreases with the slow scale rate, and the effects of the fast-scale rate outweigh the effects of the slow-scale rate in a short period.
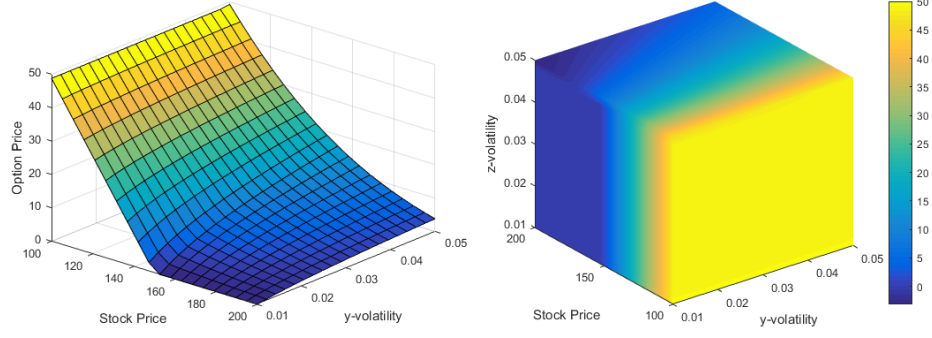
(a) 2d View of European Put Option (b) 3d View of European Put Option
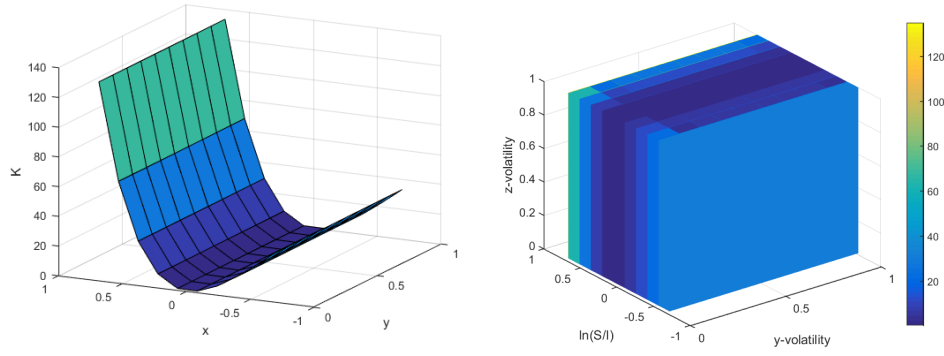
(c) 2d view of Strike Price of Variance Swap

(d) 3d view of strike price

Figure 3.7: Option Price
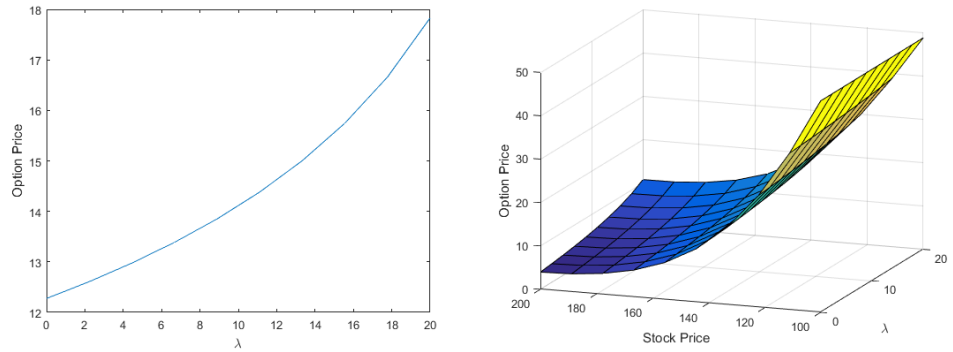
(a) 1d View of Jump Effect (b) 2d View of Jump Effect

Figure 3.8: The Effect of Jump Intensity Rate λ

We also study the fair strike price of variance swap. *Figure 3.11* shows the relationship between the strike price and the maturity time of variance swap, which is anti-correlated due to the introduction of fast and slow scale volatilities. The fast scale and slow scale rate chosen in this analysis is $\xi = 0.01$ and $\sigma = 0.1$ separately. The result verifies that volatility provides a measure of risk exposure.
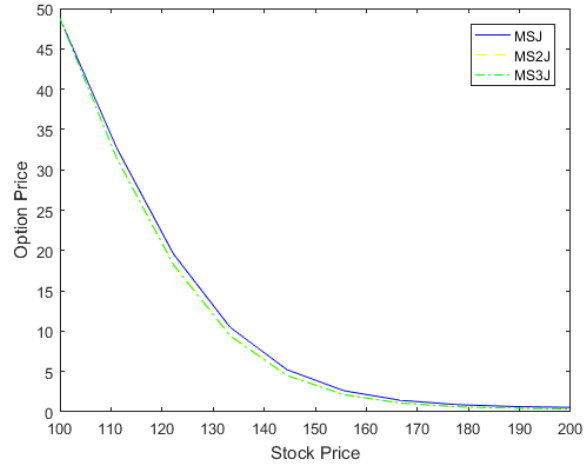
Figure 3.9: Option price Comparison with Jump including in Different Process

(a) Fast-scale Rate Effect   (b) Slow-scale Rate Effect
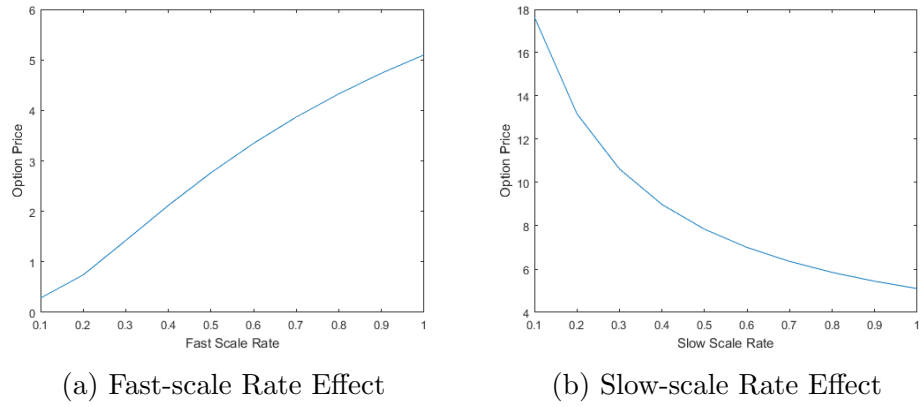
Figure 3.10: Effects of Fast-scale Rate and Slow-scale Rate

The longer the investors hold the contract, the higher risk they are exposed.
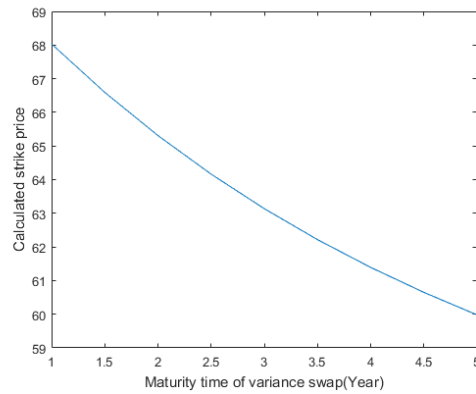
Figure 3.11: Relationship between T and strike price

3.5 Concluding Remarks

In this chapter, we apply the finite element method and the dimension reduction technique to obtain the approximate solution for the price of classical European options and the fair strike price for the prices of variance swaps under both multi-scale stochastic volatility and jump diffusion process. The time scale rate of stochastic volatility is used to describe the long term and short term perturbation of volatility processes. Our numerical results compare well with those by Monte Carlo simulation. Also, we find that the option price increases with the jump rate and volatility value, which is in line with the reality. In terms of the effects of multi-scale volatility, it is a combination result. As assumed in our model, the volatility of the stock process is driven by both fast scale volatility and slow scale volatility. The fast scale volatility is related to the short term volatility with high frequency, while the slow scale volatility is related to the long term volatility and is more smooth. The option price increases with the fast-scale rate and decrease with the slow scale rate. The effect of the slow scale volatility outweighs the effect of the fast scale volatility in a long run. Also, the strike price of variance swap is anti-correlated with the maturity time. Volatility is a measure of risk, and the strike price falls when the maturity time increases. The significance of this work is in two aspects. First of all, the exact solution can only be obtained for specified models. For most PDE, especially the high dimensional ones, closed form solution is hard to obtain, which makes the numerical approach necessary. Besides, even though most work has already considered the stochastic volatility, multi-factors in volatility has not yet been tackled due to the high dimensional difficulties. We combine both multi-scale rate and jump process to make the result more reliable. Furthermore, the numerical method and the dimension reduction technique are established in this chapter and it can also be applied to solve some other three dimensional pricing problems.

CHAPTER 4

The Study of Utility Valuation of Single-name Credit Derivatives with the Fast-scale Stochastic Volatility Correction

†

4.1 General

In this work we study the credit risk pricing problem in the framework of the structural model and utility-based portfolio selection, as the payoff of financial derivatives might be replicated by varying trading strategies of the underlying assets in a complete financial market. The subject of portfolio optimization has a long history dated back to 1971 [81], in which the author provided an explicit scheme to allocate investment capital between risky stocks and riskless bond. Within the framework, the underlying asset was driven by a stochastic process, which was later known as the Black-scholes model. Nonetheless, the main disadvantage of the Black-sholes and Merton's model is the over-restrictive assumptions, especially the assumption of constant interest rate and constant volatility. A great number of extensions had been made in recent years. Heston (1993) [10] took into account the stochastic volatility and derived a semi-analytic solution for the European call option by introducing a characteristic function, allowing the arbitrary correlation between the volatility and asset price. Longstaff and Schwartz (1995)

incorporated stochastic short-term interest rate, which they found was negatively correlated to the asset value process [82]. Fouque et al.(2003) [73] developed an effective approximation of the option pricing problem through the incorporation of the multiscale volatility. However, attach the corresponding partial differential equation for option pricing is linear, the equation related to the optimal control problem is non-linear. For this reason, the asympotic theory was extended to estimate the non-linear pricing problem by Fouque et al. (2015) [83].

The valuation mechanism used in our work is called indifference prices. The so called indifference price is the amount of capital that the investor pays today, so that difference between holding or not holding the derivatives is trivial. The indifference approach was first introduced by Hodges and Neuberger(1989) [84] and extended by Davis and Yoshikawa (2012) [85]. Its mechanism is based on the utility function that is a twice continuously-differentiable one strictly increasing and concave. Herein we consider the risk attitude of individuals by applying the utility based models, and specifically assess the single-name credit default swap (CDS) that could be treated as an insurance against the default of a reference entity. CDS is written on a single-bond issued by a reference entity. The buyer pays the seller a risk premium regularly and they in turn will get compensation if default happens. More details can be found in the work of Papageorgiou and Sircar(2008) [86]. In comparison with the aforementioned work, our work mainly features the following aspects. Firstly, we study the credit-derivatives pricing considering the impact of both the default risk and fast-scale stochastic volatility. Then, the problem is solved within the framework of utility-based portfolio selection, which will lead to a high dimensional non-linear partial differential equation (PDE). As high dimensional non-linear PDEs are hard to solve via existing methods, we then apply asymptotic approximation to reduce the high dimensional non-linear PDE into low dimensional PDEs. Then, we present and analyse our results in two specific cases and numerically analyse them.

4.2   Mathematical Model

Generally, there are two approaches for pricing credit derivatives, including the structural model and the intensity-based model. Our work here is mainly based on the intensity-based model(or reduced form model), in which defaults happen at the jump process of Poisson intensity . We start our model with simple single-name default-able bonds with fast stochastic volatilities and then extend it to multi-name and multi-scale cases.

Unlike the traditional structural model, our model is based on the assumption that default happens at an unpredictable stopping time τ with stochastic intensity process λ, which incorporates information from the firm's stock price S and is called a hybrid model. The stock price S follows a geometric Brownian motion with the intensity process $\lambda(Z_t)$,where $\lambda(\cdot)$ is a non-negative, locally Lipschitz, smooth and bounded function. Our model takes the following form:

$$\frac{dS_t}{S_t} = \mu(Y_t)dt + \sigma(Y_t)dW_t, \tag{4.1}$$

$$dY_t = \frac{1}{\xi}b(Y_t)dt + \frac{1}{\sqrt{\xi}}a(Y_t)dW_t^{(1)}, \tag{4.2}$$

$$dZ_t = g(Z_t)dt + c(Z_t)dW_t^{(2)}, \tag{4.3}$$

where the Browning motion $W_t, W_t^{(1)}, W_t^{(2)}$ are correlated as follows:

$$Cov(W_t, W_t^{(1)}) = \rho_1, Cov(W_t, W_t^{(2)}) = \rho_2, Cov(W_t^{(1)}, W_t^{(2)}) = \rho_{12}, \tag{4.4}$$

in which ρ_1 measures the correlation between the Brownian motion for the volatility Y and the Brownian motion for the stock prices, ρ_2 measures the instantaneous correlation between the Brownian motion for the stock price S and the Brownian motion for the intensity process Z, and they satisfy $|\rho_1| < 1, |\rho_2| < 1, |\rho_{12}| < 1$, and $1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$. When the parameter ξ is small, the stochastic processes Y_t and Z_t represent the fast volatility process and the intensity process, respectively. Here we assume that $Y_{\frac{t}{\xi}}^{(1)}$ is an ergodic diffusion process and has the same unique invariant distribution as Y_t , and for more details we refer the reader to Section 4 of the reference by Fouque et al. [77]. The drift part of dY_t is always assumed to be mean-reverted with the long term drift θ, while the volatility of volatility could be a constant σ so that the underlying distribution of dY_t is a normal distribution. However, other specific forms can also be fit in volatility, like the CIR process, the $\frac{3}{2}$ stochastic volatility process and the $\frac{4}{2}$ stochastic volatility process. In our work, we assume the constant volatility of volatility for simplicity. The default time τ of the firm is defined by the first time when the cumulated intensity reaches the random threshold ε.

$$\tau_t = inf\{s \geq t : \int_t^s \lambda(Z_s)ds = \varepsilon\}. \tag{4.5}$$

4.2.1 Maximal Expected Utility Problem

Let X_t be the wealth process and π_t denote the money we invest in the stock at time t, where $t \in [0, T], t < \tau \wedge T$, then the wealth process is as follows:

$$
\begin{aligned}
dX_t &= \pi_t \frac{dS_t}{S_t} + r(X_t - \pi_t)dt \\
&= (rX_t + \pi_t(\mu(Y_t) - r))dt + \pi_t \sigma(Y_t)dW_t, \tag{4.6}
\end{aligned}
$$

where π_t is $\mathcal{F}_t$-measurable and satisfies the integrability constraint $E \int_0^T \pi_s^2 ds < \infty$. Under the utility form $\tilde{U}(X)$, the maximum expected utility payoff takes the general form of

$$
sup_{\pi_t \in \mathcal{A}} E\{\tilde{U}(e^{-rT}X_T)\mathbf{1}_{\{\tau > T\}} + \tilde{U}(e^{-r\tau}X_\tau)\mathbf{1}_{\{\tau \le T\}}\}, \tag{4.7}
$$

where $\mathcal{A}$ is the set of π.

To simplify the formulation, we denote $e^{-rT}X_t$ by X_t and $\mu - r$ by μ, then the wealth process can be described by

$$
dX_t = \pi_t \mu(Y_t, Z_t)dt + \pi_t \sigma(Y_t, Z_t)dW_t^{(1)}. \tag{4.8}
$$

If default happens, stock of the firm cannot be traded, and investors have to liquidate holdings in the stock and deposit them in the bank account. For simplicity, we assume that the investors get full amount of the liquidated pre-default stocks and invest all of them into the bank account. Therefore, we obtain

$$
X_T = X_\tau e^{r(T - \tau)}. \tag{4.9}
$$

The problem here is to maximize the expected utility payoff at time zero, which takes the form as follows:

$$
V(t, x, y, z) = sup_{\pi \in \mathcal{A}} E\{\tilde{U}(X_T)\mathbf{1}\{\tau_t > T\} + \tilde{U}(X_{\tau_t})\mathbf{1}\{\tau_t \le T\} \mid X_t = x, Y_t = y, Z_t = z\} \tag{4.10}
$$

Proposition 4.1. *The HJB equation of the value function is*

$$
V_t + \frac{1}{\xi}\mathcal{L}\dagger V + \mathcal{L}\ddagger V + \frac{1}{\sqrt{\xi}}\rho_{12}a(y)c(z)V_{yz} + max\{\pi\mu(y)V_x + \frac{1}{2}\pi^2\sigma^2(y)V_{xx}
$$
$$
+ \frac{1}{\sqrt{\xi}}\pi\rho_1\sigma(y)a(y)V_{xy} + \rho_2 V_{xz}\pi\sigma(y)c(z)\} + \lambda(z)(\tilde{U}(x) - V) = 0 \tag{4.11}
$$

with $V(T, x, y, z) = \tilde{U}(x)$ and the operators $\mathcal{L}\dagger$ and $\mathcal{L}\ddagger$ are defined by

$$\mathcal{L}\dagger = b(y)\frac{\partial}{\partial y} + \frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2} \tag{4.12}$$

$$\mathcal{L}\ddagger = g(z)\frac{\partial}{\partial z} + \frac{1}{2}c^2(y)\frac{\partial^2}{\partial z^2}. \tag{4.13}$$

where x represents the wealth process, y is a stochastic volatility process, and z is an intensity process.

Proof. The proof follows by the extension of the arguments used in Theorem 4.1 of Duffie and Zariphopoulou (1993) [87] and thus is omitted here. For more details and applications, we refer the reader to Sircar and Zariphopoulou (2007) [88] , Sircar and Zariphopoulou(2010) [89], and Brémand (1981) [90]. □

4.2.2 Bond Holder's Problem and Indifference Price

In this section we assume that the investor owns a bond of the firm, which is defaultable and pays 1 dollar at maturity. We then construct a similar problem, i.e.,

$$U(t, x, y, z) = sup_{\pi \in \mathcal{A}}E\{\tilde{U}(X_T+c)\mathbf{1}\{\tau_t > T\}+\tilde{U}(X_{\tau_t})\mathbf{1}\{\tau_t \leq T\} \mid X_t = x, Y_t = y, Z_t = z\} \tag{4.14}$$

where c denotes e^{-rT} .

Proposition 4.2. *The HJB equation of Bond Holder's value function is*

$$U_t + \frac{1}{\xi}\mathcal{L}\dagger U + \mathcal{L}\ddagger U + \frac{1}{\sqrt{\xi}}\rho_{12}a(y)c(z)U_{yz} + max\{\pi\mu(y)U_x + \frac{1}{2}\pi^2\sigma^2(y)U_{xx}$$

$$+\frac{1}{\sqrt{\xi}}\pi\rho_1\sigma(y)a(y)U_{xy} + \rho_2 U_{xz}\pi\sigma(y)c(z)\} + \lambda(z)(\tilde{U}(x) - U) = 0, \tag{4.15}$$

with $U(T, x, y, z) = \tilde{U}(x + c)$.

We can then have the following definition

Definition 1.1. *The indifference price to an investor is defined at time zero by*

$$V(0, x, y, z) = U(0, x - p_0, y, z), \tag{4.16}$$

which aims to keep the utility indifference between holding or not holding the bond. The bond holder should lower the initial wealth level. And the yield spread is defined as

$$y_0(T) = -\frac{1}{T}log(p_0(T)) - \gamma, \tag{4.17}$$

which is non-negative for all $T > 0$ and $p_0(T)$ represents the indifference price at time T.

4.3 Asymptotic approximation

We start our analysis under exponential utility, as we found that the analytic form of solution can be obtained for an exponential affine structure. Analysis for the problem with the constant-relative risk aversion(CRRA) utility is shown in section 4.6. By the necessary condition for extreme values, we obtain the maximizer π^* for the optimization problem (4.11), namely

$$\pi^* = -\frac{\frac{1}{\sqrt{\xi}}\rho_1\sigma(y)a(y)V_{xy} + \mu(y)V_x + \rho_2\sigma(y)c(z)V_{xz}}{\sigma^2(y)V_{xx}}. \tag{4.18}$$

Substituting (4.18) into (4.11), we obtain the following non-linear PDE,

$$
V_t + \frac{1}{\xi}\mathcal{L}\dagger V + \mathcal{L}\ddagger V + \frac{1}{\sqrt{\xi}}\rho_{12}a(y)c(z)V_{yz}
$$
$$
- \frac{[\theta(y)V_x + \frac{1}{\sqrt{\xi}}\rho_1 a(y)V_{xy} + \rho_2 c(z)V_{xz}]^2}{2V_{xx}} + \lambda(z)(-e^{-\gamma x} - V) = 0, \tag{4.19}
$$

where

$$\theta(y) = \frac{\mu(y)}{\sigma(y)}. \tag{4.20}$$

It is hard to get the explicit solution of the non-linear PDE. Thus, we use the perturbation method to solve the problem.

Firstly, we expand the V as follows

$$V^\xi = V^{(0)} + \xi^{1/2}V^{(1)} + \xi V^{(2)} + \xi^{3/2}V^{(3)} + \cdots \tag{4.21}$$

We assume that the fast-scale correcting rate ξ is positive and $\xi << 1$. According to Fouque [91], the fast mean reverting stochastic volatility with small time-scale can be viewed as a singular perturbation. Thus, asymptotic approximation can be applied to approximate the solution of (4.19), and according to the perturbation theory, the asymptotic solution of (4.19) consists of the leading term and a first correction term.

Substituting (4.21) into (4.19) and then extracting the coefficient of the term

ξ^{-1} , we obtain

$$\mathcal{L}\dagger V^{(0)} + \frac{(\rho_1 a(y)V_{xy}^{(0)})^2}{2V_{xx}^{(0)}} = 0. \tag{4.22}$$

As $\mathcal{L}\dagger$ is a differential operator with respect to y as defined in (4.12), we can prove that $V^{(0)}$ is independent of y.

Similarly, by extracting the coefficients of $\xi^{-\frac{1}{2}}$, we obtain

$$\mathcal{L}\dagger V^{(1)} + \rho_{12}a(y)c(z)V_{yz}^{(0)} - \frac{\rho_1 \rho_2 a(y)c(z)\theta(y)V_z^{(0)}V_{xz}^{(0)}V_{xy}^{(0)}}{2V_{xx}^{(0)}} + \frac{(\rho_1 a(y)V_{xy}^{(0)})^2}{2V_{xx}^{(0)}} * \frac{V_{xx}^{(1)}}{V_{xx}^{(0)}} = 0 \tag{4.23}$$

We can prove that $V^{(1)}$ is independent of y, which means $V^{(0)}$ and $V^{(1)}$ are functions of t and x. The variable y is involved only in the expansion of the term $V^{(2)}$.

By extracting the coefficient of the term ξ^0, we obtain

$$V_t^{(0)} + \mathcal{L}\dagger V^{(2)} + \mathcal{L}\ddagger V^{(0)} + NL(1) + \lambda(z)(-e^{-rx} - V^{(0)}) = 0. \tag{4.24}$$

By extracting the coefficient of the term $\xi^{\frac{1}{2}}$, we obtain

$$V_t^{(1)} + \mathcal{L}\dagger V^{(3)} + \mathcal{L}\ddagger V^{(1)} + NL(2) - \lambda(z)V^{(1)} = 0. \tag{4.25}$$

Now we consider the expansion about $NL(i)(i = 1, 2)$. By using the Taylor expansion and the fact that $V^{(0)}$ and $V^{(1)}$ are independent of y, we get

$$
\begin{aligned}
NL(i) &= -\frac{[\theta(y)V_x + \frac{1}{\sqrt{\xi}}\rho_1 a(y)V_{xy} + \rho_2 c(z)V_{xz}]^2}{2V_{xx}} \\
&= -[\theta(y)(V_x^{(0)} + \sqrt{\xi}V_x^{(1)}) + \frac{1}{\sqrt{\xi}}\rho_1 a(y)(V_{xy}^{(0)} + \sqrt{\xi}V_{xy}^{(1)}) + \xi V_{xy}^{(2)} \\
&\quad + \rho_2 c(z)(V_{xz}^{(0)} + \sqrt{\xi}V_{xy}^{(1)})]^2 \frac{1}{2V_{xx}^{(0)}}(1 - \sqrt{\xi}\frac{V_{xx}^{(1)}}{V_{xx}^{(0)}} - \xi\frac{V_{xx}^{(1)}}{V_{xx}^{(0)}}) \\
&= -\frac{1}{2V_{xx}^{(0)}}[\theta(y)V_x^{(0)} + \rho_2 c(z)V_{xz}^{(0)}]^2 - \sqrt{\xi}\{-\frac{V_{xx}^{(1)}}{2(V_{xx}^{(0)})^2}[\theta(y)V_x^{(0)} + \rho_2 c(z)V_{xz}^{(0)}]^2 \\
&\quad + \frac{1}{V_{xx}^{(0)}}[\theta(y)V_x^{(0)} + \rho_2 c(z)V_{xz}^{(0)}][\theta(y)V_x^{(1)} + \rho_1 a(y)V_{xy}^{(2)}]\} \tag{4.26}
\end{aligned}
$$

Then we have

$$NL(1) = -\frac{1}{2V_{xx}^{(0)}}[\theta(y)V_x^{(0)} + \rho_2 c(z)V_{xz}^{(0)}]^2, \tag{4.27}$$

and

$$NL(2) = \frac{V_{xx}^{(1)}}{2(V_{xx}^{(0)})^2}[\theta(y)V_x^{(0)} + \rho_2 c(z)V_{xz}^{(0)}]^2$$
$$- \frac{1}{V_{xx}^{(0)}}[\theta(y)V_x^{(0)} + \rho_2 c(z)V_{xz}^{(0)}][\theta(y)V_x^{(1)} + \rho_1 a(y)V_{xy}^{(2)} + \rho_2 c(z)V_{xz}^{(1)}] \quad (4.28)$$

4.3.1 Analysis of the Zero-strategy leading term

From Fredholm's Alternative solvability condition specified in equation (4.24) in Fouque et al. (2011) [77], we obtain

$$V_t^{(0)} + \mathcal{L}\ddagger V^{(0)} - \frac{(\hat{\theta}V^{(0)} + \rho_2 c(z)V_{xz}^{(0)})^2}{2V_{xx}^{(0)}} + \lambda(-e^{-\gamma x} - V^{(0)}) = 0, \quad (4.29)$$

where

$$V(t, x, y, z) = -e^{-\gamma x} \quad (4.30)$$

The equation(4.24) can be simplified by a distortion scaling

$$V^{(0)}(t, x, z) = -e^{-\gamma x}M(t, z)^{\frac{1}{1-\rho_2^2}}, \quad (4.31)$$

to become

$$M_t + \tilde{\mathcal{L}}\ddagger M - (1 - \rho_2^2)(\frac{\theta^2}{2} + \lambda)M - \lambda(1 - \rho_2^2)M^\alpha = 0, \quad (4.32)$$

where

$$\alpha = \frac{\rho_2^2}{\rho_2^2 - 1}, \quad \tilde{\mathcal{L}}\ddagger = \mathcal{L}\ddagger - \rho_2\hat{\theta}c(z)\frac{\partial}{\partial z}. \quad (4.33)$$

The only difference between holding or not holding the bond is the initial condition of the leading term. The differential equation follows:

$$U_t^{(0)} + \mathcal{L}\ddagger U^{(0)} - \frac{(\hat{\theta}U^{(0)} + \rho_2 c(z)U_{xz}^{(0)})^2}{2U_{xx}^{(0)}} + \lambda(-e^{-\gamma x} - U^{(0)}) = 0, \quad (4.34)$$

where

$$U(t, x, y, z) = -e^{-\gamma(x+c)} \quad (4.35)$$

The above equation can be simplified by a distortion scaling

$$U^{(0)}(t, x, z) = -e^{-\gamma(x+c)}N(t, z)^{\frac{1}{1-\rho_2^2}}, \quad (4.36)$$

to become

$$N_t + \tilde{\mathcal{L}}\ddagger N - (1 - \rho_2^2)(\frac{\hat{\theta}^2}{2} + \lambda)N - \lambda(1 - \rho_2^2)e^{\lambda c}N^\alpha = 0, \qquad (4.37)$$

where

$$\alpha = \frac{\rho_2^2}{\rho_2^2 - 1}, \quad \tilde{\mathcal{L}}\ddagger = \mathcal{L}\ddagger - \rho_2\hat{\theta}c(z)\frac{\partial}{\partial z}. \qquad (4.38)$$

4.3.2   Analysis of the fast modification term

Firstly, we give the following notations

$$\phi_1 = -\frac{\theta(y)V_x^{(0)} + \rho_2 c(z)V_{xz}^{(0)}}{V_{xx}^{(0)}}\frac{\partial}{\partial x}, \qquad (4.39)$$

$$\phi_2 = [\frac{\theta(y)V_x^{(0)} + \rho_2 c(z)V_{xz}^{(0)}}{V_{xx}^{(0)}}]^2\frac{\partial^2}{\partial x^2}. \qquad (4.40)$$

Thus, the non-linear term of (4.25) can be rewritten as

$$\mathcal{L}\dagger V^{(3)} + V_t^{(1)} + \mathcal{L}\ddagger V^{(1)} + \frac{1}{2}\phi_2 V^{(1)} + \theta\phi_1 V^{(1)} + \rho_1 a\phi_1 V_y^{(2)} - \lambda(z)V^{(1)} + \rho_2 c\phi_1 V_z^{(0)} = 0. \qquad (4.41)$$

Similarly, by using ϕ_1 and ϕ_2, equation (4.24) can be written as

$$\mathcal{L}\dagger V^{(2)} + V_t^{(0)} + \mathcal{L}\ddagger V^{(0)} - \lambda(z)V^{(0)} + \phi_2 V^{(0)} + \theta\phi_1 V^{(1)} + \rho_2 c\phi_1 V_z^{(0)} = \lambda(z)e^{-\gamma x}. \qquad (4.42)$$

By using the Fredholm Alternative theorem as before, we obtain

$$V_t^{(1)} + \mathcal{L}\ddagger V^{(1)} + \frac{1}{2}\hat{\phi}_2 V^{(1)} + \theta\hat{\phi}_1 V^{(1)} - \lambda(z)V^{(1)} + \rho_2 c\hat{\phi}_1 V_z^{(0)} = -\rho_1 a\hat{\phi}_1 V_y^{(2)}. \qquad (4.43)$$

$$V_t^{(0)} + \mathcal{L}\ddagger V^{(0)} + \frac{1}{2}\hat{\phi}_2 V^{(0)} + \theta\hat{\phi}_1 V^{(0)} - \lambda(z)V^{(1)} + \rho_2 c\hat{\phi}_1 V_z^{(0)} = \lambda e^{-\gamma x}. \qquad (4.44)$$

By comparing the above two equations , we establish that

$$V^{(1)} = -(T - t)\rho_1 a\hat{\phi}_1 V_y^{(2)} + c(t, x), \qquad (4.45)$$

where $V^{(2)}$ is a function of $V^{(0)}$ and $c(t, x)$ can be determined by substituting (4.45) into (4.43) .

4.4 Analysis of Fast-scale Correction under the Exponential Utility Assumption

For simplification of the problem, we assume λ to be a constant. Firstly, we consider our problem under the fast mean-reverting stochastic volatility, namely assuming that the volatility of the stock process is only related to Y. We then have the following model:

$$\frac{dS_t}{S_t} = \mu(Y_t)dt + \sigma(Y_t)dW_t, \tag{4.46}$$

$$dY_t = \frac{1}{\xi}b(Y_t)dt + \frac{1}{\sqrt{\xi}}a(Y_t)dW_t^{(1)}. \tag{4.47}$$

4.4.1 Fast-scale expansion for single name derivatives

The HJB equation (4.11) is transformed into the following form

$$V_t^\xi + \frac{1}{\xi}\mathcal{L}\dagger_0 V^\xi + \lambda(-e^{-\gamma x} - V) + \mathcal{F}V = 0, \tag{4.48}$$

where

$$\mathcal{F}V = sup_{\pi_t \in \mathcal{A}}[\pi_t \mu(y)V_x + \frac{1}{2}(\pi_t)^2\sigma(y)^2 V_{xx} + \pi_t\frac{1}{\sqrt{\xi}}\rho_1 a(y)\sigma(y)V_{xy}] \tag{4.49}$$

By solving the optimization problem in (4.49) , we obtain π_t^* as follows

$$\pi_t^* = -\frac{\mu(y)}{\sigma^2(y)}\frac{V_x}{V_{xx}} - \frac{1}{\sqrt{\xi}}\rho_1\frac{a(y)}{\sigma(y)}\frac{V_{xy}}{V_{xx}}. \tag{4.50}$$

Substituting (4.50) into (4.48),the non-linear equation becomes

$$V_t^\xi + \frac{1}{\xi}\mathcal{L}\dagger_0 V^\xi - \frac{(\theta(y)V_x^\xi + \frac{\rho_1 a(y)}{\sqrt{\xi}}V_{xy}^\xi)^2}{2V_{xx}^\xi} + \lambda(-e^{-rx} - V) = 0, \tag{4.51}$$

where

$$\theta(y) = \frac{\mu(y)}{\sigma(y)}. \tag{4.52}$$

Then we can look for an expansion of the value function:

$$V^\xi = V^{(0)} + \sqrt{\xi}V^{(1)} + \xi V^{(2)} + \xi^{3/2}V^{(3)} + \cdots . \tag{4.53}$$

By Substituting (4.53)into(6.31) and collecting the coefficients of the terms ξ^{-1} and $\xi^{-\frac{1}{2}}$, we can get the conclusion that $V^{(0)}$ and $V^{(1)}$ are independent of Y. From the coefficients of the constant term and the term ξ^{-1}, we get the following two equations:

$$V_t^{(0)} + \mathcal{L}\dagger_0 V^{(2)} - \frac{1}{2}\theta^2(y)\frac{(V_x^{(0)})^2}{V_{xx}^{(0)}} - \lambda V^{(0)} = \lambda e^{-rx}, \qquad (4.54)$$

$$V_t^{(1)} + \mathcal{L}\dagger_0 V^{(3)} - NL(1) - \lambda V^{(1)} = 0, \qquad (4.55)$$

where

$$NL(1) = -\frac{\theta(y)}{V_{xx}^{(0)}}V_x^{(0)}[\lambda(y)V_x^{(1)} + \rho_1 a(y)V_{xy}^{(2)}] + \frac{V_{xx}^{(1)}}{2(V_x^{(0)})^2}\theta^2(y)(V_x^{(0)})^2. \qquad (4.56)$$

From Fredholm's alternative solvability condition, we get

$$V_t^{(0)} - \frac{1}{2}\hat{\theta}^2\frac{(V_x^{(0)})^2}{V_{xx}^{(0)}} - \lambda V^{(0)} = \lambda e^{-rx}, \qquad (4.57)$$

$$V_t^{(1)} - <NL(1)> -\lambda V^{(1)} = 0. \qquad (4.58)$$

where $< \cdot >$ denotes the average of y. From equation(4.57),we get the leading term $V^{(0)}$,and from (4.44), we can get the relationship between $V^{(0)}$ and $V^{(1)}$,and then we can get the approximation of V^ξ.

Proposition 4.3. *The explicit solution of equation (4.57) is*

$$V^{(0)}(t,x) = -\frac{\lambda}{\frac{1}{2}\hat{\theta}^2 + \lambda}e^{-\gamma x} + (1 - \frac{\lambda}{\frac{1}{2}\hat{\theta}^2 + \lambda})e^{-(\frac{1}{2}\hat{\theta}^2+\lambda)(T-t)}e^{-\gamma x}, \qquad (4.59)$$

where $\hat{\theta}$ is the average value of $\theta(y)$ with the distribution of Π, namely

$$\hat{\theta} = \int \theta(y)\Pi(dy) \qquad (4.60)$$

Proof. We firstly transform the PDE by averaging $\theta(y)$. Because $V^{(0)}$ is independent of y, we get the following PDE from (4.57),

$$V_t^{(0)} - \frac{1}{2}\hat{\theta}^2\frac{(V_x^{(0)})^2}{V_{xx}^{(0)}} - \lambda V^{(0)} = \lambda e^{-\gamma x}, \quad V_T^{(0)} = e^{-\gamma x} \qquad (4.61)$$

By making the substitution of $V_T^{(0)} = -e^{-\gamma x} M$, we get the following ODE,

$$M_t - (\lambda + \frac{1}{2}\hat{\theta}^2)M = -\lambda, \quad M_T = 1 \tag{4.62}$$

Then we can obtain the solution of (4.57) by solving the above equation. □

We then introduce

$$R^{(0)} = -\frac{V_x^{(0)}}{V_{xx}^{(0)}} \tag{4.63}$$

$$D_k = (R^{(0)})^k \frac{\partial^k}{\partial x^k}, k = 1, 2, \cdots \tag{4.64}$$

$$\mathcal{L}\dagger_{t,x,y}^e = \frac{\partial}{\partial t} + \frac{1}{2}\theta^2(y)D_2 + \theta^2(y)D_1 - \lambda \tag{4.65}$$

$$\mathcal{L}\dagger_{t,x}^e = \frac{\partial}{\partial t} + \frac{1}{2}\hat{\theta}^2 D_2 + \hat{\theta}^2 D_1 - \lambda \tag{4.66}$$

Equations (4.54) and (4.57) become

$$\mathcal{L}\dagger_0 V^{(2)} + \mathcal{L}\dagger_{t,x,y} V^{(0)} = \lambda e^{-\gamma x}, \tag{4.67}$$

$$\mathcal{L}\dagger_{t,x} V^{(0)} = \lambda e^{-\gamma x}. \tag{4.68}$$

Subtracting (4.67) by (4.68), we get

$$V^{(2)} = -\eta(y)(\frac{1}{2}D_2 + D_1)V^{(0)}, \tag{4.69}$$

$$\eta(y) = \mathcal{L}\dagger_0^{-1}(\theta^2(y) - \hat{\theta}). \tag{4.70}$$

Substituting (4.69) into (4.56), we can get the following proposition

Proposition 4.4. *The value of the fast modification form is the solution of the equation below,*

$$\mathcal{L}\dagger_{t,x,y}^e V^{(1)} = \frac{1}{2}\rho_1 B D_1^2 V^{(0)}(t, x), \quad V^{(1)}(T, x) = 0, \tag{4.71}$$

where $B = \theta(y)a(y)\eta(y)$.

Proof. As $D_2 = -D_1$, we have

$$V^{(2)} = -\eta(y)(\frac{1}{2}D_2 + D_1)V^{(0)} = -\frac{1}{2}\eta(y)D_1 V^{(0)}. \tag{4.72}$$

Based on (4.58) and (4.56), we have

$$V_t^{(1)} - \langle \frac{V_x^{(0)}}{V_{xx}^{(0)}}\theta(y)[V_x^{(1)} + \rho_1 a(y)V_{xy}^{(2)}] - \frac{(V_x^{(0)})^2}{2(V_{xx}^{(0)})^2}V_{xx}^{(1)}\theta^2(y)\rangle - \lambda V^{(1)}$$

$$= V_t^{(1)} - \langle -\theta(y)D_1 V_x^{(1)} - \rho_1 a(y)\theta(y)D_1 V_y^{(2)} - \frac{1}{2}\theta^2(y)D_2 V^{(1)}\rangle - \lambda V^{(1)}$$

$$= V_t^{(1)} + \hat{\theta}(y)D_1 V^{(1)} - \frac{1}{2}\rho_1 B D_1^2 V^{(0)} + \frac{1}{2}\hat{\theta}^2(y)D_2 V^{(1)} - \lambda V^{(1)}, \qquad (4.73)$$

where $B = \langle a(y)\theta(y)\eta'(y)\rangle$. $\qquad\qquad\square$

Lemma 4.1. *The operators $\mathcal{L}\dagger_{t,x}^e$ and D_1 acting on smooth functions of (t,x) commute:*

$$\mathcal{L}\dagger_{t,x}^e D_1 = D_1 \mathcal{L}\dagger_{t,x}^e. \qquad (4.74)$$

Proof.

$$D_2 D_1 - D_1 D_2 = (R^{(0)})^2\frac{\partial^2}{\partial x^2}(R^{(0)w_x}) - R^{(0)}\frac{\partial}{\partial x}((R^{(0)})^2 w_{xx}) = (R^{(0)})^2 R_{xx}^{(0)}w_x \quad (4.75)$$

$$\begin{aligned}
\mathcal{L}\dagger_{t,x}^e D_1 w &= (\frac{\partial}{\partial t} + \frac{1}{2}\hat{\theta}^2 D_2 + \hat{\theta}D_1 - \lambda)D_1 w \\
&= D_1\frac{\partial}{\partial t} + \frac{1}{2}\hat{\theta}^2 D_1 D_2 + \hat{\theta}D_1^2 - \lambda D_1)w + (R_t^{(0)} + \frac{1}{2}\hat{\theta}^2(R^{(0)})^2 R_{xx}^{(0)})w_x \\
&= D_1 \mathcal{L}\dagger_{t,x}^e w. \qquad (4.76)
\end{aligned}$$

$\qquad\qquad\square$

From lemma 4.1 we can draw the conclusion that $\mathcal{L}\dagger_{t,x}^e(D_1^k V^{(0)}) = D_1^k\mathcal{L}\dagger_{t,x}^e V^{(0)}$, which leads to the following proposition.

Proposition 4.5. *The solution of (4.71) is*

$$V^{(1)} = -(T-t)\frac{1}{2}\rho_1 B D_1^2 V^{(0)}(t,x) + c(t,x), \qquad (4.77)$$

where $B = \theta(y)a(y)\eta'(y)$, and

$$c(t,x) = (\frac{M'}{N'}(T-t) + \frac{M'}{N'^2} - \frac{M'}{N'^2}e^{N'(T-t)})e^{-\gamma x}, \qquad (4.78)$$

$$M' = \frac{1}{2}\rho_1 B\lambda\gamma^2, \qquad (4.79)$$

$$N' = \frac{1}{2}\hat{\theta}^2(R^{(0)})^2 - \hat{\theta}R^{(0)}\gamma - \lambda. \qquad (4.80)$$

Proof. We firstly assume that the solution of (4.71) is

$$V^{(1)} = -(T-t)\frac{1}{2}\rho_1 BD_1^2 V^{(0)}(t,x) + c(t,x). \tag{4.81}$$

Substituting (4.81) into (4.71), we obtain

$$\frac{1}{2}\rho_1 BD_1^2 V^{(0)} - (T-t)\frac{1}{2}\rho_1 BD_1^2 \mathcal{L}\dagger_{t,x} V^{(0)} + \mathcal{L}\dagger_{t,x} c(t,x) = \frac{1}{2}\rho_1 BD_1^2 V^{(0)}. \tag{4.82}$$

Then we obtain

$$\mathcal{L}\dagger_{t,x} c(t,x) = (T-t)\frac{1}{2}\rho_1 BD_1^2 \mathcal{L}\dagger_{t,x} V^{(0)}. \tag{4.83}$$

Because $\mathcal{L}\dagger_{t,x} V^{(0)} = \lambda e^{-\gamma x}$, we obtain the PDE as follows

$$\mathcal{L}\dagger_{t,x} c(t,x) = (T-t)\frac{1}{2}\rho_1 B\gamma^2 \lambda e^{-\gamma x}, \quad c(T,x) = 0. \tag{4.84}$$

Assume $c(t,x) = A(t)e^{-\gamma x}$, then we obtain

$$A_t + N'A = (T-t)M', \quad A(T,x) = 0, \tag{4.85}$$

where

$$M' = \frac{1}{2}\rho_1 B\lambda\gamma^2, \quad N' = \frac{1}{2}\hat{\theta}^2(R^{(0)})^2 - \hat{\theta}R^{(0)}\gamma - \lambda. \tag{4.86}$$

The terminal condition here is arised from the condition $V^{(1)}(T,x) = c(T,x) = 0$. By solving the ODE for A, we get

$$A = \frac{M'}{N'}(T-t) + \frac{M'}{N'^2} - \frac{M'}{N'^2}e^{N'(T-t)}, \tag{4.87}$$

and

$$c(t,x) = \left(\frac{M'}{N'}(T-t) + \frac{M'}{N'^2} - \frac{M'}{N'^2}e^{N'(T-t)}\right)e^{-\gamma x}. \tag{4.88}$$

$\square$

From the expansion (4.53), and the solution of $V^{(0)}$, $V^{(1)}$ and $V^{(2)}$, we obtain

$$V^{(\xi)} = V^{(0)} + \sqrt{\xi}V^{(1)} + \xi V^{(2)} + o(\xi^{\frac{3}{2}})$$
$$= (1 - \sqrt{\xi}\frac{1}{2}(T-t)\rho_1 BD_1^2)V^{(0)}(t,x) + \sqrt{\xi}c(t,x) + o(\xi^{\frac{3}{2}}). \tag{4.89}$$

Then we analyse the approximation of the maximizer π^* as given in (4.50).

Using Taylor expansion, we get

$$
\begin{aligned}
\frac{V_x}{V_{xx}} &= \frac{V_x^{(0)} + \sqrt{\xi}V_x^{(1)}}{V_{xx}^{(0)} + \sqrt{\xi}V_{xx}^{(1)}} \\
&= \frac{1}{V_{xx}^{(0)}}(V_x^{(0)} + \sqrt{\xi}V_x^{(1)})(1 - \sqrt{\xi}\frac{V_{xx}^{(1)}}{V_{xx}^{(0)}}) \\
&= \frac{V_x^{(0)}}{V_{xx}^{(0)}} + \sqrt{\xi}(\frac{V_x^{(1)}}{V_{xx}^{(0)}} - \frac{V_x^{(0)}V_{xx}^{(1)}}{(V_{xx}^{(0)})^2}) \\
&= \frac{V_x^{(0)}}{V_{xx}^{(0)}} + \frac{\sqrt{\xi}}{V_{xx}^{(0)}}(c_x + R^{(0)}c_{xx}) \\
&= \frac{V_x^{(0)}}{V_{xx}^{(0)}} - \sqrt{\xi}\frac{1}{V_x^{(0)}}(D_1 + D_2)c + o(\xi),
\end{aligned}
\tag{4.90}
$$

and

$$
\begin{aligned}
\frac{V_{xy}}{V_{xx}} &= \frac{V_{xy}^{(0)} + \sqrt{\xi}V_{xy}^{(1)} + \xi V_{xy}^{(2)}}{V_{xx}^{(0)} + \sqrt{\xi}V_{xx}^{(1)} + \xi V_{xx}^{(2)}} \\
&= \xi\frac{V_{xy}^{(2)}}{V_{xx}^{(0)}} = -\xi\frac{1}{V_x^{(0)}}\eta(y)\frac{1}{2}D_2 D_1 V_y^{(0)}.
\end{aligned}
\tag{4.91}
$$

Substituting the above into (4.50) yields

$$
\pi^* = -\frac{\theta(y)}{\delta(y)}\frac{V_x^{(0)}}{V_{xx}^{(0)}} + \frac{\sqrt{\xi}}{V_x^{(0)}}[\frac{\theta(y)}{\delta y}(D_1 + D_2)c + \rho_1\eta(y)\frac{1}{2}D_1 D_2 V_y^{(0)}]
\tag{4.92}
$$

Similarly, the solution of the bond holders' problem is given in the following properties,

Proposition 4.6. *The leading term of the bond holder's problem is*

$$
U^{(0)} = -\frac{\lambda}{\frac{1}{2}\hat{\theta}^2 + \lambda}e^{-\gamma x} + (1 - \frac{\lambda e^{\gamma c}}{\frac{1}{2}\hat{\theta}^2 + \lambda})e^{-(\frac{1}{2}\hat{\theta}^2 + \lambda)(T-t) - \gamma(x+c)}
\tag{4.93}
$$

where $\hat{\theta}$ is the average of $\theta(y)$ with respect to the distribution Π, namely

$$
\hat{\theta} = \int \theta(y)\Pi(dy).
\tag{4.94}
$$

The fast-scale modification term of the bond holder's problem is

$$
(1 - \sqrt{\xi}\frac{1}{2}(T-t)\rho_1 BD_1^2)U^{(0)}(t,x) + \sqrt{\xi}C(t,x) + o(\xi^{\frac{3}{2}}),
\tag{4.95}
$$

where

$$C(t,x) = (\frac{M'}{N'}(T-t) + \frac{M'}{N'^2} - \frac{M'}{N'^2}e^{D(T-t)})e^{-\gamma x}. \tag{4.96}$$

So the approximation of the bond holder's value function is

$$U^{(\xi)} = U^{(0)} + \sqrt{\xi}U^{(1)} + \xi U^{(2)} + o(\xi^{\frac{3}{2}})$$
$$= (1 - \sqrt{\xi}\frac{1}{2}(T-t)\rho_1 BD_1^2)U^{(0)}(t,x) + \sqrt{\xi}C(t,x) + o(\xi^{\frac{3}{2}}). \tag{4.97}$$

4.5 Numerical Study of Exponential Utility

4.5.1 Analysis of the Value Function

The utility we use for Bond seller is exponential and is given by

$$V(x) = -e^{-\gamma x}, \tag{4.98}$$

where $\gamma > 0$ represents the risk aversion parameter. We can prove that the utility function is concave and increasing since

$$V'(x) = \gamma e^{-\gamma x} > 0, V''(x) = -\gamma^2 e^{-\gamma x} < 0. \tag{4.99}$$

The concave property of the utility function implies that the bond seller is risk aversion. The risk aversion rate is calculated by the Arrow-Pratt index,

$$AP[U] := -\frac{U''(x)}{U'(x)} = \gamma, \tag{4.100}$$

where the larger the γ is, the higher risk averse the agent is. The risk-tolerance function at the terminal time T is

$$R(T,x) = -\frac{U'}{U''} = \frac{1}{\gamma}. \tag{4.101}$$

4.5.2 The Effect of Volatility Correction

The formulation given above is in general form. To demonstrate the result graphically, we consider a special case with all parameters specified at certain given values, the mean-reverted process with constant volatility of volatility, as given
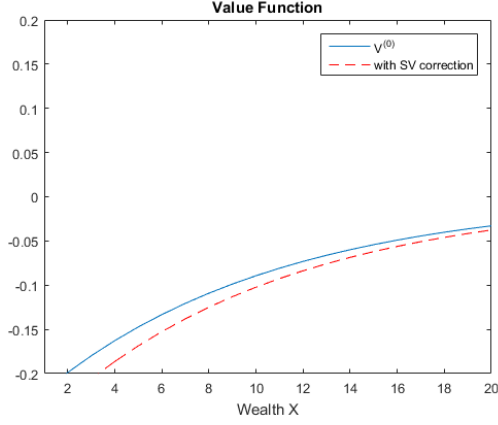
Figure 4.1: Value Function of Bond Seller
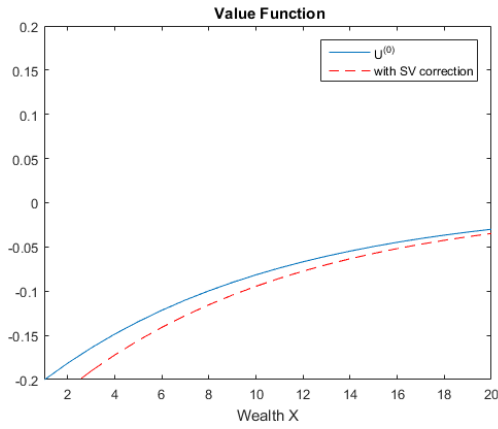
Figure 4.2: Value Function of Bond Holder

below:

$$\frac{dS_t}{S_t} = Y_t dt + \sqrt{Y_t} dW_t, \tag{4.102}$$

$$dY_t = \frac{1}{\xi}(m_1 - Y_t)dt + \sqrt{\frac{2}{\xi}}v dW_t^{(1)}. \tag{4.103}$$

If Y_t is an ergodic process, it has the distribution of $N(m_1, v^2)$. Assume that $m_1 = 0.01, v^2 = 0.25, \xi = \frac{1}{200}$. Based on the definition of $\hat{\theta}$, we obtain

$$\hat{\theta} = \frac{1}{\sqrt{2\pi}v}\int_{-\infty}^{\infty}\sqrt{y}e^{-\frac{(y-m_1)^2}{2v^2}}dy \tag{4.104}$$

According to (4.59) and (4.81), we get the solution of $V^{(0)}$, and also the fast modification term of $V^{(1)}$. We then calculate the utility term as $V^{(0)} + \sqrt{\xi}V^{(1)}$. The approximations to the value functions for bond seller and bond holder are plotted respectively in figure 4.1 and figure 4.2.
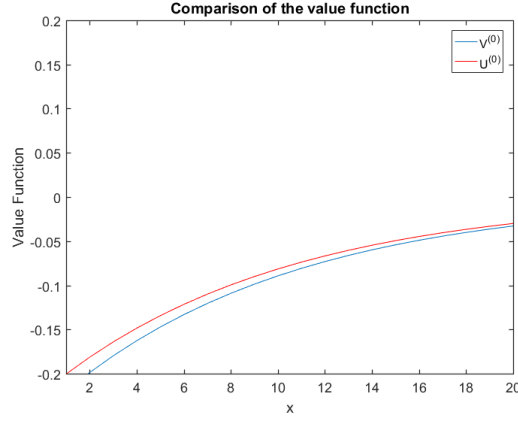
Figure 4.3: Leading Term Value Function
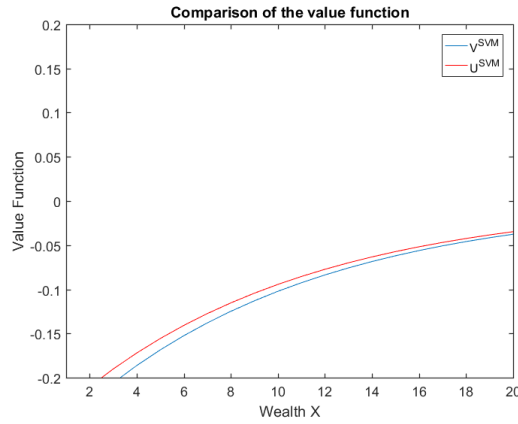
Figure 4.4: SV Modification Value Function

Also, since the bond pays \$1 on maturity date T if the firm has survived till then, the bond seller's value function will be higher than the bond holder's value function. The comparison of the value functions of bonder seller and bond holder are shown in figure 4.3 and figure 4.4. The Stochastic Volatility Model(SVM) in figure 4.4 represents the Stochastic Volatility(SV) modification form,

The approximate indirect utilities $V^{(0)}$ or $V^{(0)}+\sqrt{\xi}V^{(1)}$ can also be represented by their certainty equivalents $U^{-1}(V^{(0)})$ and $U^{-1}(V^{(0)}+\sqrt{\xi}V^{(1)})$, which are shown in figure 4.5 and figure 4.6, In figure 4.1 and figure 4.2, the original value function is denoted by blue solid line, while the dashed blue line is the value function with stochastic volatility correction. We can see clearly that the correction line is a little lower than the original line. In figure 4.3 and figure 4.4, we make a comparison of the value function for holding and not holding the bond. Figure 4.3 shows the relationship of the value functions before SV correction while figure 4.4 shows the relationship of bond holder and bond seller's value functions after SV correction. Figure 4.5 and figure 4.6 show the certainty equivalent before or
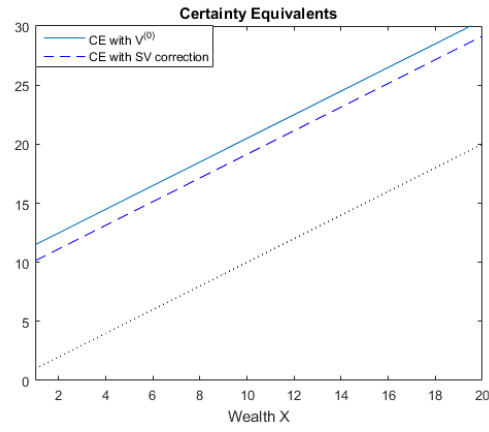
Figure 4.5: Certainty Equivalents of Bond Seller
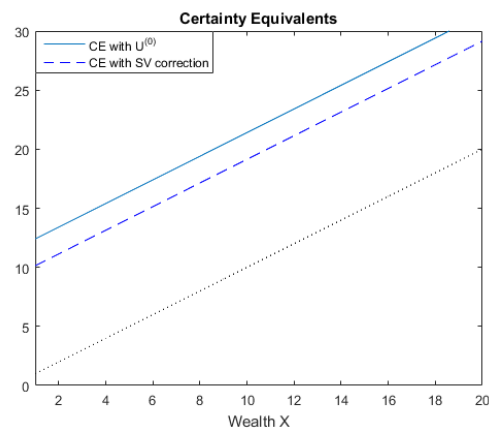
Figure 4.6: Certainty Equivalents of Bond Holder

after the correction. The certainty equivalent is solved from the indifferent utility function, which is increasing with the wealth. The solid line in figure 4.5 and figure 4.6 shows the certainty equivalent before the correction and the dashed line shows the quantity after the correction.

Therefore, we can draw the conclusion that by adding a stochastic volatility process into the model (4.103), the investor becomes more risk adverse. The stochastic Volatility is lower than both the utility function and the certainty equivalent. Also, as the bond holder will get a fixed pay at the maturity date if default does not happen, the value function of the bond holder will be a little higher than that of the bond seller. That is why we give the definition of indifference price p_0. By cutting down the initial wealth of bond holder, the expected utility of bond holder should be the same as that of the bond seller. In the following subsection, we will analyse the indifference and yield spread numerically.

4.5.3   Analysis of yield spread

According to the definition 4.1, it is easy to calculate p_0 and the yield spread. Without the modification term, the indifference price $p_0^{(0)}$ is given by

$$p_0^{(0)} = e^{-rT} + \frac{1}{\gamma} ln \frac{u - (1-u)e^{-(\frac{1}{2}\hat{\theta}^2 + \lambda)T}}{ue^{\gamma c} - (1 - ue^{\gamma c})e^{-(\frac{1}{2}\hat{\theta}^2 + \lambda)T}}, \tag{4.105}$$

where

$$u = \frac{\lambda}{\frac{1}{2}\hat{\theta}^2 + \lambda}. \tag{4.106}$$

If γ takes the value of 0.05, 0.1, 0.25, 0.5 and 0.75 respectively, we obtain the profile of yield spread $y_0(T) = -\frac{1}{T} log(p_0^{(0)}(T)) - r$ as shown in Figure 7. It is noted that the yield spread is not flat even though the intensity is a constant and this is due to the effect of the intensity rate λ upon T. When T goes to infinity, yield spread will converge to a long time level and become flat. As we can see from figure4.7, the yield spread for the investor is upward sloping and is approximated to a long time level due to the different maturity time.

4.6   Numerical Study of CRRA Utility

The utility we use from Bond seller is exponential and given by

$$V(x) = c_0 \frac{x^{1-\gamma}}{1-\gamma}, \tag{4.107}$$
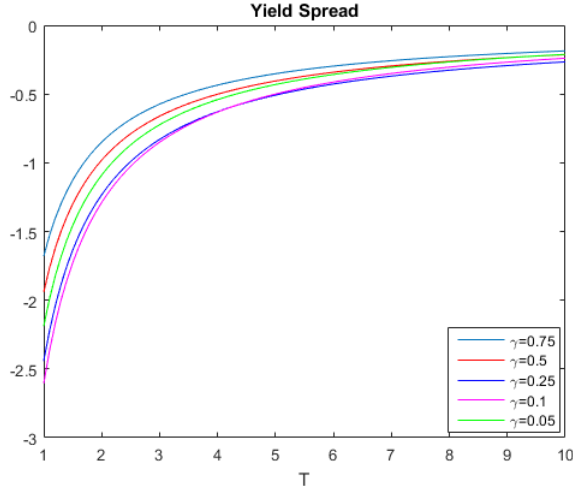
Figure 4.7: Yield Spread

where $\gamma > 0$ represents the risk aversion parameter. We can prove that the utility function is concave and increasing since

$$V'(x) = x^{-\gamma} > 0, V''(x) = -\gamma x^{-\gamma-1} < 0. \tag{4.108}$$

The concave property of the utility function implies that the bond seller is risk aversion. The risk aversion rate is calculated by the Arrow-Pratt index,

$$AP[U] := -\frac{U''(x)}{U'(x)} = \gamma/x, \tag{4.109}$$

where the larger the γ is, the higher risk averse the agent is. And the risk-tolerance function at terminal time T is

$$R(T, x) = -\frac{U'}{U''} = \frac{1}{\gamma}x. \tag{4.110}$$

In order to obtain the first correction of the CRRA utility, we solve the following parabolic equation numerically,

$$V_t^{(1)} + \frac{1}{2\gamma^2}\hat{\theta}^2 x^2 V_{xx}^{(1)} + \frac{1}{\gamma}\hat{\theta}^2 x V^{(1)} - \lambda V^{(1)} = f(T-t, x), \tag{4.111}$$

with $f(T-t, x) = (T-t)\frac{1}{2}\rho_1 B\lambda c_0 D_1^2 \frac{x^{1-\gamma}}{1-\gamma}$, $c(T, x) = 0$, and terminal condition $V^{(1)}(T, x) = 0$. Let $\tau = T - t$, we can obtain the weak form of (6.19),

$$(V_\tau^{(1)}, V) + \frac{1}{2}\hat{\theta}^2(x^2 V_x^{(1)}, U_x) - \hat{\theta}^2\frac{1}{\gamma}(1 + \frac{1}{\gamma^2})(xV_x^{(1)}, U) + \lambda(V^{(1)}, U) = (f(\tau, x), U). \tag{4.112}$$

The basis function $V^{(1)}$, the test function U and the function f can be approximated by the following form

$$V^{(1)}(x, \tau) = \sum_{i=1}^{N} u_i(\tau) \Phi_i(x)$$

$$U(x) = \sum_{j=1}^{N} v_j \Phi_j(x),$$

$$f(x, \tau) = \sum_{i=1}^{N} f_i(\tau) \tag{4.113}$$

We then obtain the systems of ODEs,

$$M\dot{U} + RU = F,$$
$$U(0) = 0, \tau \in [0, T] \tag{4.114}$$

where

$$M = (\Phi_i, \Phi_j);$$

$$R = \frac{1}{2}\hat{\theta}^2(x_i^2 \frac{\partial \Phi_i}{\partial x}, \frac{\partial \Phi_j}{\partial x}) - \hat{\theta}^2 \frac{1}{\gamma}(1 + \frac{1}{\gamma^2})(x_i \frac{\partial \Phi_i}{\partial x}, \Phi_j) + \lambda(\Phi_i, \Phi_j);$$

$$F = (f_i, \Phi_j) \tag{4.115}$$

We then apply the backward Euler method to solve the above dynamic ODE system and obtain

$$(\frac{M}{\Delta t} + R)U_{n+1} = M\frac{U_n}{\Delta t} + F. \tag{4.116}$$

4.7 Conclusion and Future Work

In this paper we study the single-name bond under the stochastic intensity and the stochastic volatility. In order to solve the non-linear PDE, we use the method of asymptotic approximation. We establish the expression of leading term $V^{(0)}$, and fast-scale modification term $V^{(1)}$. By comparing the leading term and the utility with fast scale modification, we can draw the conclusion that by considering the effects of the fast-scale volatility, investors become more and more risk aversive, which lowers down their utility and increases the certainty equivalents. Also, according to the analysis, we prove that the yield spread of the investor goes up with the maturity time and converges to a long time level. The advantage of the

asymptotic method is that it reduces the high dimensional problem into a lower dimensional problem, which is relatively easy to solve. However, the limitation of this approach is that it only works for a specific utility model, and for other utilities, the analytic solutions may not be obtained so that numerical method is needed. In our future research, the effect of multi-scale volatility and stochastic interest rate will be taken into consideration.

CHAPTER 5

Variance Swap Pricing Under Multifactor Stochastic Process

5.1 General

Variance and volatility swap is a well-known financial derivative which allows investors to trade the realized volatility against the current implied volatility. Long variance position will benefit when the realized volatility is higher than the strike price, while the short variance position will benefit when the realized volatility is lower than the strike price. The first volatility derivative was traded in 1998 and flourished recently. Demeterfi et al.(1999) listed two main reasons to trade volatility derivatives [16], such as the variance swap and the volatility swap. Firstly, investigators may take long/short position of the variance swap to hedge the risk exposure of trading volatility. Secondly, the variance swap provides a possibility to speculate the spread of the realized volatility and the implied volatility.

A lot of attempts have been made to value the variance swap both numerically and analytically. Carr and Madan showed in their work that the price of a volatility product can be replicated by a static position of call and put options [92]. Ian Martin proposed a simple variance swap by letting the denominator of the variance payoff be a forward price geometrically increased with time, and derived the analytic solution following the work of Carr and Madan [93]. Broadie and Jain investigated the analytic approximation of the fair strike price of a continuous sampling variance swap driven by both the Merton jump and the stochastic volatility process [94]. Numerical algorithms have also been applied to study the option pricing problem of the variance swap [18,63]. Little&Pant(2001) [63] applied the finite difference method(FDM) to solve the variance swap problem based on the constant volatility assumption [3], in which a two-dimensional(2D)

problem is reduced to a one-dimensional partial differential equation system.

This chapter studies the pricing of the discrete sampling variance swap taking into account the effect of imposing multi-scale stochastic volatility into the stochastic process. A dimension reduction technique is applied along with the generalized Fourier transform to solve the underlying partial differential equations, and the results show that the proposed model can capture the effects of the fast and slow scale volatilities. Closed form solutions are obtained in both partial correlated and full correlated stochastic volatility. Monte Carlo simulation is also applied as a benchmark and we find that our approach is more efficient than Monte Carlo simulation.

The rest of the chapter is organized as follows. Section 5.2 describes the model of the underlying asset price, with the volatility following a multi-scale stochastic process. Section 5.3 presents the algorithm used for this model. Numerical results are presented in section 5.4, followed by conclusions in section 5.5.

5.2 Model Setup

The price of stock is assumed to follow the following stochastic process,

$$dS = \mu S dt + f(y_1, y_2) S dw_t^{(0)}, \tag{5.1}$$

where $f(y_1, y_2)$ are functions of two factors y_1 and y_2 respectively representing fast and slow scale volatilities driven by the following processes,

$$dy_1 = \frac{1}{\xi}\alpha(y_1)dt + \frac{1}{\sqrt{\xi}}\beta(y_1)dw_t^{(1)}, \tag{5.2}$$

$$dy_2 = \sigma c(y_2)dt + \sqrt{\sigma}g(y_2)dw_t^{(2)}. \tag{5.3}$$

The concept of fast-scale and slow-scale is distinguished by the frequencies of the observed volatility data, and it is suggested to consider them simultaneously by Chacko and Viceria[2005] [37]. In addition, we assume that the Brownian motion $(w_t^{(0)}, w_t^{(1)}, w_t^{(2)})$ are correlated with the following correlation $Cov(w_t^{(0)}, w_t^{(1)}) = \rho_1, Cov(w_t^{(0)}, w_t^{(2)}) = \rho_2$ and $Cov(w_t^{(1)}, w_t^{(2)}) = \rho_{12}$, with $\rho_1, \rho_2, \rho_{12}$ satisfying $|\rho_1| < 1, |\rho_2| < 1, |\rho_{12}| < 1, 1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$. Under the risk-neutral

assumption,the dynamic processes for S, y_1 and y_2 can be described as follows

$$dS = rSdt + f(y_1, y_2)Sdw^{*(0)},$$

$$dy_1 = (\frac{1}{\xi}\alpha(y_1) - \frac{1}{\sqrt{\xi}}\Lambda(y_1)\beta(y_1))dt + \frac{1}{\sqrt{\xi}}\beta(Y_1)dw^{*(1)},$$

$$dy_2 = (\sigma c(y_2) - \sqrt{\sigma}\tau(y_1, y_2)g(y_2))dt + \sqrt{\sigma}g(y_2)dw^{*(2)}, \qquad (5.4)$$

where r is the risk-free interest rate, the functions $\tau = \rho_2\frac{\mu-r}{f}+\bar{\rho}_{12}\varepsilon+\sqrt{1-\rho_1^2+\bar{\rho}_{12}^2}\eta$ and $\Lambda = \rho_1\frac{\mu-r}{f} + \sqrt{1-\rho_1^2}\varepsilon$ denote market prices of volatility.

There are two steps to convert (5.1) and (5.2) to (5.4). The first step is to change the correlated Brownian motion $(w_t^{(0)}, w_t^{(1)}, w_t^{(2)})$ into $(w_t^{(0)}, w_t^{\perp(1)}, w_t^{\perp(2)})$,where the second set is a orthogonal set, from which we can decompose the first set into the following form,

$$w_t^{(1)} = \rho_1 w_t^{(0)} + \sqrt{1-\rho_1^2}w_t^{\perp(1)},$$

$$w_t^{(2)} = \rho_2 w_t^{(0)} + \bar{\rho}_{12}w_t^{\perp(1)} + \sqrt{1-\rho_2^2-\rho_{12}^2}w_t^{\perp(2)}, \qquad (5.5)$$

where it is easy to prove that $\rho_{12} = \rho_1\rho_2 + \bar{\rho}_{12}\sqrt{1-\rho_1^2}$,

The second step is to change the set $(w_t^{(0)}, w_t^{\perp(1)}, w_t^{\perp(2)})$ to $(w_t^{*(0)}, w_t^{*(1)}, w_t^{*(2)})$, where the third set is the Brownian motion under the risk neutral assumption. Applying the Girsanov Theorem, we obtain

$$w_t^{*\perp(0)} = w_t^{(0)} + \int_0^t \frac{\mu-r}{f}du,$$

$$w_t^{*\perp(1)} = w_t^{\perp(1)} + \int_0^t \varepsilon du,$$

$$w_t^{*\perp(0)} = w_t^{\perp(2)} + \int_0^t \eta du \qquad (5.6)$$

where $\frac{\mu-r}{f}, \varepsilon, \eta$ are assumed to be the market price of risk. Substituting (5.5) and

(5.6) into (5.4),we obtain

$$dS = rSdt + f(y_1, y_2)Sdw^{\perp(0)},$$

$$dy_1 = (\frac{1}{\xi}\alpha(y_1) - \rho_1\frac{\beta}{\sqrt{\xi}}\frac{\mu-r}{f} - \frac{\beta}{\sqrt{\xi}}\sqrt{1-\rho_1^2}\varepsilon)dt$$

$$+\frac{\beta}{\sqrt{\xi}}(\rho_1 dw^{*\perp(0)} + \sqrt{1-\rho_1^2}dw^{*\perp(1)}),$$

$$dy_2 = (\sigma c(y_2) - \sqrt{\delta}g(\rho_2\frac{\mu-r}{f} + \bar{\rho}_{12}\varepsilon + \sqrt{1-\rho_1^2+\bar{\rho}_{12}^2}\eta))dt$$

$$+\sqrt{\sigma}g(\rho_2 dw^{*\perp(0)} + \bar{\rho}_{12}dw^{*\perp(1)} + \sqrt{1-\rho_1^2-\bar{\rho}_{12}^2}dw^{*\perp(2)}) \qquad (5.7)$$

Then we obtain (5.4) with

$$\Lambda = \rho_1\frac{\mu-r}{f} + \sqrt{1-\rho_1^2}\varepsilon,$$

$$\tau = \rho_2\frac{\mu-r}{f} + \bar{\rho}_{12}\varepsilon + \sqrt{1-\rho_1^2+\bar{\rho}_{12}^2}\eta.$$

Variance and volatility swap are well known financial derivatives which allow investors to trade the realized volatility and unrealized variance against the current implied volatility. Different from the European options, variance swaps and volatility swaps are time-dependent.

To price the variance swap, we must distinguish the realized volatility from the implied volatility. The realized volatility is calculated by applying the historical data of option prices, while the implied one is derived from the prices of options. The realized volatility is commonly calculated by the following two formulas, and the two forms of approximation makes no difference if we assume the stock price is a stochastic process driven by a Brownian motion.

We apply the dimensional reduction technique, as in Little&Pant[2001] [63], by introducing a new variable I_t driven by the underlying process

$$I_t = \int_0^t \delta(t_{i-1} - \tau)S_\tau d\tau, \qquad (5.8)$$

where δ is the Dirac delta function, which means $I_t = 0$ if $t < t_{i-1}$, and $I_t = S_{i-1}$ if $t \geq t_{i-1}$. The terminal condition then becomes

$$U_i(T, S, Y, Z, I) = (\frac{S_i}{I_i} - 1)^2. \qquad (5.9)$$

In the next section, we will show that the use of I will reduce the dimension of the problem and simplify the problem consequently.

5.3 Pricing variance swaps under the multi-factor Heston model

In this section, we derive a semi-closed form solution for the fair strike price $Kvar$ of a variance swap under a multi-factor Heston model. $Kvar$ is approximated by the expectation of the realized and unrealised variance,

$$Kvar = E_0^T[\delta_R^2] = \sum_{i=1,\cdots,N} f_k, \tag{5.10}$$

with $f_k = e^{r\Delta t} E_0^T[\left(\frac{S_k - S_{k-1}}{S_{k-1}}\right)^2]$.

First we assume $T = k * \Delta t, k = 1, \cdots, N$. As mentioned before, the expectation value of the realized variance can be reduced to calculating N expectations of $\left(\frac{S_k}{I_k} - 1)^2\right)$, with $t_k = k * \Delta t$. There are two different situations:

If $k = 1$, and $T = \Delta t$, we calculate f_0 by the expectation

$$E_0^T\left[\left(\frac{S_1}{I_1} - 1\right)^2\right] \tag{5.11}$$

If $i = 2, \cdots, N$, and $T = N\Delta t$, we calculate f_k by the expectation

$$E_0^T\left[E_{i-1}^T\left[\left(\frac{S_i}{Ii} - 1\right)^2\right]\right]. \tag{5.12}$$

According to the definition of I, we deduce our problem into a two stage PDE system, as detailed in the following subsections.

5.3.1 Partial Correlated Volatility

In this section, we assume that there are n different stochastic volatilities with no correlation. When $n = 1$, our model is exactly the same as the Heston model in [19]; when $n = 2$, it reduces to Heston's two-factor model. To be more specific, let $\alpha_i(_i) = k_i(a_i - b_i * y), i = 1, \cdots, n, \beta_i(_i) = \delta_i \sqrt{y_i}$, and assume both the fast-scale process and the slow scale process to be the mean reverted CIR process.

Then the model can be rewritten as

$$dS = rSdt + \sum_{i=1}^{n} f_i(y_i)Sdw_s^{(i)},$$

$$dy_i = (\frac{1}{\xi}\alpha_i(y_i) - \frac{1}{\sqrt{\xi}}\Lambda(y_i)\beta_i(y_i)dt + \frac{1}{\sqrt{\xi}}\beta_i(y_i)w_y^i,$$

with $Cov(w_i^{(s)}, w_i^{(y)}) = \rho_i$ and $Cov(w_i^{(y)}, w_j^{(y)}) = 0, when i \neq j.$ Here we assume that the stochastic volatility is a mean-reverted process with the market price of volatility $\Lambda = \frac{\beta_i}{\sqrt{\xi}\delta_i}$, therefore, the specific case of stochastic process can be rewritten in the form of

$$dy_i = k_i^*(a_i^* - b_i^* y_i)dt + \delta_i\sqrt{y_i}w_y^i,$$

where $a_i^* = \frac{1}{\xi}\frac{ka}{k+1}, k^* = \frac{1}{\xi}(k+1)$, and $\delta^* = \frac{\delta_i}{\sqrt{\xi}}$

(A) The first stage of calculation

When $T - \Delta t < t < T$, according to the Feynman-Kac theorem under the risk neutral assumption, we obtain the following PDE,

$$U_t + \frac{1}{2}\sum_{i=1}^{n} y_i S^2 U_{SS} + rSU_S - rU + \sum_{i=1}^{n}\left[\frac{1}{2}\delta_i^{*2}y_i U_{y_i y_i} + k_i^*(a_i^* - b_i^* * y_i)U_{y_i} + \rho_i\delta_i^* y_i SU_{Sy_i}\right] = 0,$$

$$(5.13)$$

with the terminal condition

$$U = (\frac{S}{I} - 1)^2. \tag{5.14}$$

Let $\tau = T - t, x = ln(S)$, then (6.19) can be converted to the following form

$$U_\tau - \frac{1}{2}\sum_{i=1}^{n} y_i U_{xx} - (r - \frac{1}{2}\sum_{i=1}^{n} y_i U_x + rU - \sum_{i=1}^{n}\left[\frac{1}{2}\delta_i^{*2}y_i U_{y_i y_i} + k_i^*(a_i^* - b_i^* * y_i)U_{y_i} + \rho_i\delta_i^* y_i U_{xy_i}\right] = 0,$$

$$(5.15)$$

with $0 < \tau < \Delta t$ and the initial condition $U(x, y_i, 0) = (\frac{e^x}{I} - 1)^2$. Taking the Fourier transform with respect to x, equation (6.31) becomes

$$V_\tau + \sum_{i=1}^{n}\left[-\frac{1}{2}\delta_i^2 y_i V_{y_i y_i} + (k_i^* a_i - (b_i - jw\rho_i\delta_i^*)) y_i V_{y_i} + \frac{1}{2}(r - rjw + (w^2 + jw)y_i)) V\right] = 0,$$

$$(5.16)$$

where $V(w, y_i, t) = \mathcal{F}(U(x, y_i, t))$, and j is the complex number and $j^2 = -1$.

We assume that the solution of (5.16) is an affine process with the structure of

$$V = e^{C(w,\tau) + \sum_{i=1}^{n} D_i(w,\tau)y_i}V(w, y_i, 0). \tag{5.17}$$

Substituting (5.17) into (5.16), we obtain the following ODEs

$$\frac{\partial D_i}{\partial \tau} = \frac{1}{2}\delta_i^{*2}D_i^2 + (jw\rho_i\delta_i^* - k_i^*b_i^*)D_i - \frac{1}{2}(w^2 + jw), i = 1, \cdots, n \qquad (5.18)$$

$$\frac{\partial C}{\partial \tau} = \sum_i k_i^* a_i^* D_i - (r + rjw), \qquad (5.19)$$

with initial conditions $C(w, 0) = 0, D_i(w, 0) = 0$. The above ODEs can be solved analytically to yield

$$D_i(\tau) = \frac{A_i + B_i}{\delta_i^{*2}} \frac{1 - e^{B_i\tau}}{1 - g_i e^{B_i\tau}}, \qquad (5.20)$$

where $A_i = -k_i^*(jw\rho_i\delta_i^* - b_i), B_i = \sqrt{A_i^2 + \delta_i^{*2}k_i^*(w^2 + jw)}, g_i = \frac{A_i + B_i}{A_i - B_i}$.

$$D_\tau = AD^2 - BD + C, \qquad (5.21)$$

where $A = \frac{1}{2}\delta^2, B = -(jw\rho\delta - kb), C = -\frac{1}{2}(w^2 + jw)$. By completing square

$$D_\tau = A(D^2 - \frac{B}{A}D) + C$$

$$= A[(D^2 - \frac{B}{A}D)^2 - \frac{B^2}{4A^2}] + C$$

$$= A[(D^2 - \frac{B}{A}D)^2 - \frac{B^2 - 4AC}{4A^2}]$$

$$= A[(D^2 - \frac{B}{A}D) + \frac{\sqrt{B^2 - 4AC}}{4A^2}][(D^2 - \frac{B}{A}D) - \frac{\sqrt{B^2 - 4AC}}{4A^2}].$$

By simplification, we obtain

$$\frac{dD}{A[(D^2 - \frac{B}{A}D) + \frac{\sqrt{B^2-4AC}}{4A^2}][(D^2 - \frac{B}{A}D) - \frac{\sqrt{B^2-4AC}}{4A^2}]} = d\tau \qquad (5.22)$$

$$\frac{1}{A}\frac{A}{\sqrt{B^2 - 4AC}}\left[\frac{dD}{D - \frac{B+\sqrt{B^2-4AC}}{2A}} - \frac{dD}{D - \frac{B-\sqrt{B^2-4AC}}{2A}}\right]. \qquad (5.23)$$

Thus,

$$\frac{2AD - (B + \sqrt{B^2 - 4AC})}{2AD - (B - \sqrt{B^2 - 4AC})} = C_0 e^{\sqrt{B^2-4AC}\tau}, \qquad (5.24)$$

where C_0 can be denoted by the initial condition $D(0, w) = 0$,

$$C_0 = \frac{B + \sqrt{B^2 - 4AC}}{B - \sqrt{B^2 - 4AC}}. \qquad (5.25)$$

Substituting A, B and C into (5.20), and solving for D, we obtain

$$D = \frac{a+b}{\delta^2} \frac{1 - e^{b\tau}}{1 - ge^{b\tau}}, \tag{5.26}$$

with $g = \frac{a+b}{a-b}$.

In order to obtain the solution of $U(x, y_i, \tau)$, we perform the inverse Fourier transformation and obtain

$$U(x, y_i, \tau) = \mathcal{F}^{-1}[V(w, y, z, \tau)] = \mathcal{F}^{-1}[e^{C(w,\tau) + \sum_{i=1}^{2} D_i(w,\tau)y_i} \mathcal{F}[(\frac{e^x}{I} - 1)])^2], \tag{5.27}$$

From the generalized Fourier transform defined by

$$\mathcal{F}[e^{imx}] = 2\pi\delta_m(w), \tag{5.28}$$

with $\delta_m(w)$ satisfying

$$\int \delta_m \phi(x) dx = \phi(m), \tag{5.29}$$

after some derivations, we obtain

$$
\begin{aligned}
U(x, y_i, \tau) &= \int e^{C(w,\tau) + \sum_{i=1}^{2} D_i(w,\tau)y_i} \frac{1}{2\pi} \left[\frac{\delta_{-2j}(w)}{I^2} - 2\frac{\delta_{-j}(w)}{I} + \delta_0(w) \right] dw \\
&= \frac{e^{2x}}{I^2} e^{C(-2j,\tau) + \sum_{i=1}^{2} D_i(-2j,\tau)y_i} - 2\frac{e^x}{I} e^{C(-j,\tau) + \sum_{i=1}^{2} D_i(-j,\tau)y_i} + C(0, \tau) \\
&= \frac{e^{2x}}{I^2} e^{C(-2j,\tau) + \sum_{i=1}^{2} D_i(-2j,\tau)y_i} - 2\frac{e^x}{I} e^{C(-j,\tau)} + e^{-r\Delta}. \tag{5.30}
\end{aligned}
$$

with $D_i(-j, y_i, \tau) = 0. \frac{\partial C_i(0, y_i, \tau)}{\partial \tau} = -r$.

(B)   The second stage of calculation

When $0 < t < T - \Delta t$, we have $\Delta t < \tau < T$. We also know that $I_i = S_{i-1}$ at time t_{i-1}. Thus, the maturity condition at time t_{i-1} reduces to

$$F(y_i) = e^{C(-2j,\Delta t) + \sum_{i=1}^{2} D_i(-2j,\Delta t)y_i} - 2e^{C(-j,\Delta t)} + 1, \tag{5.31}$$

We assume that y_1 and y_2 are independent processes, then,

$$f_k = E_0^T(F(y_i)) = e^{C(-2j,\Delta t)} g_1 g_2 - 2e^{C(-j,\Delta t)} + 1, \tag{5.32}$$

with $g_i = E_0^T(e^{D_i(-2j,\Delta t)y_i})$. According to the Feynman-Kac formula, $f_i, i = 1, 2$ can be obtained by solving the following PDE,

$$\frac{\partial g_i}{\partial \tau} = \frac{1}{2}\delta_i^{*2}\frac{\partial g_i^2}{\partial y_i^2} + k_i^*(a_i^* - b_i^* y_i)\frac{\partial g_i}{\partial y}, \tag{5.33}$$

$$g_i(y_i, \Delta t) = e^{D_i(-2j,\Delta t)y_i}, \Delta t < \tau < T. \tag{5.34}$$

Similarly, we assume that the solution of (6.46) has an affine form of

$$g_i = e^{L_i + H_i y_i}. \tag{5.35}$$

Substituting (6.48) into (6.46), we obtain the following ODEs,

$$\frac{\partial L_i}{\partial t} = k^* a_1^* H_i, \tag{5.36}$$

$$\frac{\partial H_i}{\partial t} = -k^* H_i + \frac{1}{2}\sigma^{*2}H_i^2, \tag{5.37}$$

with the initial condition $L_i(D_i(-2j, \Delta t), \Delta t) = 0$ and $H_i(D_i(-2j, \Delta t), \Delta t) = D_i(-2j, \Delta t)$. By solving the equation, we obtain

$$H_i = \frac{2k_i^*}{\sigma_i^{*2}}\frac{e^{-l\tau}}{e^{-k_i^*\tau} - c_0}, \tag{5.38}$$

$$L_i = \frac{-2k_i^* a_i^*}{\sigma_{*i}^2}ln(1 - \frac{e^{-k_i^*\tau}}{c_0}), \tag{5.39}$$

with $c_0 = 1 - \frac{2k_i^*}{\sigma_i^2 D_i(-2j,\Delta t)}$.

Thus, the fair strike price is

$$Kvar = E_0^T[\delta_R^2] = \frac{100^2}{T}\left[f_1 + \sum_{k=2}^{N} f_k\right], \tag{5.40}$$

where

$$f_1 = e^{C(-2j,\Delta)+\sum_{i=1}^2 D_i(-2j,\Delta)y_i} - 2e^{C(-j,\Delta)} + 1, \tag{5.41}$$

$$f_k = E_0^T(F(y_i)) = e^{C(-2j,k\Delta t)}g_1 g_2 - 2e^{C(-j,k\Delta t)} + 1, \tag{5.42}$$

with $g_i, i = 1, \cdots, n$ calculated by (6.48).

5.3.2   Full Correlated Stochastic Volatility

In model (6.2), it is more realistic to assume that the fast-scale volatility and slow-scale volatility are not mutually independent. However, the derivation of the solution to the problem becomes more complex.

(A)   The first stage of calculation

To specify our algorithm in detail, we assume that there are two volatilities. When $T - \Delta t < t < T$, according to the Feynman-Kac theorem under the risk neutral assumption, we obtain the following PDE,

$$U_t + rSU_S - rU + \frac{1}{2}\delta_1^*\delta_2^*\rho_{12}\sqrt{y_1 y_2}U_{y_1,y_2} +$$

$$\sum_{i=1}^{2}\left[\frac{1}{2}\delta_i^{*2}y_i U_{y_i y_i} + k_i^*(a_i^* - b_i^* * y_i)U_y + \rho_i\delta_i^* y_i SU_{Sy_i} + \frac{1}{2}y_i S^2 U_{SS}\right] = 0, \quad (5.43)$$

Let $\tau = T - t, x = ln(S)$, then (5.43) can be converted to the following form

$$U_\tau - rU_x + rU - \frac{1}{2}\delta_1^*\delta_2^*\rho_{12}\sqrt{y_1 y_2}U_{y_1,y_2} -$$

$$\sum_{i=1}^{2}\left[\frac{1}{2}\delta_i^{*2}y_i U_{y_i y_i} + k_i^*(a_i^* - b_i^* * y_i)U_{y_i} + \rho_i\delta_i^* y_i U_{xy_i} + \frac{1}{2}y_i U_{xx} - \frac{1}{2}y_i\right] = 0, \quad (5.44)$$

By the Fourier transform with respect to x, from (5.44) we obtain

$$V_\tau - \frac{1}{2}\delta_1\delta_2\rho_{12}\sqrt{y_1 y_2}V_{y_1,y_2} + \sum_{i=1}^{2}\left[\frac{1}{2}\left(r - rjw + (w^2 + jw)y_i)\right)V\right]$$

$$+ \sum_{i=1}^{2}\left[-\frac{1}{2}\delta_i^{2*}y_i V_{y_i y_i} + (k_i^* a_i^* - (b_i^* - jw\rho_i\delta_i^*)) y_i V_{y_i}\right] = 0, \quad (5.45)$$

where $V = \mathcal{F}(U(x, y_1, y_2, 0))$ is the Fourier transform of U with respect to x. Substituting (5.17) into (5.45), we find that the D_i is exactly the same as (5.20), while C_i satisfies the following ODE,

$$\frac{\partial C}{\partial \tau} = \sum_i k_i^* a_i^* D_i - (r + rjw) + \frac{1}{2}\delta_1^*\delta_2^*\rho_{12}\sqrt{y_1 y_2}D_1 D_2, i = 1, 2 \quad (5.46)$$

with initial conditions $C_i(w, 0) = 0$.

(B)   The second stage of calculation

When $0 < t < T - \Delta t$, we have $\Delta t < \tau < T$. We also know that $I_i = S_{i-1}$ at time t_{i-1}. Thus, the maturity condition at time t_{i-1} reduces to

$$F(y_i) = e^{C(-2j,\Delta t) + \sum_{i=1}^{2} D_i(-2j,\Delta t)y_i} - 2e^{C(-j,\Delta t)} + 1, \tag{5.47}$$

As y_1 and y_2 are not independent processes, we have

$$
\begin{aligned}
f_k &= E_0^T(F(y_i)) = E_0^T(e^{C(-2j,\Delta t) + \sum_{i=1}^{2} D_i(-2j,\Delta t)y_i}) - 2E_0^T(e^{C(-j,\Delta t)}) + 1 \\
&= E_0^T(e^{C(-2j,\Delta t)})E_0^T(e^{\sum_{i=1}^{2} D_i(-2j,\Delta t)y_i}) - 2E_0^T(e^{C(-j,\Delta t)}) + e^{-r\Delta t},
\end{aligned}
\tag{5.48}
$$

with

$$E_0^T(e^{C(-2j,\Delta t)}) = e^{E_0^T(C(-2j,\Delta))}, \tag{5.49}$$

$$E_0^T(e^{C(-j,\Delta t)}) = e^{E_0^T(C(-j,\Delta t))}, \tag{5.50}$$

where $C(w,\tau)$ satisfies

$$\frac{\partial C}{\partial \tau} = \sum_i k_i^* a_i^* D_i - (r + rjw) + \rho_{12}\frac{1}{2}\delta_1^*\delta_2^* E_0^T(\sqrt{y_1 y_2})D_1 D_2. \tag{5.51}$$

As $\sqrt{y_1 y_2}$ is not an affine structure, based on the work of [49], the variable $y_i(t)$ is approximated by the normal distribution with

$$
\begin{aligned}
E(y_i(t)) &= c_i(t)(d_i + \lambda_i(t)), \\
Var(y_i(t)) &= c_i^2(t)(2d_i + 4\lambda_i(t)),
\end{aligned}
\tag{5.52}
$$

where $c_i(t) = \frac{\delta_i^{*2}}{4k_i^*}(1 - e^{-k_i^* t})$, $d_i(t) = \frac{4k_i^* a_i}{\delta^{*2}}$, $\lambda_i(t) = \frac{4a_{*i}y_0}{\delta_i^2}\frac{e^{-k_i^* t}}{1 - e^{-k_i^* t}}$.
   Therefore,

$$E(\sqrt{y_i(t)}) = c_i(d_i + \lambda_i - 1) + \frac{c_i d_i}{2(d_i + \lambda_i)} \tag{5.53}$$

$$Var(\sqrt{y_i(t)}) = c_i - \frac{c_i d_i}{2(d_i + \lambda_i)} \tag{5.54}$$

Proof. According to the Taylor expansion, $Var(\sqrt{y_i(t)})$ can be approximated by

$$Var(\sqrt{y_i(t)}) \approx \frac{Var(y_i(t))}{4E(y_i(t))} = c_i - \frac{c_i d_i}{2(d_i + \lambda_i)}; \tag{5.55}$$

As $Var(\sqrt{y_i(t)}) = E(y_i(t)) - E^2(\sqrt{y_i(t)})$, we obtain

$$E(\sqrt{y_i}(t)) = \sqrt{E(y_i(t)) - Var(\sqrt{y_i(t)})} = c_i(d_i + \lambda_i - 1) + \frac{c_i d_i}{2(d_i + \lambda_i)}, \quad (5.56)$$

$\square$

Let $Z = \sum_{i=1}^{2} D_i(-2j, r) y_i$. Now we discuss the approximation of $E_0^T(e^Z)$,

$$E_0^T(e^Z) = \int_{i=-\infty}^{\infty} e^Z e^{-(\frac{Z-E(Z)}{Var(Z)})^2} \approx e^{E(Z)+\frac{1}{2}Var(Z)}, \quad (5.57)$$

with

$$E(Z) = D_i(-2j, r) \sum_{i=1}^{2} E(y_i) = D_i(-2j, r) \sum_{i=1}^{2} c_i(t)(d_i + \lambda_i(t)), \quad (5.58)$$

$$Var(Z) = \sum_{i=1}^{2} D_i^2(-2j, r) Var(y_i) + 2D_1(-2j, r)D_2(-2j, r)\rho_{12} Cov(y_1, y_2)$$

$$= \sum_{i=1}^{2} D_i^2(-2j, r) c_i^2(t)(2d_i + 4\lambda_i(t))$$

$$+ 2D_1(-2j, r)D_2(-2j, r)\rho_{12}\sqrt{c_1^2(t)(2d_1 + 4\lambda_1(t))}\sqrt{c_2^2(t)(2d_2 + 4\lambda_2(t))}$$
$$(5.59)$$

Thus, the fair strike price

$$Kvar = E_0^T[\delta_R^2] = \frac{100^2}{T}\left[f_1 + \sum_{k=2}^{N} f_k\right], \quad (5.60)$$

where

$$f_1 = e^{C(-2j,\Delta)+\sum_{i=1}^{2} D_i(-2j,\Delta)y_i} - 2e^{C(-j,\Delta)} + 1, \quad (5.61)$$

$$f_k = E_0^T(e^{C(-2j,\tau)+\sum_{i=1}^{2} D_i(-2j,\tau)y_i}) - 2E_0^T(e^{C(-j,\tau)}) + 1, \quad (5.62)$$

5.4 Numerical result

5.4.1 Partial Correlation Stochastic Volatility

The parameters shown in table 5.1 is calibrated by Christoffersen and Heston [78], from which they distinguished the two factors model by the principal analysis. We calculate the fair strike price $Kvar$ by (5.60), and compare our result with the

Table 5.1: Calibrated Parameters

b_1	a_1	σ_1	ρ_1
0.1500	0.0059	1.9829	-0.9902

b_2	a_2	σ_2	ρ_2
0.2335	0.1621	0.1971	-0.8918

approximation value by the continuous model by Swishchuk formula [18]. The Swishchuk formula is extended to incorporate two factors, as shown in (5.63). From figure 5.1, we find that the result from our approach is reasonable and the fair strike price(Kvar) will approach 270 in a long run.

$$Kvar = \sum_{i=1}^{2} y_i * \frac{1 - e^{-k_i * T}}{k_i * T} + a_i * \frac{1 - (1 - e^{-k_i * T})}{k_i * T}; \qquad (5.63)$$
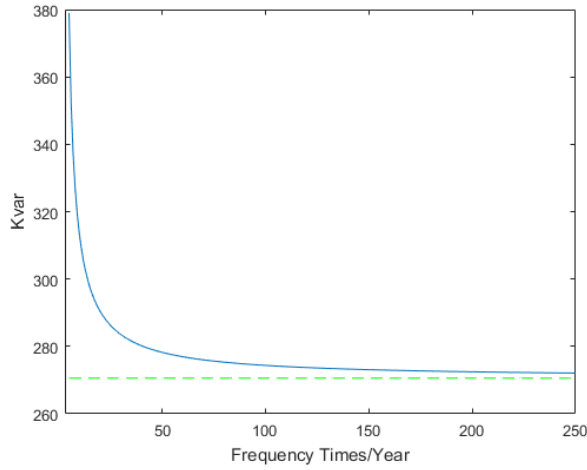
Figure 5.1: Strike Price of Variance Swap

We also investigate the effect of fast-scale rate ξ and the slow-scale rate σ. In table 5.2, we assume that it is daily sampled with $AF = 252$, and we find that the effect of the scale-rate of the stochastic volatility is significant. The fair strike price increases with the fast-scale rate ξ and decreases with the slow scale rate σ, and the effect of the slow-scale rate outweighs the effect of the fast-scale rate.

5.4.2 Full Correlation Stochastic Volatility

In order to study the validation of our result, we firstly set $\rho_{12} = 0$, thus the full correlated model reduces to a partial correlated model. We obtain the result as shown in figure 5.2, and compare the result calculated by the second approximation(full correlated volatility) with the solution calculated by the first approxi-

Table 5.2: The effects of fast and slow scale rate

	$\xi = 1$	$\xi = 0.5$	$\xi = 0.1$	$\xi = 0.05$	$\xi = 0.01$
$\sigma = 1$	277.9368	278.8180	280.4146	284.5397	379.3164
$\sigma = 0.5$	237.5771	238.4511	240.0597	244.1794	338.9487
$\sigma = 0.1$	195.7911	196.6649	198.2734	202.3927	297.1543
$\sigma = 0.05$	152.5218	153.3956	155.3956	159.1228	253.8764
$\sigma = 0.01$	107.7001	108.5738	108.5738	114.3005	209.0458

mation(partial correlated volatility). The numerical integration of the C_i term is calculated by the trapezoidal rule by MATLAB. We also compare our results with those obtained from Monte Carlo simulation and the continuous approximation (5.63). For the reason that the Monte Carlo simulation serves as a benchmark in this section, we simply use the Euler Maruyama method without considering the variance reduction technique. We find that our results are in good agreement with those obtained by other methods, and the speed of our method is thousand times faster than any kinds of numerical methods. The parameters we use is this section is $k_i = 11.35, \rho_i = -0.64, y_i = 0.005, a_i = 0.022, b_i = 1, sigma_i = 0.618$, which satisfy Feller's condition $2ka >> \sigma^2$. In this section, we also study the
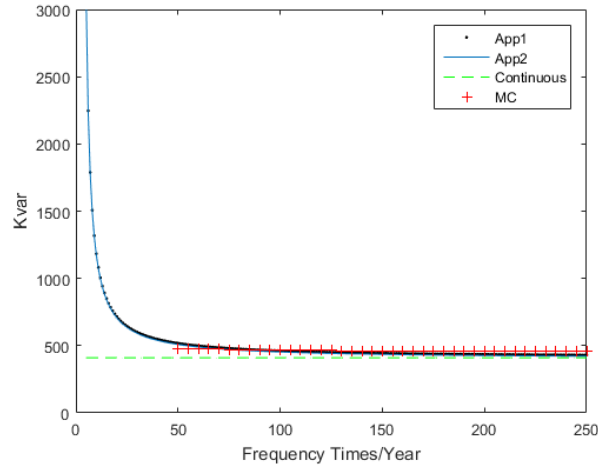
Figure 5.2: Strike Price of Variance Swap

effect of the correlation. As shown in table 5.3, we compare the monthly, weekly, and daily sampled fair strike price with different correlation rate ρ_{12}. We find that the incorporation of correlation rate will slightly change the value of fair strike price.

Table 5.3: The Effect of Correlation

	$\rho_{12} = 0$	$\rho_{12} = -0.5$	$\rho_{12} = 0.5$
$AF = 12$	994.9997	994.5535	995.4609
$AF = 52$	506.6148	506.574	506.6557
$AF = 252$	421.345	421.3383	421.3518

5.5 Conclusion

This chapter incorporates both fast-scale volatility and slow-scale volatility to study the pricing of the discrete sampled variance swap problem. A semi-closed form solution is obtained by applying the generalized Fourier transform in both the partial correlated model and the full correlated model. The Monte Carlo simulation result and approximation of the continuous variance swap price are used for the verification of our formula. we find that the effect of scale-rate is significant. The fair strike price increases with the fast scale rate and decreases with the slow scale rate. The effect of fast scale rate surpasses the effect of slow scale rate in a short run. We also study the effect of the correlation rate, and find that the effect of the correlation rate, though small, does exist. Negative correlation between two volatility processes will lower down the fair strike price, while the fair strike price increases slightly when the correlation value between the two volatilities is positive.

CHAPTER 6

The Variance Swap Pricing Under Hybrid Jump Model

6.1 General

This chapter investigates the pricing of discretely sampled variance swaps driven by a generalized stochastic model taking into account both stochastic volatility and jump. By proper selection of parameters, our model includes various existing models as special cases, including the CIR model, the Heston-CIR model, and the multifactor-CIR model. We deal with the integral term arising from the jump diffusion process with the characteristic function through Fourier convolution, and a semi-analytic solution is derived for pricing variance swap based on a generalized high-dimensional hybrid model. The effects of stochastic interest rate, stochastic volatility and jump rate are studied in this chapter.

The contribution of this work includes the following three aspects. Firstly, we consider a more general model. With proper selection of parameters, our proposed model covers various existing models as special cases, including the jump diffusion model, the CIR interest rate model [21], the one-factor Heston-CIR model, and the multi-factor-CIR Heston model. Besides, we take into consideration not only the jump diffusion effects, but also the stochastic interest rate and the multi-factor stochastic volatility process in the model. Different from Brodie&Jain's work [94], a semi-analytic solution of the discrete sampling variance swap is derived by relating the associated partial integral differential equation with the generalized Fourier transform. The integral term arising from jump diffusion is solved by the Fourier convolution and the characteristic function. Furthermore, inclusion of multi-factor processes results in a high dimensional partial differential equation(PDE). We successfully reduce the dimension of the equation by embedding our problem into the framework of Little and Pant(2001). The skew effects

of correlation between different volatility processes are also investigated. To be more specific in detail, the payoff function of the variance swap is treated as a function of the current stock price and the previous stock price, with the former following a stochastic process, while the later being determined at the current time. In this case, the n-dimensional PDE is reduced to a $n-1$ dimensional PDE in two different periods. We then apply the generalized Fourier transformation based on the Cox and Ross work [21] to solve the first stage PDE system. In comparison, the work by Lian and Zhu [19] only solves the problem involving only the one-factor CIR process. For the partial correlated model, a semi-closed form solution is derived by the assumption of affine structure. However, when the model is fully correlated, non-affine item is included in our model, and in this case, we approximate the expectation of the non-affine term utilizing the result of Grzelak and Oosterlee's work [49].

The rest of the chapter is organised as follows. In section 6.2, we first present the models for stock price, volatility and interest rate, taking into consideration of volatility and jump, then demonstrates the change of measure under the risk-neutral assumption. In section 6.4, a semi-analytic solution is derived by applying the generalized Fourier transform. Numerical results are given in section 6.5, followed by a conclusion in section 6.6.

6.2 Mathematical Modelling

The price of stock is assumed to follow the following stochastic process,

$$dS = \mu S dt + f(y_i) S dw_s + S dJ^S, i = 1, \cdots, n \tag{6.1}$$

It should be addressed that the return rate μ of the stock price is not necessary equal to the risk-free rate r before the risk-neutral adjustment. By a careful selection of the market price of the volatility term, it turns to r, and thus the discount stock process becomes a martingale. We can apply the Feynman-Kac theorem to obtain the associated PDE if the underlying process is a martingale. $f_i(y_i)$ is a function of volatility, and we assume that the volatility is driven by more than one stochastic process.

$$dy_i = \alpha_i(y_i)dt + \beta_i(y_i)dw^i_{y_i}, (i = 1, \cdots, n). \tag{6.2}$$

The interest rate is assumed to follow the stochastic process

$$dr = m(r)dt + n(r)dw_r. \tag{6.3}$$

The fast-scale and slow-scale volatilities are distinguished by the frequencies of the observed volatility data, and have to be considered simultaneously as suggested by Chacko and Viceria[2005] [37]. In addition, we assume that the Brownian motions (w_s, w_{y_i}, w_r) are correlated with the following correlation matrix

$$C = \begin{bmatrix} 1 & \rho_{sy_1} & \cdots & \rho_{sy_n} & \rho_{sr} \\ \rho_{sy_1} & 1 & \cdots & \rho_{y_1 y_n} & \rho_{y_1 r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{sy_n} & \rho_{y_n y_1} & \cdots & 1 & \rho_{y_n r} \\ \rho_{sr} & \rho_{y_1 r} & \cdots & \rho_{y_n r} & 1 \end{bmatrix} \tag{6.4}$$

with $[\rho_{sy_i}, \rho_{sr}, \rho_{y_i y_j}, \rho_{y_i r}]$ satisfying $|\rho_{sy_i}| < 1, |\rho_{sr}| < 1, |\rho_{y_i y_j}| < 1, |\rho_{yr}| < 1, |\rho_{zr}| < 1$, along with the positive definite property, $Def(C) > 0$. According to the Cholesky decomposition $C = LL^T$, we obtain the lower triangle matrix L

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \rho_{sy_1} & \bar{\rho_{y_1}} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \rho_{sy_i} & \bar{\rho_{y_1 y_i}} & \cdots & \bar{\rho_{y_i}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \rho_{sr} & \bar{\rho_{y_1 r}} & \cdots & \bar{\rho_{y_i r}} & \cdots & \bar{\rho_r} \end{bmatrix}, \tag{6.5}$$

where

$$\bar{\rho_{y_1}} = \sqrt{1 - \rho_{sy_1}^2}$$

$$\bar{\rho_{y_i}} = \sqrt{1 - \rho_{sy_i}^2 - \sum_{k=1}^{i-1} \bar{\rho_{y_k y_i}}^2}$$

$$\bar{\rho_r} = \sqrt{1 - \rho_{sr}^2 - \sum_{k=1}^{n} \bar{\rho_{y_k r}}^2}$$

$$\bar{\rho_{y_j y_i}} = \frac{\rho_{y_i y_j} - \rho_{sy_i} \rho_{sy_j} - \sum_{k=1}^{i-1} \bar{\rho}_{y_j y_k} \bar{\rho}_{y_k y_i}}{\bar{\rho}_{y_i}}, i > j$$

$$\bar{\rho_{y_i r}} = \frac{\rho_{y_i r} - \rho_{sr} \rho_{sy_i} - \sum_{k=1}^{j-1} \rho_{y_i y_k} \bar{\rho}_{y_k r}}{\bar{\rho}_r}$$

Let $\Upsilon = [dw_s, d\mathbf{w}_y, dw_r]^T$ with $\mathbf{w}_y = [w_{y_1}, w_{y_2}, \cdots, w_{y_n}]^T$. By implementing the numeraire change from measure P to measure Q, we obtain the risk neutral vector satisfying

$$\hat{\Upsilon} = \Upsilon + \Pi dt \tag{6.6}$$

where $\hat{\Upsilon} = [d\hat{w}_s, d\hat{\mathbf{w}}_y, d\hat{w}_r]^T$, $\Pi = [\frac{\mu - r}{f}, \mathbf{\Gamma}(t), \gamma_r(t)]^T$, $\mathbf{\Gamma}(t) = [\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t)]$ denote the market price of risk from stochastic volatility, $\gamma_r(t)$ denotes the market price of risk from stochastic interest rate. Similar technique was applied in [95] [96] [97].

Under the risk-neutral measure and the above adjustment, (6.1)-(6.3) can be rewritten as

$$\mathbf{D} = \mathbf{U}^p dt + \Sigma \Upsilon = \mathbf{U}^Q dt + \Sigma \hat{\Upsilon}, \tag{6.7}$$

where $\mathbf{D} = [dS/S, d\mathbf{y}, dr]^T$ denotes the change of the underlying process, $\mathbf{U}^P = [\mu, \alpha(\mathbf{y}), m(r)]^T$ denotes the drift part under the measure P, $\mathbf{U}^Q = \mathbf{U}^P - \Pi$ denotes the drift under the risk neutral measure Q. Σ is an $n \times n$ matrix which denotes the volatility part

$$\Sigma = \begin{bmatrix} f(y_i) & 0 & 0 \\ 0 & \mathbf{B}(\mathbf{y}) & 0 \\ 0 & 0 & n(r) \end{bmatrix}, \tag{6.8}$$

with

$$\mathbf{B}(\mathbf{y}) = \begin{bmatrix} \beta_1(y_1) & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \beta_n(y_n) \end{bmatrix}. \tag{6.9}$$

Now we change the numeraires form Q to Q^T together with the orthogonal decomposition. Let $\Upsilon^* = [dw_s^*, dw_y^*, dw_z^*, dw_r^*]^T$ denote the orthogonal vector,

$$\hat{\Upsilon} = L\Upsilon^* \tag{6.10}$$

Note that the numeraire under Q is $e^{\int r(s)ds}$, the numeraire under the T forward measure Q^T is $A(t, T)e^{-B(t,T)r(t)}$. T-forward measure is a pricing measure absolutely continuous and under which the pricing process is a martingale; however, rather than using the money market as numeraire, it uses a bond with maturity T.

Thus, the drift part $\mathbf{U}^T$ can be obtained by the formula below

$$\mathbf{U}^T = \mathbf{U}^Q + \Sigma C \Sigma^{Q^T}, \tag{6.11}$$

where $\Sigma^{Q^T} = [0, 0, \cdots, -B(t, T)n(r)]^T$. Therefore, the SDE can be rewritten

by the following forms under the measure Q^T,

$$dS = (r - \rho_{sr}B(t,T)n(r))) \, S dt + f(y_i)S dw_s^* + S dJ^s, \tag{6.12}$$

$$d\mathbf{y} = (\alpha(\mathbf{y}) - \Lambda(\mathbf{y},r)\beta(\mathbf{y})) \, dt + \beta(\mathbf{y})d\mathbf{w}_y^*, \tag{6.13}$$

$$dr = (m(r) - (\gamma_r(r) + B(t,T)n(r)) \, n(r)) \, dt + n(r)dw_r^* \tag{6.14}$$

with $\Lambda = [\lambda_1(y_1, r), \cdots, \lambda_n(y_n, r)]$ and $\lambda_i(y_i, r) = \gamma_i(y_i) + \rho_{y_i r}B(t,T)n(r)$.

Different from the European options, variance swaps and volatility swaps are time-dependent. The payoff function of a variance swap is as shown below

$$V(T, x, y, z) = L * E^Q(\sigma_R^2 - K). \tag{6.15}$$

However, under the risk-neutral assumption, we are more interested in the fair strike price $Kvar$ of the variance swap, which can be calculated by taking the expectation of the realized variance. In our work, we assume that the realized variance is calculated discretely by the formula below

$$\sigma_R^2 = \frac{AF}{N} \sum_{i=0}^{N-1} \left(\frac{S_{i+1} - S_i}{S_i} \right)^2. \tag{6.16}$$

For the reason that our general model will result in a high dimensional problem, we apply the dimensional reduction technique, as in Little&Pant[2001] [63], by introducing a new variable I_t driven by the underlying process

$$I_t = \int_0^t \delta(t_{i-1} - \tau)S_\tau d\tau, \tag{6.17}$$

where δ is the Dirac delta function. I_t is only related with the value of S_t at time t_{i-1}, which means $I_t = 0$ if $t < t_{i-1}$, and $I_t = S_{i-1}$ if $t \geq t_{i-1}$. The terminal condition under the fast and slow scale correction becomes

$$U_t(T, S, Y, Z, I) = (\frac{S_t}{I_t} - 1)^2. \tag{6.18}$$

Let $x = log(S), \tau = T - t$, then according to the standard no-arbitrage argument, the following PDE can be derived from the Feynman-Kac theorem [7]

$$\mathcal{L}U + U_I = 0, \tag{6.19}$$

for $t \in [0, T]$, where the operator $\mathcal{L}$ is given by

$$\mathcal{L} = \partial_\tau - (\tilde{\mathbf{U}})^T \mathbf{P} - \mathbf{P}^T \Sigma \mathbf{C} \Sigma \mathbf{P}, \tag{6.20}$$

in which $\tilde{U} = U^T - [-\frac{1}{2}f^2 + \lambda(1 - E(e^z))]$

$$\mathbf{P} = \begin{bmatrix} \partial_s & 0 & 0 \\ 0 & \partial_\mathbf{y} & 0 \\ 0 & 0 & \partial_r \end{bmatrix}, \tag{6.21}$$

with

$$\partial_\mathbf{y} = \begin{bmatrix} \partial_{y_1} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \partial_{y_n} \end{bmatrix}. \tag{6.22}$$

With the incorporation of a new variable I, the above PDE(6.19) can be equivalently expressed by the following system of equations,

$$\begin{cases} \mathcal{L}U = 0, \\ U_i(S, \mathbf{y}, r) = (\frac{S}{I} - 1)^2, t_{i-1} \le t \le t_i \end{cases} \tag{6.23}$$

and

$$\begin{cases} \mathcal{L}U = 0, \\ lim_{t \uparrow t_{i-1}} U_i(S, \mathbf{y}, r) = lim_{t \downarrow t_{i-1}} U_i(S, \mathbf{y}, r), 0 \le t \le t_{i-1} \end{cases} \tag{6.24}$$

A semi-analytic solution will be derived by the generalized Fourier transformation in the subsequent section.

6.3 Algorithm of Partial Correlation Case

To obtain a solution of the model given by (6.1) to (6.3), we firstly assume a partial correlated case with the correlation matrix

$$C = \begin{bmatrix} 1 & \rho_{sy_1} & \cdots & \rho_{sy_n} & 0 \\ \rho_{sy_1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{sy_n} & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{6.25}$$

The stochastic processes are assumed to be a multi-factor stochastic volatility process with a mean reverted drift $\alpha(\mathbf{y}) = \mathbf{a}(\mathbf{m} - \mathbf{y})$, and a 1/2 volatility of

volatility $\beta(\mathbf{y}) = \mathbf{b}\sqrt{\mathbf{y}}$, where $\mathbf{a} = [a_1, a_2, \cdots, a_n]^T, \mathbf{m} = [m_1, m_2, \cdots, m_n]^T$ and $\mathbf{b} = [b_1, b_2, \cdots, b_n]^T$ are n dimensional vectors. The stochastic interest rate process is a CIR process with $m = k^*(\theta - r(t)), n = \eta\sqrt{r(t)}$, and $B(t, T)$ is assumed to have the following specific form [49]

$$B(t,T) = \frac{2\left(e^{(T-t)\sqrt{k^{*2}+2\eta^2}} - 1\right)}{2\sqrt{k^{*2}+2\eta^2} + \left(k^* + \sqrt{k^{*2}+2\eta^*}\right)\left(e^{(T-t)\sqrt{k^{*2}+2\eta^2}} - 1\right)} \quad (6.26)$$

Let $\gamma_i(y_i) = \lambda_i \frac{\sqrt{y_i}}{b_i}$ and $\gamma_r(r) = \Lambda_r \frac{\sqrt{r}}{\eta}$

$$dS = \left(r - \frac{1}{2}(\sum_1^n \sqrt{y_i})^2 + \lambda(1 - E(e^z))\right)dt + \sum_{i=1}^n \sqrt{y_i}dw_s^*, \quad (6.27)$$

$$dy_i = a_i^*(m_i^* - y_i)dt + b_i\sqrt{y_i}dw_{yi}^*, i = 1, \cdots, n \quad (6.28)$$

$$dr = \left(k^*(\theta^* - r) - B(t,T)\eta^2 r\right)dt + \eta\sqrt{r}dw_r^*, \quad (6.29)$$

where $a_i^* = a_i + \lambda_i, m_i^* = \frac{a_i m_i}{a_i + \lambda_i}, i = 1, \cdots, n$, and $k^* = k + \Lambda_r, \theta^* = \frac{k\theta}{r + \Lambda_r}$ When $n = 1$ and $\lambda = 0$, the model reduces to the model of Cao & Lian [20].

6.3.1 The First Stage of Calculation

When $T - \Delta t < t < T$, or $0 < \tau < \Delta t$, (6.19) can be expanded as the following PDE

$$U_\tau - [r - \frac{1}{2}(\hat{\mathbf{y}}^\mathbf{T}\hat{\mathbf{y}})^2 + \lambda(1 - E(e^z))]U_x - \frac{1}{2}((\hat{\mathbf{y}}^\mathbf{T}\hat{\mathbf{y}}))^2 U_{xx} - [k^*(\theta^* - r) -$$
$$B(T - \tau, 0)\eta^2 r]U_r - \frac{1}{2}\eta^2 r U_{rr} - \lambda \int_R [u(x + \eta) - u(x)]\Gamma(d\eta) - \mathcal{L}_y U = 0 \quad (6.30)$$

with $\hat{y} = [\sqrt{y_1}, \sqrt{y_2}, \cdots, \sqrt{y_n}]^T$,

$$\mathcal{L}_y = \sum_{i=1}^n \left\{ [a_i^*(m_i^* - y_i)]\partial_{y_i} + \frac{1}{2}b_i^2 y_i \partial_{y_i y_i} + \rho_{\mathsf{sy}_i} b_i \sqrt{y_i}(\sum_{i=1}^n \sqrt{y_i})\partial_{xy_i} \right\}$$

and the initial condition $U(0, x, y_i, z) = (\frac{e^x}{I} - 1)^2$.

Let V be the Fourier transform of U with respect to x, i.e. $V = \mathcal{F}(U)$, then,

by taking the Fourier transform of (6.30) , we obtain the following PDE,

$$V_\tau = \left\{ [r - \frac{1}{2}(\hat{\mathbf{y}}^\mathbf{T}\hat{\mathbf{y}})^\mathbf{2} + \lambda(1 - E(e^z))](jw) - \frac{1}{2}(\hat{\mathbf{y}}^\mathbf{T}\hat{\mathbf{y}})^2 w^2 - \lambda + \lambda\phi_\eta(w) \right\} V +$$

$$\left\{ k^*(\theta^* - r) - B(T - \tau, 0)\eta^2 r \right\} V_r + \frac{1}{2}\eta^2 r V_{rr} + \tilde{\mathcal{L}}_y V,$$

$$(6.31)$$

where

$$\tilde{\mathcal{L}}_y = \sum_{i=1}^n \left\{ [a_i^*(m_i^* - y_i) + \rho_{\mathsf{sy}_i} b_i \sqrt{y_i}(\sum_{i=1}^n \sqrt{y_i})(jw)]\partial_{y_i} + \frac{1}{2}b_i^2 y_i \partial_{y_i y_i} \right\}$$

and the initial condition is $V(0, w, \mathbf{y}, r) = \mathcal{F}\left((\frac{e^x}{I} - 1)^2 \right)$. $\phi_\eta(w) = \int_R e^{iw\eta}\Gamma(d\eta)$ denotes the characteristic function of the underlying process of the jump size. The commonly used jump model includes Merton's jump model and Kou's double exponential model, as shown in table 6.1. Merton's model assumes that the jump size follows a normal distribution, while the jump size of Kou's model is assumed to be a double exponential distribution. The generalized Fourier transform of the integral term arising from the jump diffusion process is equivalent to the characteristic function of the underlying distribution of jump size:

$$\mathcal{F} \int_R U(x + \eta)\Gamma(d\eta) = \mathcal{F} \int_R U(x + \eta)p(\eta)d\eta$$

$$= \int_R \int_R U(x + \eta)p(\eta)e^{-iwx}d\eta dx$$

$$= \int_R p(\eta) \int_R U(x + \eta)e^{-iwx}dx d\eta$$

$$= \int_R p(\eta) \int_R U(y)e^{-iw(y-\eta)}dy d\eta$$

$$= \int_R p(\eta)e^{iw\eta}d\eta \int_R U(y)e^{-iwy}d\eta$$

$$= \phi_\eta(w)V \qquad\qquad (6.32)$$

Similar results can be obtained from the Fourier convolution theorem if we let

$p(\eta) = g(-s).$

$$\mathcal{F} \int_R U(x + \eta)\Gamma(d\eta) = \mathcal{F} \int_R U(x + \eta)p(\eta)d\eta$$

$$= - \int_R \int_R U(x - s)g(-s)e^{-iwx}dsdx$$

$$= -\mathcal{F}(U(x) \otimes g(-s))$$

$$= -V(w)\mathcal{F}(g(-s))$$

$$= \phi_\eta(w)V(w) \qquad (6.33)$$

Table 6.1: Jump Model

Model	$\Gamma(d\eta)$	$\phi_\eta(w)$
Merton	$\frac{exp(-(\eta-\mu)^2)}{\sqrt{2\pi}\delta}d\eta$	$exp(j\mu w - \frac{w^2}{2}\delta^2)$
Kou	$p\lambda_1 exp(-\lambda_1\eta)\mathcal{I}_{\eta>0} + (1-p)\lambda_2 exp(\lambda_2\eta)\mathcal{I}_{\eta<0}d\eta$	$\frac{p\lambda_1}{\lambda_1-jw} + \frac{(1-p)\lambda_2}{\lambda_2+jw}$

By assuming that the solution has affine structure and following the procedure of Heston [10], the solution can be assumed to have the following form

$$V(\tau, w, \mathbf{y}, r) = exp(C(w, \tau) + \mathbf{D^T}(w, \tau)\mathbf{y} + E(w, \tau)r)V(0, w, \mathbf{y}, r), \qquad (6.34)$$

Then by substituting (6.34) into (6.31), we obtain the following ODEs,

$$D_\tau = -\frac{1}{2}(jw + w^2) + \frac{1}{2}b_i^2 D_i^2 + (\rho_{sy_i}b_i jw - \alpha_i^* m_i^*)D_i, \quad i = 0\cdots n \qquad (6.35)$$

$$E_\tau = wj + \frac{1}{\eta^2}E^2 - (k^* + B(T - \tau, 0)\eta^2)E \qquad (6.36)$$

$$C_\tau = \sum_{i=1}^{n} \alpha_i^* m_i^* D_i + k^*\theta^* E + \lambda(1 - E(e^x))(jw) - \lambda+$$

$$\lambda\phi_\eta(w) + \sum_{i=1}^{n}\sum_{j\neq i}\left\{-\sqrt{y_i y_j}(jw + \frac{1}{2}w^2) + b_i\rho_{sy_i}\sqrt{y_i y_j}(jw)D_i\right\}, \quad i = 0\cdots n$$

$$(6.37)$$

with initial conditions $C(w, 0) = 0, D_i(w, 0) = 0, E(w, 0) = 0$. The E can be solved numerically by using MATLAB. While D can be solved analytically to yield

$$D_i(\tau) = \frac{A_i + B_i}{b_i^{*2}} \frac{1 - e^{B_i\tau}}{1 - g_i e^{B_i\tau}}, \qquad (6.38)$$

where $A_i = -\alpha_i^*(jw\rho_{sy_i]}b_i^* - m_i^*), B_i = \sqrt{A_i^2 + b_i^{*2}\alpha_i^*(w^2 + jw)}, g_i = \frac{A_i+B_i}{A_i-B_i}$. The

calculation of C will be given in detail in the second stage of calculation.

In order to obtain the solution of $U(\tau, x, \mathbf{y}, r)$, we perform the inverse Fourier transformation and obtain

$$U(\tau, x, \mathbf{y}, r) = \mathcal{F}^{-1}[V(\tau, w, \mathbf{y}, \mathbf{r})] = \mathcal{F}^{-1}[exp(C(w, \tau) + \mathbf{D^T}(w, \tau)\mathbf{y} + E(w, \tau)r)]U_0,$$
(6.39)

with

$$U_0 = \mathcal{F}^{-1}(V_0) = \mathcal{F}^{-1}\left\{\mathcal{F}[(\frac{e^x}{I} - 1))^2]\right\};$$
(6.40)

Based on the generalized Fourier transform,

$$\mathcal{F}[e^{imx}] = \delta_m(w),$$
(6.41)

with $\delta_m(w)$ satisfying

$$\int \delta_m\phi(x)dx = \phi(m),$$
(6.42)

after some derivation, we obtain

$$U(x, y_i, \tau) = \int exp(C(w, \tau) + \mathbf{D^T}(w, \tau)\mathbf{y} + E(w, \tau)r)\left[\frac{\delta_{-2j}(w)}{I^2} - 2\frac{\delta_{-j}(w)}{I} + \delta_0(w)exp(jwx)\right]d$$

$$= \frac{e^{2x}}{I^2}exp(\tilde{C}(\tau) + \tilde{\mathbf{D}}(\tau)^{\mathbf{T}}\mathbf{y} + \tilde{E}(\tau)r) - 2\frac{e^x}{I}exp(\hat{C}(\tau) + \hat{E}(\tau)r) + 1.$$
(6.43)

where $\tilde{C}(\tau), \tilde{\mathbf{D}}(\tau)$ and $\tilde{E}(\tau)$ denote $C(-2j, \tau), \mathbf{D}(-2j, \tau), E(-2j, \tau)$ respectively, whereas $\hat{C}(\tau)$ and $\hat{E}(\tau)$ denote $C(-i, \tau)$ and $D(-i, \tau)$ respectively.

6.3.2   The Second Stage of Calculation

When $0 < t < T - \Delta t$, and $\Delta t < \tau < T$, we know that $I_i = S_{i-1}$ at time t_{i-1}. Thus, the maturity condition at time t_{i-1} reduces to

$$F(y_i) = exp(C(\tilde{\Delta}t) + \tilde{\mathbf{D}}(\Delta t)^T\mathbf{y} + \tilde{E}(\Delta t)r) - 2exp(\hat{C}(\Delta t) + \hat{E}(\Delta t)r) + 1. \quad (6.44)$$

For the reason that the correlated matrix is assumed to be partial correlated, we obtain

$$f_k = E_0^T(F(y_i)) = e^{\tilde{C}(\Delta t)}\prod_{i=1}^{n}g_i\tilde{h} - 2e^{\hat{C}(\Delta t)}\hat{h} + 1,$$
(6.45)

with $g_i = E_0^T(e^{\hat{D}_i(\Delta t)y_i})$. According to the Feynman-Kac formula [7], $g_i, i = 1, \cdots, n$, can be obtained by solving the following PDE,

$$\frac{\partial g_i}{\partial \tau} = \frac{1}{2}b_i^{*2}\frac{\partial g_i^2}{\partial y_i^2} + \alpha_i^*(m_i - y_i)\frac{\partial g_i}{\partial y}, \tag{6.46}$$

$$g_i(y_i, \Delta t) = e^{\tilde{D}_i(\Delta t)y_i}, \Delta t < \tau < T. \tag{6.47}$$

Similarly, we assume that the solution of (6.46) has an affine form of

$$g_i = e^{L_i + H_i y_i}. \tag{6.48}$$

Substituting (6.48) into (6.46), we obtain the following ODEs,

$$\frac{\partial L_i}{\partial t} = \alpha_i^* m_i^* H_i, \tag{6.49}$$

$$\frac{\partial H_i}{\partial t} = -\alpha_i^* + \frac{1}{2}b_i^{*2}H_i^2, \tag{6.50}$$

with the initial condition $L_i(\tilde{D}_i(\Delta t), \Delta t) = 0$ and $H_i(\tilde{D}_i(\Delta t), \Delta t) = \tilde{D}_i(\Delta t)$. By simple derivation, we obtain

$$H_i = \frac{2\alpha_i^*}{b_i^{*2}}\frac{e^{-l\tau}}{e^{-\alpha_i^*\tau} - c_0}, \tag{6.51}$$

$$L_i = \frac{-2\alpha_i^* m_i^*}{b_i^{*2}}ln(1 - \frac{e^{-\alpha_i^*\tau}}{c_0}), \tag{6.52}$$

with $c_0 = 1 - \frac{2\alpha_i^*}{b_i^{*2}\tilde{D}_i(\Delta t)}$. Similarly, $h(w, \Delta t), w = -j, -2j$ can be calculated through the assumption

$$h = e^{M + Nr}, \tag{6.53}$$

which can be resolved trough the following PDE

$$\frac{\partial h}{\partial \tau} = \frac{1}{2}\eta^{*2}\frac{\partial^2 h}{\partial r^2} - \left\{k^*(\theta^* - r) - B(T - \tau, 0)\eta^2 r\right\}\frac{\partial h}{\partial r}, \tag{6.54}$$

$$h(w, \Delta t, r) = e^{D(w, \Delta t)}r, \Delta t < \tau < T, \tag{6.55}$$

when $w = -j$, $h = \hat{h}$, when $w = -2j$, $h = \tilde{h}$.

By some derivation, we obtain

$$M = \frac{2k^*}{\eta^{*2}}\frac{e^{-l\tau}}{e^{-\theta^*\tau} - c_1}, \tag{6.56}$$

$$L = \frac{-2\theta^* m^*}{\eta^{*2}}ln(1 - \frac{e^{-m^*\tau}}{c_1}), \tag{6.57}$$

with $c_1 = 1 - \frac{2\theta^*}{\eta^{*2}D(w,\Delta t)}$, and C satisfies

$$
\begin{aligned}
C_\tau \quad &= \sum_{i=1}^n \alpha_i^* m_i^* D_i + k^* \theta^* E + \lambda(1 - E(e^x))(jw) - \lambda + \lambda\phi_\eta(w) \\
&+ \quad \sum_{i=1}^n \sum_{j\neq i} E_0^T \left\{ -\sqrt{y_i y_j}(jw + \tfrac{1}{2}w^2) + b_i \rho_{sy_i} \sqrt{y_i y_j}(jw)D_i \right\} \\
&= \sum_{i=1}^n \alpha_i^* m_i^* D_i + k^* \theta^* E + \lambda(1 - E(e^x))(jw) - \lambda + \lambda\phi_\eta(w) \\
&+ \quad \sum_{i=1}^n \sum_{j\neq i} \left\{ -(jw + \tfrac{1}{2}w^2)E_0^T(\sqrt{y_i y_j}) + (jw)b_i \rho_{sy_i} E_0^T(\sqrt{y_i y_j}D_i) \right\} \quad (6.58)
\end{aligned}
$$

Based on the independence property, we obtain

$$
E(\sqrt{y_i y_j}) = E(\sqrt{y_i})E(\sqrt{y_j}) \tag{6.59}
$$

with $E(\sqrt{y_i(t)})$ determined by the following form

$$
E(\sqrt{y_i(t)}) = c_i(d_i + \lambda_i - 1) + \frac{c_i d_i}{2(d_i + \lambda_i)}, \tag{6.60}
$$

with $c_i(t) = \frac{\delta_i^2}{4k_i^*}(1 - e^{-k_i^* t})$, $d_i(t) = \frac{4k^* a_i}{\delta^{*2}}$, $\lambda_i(t) = \frac{4a_i y_0}{\delta_i^2} \frac{e^{-k_i^* t}}{1 - e^{-k_i^* t}}$. Thus, the ODE (6.58) can be solved numerically by using Matlab.

6.4 Numerical Result of One Factor Model

If $\lambda = 0$, $i = 0$, our model reduces to the basic model studied in Little&Pant [63]. If $\lambda = 0$, $i = 1$, it reduces to the one factor stochastic volatility model in [19]. However, compared to the aforementioned work, our model is more general and realistic by considering both the jump process and the stochastic interest rate. Also, our stochastic volatility can be extended to a multifactor case. As illustrated in Heston [78], one factor stochastic models are cannot capture the phenomenon of option smirk, and at least two factors are needed for a more realistic model. In this section, we will compare the numerical results obtained respectively by the one-factor stochastic model and the two-factors stochastic volatility model.

6.4.1 Study of Stochastic Interest Rate Effects

Let $\lambda = 0$, and $i = 1$, our model reduces to a Heston-CIR model. The parameters of the stochastic interest rate process and the stochastic volatility process are presented in table 6.2, which satisfy Feller's condition. The correlation of the stochastic volatility process and the stock process is assumed to be -0.4. Figure 6.1 shows the fair strike price with different long term interest rate θ, and figure

6.2 shows the fair strike price for different volatility of the interest process. It is noted that the effects of stochastic interest rate is very small.

Table 6.2: Calibrated Parameters

k	θ	σ_r	r_0
1.2	0.05	0.01	0
a_1	m_1	b_1	y_0
2	0.05	0.05	0
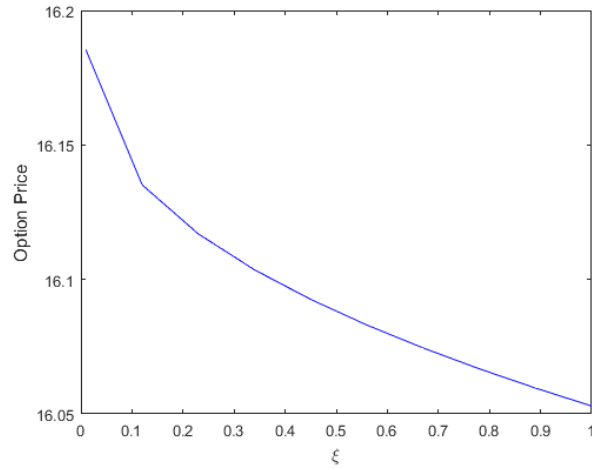
Figure 6.1: The Effects of Long-term Interest Rate
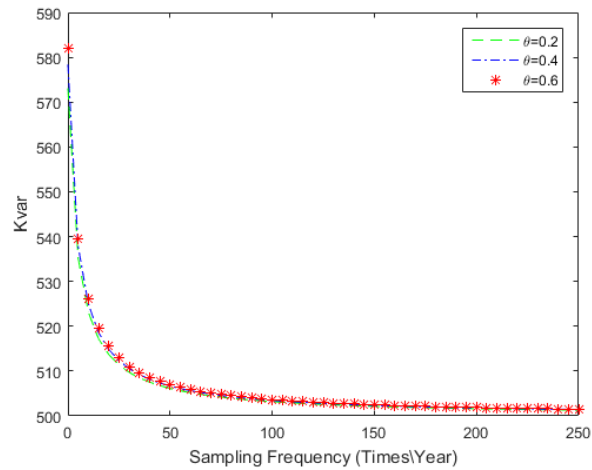
Figure 6.2: The Effects of Volatility of CIR Process

6.4.2 Effects of Jumps on Fair Strike Price

In this subsection, λ is chosen from a range of numbers. We study both the double exponential jump and the Merton jump, and the parameters related to the jump processes are presented in Table 6.3. The results for different λ values are compared in Figure 6.3 and Figure 6.4, from which we know that the jump rate has a significant effect on the fair strike price. No matter which distribution we choose, the fair strike price increases dramatically with the jump rate. This result is in line with the result in [94].

Table 6.3: Calibrated Parameters of Jump Diffusion Process

Kou	λ_1	λ_2	p
	40	12	0.3
$Merton$	μ	δ	
	-0.05	0.086	
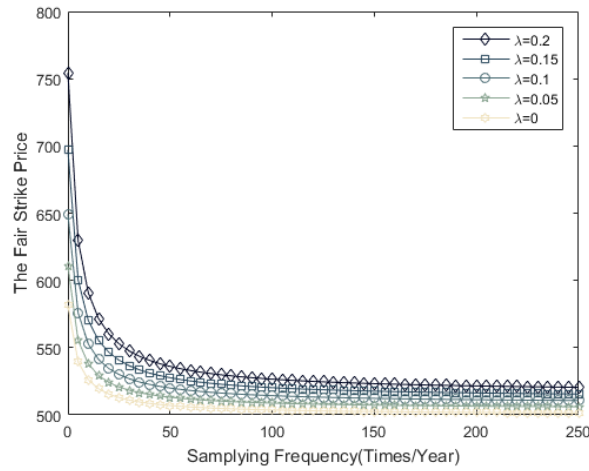
Figure 6.3: The Effects of Jump Intensity with Double Exponential Distribution

6.5 Numerical Result of Multifactor-CIR Model

In this subsection, assume that $i = 2$ and $\lambda = 0$, then our model reduces to a two-factor-CIR model. We compare our numerical result with the result of the MC simulation with 200,000 paths and the initial value of the stock price is assumed to be 1. The stochastic process is discretized by the Euler-Maruyama scheme, and the parameters are selected as in table 6.4: $b_1 = 0.1500$, $a_1 = 0.0059$, $\sigma_1 = 1.9829$, $\rho_1 = -0.9902$, $b_2 = 0.2335$, $a_2 = 0.1621$, $\sigma_2 = 0.1971$, $\rho_2 = -0.8918$. The red
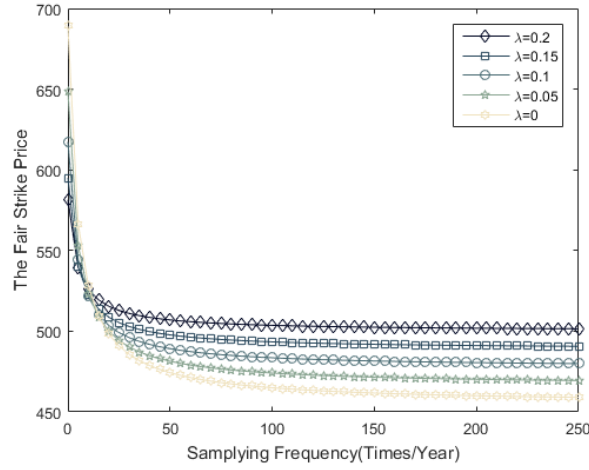
Figure 6.4: The Effects of Jump Intensity with Normal Distribution
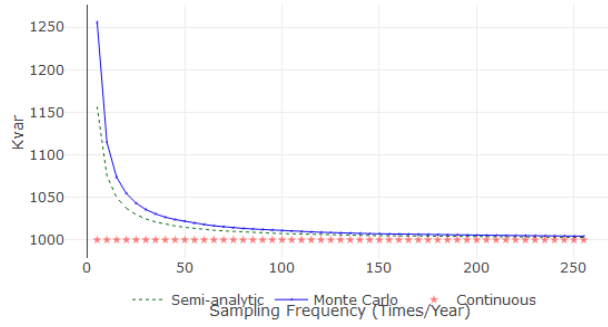
Figure 6.5: Comparison of Three Methods

horizontal line of figure 6.5 is calculated by the continuous approximation of $Kvar$ as in [94]. For a more realistic situation, we absorb the jump into the stock process, and apply the parameters calibrated by Heston et. al(Table6.4). [78].

We assume that the stochastic volatility process is independent of each other. According to L'Hopital rule, for the reason that $\tilde{C}(\Delta t) = 0, \tilde{\mathbf{D}}(\Delta t) = 0, \tilde{E}(\Delta t) = 0, \hat{C}(\Delta t) = 0, \hat{E}(\Delta t) = 0$, we obtain

$$lim_{\Delta t \to 0} \frac{exp(C(\tilde{\Delta}t) + \tilde{\mathbf{D}}(\Delta t)^T \mathbf{y} + \tilde{E}(\Delta t)r) - 2exp(\hat{C}(\Delta t) + \hat{E}(\Delta t)r) + 1}{\Delta t}$$

$$= lim_{\Delta t \to 0} \left(C'(\tilde{\Delta}t) + \tilde{\mathbf{D}}'(\Delta t)\mathbf{y} + \tilde{E}'(\Delta t)r - 2(\hat{C}'(\Delta t) + \hat{E}'(\Delta t)r) \right)$$

$$= lim_{\Delta t \to 0} \left(\tilde{C}'(\Delta t) - 2\hat{C}'(\Delta t) \right) + lim_{\Delta t \to 0} \left(\tilde{E}'(\Delta t) - 2\hat{E}'(\Delta t) \right) + lim_{\Delta t \to 0} \tilde{\mathbf{D}}'(\Delta t)\mathbf{y} \quad (6.61)$$

From the Taylor expansion and the Merton jump assumption, we can verify that

$$
\begin{aligned}
lim_{\Delta t \to 0}\left(\tilde{C}'(\Delta t) - 2\hat{C}'(\Delta t)\right) &= \lambda + \lambda(E(e^{2x}) - E(e^x)) \\
&= \lambda + \lambda(-1 + E(X^2)) \\
&= \lambda(E^2(x) + Var(x)) \\
&= \lambda(\mu^2 + \delta^2)
\end{aligned}
\tag{6.62}
$$

and

$$
lim_{\Delta t \to 0}\left(\tilde{E}'(\Delta t) - 2\hat{E}'(\Delta t)\right) = -2j * j - 2 * (-j) * j = 0,
\tag{6.63}
$$

According to the expression of D_i, we obtain

$$
lim_{\Delta t \to 0}\tilde{\mathbf{D}}'(\Delta t)\mathbf{y} = \mathbf{y},
\tag{6.64}
$$

Thus, due to the property of the variance process, the limit (6.61) is equivalent to $\lambda(\mu^2 + \sigma^2) + \mathbf{y}$. For a continuous case, we have

$$
\begin{aligned}
Kvar = lim_{n \to \infty}\frac{AF}{N}\sum_{i=1}^{i=n} f_i &= lim_{\Delta t \to 0}\frac{1}{T}\sum_{i=1}^{i=n}\frac{1}{\Delta t} * \Delta t * f_i \\
&= \frac{1}{T}\int_0^T (\lambda(\mu^2 + \delta^2) + E(y_t))dt \\
&= \frac{1}{T}\int_0^T \left(\lambda(\mu^2 + \sigma^2) + \sum_{j=1}^{m}(y_j e^{-a_j t} + m_j(1 - e^{-a_j(i-1)t}))\right) dt \\
&= \lambda(\mu^2 + \sigma^2) + \sum_{j=1}^{m}(y_j\frac{1 - e^{a_j T}}{a_j T} + m_j(1 - \frac{1 - e^{a_j T}}{a_j T})).
\end{aligned}
\tag{6.65}
$$

This result is in line with the result in Brodie et. al. [94], but proved by the result of Fourier transform. The above result is obtained by the assumption of the Merton jump. If the underlying process is a double exponential process instead of a normal distribution, we derive that

$$
Kvar = \lambda\left((\frac{p}{\lambda_1^2} + \frac{q}{\lambda_2^2})(p + q + 1) - \frac{4pq}{\eta_1\eta_2}\right) + \sum_{j=1}^{m}(y_j\frac{1 - e^{a_j T}}{a_j T} + m_j(1 - \frac{1 - e^{a_j T}}{a_j T})).
\tag{6.66}
$$

The continuous fair strike price with Merton jump satisfies equation (6.67). We then compare the result with $\lambda = 0$ and $\lambda = 0.1$ in Figure 6.6. It is noted that

the inclusion of the jump diffusion process shifts the fair strike price up discretely and continuously.

Table 6.4: Calibrated Parameters

b_1	a_1	σ_1	ρ_1
0.1500	0.0059	1.9829	-0.9902
b_2	a_2	σ_2	ρ_2
0.2335	0.1621	0.1971	-0.8918

$$Kvar = \lambda(\mu^2 + \sigma^2) + \sum_{j=1}^{m}(y_j\frac{1 - e^{a_jT}}{a_jT} + m_j(1 - \frac{1 - e^{a_jT}}{a_jT})), \qquad (6.67)$$
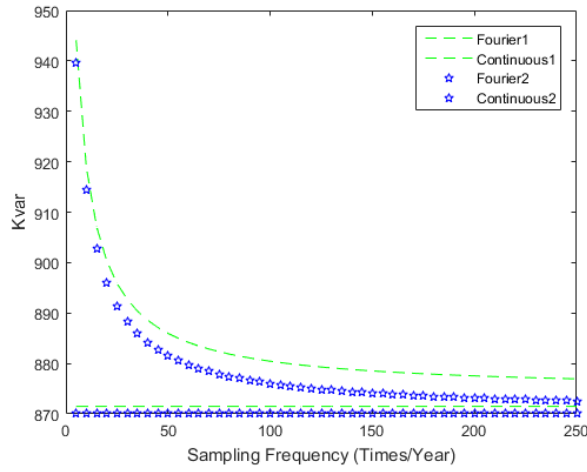
Figure 6.6: Comparison of Jump Effects in Multi-factor-CIR Model

6.6 Concluding Remark

In this chapter, we study the pricing of variance swaps in a generalized hybrid financial model. A semi-analytic solution is obtained by using the generalized Fourier transform. With proper selection of parameters, our model includes various existing financial models as special cases, including the CIR model, the hybrid Heston-CIR model, the multi-factor-CIR model and the jump model. Monte Carlo simulation and approximation of continuous strike price is set as a benchmark of our numerical results, and we find that our model is more accurate and efficient with the discrete sampled variance swap. The effects of stochastic interest rare and the effect of jump are also studied in our work, We find that

compared to the stochastic interest rate, the effect of jump is more significant, and this result is in line with Brodie et al. 2008 [94], in which the jump effects are studied via probability technique with continuous sample assumption. In this work, we deal with the integral term arising from the jump diffusion process with the characteristic function via Fourier convolution.

CHAPTER 7

Summary and Future Research

7.1   Summary

In this thesis, we study various financial derivatives pricing problem via establishing a generalized hybrid multi-factor stochastic volatility model. Our generalized model takes into account of stochastic interest rate, multi-factor stochastic volatility rate and jump diffusion process, and include various existing models as special cases. Our hybrid model results in a high-dimensional partial differential equation under risk-neutral assumption. Various approaches have then been applied to get the approximate solutions. Various results has been obtained and summarized as below.

(i) The finite element method has been applied to obtain the approximate solution of classical European option and the fair strike price of variance swaps under both multi-scale stochastic volatility and jump diffusion process. The time scale rate of stochastic volatility is used to describe the long term and short term perturbation of volatility process. Consequently, the option price increases with the jump rate, while the effects of multi-scale volatility is a combination result of both fast scale volatility and slow scale volatility. The option price increases with the fast-scale rate and decreases with the slow scale rate. The effect of slow scale volatility outweighs the effect of fast scale volatility in a long run. Also, the strike price of variance swap is anti-correlated with the maturity time. The significance of the numerical approach is mainly in two aspects. Firstly, the work establishes a generalized hybrid model, which takes account of stochastic interest rate, multi-factor stochastic volatility rate and jump diffusion process. Comparison has been made between our hybrid model and existing models. Our hybrid model results in a high-dimensional partial differential equation under risk-neutral assumption. Various approaches have been applied to get the approximate

solution. Exact solutions have been obtained for some models. Secondly, even though most existing literatures have already considered the stochastic volatility, multi-factors in volatility have not yet been tackled due to the high dimensional difficulties, while we combine both multi-scale rate and jump process in our work to make the result more reliable.

(ii) In addition to evaluating the traditional European option, the hybrid model can also be applied to study the portfolio selection and optimization problems. The single-name bond under the stochastic intensity and the stochastic volatility has been studied in this thesis. The non-linear PDE arising from the optimal problem has been studied by the method of asymptotic approximation. The approximated solution has been decomposed into the leading term $V^{(0)}$, and the fast-scale modification term $V^{(1)}$. Consequently, we find that consideration of the fast-scale volatility will lower down the utility and increases the certainty equivalents. We prove that the yield spread of the investor goes up with the maturity time and converges to a long time level.

(iii) Besides the traditional option pricing problem, the pricing of variance swap in our generalized hybrid financial model has been studied by the generalized Fourier transform. With proper selection of parameters, our hybrid model reduces to the CIR model, the hybrid Heston-CIR model, multi-factor-CIR model. The jump diffusion process is also considered in our research, and the integral term arising from the jump diffusion process has been solved via Fourier convolution. The effects of stochastic interest rate and the effect of jump have also been studied in our work. We find that the effects the jump term and stochastic volatility is vital, in comparison with the stochastic interest rate. It is also shown that our numerical result converges to continuous model of variance swap.

In brief, in this thesis, a hybrid model is studied and applied to study both the derivative pricing and credit risk optimization. The study is significant because the hybrid model is more realistic and applicable to many other financial area. However, the consideration of more factors will add the complexity of the problem. In this research, multiple approaches are applied and the results are compared to show the accuracy and efficiency of the problem.

7.2   Future research directions

In this thesis, our main objective is to develop a hybrid financial model for option pricing. It is observed that our model is more effective and realistic. However, further improvements could be made as detailed below.

(i) In spite of the option pricing problem and credit risk optimization problem, our model can also been applied to other portfolio optimization problems, such as the time inconsistent mean-variance problem;

(ii) For the reason that the real financial markets is dynamic rather than static. More accurate and practical models such as regime switching model should also been taken into consideration in our future research;

(iii) Fractional Brownian motion will be utilized in our future research by taking the long-memory and short-memory effects into consideration;

(iv) The model calibration of our hybrid model and the effects of the multi-factor and jump term on implied volatility should also been taken into consideration in our future research.

Bibliography

[1] Paul H Cootner. The random character of stock market prices. 1964.

[2] John C Cox, Stephen A Ross, and Mark Rubinstein. Option pricing: A simplified approach. *Journal of financial Economics*, 7(3):229–263, 1979.

[3] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The journal of political economy*, pages 637–654, 1973.

[4] Phelim P Boyle. Options: A monte carlo approach. *Journal of financial economics*, 4(3):323–338, 1977.

[5] Mark Broadie and Paul Glasserman. Estimating security price derivatives using simulation. *Management science*, 42(2):269–285, 1996.

[6] Francis A Longstaff and Eduardo S Schwartz. Valuing american options by simulation: a simple least-squares approach. *The review of financial studies*, 14(1):113–147, 2001.

[7] Steven E Shreve. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer Science & Business Media, 2004.

[8] John Hull and Alan White. Pricing interest-rate-derivative securities. *The Review of Financial Studies*, 3(4):573–592, 1990.

[9] John C Cox, Jonathan E Ingersoll Jr, and Stephen A Ross. A theory of the term structure of interest rates. In *Theory of Valuation*, pages 129–164. World Scientific, 2005.

[10] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343, 1993.

[11] Elias M Stein and Jeremy C Stein. Stock price distributions with stochastic volatility: an analytic approach. *The review of financial studies*, 4(4):727–752, 1991.

[12] Steven L Heston. A simple new formula for options with stochastic volatility. 1997.

[13] Martino Grasselli. The 4/2 stochastic volatility model: a unified approach for the heston and the 3/2 model. *Mathematical Finance*, 27(4):1013–1034, 2017.

[14] Robert C Merton. Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, 3(1-2):125–144, 1976.

[15] Steven G Kou. A jump-diffusion model for option pricing. *Management science*, 48(8):1086–1101, 2002.

[16] Kresimir Demeterfi, Emanuel Derman, Michael Kamal, and Joseph Zou. More than you ever wanted to know about volatility swaps. *Goldman Sachs quantitative strategies research notes*, 41, 1999.

[17] H Windcliff, PA Forsyth, and KR Vetzal. Pricing methods and hedging strategies for volatility derivatives. *Journal of Banking & Finance*, 30(2):409–431, 2006.

[18] Anatoliy Swishchuk. Modeling of variance and volatility swaps for financial markets with stochastic volatilities. *WILMOTT magazine*, 2:64–72, 2004.

[19] Song-Ping Zhu and Guang-Hua Lian. A closed form exact solution for pricing variance swaps with stochastic volatility. *Mathematical Finance*, 21(2):233–256, 2011.

[20] Jiling Cao, Guanghua Lian, and Teh Raihana Nazirah Roslan. Pricing variance swaps under stochastic volatility and stochastic interest rate. *Applied Mathematics and Computation*, 277:72–81, 2016.

[21] John C Cox, Jonathan E Ingersoll Jr, and Stephen A Ross. A theory of the term structure of interest rates. *Econometrica: Journal of the Econometric Society*, pages 385–407, 1985.

[22] Malik Magdon-Ismail. The equivalent martingale measure: an introduction to pricing using expectations. *IEEE Transactions on Neural Networks*, 12(4):684–693, 2001.

[23] Helyette Geman, Nicole El Karoui, and Jean-Charles Rochet. Changes of numeraire, changes of probability measure and option pricing. *Journal of Applied probability*, 32(2):443–458, 1995.

[24] Igor Vladimirovich Girsanov. On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory of Probability & Its Applications*, 5(3):285–301, 1960.

[25] AA Markov. Rasprostranenie zakona bolshih chisel na velichiny, zavisyaschie drug ot druga, izvestiya fiziko-matematicheskogo obschestva pri kazanskom universitete, 2-ya seriya 15, 135–156. *Russian. English translation:Extension of the law of large numbers to dependent quantities*, 1906.

[26] Pierre Del Moral. Feynman-kac formulae. In *Feynman-Kac Formulae*, pages 47–93. Springer, 2004.

[27] Mark Rubinstein. Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active cboe option classes from august 23, 1976 through august 31, 1978. *The Journal of Finance*, 40(2):455–480, 1985.

[28] Jens Carsten Jackwerth and Mark Rubinstein. Recovering probability distributions from option prices. *The Journal of Finance*, 51(5):1611–1631, 1996.

[29] Bruno Dupire. Pricing and hedging with smiles. *Mathematics of derivative securities*, 1(1):103–111, 1997.

[30] John C Cox and Stephen A Ross. The valuation of options for alternative stochastic processes. *Journal of financial economics*, 3(1-2):145–166, 1976.

[31] Helyette Geman and Yih Fong Shih. Modeling commodity prices under the cev model. *The Journal of Alternative Investments*, 11(3):65–84, 2009.

[32] John Hull and Alan White. The pricing of options on assets with stochastic volatilities. *The journal of finance*, 42(2):281–300, 1987.

[33] Louis O Scott. Option pricing when the variance changes randomly: Theory, estimation, and an application. *Journal of Financial and Quantitative analysis*, 22(4):419–438, 1987.

[34] Kenneth R French, G William Schwert, and Robert F Stambaugh. Expected stock returns and volatility. *Journal of financial Economics*, 19(1):3–29, 1987.

[35] G William Schwert. Why does stock market volatility change over time? *The journal of finance*, 44(5):1115–1153, 1989.

[36] John Y Campbell, Sanford J Grossman, and Jiang Wang. Trading volume and serial correlation in stock returns. *The Quarterly Journal of Economics*, 108(4):905–939, 1993.

[37] George Chacko and Luis M Viceira. Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *Review of Financial Studies*, 18(4):1369–1402, 2005.

[38] Nigel Clarke and Kevin Parrott. Multigrid for american option pricing with stochastic volatility. *Applied Mathematical Finance*, 6(3):177–195, 1999.

[39] Jean-François Muzy, Jean Delour, and Emmanuel Bacry. Modelling fluctuations of financial time series: from cascade process to stochastic volatility model. *The European Physical Journal B-Condensed Matter and Complex Systems*, 17(3):537–548, 2000.

[40] Jean-Pierre Fouque and Xianwen Zhou. Perturbed gaussian copula. *Advances in Econometrics*, 22:103–121, 2008.

[41] Dilip B Madan and Eugene Seneta. Chebyshev polynomial approximations and characteristic function estimation. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 163–169, 1987.

[42] Dilip B Madan and Eugene Seneta. The variance gamma (vg) model for share market returns. *Journal of business*, pages 511–524, 1990.

[43] Ernst Eberlein, Ulrich Keller, et al. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.

[44] Ole E Barndorff-Nielsen. Normal inverse gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of statistics*, 24(1):1–13, 1997.

[45] Darrell Duffie, Jun Pan, and Kenneth Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376, 2000.

[46] E Pillay and JG OHara. Fft based option pricing under a mean reverting process with stochastic volatility and jumps. *Journal of Computational and Applied Mathematics*, 235(12):3378–3384, 2011.

[47] J Andreasen. Closed form pricing of fx options under stochastic rates and volatility. In *Presentation at Global Derivatives Conference*, 2006.

[48] Alexander Giese. On the pricing of auto-callable equity securities in the presence of stochastic volatility and stochastic interest rates. In *Presentation at MathFinance Workshop: Derivatives and Risk Management in Theory and Practice, Frankfurt*, 2004.

[49] Lech A Grzelak and Cornelis W Oosterlee. On the heston model with stochastic interest rates. *SIAM Journal on Financial Mathematics*, 2(1):255–286, 2011.

[50] Ronald Newbold Bracewell and Ronald N Bracewell. *The Fourier transform and its applications*, volume 31999. McGraw-Hill New York, 1986.

[51] Henri Poincaré. *Calcul des probabilités*. Gauthier-Villars, 1912.

[52] J Michael Harrison and David M Kreps. Martingales and arbitrage in multi-period securities markets. *Journal of Economic theory*, 20(3):381–408, 1979.

[53] Gurdip Bakshi and Dilip Madan. Spanning and derivative-security valuation. *Journal of financial economics*, 55(2):205–238, 2000.

[54] Peter Carr and Dilip Madan. Option valuation using the fast fourier transform. *Journal of computational finance*, 2(4):61–73, 1999.

[55] Peter Carr and Liuren Wu. The finite moment log stable process and option pricing. *The journal of finance*, 58(2):753–777, 2003.

[56] Daniel J Duffy. *Finite Difference methods in financial engineering: a Partial Differential Equation approach*. John Wiley & Sons, 2013.

[57] John Hull and Alan White. An analysis of the bias in option pricing caused by a stochastic volatility. *Advances in futures and options research*, 3(1):29–61, 1988.

[58] John Hull and Alan White. Valuing derivative securities using the explicit finite difference method. *Journal of Financial and Quantitative Analysis*, 25(1):87–100, 1990.

[59] Rama Cont and Ekaterina Voltchkova. A finite difference scheme for option pricing in jump diffusion and exponential lévy models. *SIAM Journal on Numerical Analysis*, 43(4):1596–1626, 2005.

[60] Lixin Wu and Yue-Kuen Kwok. A front-fixing finite difference method for the valuation of american options. *Journal of Financial Engineering*, 6(4):83–97, 1997.

[61] Samuli Ikonen and Jari Toivanen. Operator splitting methods for american option pricing. *Applied mathematics letters*, 17(7):809–814, 2004.

[62] Leif Andersen and Jesper Andreasen. Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing. *Review of Derivatives Research*, 4(3):231–262, 2000.

[63] Thomas Little and Vijay Pant. A finite-difference method for the valuation of variance swaps. In *Quantitative Analysis In Financial Markets: Collected Papers of the New York University Mathematical Finance Seminar (Volume III)*, pages 275–295, 2001.

[64] Yves Achdou and Olivier Pironneau. Finite element methods for option pricing. *Université Pierre et Marie Curie*, 2007.

[65] Pavlo Kovalov, Vadim Linetsky, and Michael Marcozzi. Pricing multi-asset american options: A finite element method-of-lines with smooth penalty. *Journal of Scientific Computing*, 33(3):209–237, 2007.

[66] Song Wang. A novel fitted finite volume method for the black–scholes equation governing option pricing. *IMA Journal of Numerical Analysis*, 24(4):699–720, 2004.

[67] Lutz Angermann and Song Wang. Convergence of a fitted finite volume method for the penalized black–scholes equation governing european and american option pricing. *Numerische Mathematik*, 106(1):1–40, 2007.

[68] Jean-Pierre Fouque, George Papanicolaou, and K Ronnie Sircar. Asymptotics of a two-scale stochastic volatility model. *Equations aux derivees partielles et applications, in honour of Jacques-Louis Lions*, pages 517–525, 1998.

[69] Peter Cotton, Jean-Pierre Fouque, George Papanicolaou, and Ronnie Sircar. Stochastic volatility corrections for interest rate derivatives. *Mathematical Finance*, 14(2):173–200, 2004.

[70] Jean-Pierre Fouque, George Papanicolaou, and K Ronnie Sircar. From the implied volatility skew to a robust correction to black-scholes american option prices. *International Journal of Theoretical and Applied Finance*, 4(04):651–675, 2001.

[71] Eric Fournie, Jérôme Lebuchoux, and Nizar Touzi. Small noise expansion and importance sampling. *Asymptotic Analysis*, 14(4):361–376, 1997.

[72] K Ronnie Sircar, George C Papanicolaou, et al. Stochastic volatility, smile & asymptotics. *Applied Mathematical Finance*, 6:107–145, 1999.

[73] Jean-Pierre Fouque, George Papanicolaou, Ronnie Sircar, and Knut Solna. Multiscale stochastic volatility asymptotics. *Multiscale Modeling & Simulation*, 2(1):22–42, 2003.

[74] Jean-Pierre Fouque, Ronnie Sircar, and Knut Sølna. Multiname and multi-scale default modeling. *Multiscale Modeling & Simulation*, 7(4):1956–1978, 2009.

[75] Jean-Pierre Fouque and Chuan-Hsiang Han. Asian options under multiscale stochastic volatility. *Contemporary Mathematics*, 351:125–138, 2004.

[76] Jean-Pierre Fouque, Matthew Lorig, and Ronnie Sircar. Second order multiscale stochastic volatility asymptotics: stochastic terminal layer analysis and calibration. *Finance and Stochastics*, 20(3):543–588, 2016.

[77] Jean-Pierre Fouque, George Papanicolaou, Ronnie Sircar, and Knut Solna. *Multiscale stochastic volatility for equity, interest rate, and credit derivatives.* Cambridge University Press, 2011.

[78] Peter Christoffersen, Steven Heston, and Kris Jacobs. The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. *Management Science*, 55(12):1914–1932, 2009.

[79] Desmond J Higham. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM review*, 43(3):525–546, 2001.

[80] J-P Fouque and Yuri F Saporito. Heston stochastic vol-of-vol model for joint calibration of vix and s&p 500 options. *Quantitative Finance*, pages 1–14, 2018.

[81] Robert C Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory*, 3(4):373–413, 1971.

[82] Francis A Longstaff and Eduardo S Schwartz. A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3):789–819, 1995.

[83] Jean-Pierre Fouque, Ronnie Sircar, and Thaleia Zariphopoulou. Portfolio optimization and stochastic volatility asymptotics. *Mathematical Finance*, 2015.

[84] MR Grasselli and TR Hurd*. Indifference pricing and hedging for volatility derivatives. *Applied Mathematical Finance*, 14(4):303–317, 2007.

[85] Mark Davis and Daisuke Yoshikawa. An equilibrium approach to indifference pricing. *Available at SSRN 1568856*, 2012.

[86] Evan Papageorgiou and Ronnie Sircar. Multiscale intensity models for single name credit derivatives. *Applied Mathematical Finance*, 15(1):73–105, 2008.

[87] Darrell Duffie and Thaleia Zariphopoulou. Optimal investment with undiversifiable income risk. *Mathematical Finance*, 3(2):135–148, 1993.

[88] Ronnie Sircar and Thaleia Zariphopoulou. Utility valuation of credit derivatives: Single and two-name cases. pages 279–301, 2007.

[89] Ronnie Sircar and Thaleia Zariphopoulou. Utility valuation of multi-name credit derivatives and application to cdos. *Quantitative Finance*, 10(2):195–208, 2010.

[90] Pierre Brémaud. Point processes and queues. 1981.

[91] George Papanicolaou, Jean-Pierre Fouque, Knut Solna, and Ronnie Sircar. Singular perturbations in option pricing. *SIAM Journal on Applied Mathematics*, 63(5):1648–1665, 2003.

[92] Peter Carr and Dilip Madan. Towards a theory of volatility trading. *Volatility: New estimation techniques for pricing derivatives*, (29):417–427, 1998.

[93] Ian Martin. Simple variance swaps. Technical report, National Bureau of Economic Research, 2011.

[94] Mark Broadie and Ashish Jain. The effect of jumps and discrete sampling on volatility and variance swaps. *International Journal of Theoretical and Applied Finance*, 11(08):761–797, 2008.

[95] Bin Chen, Lech Grzelak, and Cornelis Oosterlee. Calibration and monte carlo pricing of the sabr-hull-white model for long-maturity equity derivatives. 2011.

[96] Carl Chiarella and Boda Kang. The evaluation of american compound option prices under stochastic volatility and stochastic interest rates. *Journal of Computational Finance*, 2013.

[97] Tinne Haentjens and Karel J Hout. Adi finite difference schemes for the heston-hull-white pde. *arXiv preprint arXiv:1111.4087*, 2011.

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Appendix 1. Statement of Candidate's Contributions to Joint-Authored published

To Whom It May Concern,

I, Shican Liu, made major contributions in the design of the research work, development of theories, analysis of results drafting of the paper entitled 'The Study of Utility Valuation of Single-Name Credit Derivatives with the Fast-Scale Stochastic Volatility Correction', Sustainability, 10(4):1027, DOI: 10.3390/su10041027

Shican Liu   _Shican Liu_

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Yanli Zhou

Benchawan Wiwatanapataphee

YongHong Wu

Xiangyu Ge

Appendix2. Statement of Candidate's Contributions to Joined Authored Published

To whom it may concern,

I, Shican Liu, made the major contributions in the design of the research work, development of theories, analysis of results drafting of the paper *entitled 'The Study on the Pricing of Credit Risk under the Fast Stochastic volatility, International Conference in Identification, Information and knowledge in the internet of things, 2016.'*

Shican Liu   *Shican Liu*

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Xiangyu Ge

Benchawan Wiwatanapataphee

Yonghong Wu

Appendix3. Statement of Candidate's Contributions to Joined Authored Published

To whom it may concern,

I, Shican Liu, made the major contributions in the design of the research work, development of theories, analysis of results drafting of the paper entitled *'The Simulation of Stochastic Variance Swap, proceeding of ICDSBA, 2017'*

Shican Liu   *Shican Liu*

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